Harmony and Modality

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ABSTRACT. It is argued that the meaning of the modal connectives must be given inferentially, by the rules for the assertion of formulae containing them, and not semantically by reference to possible worlds. Further, harmony confers transparency on the inferentialist account of meaning, when the introduction-rule specifies both necessary and sufficient conditions for assertion, and the elimination-rule does no more than exhibit the consequences of the meaning so conferred. Hence, harmony is not to be identified with normalization, since the standard modal natural deduction rules, though normalizable, are not in this sense harmonious. Harmonious rules for modality have lately been formulated, using labelled deductive systems.

Keywords Inferentialism, Necessity, Possibility, Gentzen, Prawitz, Meaning, Labelled Deductive Systems

Shahid Rahman argues in favour of logical pluralism, which I oppose.[1] But what he really favours is an eclecticism of logics which enables different logics to be formulated and compared. That is a project I endorse. It is important to be able to identify the commonality between logics, so that one can discern where they disagree. Only then can one apply appropriate criteria to decide which logic is right. That means that we must distinguish the meanings of the logical connectives from the logical principles which they satisfy. If, as the inferentialist believes, those meanings are given by the rules of inference that may look difficult, if not impossible. The solution lies in distinguishing the rules specific to specific connectives, the operational inferences, from the generic rules, the framework within which the operators work.

[1]See, e.g., [16] § 4; [20].

1 Inferentialism

Inferentialism about the logical connectives is the claim that their meanings are given by the rules for their use. Robert Brandom places it in opposition to representationalism, the claim that "inferential relations [derive] from the contents of representings" ([5] p. 46). Arthur Prior [15] gave a famous argument to show that inferentialism must be mistaken. Rather, he said, we must first give the meanings of the connectives independently of their use in inference, for only then can we decide whether the rules are valid or not. We cannot just give rules for the connectives, trusting to faith that they are correct and hope that they will define a meaning.

J.T. Stevenson [23] interpreted Prior as demanding a truth-functional, or more generally, value-functional, explanation of the meaning of each connective. This is clearly unreasonable, as there are many non-truth-functional connectives, forming complex propositions whose truth-value is not determined by any finite set of "truth"-values of their constituents. What one might seek is a possible-worlds semantics for the connectives. Indeed, Richard Routley [21] showed that every logic has a two-valued worlds semantics, so justifying the search for a semantic articulation of the connectives in terms of an indexed map from the proposition's parts to the whole. For each n-place connective, a Routley frame consists of a set of worlds and an $n+1$-place "accessibility" or "relative possibility" relation, and the connective receives either a universal or an existential truth-condition: for example, for the case of the 1-place necessity ($\Box$) and possibility ($\Diamond$) connectives, the familiar conditions:

- $v(\Box\alpha, w) = \mathrm{T}$ if $v(\alpha, w') = \mathrm{T}$ for all w' such that Rww';
- $v(\Diamond\alpha, w) = \mathrm{T}$ if $v(\alpha, w') = \mathrm{T}$ for some w' such that Rww'.

Unless one is a modal realist, however, one might worry whether such a world-relative account of the connectives is really independent of their use in inference. The valid rules are all and only those which hold in the canonical model. Consider the canonical model of a modal logic, **S4**, say. The "worlds" of the model are simply prime filters in the Lindenbaum algebra of **S4**. But the Lindenbaum algebra is obtained by dividing the set of formulae into equivalence classes of provably equivalent formulae. And what determines whether formulae are provably equivalent in **S4** are the inference rules of **S4**. So the notion of canonical model is not independent of the specification of the inference rules. Any model will justify a corresponding set of rules.

If one believes in the real existence of possible worlds, one can rebut this scepticism: the canonical model is correct if it describes the possible worlds correctly. If not, the rules to which it is correlated must be wrong. But if there is no independent reality of possible worlds, the canonical model simply is the mathematical model cast in the attractive language of "worlds", corresponding to the particular choice of inference rules. There is no assurance of its correctness independent of the correctness of those inference rules, any more than there was with the Lindenbaum algebra itself.

So too for the ultra-reductionist strategy which identifies worlds with maximally consistent sets of sentences. The notion of consistency depends on the logical inferences which are permitted, and so again cannot constitute the grounds or justification of those rules.

Why might one be sceptical about the reality of possible-worlds talk? Take a sentence like 'Caesar killed Brutus'. The modal realist tells us that there is a possible world, indeed, an infinite class of possible worlds in which Caesar killed Brutus. Each of these worlds differs from the actual world, for in the actual world Brutus killed Caesar and not vice versa. Nonetheless, says the realist, those worlds are like our world in consisting of a maximal collection of states of affairs, the members of the infinite class in question having in common that they each contain the state of affairs of Caesar's killing Brutus. Indeed, even such a reductionist as an advocate of the Combinatorial theory of possibility (see, e.g., [1]) will attribute to each possible world in this class a "recombination" of the constituents of the actual state of affairs in which Brutus killed Caesar constituting the state of affairs of Caesar's killing Brutus—turning the tables on his assassin.

But how is such a recombination produced? Caesar did not kill Brutus, so one has to suppose that, having disassembled the fact that Brutus killed Caesar into its components, Brutus, killing and Caesar, we can somehow recombine them into a different state of affairs. We can certainly think them together. But that only gives us the thought that Caesar killed Brutus, not the corresponding state of affairs. Moreover, we do not want to attribute too much reality to the putative state of affairs of Caesar's killing Brutus, or we might find it made the corresponding proposition, 'Caesar killed Brutus', true, rather than simply possible.

