Gödel's Cantorianism

Claudio Ternullo
KGRC, University of Vienna*

"My theory is rationalistic, idealistic, optimistic, and theological".
Gödel, in Wang (1996), p. 8

"Numeros integros simili modo atque totum quoddam legibus et relationibus compositum efficere" Cantor, *De transformatione formarum ternariarum quadraticarum*, thesis III, in Cantor (1932), p. 62

1. Introductory remarks

There is no conclusive evidence, either in his published or his unpublished work, that Gödel had read, meditated upon or drawn inspiration from Cantor's philosophical doctrines. We know about his philosophical "training", and that, since his youth, he had shown interest in the work of such philosophers as Kant, Leibniz and Plato. It is also widely known that, from a certain point onwards in his life, he started reading and absorbing Husserl's thought and that phenomenology proved to be one of the most fundamental influences he was to subject himself to in the course of the development of his ideas.[1] But we do not know about the influence of Cantor's thought.

In Wang's book containing reports of the philosophical conversations the author had with Gödel, one can find only a few remarks by Gödel concerning Cantor's philosophical conceptions. Not much material do we get from the secondary literature either. For instance, if one browses through the indexes of Dawson's fundamental biography of Gödel (Dawson 1997), or those of Wang's three ponderous volumes (1974, 1987, 1996) one finds that all mentions of Cantor in those works either refer to specific points of Cantorian set theory, as discussed by the authors of these books, or, more specifically, to Gödel's paper on Cantor's continuum problem,[2] wherein

* The writing of this article has been supported by the JTF Grant ID35216 (research project "The Hyperuniverse. Laboratory of the Infinite"). A preliminary version was presented and discussed at the conference *Kurt Gödel Philosopher: from Logic to Cosmology* held in Aix-en-Provence, July, 9-11, 2013. I wish to thank Richard Tieszen for insightful comments on my presentation and Mark van Atten for pointing me to bibliographical material which has proved of fundamental importance for the subsequent writing of this article. Gabriella Crocco and Eva-Maria Engelen read earlier drafts of this work, providing me with extremely helpful comments and suggestions. Finally, I owe a debt of gratitude to Mary Leng, whose advice and help in the final stage of revision have been invaluable.

1 Wang (1996), p. 164.

2 Gödel (1947), revised and extended version, (1964).

references to Cantor, once again, are not directed at the examination of the latter's philosophical work and conceptions.[3]

As a consequence, we do not know whether Gödel had direct or indirect acquaintance with Cantor's thought and how he judged it.[4] Indeed, Gödel's ostensible lack of interest in Cantor's philosophy in the first place might be one of the reasons why the varied and multi-faceted connections between his views and Cantor's have, with a few exceptions, gone altogether unnoticed.[5] To give you a taste of what such connections look like, let me briefly anticipate some of the material, which I will discuss in greater depth in the next sections.

There is hardly need, I believe, to emphasise both Cantor's and Gödel's commitment to a peculiar form of belief in the existence of the *actual infinite*. Cantor saw transfinite numbers as a *natural* generalisation of the natural numbers, while Gödel expressed the thought that the unbounded continuation of the process of formation of such numbers would have a deep impact on most fundamental mathematical issues.

At least from a certain point on in their lives, both embraced a thoroughgoing and unabated form of *realism*. Cantor acknowledged and reconstructed Plato's characterisation of *ideas* as the ontological basis of his own transfinite numbers.[6] But while casting transfinite numbers as purely 'ideal entities', he also vested them with a *trans-subjective* meaning, as being *(meta)physically* instantiated. Thus, he could legitimately claim that his conception encompassed 'idealist' and 'realist' features.

Similarly, Gödel strove for a theoretical synthesis between the idealist and realist position and eventually found it in Husserl's conceptions. In a sense, as we shall see, Cantor's philosophical doctrines about concepts may be viewed as reaching their theoretical completion in Gödel's *conceptual realism*.