Armstrong ([2] p. 118) replies that states of affairs combine their elements by a non-mereological form of composition. But that is just a name for a problem. How is the actual state of affairs of Brutus' killing Caesar composed? What combines its elements, Brutus, killing and Caesar into the state of affairs of Brutus' killing Caesar? Armstrong ([1] p. 42; [2] p. 30) fears what he calls Bradley's regress argument: it cannot be the addition of a further element, such as instantiation, for we already have Brutus, killing, Caesar and instantiation (since Brutus killed Caesar), but we don't have Caesar killing Brutus. All Armstrong can do is invoke the magic of a non-mereological form of composition, the result of a transcendental argument based on the fact that Brutus killed Caesar. Once conceded, it is open to the modal realist or the combinatorialist to invoke it to combine the elements in a non-actual order, and behold: out of the hat comes the non-actual state of affairs of Caesar's killing Brutus.

But the problem was misconceived before the transcendental argument was invoked. For the answer to the composition question is that killing itself unites Brutus and Caesar into the fact. As a matter simply of fact, killing related Brutus and Caesar as subject and object. Relations are relations only in as much as they relate their terms. As Bradley put it rhetorically in a challenge to Russell in *Mind* in 1911:

> "What is the difference between a relation which relates in fact and one which does not relate? ... Is there anything ... in a unity besides its 'constituents', *i.e.* the terms and the relation, and, if there is anything more, in what does this 'more' consist?" ([4] p. 74; and cf. [19])

Certainly, killing might have related Caesar and Brutus as subject and object, and that possibility stands in need of explanation. But it cannot be explained by reference to a real, existing but non-actual state of affairs of Caesar's killing Brutus, for Caesar did not kill Brutus and killing did not relate Caesar to Brutus as subject and object. The modal realist claims that states of affairs are far more numerous than we ordinarily take them to be, claiming that there are states of affairs of Caesar's killing Brutus, of donkeys talking, of Othello's loving Desdemona, of Desdemona's loving Cassio and so on. As evidence, all he can offer is their necessity in explaining modal facts. We might as well explain the roundness of round squares by pointing to their obvious roundness, or Desdemona's loving Othello by pointing to Desdemona. There are no round squares, and Desdemona never existed. Such shadow objects give no explanation and simply confuse with their duplicity and duplication. Fiction, and modality, cannot be explained by simply supposing, or asserting, that what does not exist, exists after all, but in some special, non-actual, way. There is no state of affairs of Caesar's killing Brutus, not even a shadowy, non-actual one, for killing did not relate those Romans as killer and killed. Nor is there a state of affairs of Desdemona's loving Othello—we need some other account of how 'Desdemona loved Othello' seems to be true and 'Desdemona loves Cassio' to be false.

David Lewis' modal realism does not fall to this objection directly, since he does not claim that there is a non-actual state of affairs of Caesar's killing Brutus in order to explain the possibility that Caesar might have killed Brutus. Rather, he claims that there is a state of affairs consisting of Caesar's counterpart killing Brutus' counterpart, non-actual because counterparts are counterfactual. Nonetheless, the explanation still fails, for there is no reason to suppose Caesar and Brutus have such shadowy counterparts, other than the perceived need, on the theory, to postulate them to explain modal facts.

This is not to deny that the trope of possible worlds is useful, as we will see in § 4. But modal realism, and combinatorialism, are mistaken theories, and the possible worlds semantics of modal logic is no more than a useful technical device and psychological aid to further our understanding of modality. To show that an inference preserves truth-at-a-world in a suitable class of frames for a modal logic does not show that the inference is sound. Rather, it shows that that class of frames fits that mode of inference.[2] We don't have the independent means of giving meaning to the connectives which Prior demanded. The meaning of the logical connectives, at least in the case of modality, must be given in some other way.

2 Harmony

The meaning of the connectives cannot in general be given a representationalist account, as the argument in § 1 shows. So we might seek to give their meaning by the inference rules for their use. However, not just any

[2] If truth-preservation in a frame were enough, Geach's challenge to the relevance logician to exhibit a model in which $\alpha \vee \beta$ and $\neg\alpha$ were true and β false would suffice to rebut them. But whether Disjunctive Syllogism is sound is a much more difficult matter than that—see, e.g., Read [17] ch. 7.

set of rules suffices to give them meaning, as Prior's example ([15]) of 'tonk' famously revealed. 'tonk' seems to have part of the meaning of 'or', since 'α tonk β' can be inferred from α, while also appearing to have part of the meaning of 'and', since β can be inferred from 'α tonk β'. But worse, since 'α tonk β' can be inferred from α but not from β, 'α tonk β' is in fact equivalent to α, while since β but not α can be inferred from 'α tonk β', 'α tonk β' is equivalent to β. That is, the inference theory of 'tonk' presupposes that there is a proposition which is equivalent both to α and to β, for arbitrary α and β. It is no surprise, therefore, that if inference is transitive, the theory of 'tonk' permits the derivation of any proposition from any other, the absurdity which Prior exposes.

Prior's conclusion is that we need independent evidence that there is a proposition 'α tonk β' with the meaning requisite for the inference rules proposed for it to be sound. But there is a more insightful conclusion present in the observation that 'α tonk β' seems on the one hand, *via* its introduction-rule, to be equivalent to α and on the other, *via* its elimination-rule, to be equivalent to β. It is contained in that clause, 'and not from β': 'α tonk β' can be inferred, in general, from α and not, in general, from β.[3] In contrast, 'α or β' can be inferred from α or from β. The introduction-rule not only shows what is (severally) sufficient for assertion of the conclusion ('α tonk β' or 'α or β', respectively) but also shows what is (jointly) necessary: not only may 'α or β' be inferred from α or from β, it can be inferred only from one or the other; not only may 'α tonk β' be inferred from α, it may be inferred only from α—not from β.

Of course, 'α or β' (or 'α tonk β' for that matter) can be inferred by Modus Ponens, or Simplification, or any number of other inference rules from more complex formulae not specific to 'or' (or 'tonk'). 'α or β' can also be the conclusion of generic, or structural, rules. What the inference rules for a specific connective make explicit are the logical features of that connective. Hence, what is implicit in the totality of cases of the introduction-rule for a connective is that they exhaust the grounds for assertion of that specific conclusion. Brandom ([5] p. 63) misses this crucial point when he equates the introduction-rule with the set of sufficient conditions for assertion and the elimination-rule with the set of necessary consequences of that assertion. Each already contains both necessary and sufficient conditions, and it is when these do not agree that problems like those of 'tonk' arise.

[3]Of course, in special cases, 'α tonk β' can be inferred from β—when $\alpha = \beta$, for example. But inference rules are formal and general.

The elimination-rule works in the same way. What it says is that from 'α tonk β', β but only β, not α may be inferred; from 'α and β', either α or β may be inferred, but only α or β. The elimination-rule shows not only what is (severally) necessary for the assertion of the premise ('α tonk β', 'α and β', respectively) but also what is (jointly) sufficient: not only may β be inferred from 'α tonk β', but only β may be, not α; not only may either α or β be inferred from 'α and β', but they are both inferrable from it. Hence, the theory of 'and' is coherent, for introduction- and elimination-rules agree that both α and β are required for an assertion of 'α and β', as well as sufficient; while the theory of 'tonk' is not coherent, for the introduction-rule says that α is sufficient for assertion of 'α tonk β', while the elimination-rule denies this, and moreover, the introduction-rule says that β is not sufficient for assertion of 'α tonk β', while the elimination-rule says it is.

This requirement of coherence in the rules, in particular, appreciating that what the introduction- and elimination-rules say should be both sufficient and necessary is what seems to have eluded Popper and the other proponents in the 1950s of what Prior dubs the "analytical validity" view of the connectives. But it was not a point which eluded Gentzen, writing some twenty years earlier. His brief comment on the relation between the introduction- and elimination-rules is as acute as it is famous:

> "The introductions represent, as it were, the 'definitions' of the symbols concerned, and the eliminations are no more, in the final analysis, than the consequence of these definitions." ([11] p. 80)

The introduction-rules, by being all and only the rules for asserting a complex formula, already contain all that is needed for formulation of the elimination-rules. If the elimination-rules allow one to infer more than is justified by the introduction-rules, then the meaning conferred by the rules is split between introduction- and elimination-rules, and one possible result is incoherence and inconsistency, as in the case of 'tonk'. If the elimination-rules allow one to infer less than is justified by the introduction-rules, then again, the meaning conferred by the rules is opaquely spread between them: whatever meaning might be promised by the introduction-rule is then restricted and taken back by the elimination-rule. We will consider an example of the latter situation later.

The situation recommended by Gentzen has come to be known as "harmony" (see e.g., [9] p. 455). However, different articulations have been proposed of what this means—normalization, or conservative extension, for example.[4] Actually, we should stick closer to Gentzen's original formulation: the introduction- and elimination-rules are in harmony when the elimination-rules do no more than spell out what may be inferred from the assertion of the conclusion of the introduction-rules, given the grounds for its assertion.[5] That is, we may infer from an assertion all and only what follows from the various grounds for that assertion. Specifically, if the grounds for assertion of δ are Π_i, where $\{\Pi_i\colon\ i \in I\}$ is a collection of subproofs, a collection of derivations, degeneratively of formulae, warranting assertion of δ), then the harmonious form of the elimination-rule is

$$\frac{\delta \quad \left\{\begin{array}{c}(\Pi_i)\\ \gamma\end{array}\right\}}{\gamma}$$

that is, given an assertion of δ, and (multiple—there may be several cases of the introduction-rule, as in $\vee$I) derivation(s) of γ from the canonical ground(s) for asserting δ, we may infer γ and discharge the assumption of those grounds.

Taking Prior's introduction-rule for 'tonk', for example:

$$\frac{\alpha}{\alpha \text{ tonk } \beta}\ \text{tonk-I}$$

we obtain the general case of tonk-E, assuming tonk-I to exhaust the grounds for asserting 'α tonk β':

$$\frac{\alpha \text{ tonk } \beta \quad \begin{array}{c}(\alpha)\\ \vdots\\ \gamma\end{array}}{\gamma}\ \text{tonk-E}$$

In this case, we can simplify this by permuting the derivation of γ from α with the application of the rule, to give the simpler

$$\begin{array}{c}\dfrac{\alpha \text{ tonk } \beta}{\alpha}\ \text{tonk-E}\\ \vdots\\ \gamma\end{array}$$

[4] See, e.g., [10], pp. 215-20, 246-51.