In general, although the scope of their philosophical sources is wide and varied, they have a clear preference for authors belonging to the *rationalist* (as opposed to *empiricist*) tradition. In Cantor's works, one can find references to or quotations from works of Plato, Spinoza, Leibniz, Augustine, Origen, Euclid, Nicomachus of Gerasa, Boethius and others, and most of these references and quotations are generally accompanied by extolling comments and used in support of Cantor's own theses. On

3 Further information on biographical and philosophical aspects of Gödel's life can be found in Feferman's introduction to Gödel (1986), p. 1-34. A precise and exhaustive reconstruction of the development of Gödel's thought is also carried out by van Atten and Kennedy in van Atten and Kennedy (2003). None of these works mentions specific connections between Cantor and Gödel.

4 I have only found two passages in Wang (1996), where Gödel says something directly about Cantor. The first, on p. 175, concerns the philosophy of physics: "5.4.16 The heuristics of Einstein and Bohr are stated in their correspondence. Cantor might also be classified together with Einstein and me. Heisenberg and Bohr are on the other side. Bohr [even] drew metaphysical conclusions from the uncertainty principle." The second, on p. 276, is about the distinction between *set* and *class* (for whose relevance see section 6): "8.6.13 Since concepts can sometimes apply to themselves, their extensions (their corresponding classes) can belong to themselves; that is, a class can belong to itself. Frege did not distinguish sets from proper classes, but Cantor did this first." Both remarks show at least some familiarity with Cantor's writings.

5 Among these few exceptions should be counted some remarks by Wang and van Atten, in, respectively, Wang (1996) and van Atten (2009), concerning the so-called Cantor-von Neumann axiom, for which also see section 6.

6 See section 5.

the other hand, references to Aquinas, Aristotle, Locke, Hume and Kant are mostly made by Cantor with the purpose of refuting or discarding these thinkers' views.

Analogously, Gödel, at least half a century later, fosters Plato's, Leibniz's and Husserl's conceptions, that he classifies as "objectivistic," often contrasting them with what he takes to be the opposite point of view, that is philosophical "subjectivism" or conventionalism, which is represented by such authors as Kant (but only to a certain extent), Carnap and Wittgenstein. His characterisation of philosophical conceptions as dividing into "left-wing" and "right-wing" is in line with such presuppositions. It is worth quoting in full the crucial passage where such classification and its underlying rationale are introduced:

> I believe that the most fruitful principle for gaining an overall view of the possible world-views will be to divide them up according to the degree and manner of their affinity to or, respectively, turning away from metaphysics (or religion). In this way we immediately obtain a division into two groups: skepticism, materialism and positivism stand on one side, spiritualism, idealism and theology on the ôter. (Gödel (1961/?), (1995), p. 375)[7].

Both authors lived in an age of disillusionment with, if not outright refusal of, the *metaphysical* tradition. The strong pressure exerted by *positivism* on all undue "metaphysical pre-conceptions" may have been the main reason why Cantor was so philosophically meticulous in presenting his work. In a similar vein, but many years later, against all conventionalist and formalist reductions of mathematics, Gödel felt the urge to explain carefully why set-theoretic problems such as the Continuum Problem retain a meaning, if mathematics is construed as referring to an independently existing realm of objects. Cantor and Gödel never abandoned their fundamentally metaphysical outlook on mathematics, and, more generally, on the world. This attitude they pursued so coherently, that they sometimes seem to fall prey to what some commentators believe to be rationally untenable, even "bizarre," beliefs.

An integral part of this general attitude, not unsurprisingly, is their frequent appeal to and increasing fascination with *theology*, displaying itself more overtly in Cantor's work, less so in Gödel's.[8] Theology, with all its traditional theoretical artifices, gave both authors wide scope for speculations about the nature of the infinite, the set-theoretic hierarchy and its connections to the *phenomenal* world. But what is, perhaps, most interesting is their peculiar construal of theology (and theological arguments), as connected to, if not quite part of, the theory of sets. For instance, one can view the emergence of Cantor's theological *Infinitum Absolutum* in connection with the emergence of the notion of the *absolute infinity* of the *universe of sets*, on which Gödel had subsequently much to say.

Gödel's Cantorianism is transparent throughout Gödel's philosophical work, although, as I said, it has been largely overlooked by the scholarly literature. Its examination meets several purposes. First of all, I believe that its description may help us put into focus more accurately Gödel's philosophy, its developmental stages and history. As any other philosophical "transformative" conception, it may also shed

7 Throughout this article, all quotations from Gödel's published and unpublished works reproduce the established text in the II and III volume of his *Collected Works,* (Gödel (1990) and (1995)), abbreviated to (1990), (1995).

8 However, Gödel's interest in theology is noticeable in the *Max Phil* Notebooks.

new light on other important connections, such as those that bind Husserl, Frege and Gödel together. Finally, it might also help us discuss the peculiar form of realism that Gödel advocated, and that still pays an influential role in the contemporary debate on the foundations of mathematics.

A few methodological remarks: I will quote a lot of text from the primary sources, as is necessary for a study of this sort, and I will also indicate, when necessary, the relevant secondary literature. Although I will essentially be concerned with proposing connections between the two, I will also pay attention to, and comment on, the substance of Cantor's and Gödel's philosophical arguments.

2. Intra-subjective ("immanent") existence

One of the most characteristic and widely discussed traits of Gödel's thought is his *conceptual realism*. Given Gödel's own theoretical oscillations, it is perhaps somewhat problematic to say crisply what this position consists in.[9]

What seems to be fairly certain is that it includes, at least, the two following claims:

1. Concepts exist, in a way, which is similar, although not reducible to, the existence of physical objects.
2. Mathematical truths express relations of concepts.

As we shall see, both claims are in accordance, and could even be entirely derived from Cantor's conceptions.

Of Gödel's early adherence to conceptual realism, we are informed by Wang. In a letter sent to him in 1975, Gödel says: "0.1.3 I have been a conceptual and mathematical realist since about 1925."[10] Explicit references to the *existence* of concepts are contained, respectively, in two important passages of Gödel (1944) and (1951). The first one presents Gödel's argument that logical paradoxes do not affect set theory, insofar as the formation of all purely *mathematical* sets does not involve such paradoxical notions as that of a "set of all sets not belonging to themselves."[11] Sets formed through the standard iterative procedures have proved to be free from the kind of contradiction involved in the logical paradoxes.

Gödel says:

> Classes and concepts may, however, be also conceived as real objects, namely classes as "pluralities of things" or as structures consisting of a plurality of things and concepts as the properties and relations of things existing independently of our definitions and constructions. [...] They are in the same sense necessary to obtain a satisfactory system

9 See Parsons (1995), Martin (2005) and Crocco (2006) for a careful examination of the issues related to this position.

10 However, as late as 1933, Gödel stated (Gödel (1933), in (1995) p. 50): "The result of the preceding discussion is that our axioms, if interpreted as meaningful statements, necessarily presuppose a form of Platonism, which cannot satisfy any critical mind and which does not even produce the conviction that they are consistent". See Feferman's comments on this in his Introduction to Gödel (1933). But, apart from that, it seems very plausible that Gödel embraced Platonism, in at least some of its forms, at a very early stage in his career.

11 Purely mathematical sets, in ZFC (Zermelo-Fraenkel with the Axiom of Choice), or in alternative systems, with or without *urelements*, are sets formed through the iteration of the power-set operation at successor-stages and the union of all previous stages at limit-stages, starting from Ø or ur-elements.

> of mathematics as physical bodies are necessary for a satisfactory theory of our sense perceptions and in both cases it is impossible to interpret the propositions one wants to assert about these entities as propositions about the "data," i.e., in the latter case the actually occurring sense perceptions. (Gödel (1944), in (1990), p. 128)

Incidentally, in the second part of this quotation, Gödel seems to be adumbrating his conception that mathematical concepts are "grasped" (certainly "understood," but probably also "perceived") by some special faculty in the same way as physical objects are perceived by the senses.[12] In this latter as in the former case, however, our "perceptual data" would not be the result of the mere interaction between the objects and the corresponding perceiving faculty. I will return to this point later.