[5] This account of harmony conflates what, following Dummett, I called harmony and autonomy in [18] § 2.1. A connective is "autonomous" if its meaning is (wholly) given by the I-rule; it is "harmonious" if the E-rule does no more than spell out the consequences of the meaning so conferred.

This is, of course, not Prior's tonk-E rule, and does not lead to inconsistency and triviality as did his rule.

A similar simplification is possible in the case of 'and' ($\wedge$). Given that the sole ground for asserting '$\alpha \wedge \beta$' is derivations of both α and β:

$$\frac{\alpha \quad \beta}{\alpha \wedge \beta}\ \wedge\text{I}$$

it follows that one may infer from '$\alpha \wedge \beta$' whatever one may infer from the grounds for its assertion, that is, both α and β:

$$\frac{\alpha \wedge \beta \qquad \begin{matrix} \underbrace{(\alpha, \beta)} \\ \vdots \\ \gamma \end{matrix}}{\gamma}\ \wedge\text{E}$$

Again, the derivation of γ from α and β may be permuted with the application of $\wedge$E to obtain the more common form of the $\wedge$E-rule:

$$\underbrace{\frac{\alpha \wedge \beta}{\alpha} \quad \frac{\alpha \wedge \beta}{\beta}}_{}$$

$$\vdots$$

$$\gamma$$

But this permutation is not always possible, as the case of $\vee$E shows.[6]

What is important about harmony is that it locates all the meaning-conferring power in the introduction-rule (or alternatively, in the elimination-rule). That does not prevent one introducing inconsistent connectives with harmonious rules. For example, suppose one introduces a zero-place connective $\bullet$ with the rule:

$$\begin{array}{c} (\bullet) \\ \vdots \\ \dfrac{\bot}{\bullet}\ \bullet\text{I} \end{array}$$

That is, from a derivation of absurdity ($\bot$) from $\bullet$, one can infer $\bullet$.[7] Consequently, the elimination-rule will read:

$$\dfrac{\bullet \qquad \begin{array}{c} (\bullet \Rightarrow \bot) \\ \vdots \\ \gamma \end{array}}{\gamma}\ \bullet\text{E}$$

Here, $\alpha \Rightarrow \beta$ abbreviates a supposed derivation of β from α. (E.g., one might rewrite $\bullet$I as

$$\dfrac{\bullet \Rightarrow \bot}{\bullet}\ \bullet\text{I}$$

So $\bullet$E says that one may infer from $\bullet$ whatever (*viz* γ) one may infer from supposing one can infer $\bot$ from $\bullet$, which is what $\bullet$I says justifies assertion of $\bullet$.) We can see immediately from the I-rule that $\bullet$ is self-contradictory—according to the I-rule, one may assert $\bullet$ if and only if one may infer $\bot$ from it.

$\bullet$ is a proof-conditional Liar sentence. $\bullet$ normalizes in the usual way:

[6] Actually, even here permutation is possible, given sufficient constraints on the interpretation of the notion of proof, as Kneale [12] shows for $\vee$ and Copi [6] for $\exists$.

[7] $\bullet$I is not, in Dummett's terminology [10] (p. 257), "pure", in containing another connective, $\bot$. However, as Dummett argues, this is permissible provided it is not circular. First, introduce $\bot$. Then $\neg$ and $\bullet$ can be introduced subsequently, using $\bot$.

$$\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ \bullet \Rightarrow \bot\end{array}}{\bullet}\,\bullet\mathrm{I} \qquad \begin{array}{c}(\bullet \Rightarrow \bot)\\ \Pi_2\\ \gamma\end{array}}{\gamma}\,\bullet\mathrm{E} \qquad \text{reduces to} \qquad \begin{array}{c}\Pi_1\\ \bullet \Rightarrow \bot\\ \Pi_2\\ \gamma\end{array}$$

that is, maximum formulae of the form $\bullet$ (those which are both conclusion of an introduction- or $\bot$-rule and at the same time major premise of an elimination-rule) may be eliminated by replacing each leaf in Π_2 of the form $\bullet \Rightarrow \bot$ by a copy of Π_1.

We can again simplify $\bullet$E by permuting the derivation of γ with the application of the rule to obtain:

$$\begin{array}{c}\dfrac{\bullet}{\bot}\,\bullet\mathrm{E}\\ \vdots\\ \gamma\end{array}$$

Taking familiar, and harmonious, rules for negation:

$$\dfrac{\alpha \Rightarrow \bot}{\neg\alpha}\,\neg\mathrm{I} \qquad \text{and} \qquad \dfrac{\neg\alpha \quad \alpha}{\bot}\,\neg\mathrm{E}$$

we can show that $\bullet$ equivalent to its own negation:[8]

$$\dfrac{\dfrac{\dfrac{\dfrac{\bullet^1}{\bot}\,\bullet\mathrm{E}}{\neg\bullet}\,\neg\mathrm{I}(1)}{\bullet \to \neg\bullet}\,\to\mathrm{I}(1) \qquad \dfrac{\dfrac{\dfrac{\neg\bullet^2 \quad \bullet^3}{\bot}\,\neg\mathrm{E}}{\bullet}\,\bullet\mathrm{I}(3)}{\neg\bullet \to \bullet}\,\to\mathrm{I}(2)}{\bullet \leftrightarrow \neg\bullet}\,\leftrightarrow\mathrm{I}$$

Harmony cannot prevent inconsistency. But it helps to locate and identify that inconsistency. For, taking $\bullet$I as the sole introduction-rule for $\bullet$, we can already see that $\bullet \leftrightarrow \neg\bullet$, for derivation of $\bot$ from $\bullet$ is both enough to assert $\bullet$ (by $\bullet$I) and to deny it (since it entails $\bot$).