The second important passage can be found in Gödel's famous Gibbs lecture, wherein he states that:

> [...]it is correct that a mathematical proposition says nothing about the physical or psychical reality existing in space and time, because it is true already owing to the meaning of the terms occurring in it, irrespectively of the world of the real things. What is wrong, however, is that the meaning of the terms (that is, the concepts they denote) is asserted to be something man-made and consisting merely in semantical conventions. The truth, I believe, is that these concepts form an objective reality of their own, which we cannot create or change, but only perceive and describe. (Gödel (1951), in (1995), p. 320)

As far as claim (2) is concerned, that mathematical truths express relations of concepts, one must turn one's attention to the following key passage in the same work:

> Therefore a mathematical proposition, although it does not say anything about space-time reality, still may have a very sound objective content, insofar as it says something about relations of concepts. The existence of non-"tautological" relations between the concepts of mathematics appears above all in the circumstance that for the primitive terms of mathematics, axioms must be assumed, which are by no means tautologies (in the sense of being in any way reducible to $a=a$), but still do follow from the meaning of the primitive terms under consideration. (Gödel (1951), in (1995), p. 320-321).

This passage requires an extended commentary. In his Gibbs lecture, Gödel challenges two positions concerning the nature of mathematical truth. The first assumes that mathematical truths are *tautologies*, that is, are *analytic*, in the sense that they can be reduced to basic logical laws such as the *identity law*.[13] One main reason provided by Gödel for countering such conception is the following: the axioms of sets are

12 However, it is not wholly uncontroversial what Gödel thought to be the objects of "perception," whether mathematical objects or concepts ors both. For instance, in Wang (1996), p. 253, Gödel is reported to have said: "7.3.12 Sets are objects but concepts are not objects. We perceive objects and understand concepts. Understanding is a different kind of perception: it is a step in the direction of reduction to the last cause." I thank Eva-Maria Engelen for pointing me to this quotation and to the subtle difference between these two forms of "perception" in Gödel's thought.

13 No doubt, this conception has a Leibnizian ancestry, but Gödel may have also deliberately wanted to refer to the logicist standpoint. For instance, see Frege, (2007), p. 85. "Thus, arithmetic becomes only a further developed logic, every arithmetical proposition a logical law, albeit a derivative one." However, Frege never affirmed that arithmetical truths are *tautologies*. In any case, as we have seen, in the passage quoted, Gödel fosters a different notion of "analytic," meaning: "owing to the meaning of the terms occurring in it."

non-tautological, insofar as they refer to irreducible primitive concepts (such as the very concept of "set" or "plurality").

The second conception he wishes to oppose is the "conventionalist" one (due to Poincaré and Carnap), whereby the axioms, and the theorems derivable from them, only have a "conventional" character, and do not express an objective mathematical content. Against the Carnapian, Gödel claims that, although mathematical truths do not refer to any spatio-temporal property of reality, they still refer to something, namely, the *objectively given* realm of concepts itself and, furthermore, express relations among concepts.

As I said at the beginning of this section both, this position and claim (1) above had already been expressed by Cantor. In a crucial passage concerning the existence of mathematical objects, Cantor says:

> We can speak of the actuality of the integers, finite as well as infinite, in *two* senses; but strictly speaking they are the same two relationships in which in general the reality of any concepts and ideas can be considered. First, we may regard the integers as actual insofar as, on the basis of definitions, they occupy an entirely determinate place in our understanding, are well distinguished from all other parts of our thought, and stand to them in determinate relationships, and thus modify the substance of our mind in a determinate way; let us call this kind of reality of our numbers their *intrasubjective* or *immanent reality*. Cantor (1883), p. 895-896)[14]

In a footnote, Cantor gives a characterisation of concepts and ideas in terms of Spinozian *ideae verae*, which, in turn, can be assimilated to Plato's *ideas*:

> What I here call the "immanent" or "intrasubjective" reality of concepts or ideas ought to agree with the adjective "adequate" in the sense in which Spinoza uses this word when he says (*Ethica*, part II, def. IV): 'Per ideam adequatam intelligo ideam, quae, quatenus in se sine relatione ad objectum consideratur, omnes verae ideae proprietates sive denominationes intrinsecas habet [By adequate idea I mean an idea which, as far as it is considered as not having a relationship with an object, enjoys all intrinsic properties and designations of a real idea (my translation)].' (Cantor (1883), p. 918)

The notion of an *idea vera* can be glossed in the following way: an idea which does not lead to contradictions and which is, in addition, self-subsistent. Self-subsistent concepts, in Cantor's view, are t'ose concepts that truly *exist*. As Hallett carefully explains in Hallet (1984), there is no doubt that the kind of existence that Cantor is referring to here is genuine *platonic* existence.[15]

As we will see in the next section, Cantor also explains that concepts, although self-subsistent, should also be viewed as nodes in a logical network, where the new concepts are connected to the older already found to be existing. Furthermore, concepts are connected to each other in a well-determined, non-arbitrary way. That is, new concepts must have *determinate* properties, which distinguish them from, but, at the same time, connect them to older concepts.