Some have wanted not only to claim that harmony prevents inconsistency, which $\bullet$ shows is a mistaken hope, but furthermore, that lack of harmony, if not a source of incoherence and error, at least imports non-logical features and so undermines the inferentialist project. (See, e.g., [10] pp. 246 ff.) Certainly, Prior's rules for 'tonk' are incoherent and mistaken, and also

[8] Substructuralist beancounters who are worried by the double discharge of the assumption in the second proof will realise on reflection that $\bullet$E strictly requires two copies of its premise.

inharmonious. But although lack of harmony in Prior's rules serves to obscure the source of that incoherence, it is not itself the source. As the case of • shows, contradictory connectives can be governed by harmonious rules, where the necessary and sufficient conditions for their assertion are given by the I-rules, and the E-rule uses no more than the I-rules justify. And as we will see in § 3, perfectly decent connectives can be governed by inharmonious rules. Such cases are unhelpful, in obscuring the meaning of the connective, but the lack of harmony is not itself a source of incoherence, nor does it mean that the rules do not define the meaning of the connectives as logical.

3 Modality

We now turn to a classic example of such disharmony, the standard rules for '$\Box$' and '$\Diamond$' in modal logic. The pioneers in the development of natural deduction systems for modal logic were Curry ([7] ch. V) and Fitch ([8] ch. 3), both formulating systems of **S4** (variously classical and intuitionistic). Curry's rules for '$\Box$' read:

$$\frac{\alpha}{\Box\alpha}\ \Box\text{I} \qquad \text{and} \qquad \frac{\Box\alpha}{\alpha}\ \Box\text{E}$$

where in $\Box$I every assumption on which α depends must be *modal*, that is, of the form $\Box\beta$ (or $\neg\Diamond\beta$ or $\bot$, for systems containing '$\Diamond$' and/or '$\bot$'). Fitch's system has the same $\Box$E-rule, and essentially the same $\Box$I-rule, where the proof of α is a (what he termed, strict) sub-proof into which only formulae of the form $\Box\beta$ may be reiterated. Fitch adds complementary rules for '$\Diamond$', which Prawitz ([13] ch. VI) brings into a form more comparable to Curry's $\Box$-rules:

$$\frac{\alpha}{\Diamond\alpha}\ \Diamond\text{I} \qquad \text{and} \qquad \frac{\Diamond\alpha \quad \begin{matrix}(\alpha)\\ \vdots \\ \gamma\end{matrix}}{\gamma}\ \Diamond\text{E}$$

provided that in the case of $\Diamond$E, every assumption on which the minor premise γ depends, apart from α, (the so-called parametric formulae) is modal and γ is *co-modal*, that is, has the form $\Diamond\beta$ (or $\neg\Box\beta$ or $\neg\bot$).

These rules can be strengthened to yield the logic **S5**, by including formulae of the form $\neg\Box\beta$ (equivalently, $\Diamond\beta$) among the modal formulae (and identifying the classes of modal and co-modal formulae). They can also be weakened to yield the system **T** by casting $\Box$I in the slightly different form:

$$\frac{\alpha}{\Box\alpha}\ \Box\mathrm{I}$$

where the conclusion $\Box\alpha$ depends on the formulae $\Box\Gamma$ $(= \Box A_1, \ldots, \Box A_n)$ if the premise α depends on Γ $(= A_1, \ldots, A_n)$. Finally, **S4** can be weakened to **K4** and **T** can be weakened to **K** by omitting the rule $\Box$E-rule (and taking '$\Diamond$' as a defined symbol), since it is not valid in those logics.

By a clever recasting of the above formulation, preserving the essential nature of the rule-pairs, Prawitz was able to formulate systems of **S4** and **S5** which are normalizable, that is, in which any proof can be replaced by a normal proof, one which contains no "maximum formulae" (as in § 2).

Such a normalization result is sometimes identified with harmony. ([10] p. 250) Nonetheless, although normalization can be proved for (a version of) the above rules, from the viewpoint of the account of harmony given in § 2, these rules look very unsatisfactory. As formulated above, the logics **T**, **S4** and **S5** differ only in their $\Box$I-rule, sharing $\Box$E as the elimination-rule; the logics **K**, **K4** and **KB** again differ only in their $\Box$I-rules, having no $\Box$E-rule.

- First puzzle: if logics have different $\Box$I-rules, should we not expect harmony to yield different $\Box$E-rules?
- Secondly, if logics share $\Box$I-rules, should they not be the same logic unless disharmonious?—that is, if one has the $\Box$E-rule, the other not, must not at least one of them be disharmonious?

Consider now the $\Diamond$-rules, and in particular, ask what elimination-rule is justified by $\Diamond$I. From an assertion of $\Diamond\alpha$ we should be able to infer anything we can infer from what justified the assertion of $\Diamond\alpha$. But that was just α itself. That justifies $\Diamond$E as given—but without the restriction on the parametric formulae and on the conclusion γ itself. Indeed, it would warrant inferring α from $\Diamond\alpha$.[9] But that would collapse modalities. The restriction in $\Diamond$E thus serves to restrict whatever meaning $\Diamond$I might otherwise confer on '$\Diamond$'.