14 The English translation and page numbers of all Cantor's quotations from Cantor (1883) come from Ewald (1996), p. 878-920. In reproducing it, I have also kept Ewald's annotations in square brackets.

15 See Hallett (1984), p. 17.

3. Concepts as objective constructs

Both Gödel's *conceptual realism* and Cantor's *immanence conception* are committed to the view that concepts have an objective status, namely, that they are independent, to a certain extent, from our mental faculties. Yet, using our mental faculties, we can "perceive" their objectivity, through a process of logical refinement and sharpening of the properties entering their definitions.

Both Gödel and Cantor investigated this process, but neither of them ever gave a systematic account of it. Now, we could say that Husserl's phenomenology plays a "linking" role in the transmission of Cantor's doctrines to Gödel. But Husserl was not the only philosopher who held such *objectivistic* views about concepts at the time. Frege was one further major proponent of objectivism, although the influence of Frege's conceptions upon Gödel's thought might have been considerably weaker than Husserl's.[16]

For the time being, I want to focus my attention on Cantor's theory. In a footnote of the *Grundlagen*, we find a remarkable passage concerning the crucial point of the correct procedure to generate new concepts.

> The procedure in the correct formation of concepts is in my opinion everywhere the same. One posits [setzt] a thing with properties that at the outset is nothing other than a name or a sign *A*, and then in an orderly fashion gives it different, or even infinitely many, intelligible predicates whose meaning is known on the basis of ideas that are already at hand, and which may not contradict one another. In this way one determines the connection of *A* to the concepts that are already at hand, in particular to related concepts. If one has reached the end of this process, then one has met all the preconditions for awakening the concept *A* which slumbered inside us, and it comes into being accompanied by the intrasubjective reality which is all that can be demanded of a concept; to determine its transient meaning is then a matter for metaphysics. (Cantor (1883), p. 918-919)

This passage requires some detailed interpretative work. The tacit assumption we have to keep in mind preliminarily is that concepts should be viewed as "existents" in the way indicated in the preceding section, that is, insofar as they have such a high degree of *determinacy* as to be distinguished from other existing concepts, but, at the same time, be *consistent* with them. Now, the passage under consideration tells us how the formation of new concepts conforms to such requirements.

The procedure envisaged by Cantor has three parts. One first starts with the elaboration of "signs", which may have (infinitely) many properties (in other terms, satisfy (infinitely) many predicates), all of which should not be *inconsistent* with each other. If such process is carried out successfully, then one can proceed to the next stage, wherein one declares the "birth" of a new concept. In this second stage, a new concept can be successfully declared to be born if and only if the sign "created" is correlated to something "slumbering" within us, that is, if the sign is perceived as a "reminisced" concept. Only through that can one secure the grasp of the concept

16 Frege's "objectivistic" views about concepts can be found, in particular, in Frege (1918). It should be noticed that Frege thought that the main value of his work had consisted, among other things, precisely in the clarification of the essence of *concepts* (see the letter quoted by Ricketts in Ricketts and Potter (2010), p. 149).

itself, and proceed to determine all of its properties and connections with the already available concepts. I will talk about the third stage of the process in section 7.

In view of what I have said in section 2 and of the procedure described above, Cantor's doctrines on concepts can be summarized in the following way:

1. Concepts are mental constructs that have a *meaning*, which consists of all the *properties* that can be attributed to them. Meaning is independent from the structure of the associated mental construct.
2. The meaning of a concept exists independently of our minds.
3. The formation of new concepts consists in the assignment or *clarification* of the *meaning* of new "signs," and in the *study* of their *relationships* to already existing concepts.

Let me now turn back to Husserl. It may not be accidental that all the doctrines in the bullet points above are in line with Husserl's phenomenological doctrines. The relationship between Cantor and Husserl has been recently investigated by Claire Ortiz Hill. The upshot of Ortiz Hill's examination is that Husserl may have been influenced by Cantor's Platonism and by his emphasis on *objectivism* in his early years (especially at the time of composition of the *Philosophy of Arithmetic*), and maybe also in the subsequent years, when he created his phenomenological method.[17] Regardless of this, we may still acknowledge Husserl's phenomenology as a major *trait d'union* between Cantor's and Gödel's conceptions.

Now, I do not want to present well-charted facts about Gödel's adherence to phenomenology and his use of Husserl's philosophy here.[18] I am more interested in how he saw phenomenology in connection with his and Cantor's strive for conceptual objectivity.

In Gödel (1961/?), phenomenology figures prominently among the philosophical conceptions Gödel surveys.

> Now in fact, there exists today the beginning of a science which claims to possess a systematic method for such a clarification of meaning, and that is the phenomenology founded by Husserl. [...] But one must keep clearly in mind that this phenomenology is not a science in the same sense as the other sciences. Rather it is [or in any case should be] a procedure or technique that should produce in us a new state of consciousness in which we describe in detail the basic concepts we use in our thought, or grasp other basic concepts hitherto unknown to us. (Gödel (1961/?), in (1995), p. 383).

The reader will have noticed the strong Cantorian overtones of this statement. Phenomenology is said to be a technique for generating a "new state of consciousness," which allows us to describe concepts hitherto unknown to us, something which

17 The bulk of Ortiz Hill's careful work on the relationships between Cantor and Husserl can be found in Ortiz Hill (1997). See also Ortiz Hill (2004). In (1997), Hill identifies three stages of influence of Cantor's thought on Husserl. But she clearly acknowledges, although only conjecturally, that there might be a further, fourth stage, that she does not examine, which "...would consist of the assimilation of certain of Cantor's ideas into Husserl's phenomenology and extends far beyond the compass of this study. Here it would be a matter of studying the relationship between Cantor's theories and, for example, Husserl's *Mannigfaltigkeitslehre*, his theories about eidetic intuition, the phenomenological reductions, noemata, horizons, infinity, whole and part..." (p. 166).

18 On this point, see, in particular, Wang's mentioned books, Kennedy - van Atten (2003), Tieszen (2005), (2011), (2012), Hauser (2006) and Crocco (2006). Føllesdal's introduction to Gödel (1961/?), p. 364-373 also provides interesting insights.

seems to echo the "awakening" of concepts Cantor was referring to in the passage quoted above.

As already said, Gödel absorbed phenomenology especially in the late stage of his thought and found in it an answer to the problem of whether we have a method to establish the "foundedness" of concepts.

In essence, Gödel was looking for a "science" of concepts, which may have realised the Leibnizian ideals of a *characteristica universalis* and *calculus ratiocinator*. Such a science, producing a clarification of the *meaning* of concepts, he found in Husserl's phenomenology. But with this choice, it seems to me, he was also completely and determinately fulfilling Cantor's ideal, as described and encapsulated in the conception we have reviewed above.

It should be noticed that Gödel's project is always subservient to his programme of investigating new axioms of set theory. Therefore, his reliance upon the idea of finding a systematic method for analysing concepts should be seen in connection with what he says in his Cantor paper about the role of set-theoretic axioms and concepts underlying them:

> Similarly also the concept "property of set" (the second of the primitive terms of set theory) can constantly be enlarged, and furthermore concepts of "property of property of set" etc. be introduced whereby new axioms are obtained, which, however, as to their consequences for propositions referring to limited domains of sets (such as the continuum hypothesis) are contained in the axioms depending on the concept of set. (Gödel (1947), in (1990), fn. 17, p. 181).

Further, in the same work, he suggests:

> But probably there exist others based on hitherto unknown principles; also there may exist, besides the ordinary axioms, the axioms of infinity and the axioms mentioned in footnote 17, other (hitherto unknown) axioms of set theory which a more profound understanding of the concepts underlying logic and mathematics would enable us to recognize as implied by these concepts. (Gödel (1947), in (1990), p. 182).