[9] One spur to the present paper was reflecting on Dummett's difficulties with '$\Diamond$'—see [10] p. 265; cf. [18] p. 129.

Another indication that the rules are misleading comes from consideration of the proof in **T**, **S4** or **S5** of an equivalent form of the postulate (K), using '$\Diamond$', ($K_\Diamond$): $\Box(\alpha \to \beta) \to (\Diamond\alpha \to \Diamond\beta)$

$$\dfrac{\dfrac{\dfrac{\Diamond\alpha^1 \qquad \dfrac{\dfrac{\dfrac{\Box(\alpha \to \beta)^2}{\alpha \to \beta}\,\Box\mathrm{E} \qquad \alpha^3}{\beta}\to\mathrm{E}}{\Diamond\beta}\,\Diamond\mathrm{I}}{\Diamond\beta}\,\Diamond\mathrm{E}(3)}{\Diamond\alpha \to \Diamond\beta}\to\mathrm{I}(1)}{\Box(\alpha \to \beta) \to (\Diamond\alpha \to \Diamond\beta)}\to\mathrm{I}(2)$$

The proof uses $\Box$E and $\Diamond$I, in addition to $\Diamond$E. But ($K_\Diamond$) is valid in **K**, and does not depend on the reflexivity of accessibility, and so its proof should not depend on $\Box$E or $\Diamond$I. To formulate the non-reflexive logics, **K**, **K4** and **KB**, it would be necessary to drop $\Box$E and $\Diamond$I (which are valid only for reflexive logics) and so there would apparently be no introduction-rule for '$\Diamond$'. It is sometimes argued (e.g., [14] p. 243) that $\bot$ has no I-rule, which justifies *Ex Falso Quodlibet*:

$$\dfrac{\bot}{\alpha}\,\bot\mathrm{E}$$

as its elimination-rule. We certainly do not want such a rule for '$\Diamond$', so $\Diamond\alpha$ is introduced in **K** and **K4** (as noted above) by definition as $\neg\Box\neg\alpha$. Otherwise, one would at best have conferred no meaning on '$\Diamond$' to justify the elimination-rule. So the rules could not be in harmony.

Finally, consider $\Diamond$I once again. If the rules were in harmony, $\Diamond$I would give the whole meaning of '$\Diamond$'.[10] But even in reflexive logics, α is only one ground for asserting $\Diamond\alpha$. $\Diamond\alpha$ can be true without α also being true. So if $\Diamond$I is the only introduction-rule for '$\Diamond$', part of the meaning of '$\Diamond$' must reside elsewhere, that is, in the elimination-rule. So again, the rules cannot be in harmony.

4 Harmony Regained

How can we proceed to formulate rules for the normal modal logics which are in harmony—that is, in which the introduction-rule encapsulates the whole meaning of the modal operator—and in which a clear distinction can

[10] More explicitly, in the language of [18], '$\Diamond$' is not autonomous. Part of the meaning of '$\Diamond$' is given by $\Diamond$I, part by $\Diamond$E.

be made whereby each rule has a form appropriate to each logic? One way to do so is to introduce the notion of a labelled deductive system. (See, e.g., [22] ch. 4, [24]; cf. [16] § 3.2) Whether harmony can be achieved without labels is an open question. But it can certainly be achieved by the use of labels.

In a labelled deductive system, each formula receives a label. The rules can refer to the labels, can change them, and can depend on relations between labels. In fact, in characterizing modal logic, the labels are effectively world-indices, that is, we can read α_i, where 'i' is the label on the wff α, as saying that α is true at the world (with index) i.[11] Then $\Gamma \vdash \alpha$ if there is a derivation of α_0 from $\Gamma' \subseteq \Gamma$ in which every formula in Γ' has the label 0. (0 here can be understood either as an arbitrary index—validity being defined as truth-preservation at an arbitrary world—or as the base, or home, world H of the Kripke semantics.) Non-modal rules preserve labels in obvious ways—the labels are affected by, and determine the correctness of, the modal rules. In $\wedge$I and $\rightarrow$E, for example, the premises must have the same label. In $\vee$E and $\exists$E, to take another example, the conclusion has the same label as that on the minor premise (which is itself arbitrary). We need only one explicit relation between labels, which we write $i < j$, and read as saying that (world) j is "possible relative to" (world) i.

Note that such a reading is purely pedagogical. We are setting up a formal system, in which labels and '$<$' are auxiliary symbols whose meaning, if any, is conferred by the rules. If what was said in § 1 was right, there are no (other) worlds, and so the reading of $i < j$ is at best a useful metaphor. A similar point is often made about such an expression as

$$\lim_{n\rightarrow\infty} a_n,$$

where '∞' does not refer to a value of n, but indicates rather that there is no greatest value of n. One may ask how $\Box$I and $\Box$E, as formulated below, relate to our inferential practice. That is part of a wider question, how any part of the theoretical systematization of logic relates to practice. More needs to be said, but not here. *In nuce*, inferentialism is the thesis that the meaning of all the logical symbols is given by the rules for use, and by nothing else.

For $\Box\alpha$ to be true at i is for α to be true at any world j possible relative

[11] But that is, I argued in § 1, only a helpful metaphor—helpful, provided it is not taken literally.

to, or "accessible from", i. This motivates the I-rule:

$$\dfrac{\begin{array}{c}(i<j)\\ \vdots\\ \alpha_j\end{array}}{\Box\alpha_i}\ \Box\text{I}$$

Here $j \neq i$ and j must not appear in any other assumption on which α_j depends. In other words, $\Box$I combines aspects of $\to$I and $\forall$I, unsurprising given the articulation of '$\Box$' in terms of truth at all (accessible) worlds. Then $\Box$I warrants, by Gentzen considerations:

$$\dfrac{\Box\alpha_i \qquad \begin{array}{c}(i<j \Rightarrow \alpha_j)\\ \vdots\\ \gamma_k\end{array}}{\gamma_k}\ \Box\text{E}$$

That is, if assuming α true at all worlds j accessible from i allows derivation of γ at k, then so too does assertion of $\Box\alpha_i$. We can simplify this, as in the case of $\wedge$E and $\bullet$E, by inverting the derivation of γ_k with that of α_j, deducing α_j from $i<j$ directly:

$$\begin{array}{c}\dfrac{\Box\alpha_i \quad i<j}{\alpha_j}\ \Box\text{E}\\ \vdots\\ \gamma_k\end{array}$$

These rules suffice to prove (K):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Box(\alpha\to\beta)_0^1 \quad 0<1^2}{\alpha\to\beta_1}\ \Box\text{E} \qquad \dfrac{\Box\alpha_0^3 \quad 0<1^2}{\alpha_1}\ \Box\text{E}}{\beta_1}\ \to\text{E}}{\Box\beta_0}\ \Box\text{I}(2)}{\Box\alpha\to\Box\beta_0}\ \to\text{I}(3)}{\Box(\alpha\to\beta)\to(\Box\alpha\to\Box\beta)_0}\ \to\text{I}(1)$$

The fact that $\Box$E has been justified in Gentzen's manner by $\Box$I means that normalization is immediate:

$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\ i<j\Rightarrow\alpha_j\end{matrix}}{\Box\alpha_i}\,\Box\mathrm{I} \qquad \begin{matrix}(i<j\Rightarrow\alpha_j)\\ \Pi_2\\ \gamma_k\end{matrix}}{\gamma_k}\,\Box\mathrm{E} \qquad \text{reduces to} \qquad \begin{matrix}\Pi_1\\ i<j\Rightarrow\alpha_j\\ \Pi_2\\ \gamma_k\end{matrix}$$

eliminating the maximum formula $\Box\alpha_i$.

The rules for '$\Diamond$' are equally straightforward and intuitive, given the interpretation of $\Diamond\alpha$ as indicating the truth of α at some accessible world:

$$\dfrac{\alpha_j \quad i<j}{\Diamond\alpha_i}\,\Diamond\mathrm{I}$$

$\Diamond$I justifies the E-rule (cf. $\wedge$I and $\wedge$E):

$$\dfrac{\Diamond\alpha_i \qquad \begin{matrix}(\alpha_j, i<j)\\ \vdots\\ \gamma_k\end{matrix}}{\gamma_k}\,\Diamond\mathrm{E}$$

where k is any index, $i \neq j \neq k$ and 'j' occurs in no other assumptions on which γ_k depends than those displayed. Note that the restriction on j prevents inversion and simplification of the type available in $\wedge$E and $\Box$E. We can then prove ($\mathrm{K}_\Diamond$):

$$\dfrac{\dfrac{\Diamond\alpha_0^1 \qquad \dfrac{\dfrac{\dfrac{\Box(\alpha\to\beta)_0^2 \quad 0<1^3}{\alpha\to\beta_1}\,\Box\mathrm{E} \qquad \alpha_1^4}{\beta_1}\,{\to}\mathrm{E} \qquad 0<1^3}{\Diamond\beta_0}\,\Diamond\mathrm{I}}{\dfrac{\Diamond\beta_0}{\Diamond\alpha\to\Diamond\beta_0}\,{\to}\mathrm{I}(1)}\,\Diamond\mathrm{E}(3,4)}{\Box(\alpha\to\beta)\to(\Diamond\alpha\to\Diamond\beta)_0}\,{\to}\mathrm{I}(2)$$

The fact that $\Box$E and $\Diamond$I occur in this proof is no longer a problem, since as now formulated, they are valid in **K**, and do not connote reflexivity.

These rules suffice for all the theses of the logic **K**, but they do not yield $\Box\alpha\to\alpha$, or any of the stronger theses of **K4**, **S4** and so on. To strengthen **K** to obtain the other normal logics, we need to add structural rules for the manipulation of the index-relation '$<$':

$$\dfrac{\begin{matrix}(i<i)\\ \vdots\\ \alpha_j\end{matrix}}{\alpha_j}\;T \qquad \dfrac{i<j \quad j<k \quad \begin{matrix}(i<k)\\ \vdots\\ \alpha_l\end{matrix}}{\alpha_l}\;4$$

$$\dfrac{i<j \quad \begin{matrix}(j<i)\\ \vdots\\ \alpha_k\end{matrix}}{\alpha_k}\;B \qquad \dfrac{i<j \quad i<k \quad \begin{matrix}(j<k)\\ \vdots\\ \alpha_m\end{matrix}}{\alpha_m}\;5$$

The system can also be extended to predicate logic, and versions both with varying domains and constant domains (and so validating the Barcan formula) can be developed by labelling terms in a similar way. (See, e.g., Basin et al. [3])

We can still be puzzled, however, how the use of auxiliary symbols such as $<$ in the rules for '$\Box$' and '$\Diamond$' gives them the meaning of 'necessarily' and 'possibly'. The answer lies in the similarity between, say, $\Box$I and $\forall$I:

$$\frac{\alpha(b/a)}{(\forall a)\alpha}\;\forall\text{I}$$

Here the fact that 'b' is free in no assumptions on which $(\forall a)\alpha$ depends means that α holds regardless of the interpretation of 'a', and so α holds for all a, giving the necessary universal meaning to '$\forall$'. The similar condition on 'j' in $\Box$I means that α holds for all j. Then we follow out the metaphor, a picture in which worlds are "possible" relative to one another, and infer that α is true at all worlds "possible" relative to i, that is, that α is necessarily true at i, i.e., $\Box\alpha$ is true at i.

This labelled deductive system provides harmonious modal rules, and provides an answer to the first and second puzzles. The introduction- and elimination-rules for '$\Box$' and '$\Diamond$' are the same in all the normal modal logics, so we do not have the situation of the I-rule changing without a change in the elimination-rule, nor the introduction- or elimination-rule disappearing while the other remains. What changes are the structural rules governing the assumptions. The rules by which the logics differ are generic rules, rules governing the auxiliary symbols—the labels and the relation '$<$'—while the operational rules, the specific rules for the operations, remain constant. What this means is that there is throughout these logics, these theories of modality, the same sense of necessity and possibility. What is different is the logic that they satisfy, not the meaning of '$\Box$' and '$\Diamond$'. The stronger systems permit inferences involving necessity and possibility which the weaker systems do not.

We can draw an analogy with classical logic and its rivals. We do not want the intuitionist, classicist and relevantist to disagree about the meaning of '$\to$'—that way lies Carnapian tolerance, with a different logic appropriate for each different meaning. (See [20]) Rather, we want them to agree on what they disagree about, that is, to disagree about the same thing, to attribute different logics to the one connective '$\to$', with a univocal meaning. So too, for modality. Modal tolerance would be the view that each modal system was right for a different sense, or notion of modality. Sometimes that is appropriate, if one is comparing the structure of consequence in deontic logic with epistemic logic, say. But proper logical disagreement arises when the systems give a different modal theory to the same notions, necessity and possibility. Only then does the question, which modal system is right, become legitimate. We fix that univocal sense of '$\Box$' and '$\Diamond$' by the rules $\Box$I and $\Diamond$I. The elimination-rules, $\Box$E and $\Diamond$E, are consequently justified by those senses being univocal and fully specified. What is different, is the logic of the modal notions encapsulated in the structural rules.

5 Conclusion

Unless one believes in the reality of possible-world semantics, it cannot provide an independent and non-circular account of the meaning of the modal terms 'necessary' and 'possible'. Rather, the meaning of these terms must be given inferentially, by laying down rules for their use. Meaning can be usefully made transparent by requiring harmony in the rules of inference wherever possible. For if the elimination-rule is so framed that it is, in Gentzen's phrase, "no more than a consequence" of the meaning conferred by the introduction-rules, then those introduction-rules exhibit the meaning transparently, setting out expressly and fully the conditions for assertion of propositions containing the term in question. We have seen how to express those meanings harmoniously and transparently not only for '$\wedge$' and '$\bullet$', but also for the modal connectives (using auxiliary, uninterpreted labels). Necessity and possibility have been given a proof-conditional meaning in which the introduction- and elimination-rules lie in harmony, where harmony ensures transparency in the meaning conferred and whose virtue is clarity.

There are many suggestions as to the diagnosis of the fallacy in Prior's introduction of 'tonk'. What one seeks is a diagnosis which explains what was mistaken in the inferentialist views that Prior was attacking and helps to improve our understanding of inferentialism. The recognition that the introduction-rules are jointly necessary and the minor premises of the elimination-rule jointly sufficient lets us see how inconsistency can arise from a mismatch in the rules. Harmony can prevent inconsistency arising inadvertently in this way. But it cannot prevent inconsistency *tout court*, since the introduction-rule can be inconsistent in itself, as with '$\bullet$'. And that is as it should be, for sometimes we wish to study such cases, e.g., in a self-referential paradox.

Harmony guarantees normalization but not vice versa, as we have seen in the case of Prawitz' rules for '$\Box$' and '$\Diamond$'. His rules do give the (correct) meaning of '$\Box$' and '$\Diamond$', but they do so opaquely. So harmony should not be identified with normalization. Harmony for the modal rules is achieved when the full meaning of $\Box\alpha$ and $\Diamond\alpha$ is contained in the rules for their assertion. It seems that that cannot be achieved without some auxiliary apparatus. Nothing about the conditions under which α itself can be asserted can tell whether α is necessary or possible. Probably, this is what has inspired the modal sceptics. How can one tell, considering only α, whether α is possible or necessary? Certainly, if α is true it must be possible, and if false, it cannot be necessary. But these are extremal cases. For the general case, we need to resort to some metaphor. But we must be clear that it is a metaphor, and does not represent anything real. The meaning of the modalities is given by the inferential conditions for the assertion of modal formulae.

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