Conceptual objectivity, in turn, obtained through progressive logical refinements and clarifications, lies at the roots of *mathematical evidence*, the ultimate ideal that Gödel was pursuing. The belief in mathematical evidence is what makes him conjecture that undecidable statements such as CH (the Continuum Hypothesis)[19] might be settled in the future. He seems to draw such faith in mathematical evidence from phenomenology, but it is unclear whether there is any direct connection between Husserl's ideas and Gödel's belief in the solvability of all set-theoretic problems.[20]

In any case, the following statement can be read in the light of such belief:

19 The Continuum Problem is the problem of determining the cardinality of $\mathbb{R}$ (denoted c). and CH is Cantor's conjecture that $c = \aleph_1$ See footnote 27 below.

20 The problem with Gödel's claim that set-theoretic statements might be shown to have a determinate and unique truth-value as a result of conceptual refinements is explained very neatly by Hauser in Hauser (2006), p. 539-40: "On this view, the meaning of the continuum problem is tied to an unfolding of concepts through successive refinements of mathematical intuition. One difficulty is why it should lead to a unique resolution of CH, for our intuitions could conceivably evolve into different directions inducing us to formulate axioms with opposite outcomes of CH."

> The mere psychological fact of the existence of an intuition which is sufficiently clear to produce the axioms of set theory and an open series of extensions of them suffices to give meaning to the question of the truth or falsity of Cantor's continuum hypothesis. (Gödel (1964), in (1990), p. 268).

At a certain stage, this faith in the objectivity of concepts led Gödel to proceed well beyond the Cantorian (and maybe Husserlian) ideal of a clarification of their meaning, to set up a general theory of concepts, which would work as sort of axiomatised metaphysics.

In his conversations with Wang, Gödel says, among other things:

> 8.6.20 Even though we do not have a developed theory of concepts, we know enough about concepts to know that we can have also something like a hierarchy of concepts (or also of classes) which resembles the hierarchy of sets and contains it as a segment. But such a hierarchy is derivative from and peripheral to the theory of concepts; it also occupies a quite different position; for example, it cannot satisfy the condition of including the concept of *concept* which applies to itself or the universe of all classes that belong to themselves. To take such a hierarchy as the theory of concepts is an example of trying to eliminate the intensional paradoxes in an arbitrary manner. (Wang (1996), p. 278).

In the passage above, Gödel hints at connections between objects and concepts, as being reflected by the connection between sets and concepts. We will see later on how this project was further substantiated by further philosophical conceptions, which were also discussed by Cantor.

4. Anti-subjectivism

The frequent target of Cantor's philosophical invective is Kant. In Cantor's view, Kant should be held to be the main responsible for introducing and advocating *subjectivism* in philosophy. A subjectivistic conception can only give us knowledge of "appearances," not of "stable," unchanging forms, as Plato wanted. Cantor says:

> Only since the growth of modern empiricism, sensualism, and scepticism, as well as of the Kantian criticism that grows out of them, have people believed that the source of knowledge and certainty is to be found in the senses or in the so-called pure form of intuition of the world of appearances, and that they must confine themselves to them. (Cantor (1883), p. 918).

In that footnote, Cantor also contrasts Kant's conception with Plato's, Spinoza's and Leibniz's. The association of *empiricism*, *sensualism* and *scepticism*, on the one hand, and of Plato, Leibniz and Spinoza, on the other, seems, to say the least, too quick. Such schematisations may have had an echo in that presented by Gödel in Gödel (1961/?), which grouped philosophical conceptions into left-wing and right-wing. But maybe we should not ascribe too much value to such quick distinctions. Cantor contrasts platonic "objectivism" to Kantian "subjectivism" for his own purposes, that is, defending the conceptual legitimacy of the *actual infinite*, something that Kant would have certainly found preposterous. Strongly related to such "objectivistic" attitudes is also Cantor's inclination to *arithmetical purism*, which he must have subscribed to at a very early stage in his career. As a pupil of Weierstrass', Cantor had witnessed and, in a sense, had been involved in the programme of *arithmetisation* of analysis. In his work, the programme is, in a sense, further reflected by the creation of the *transfinite*, and by his general belief that numbers, both finite and transfinite, are the building blocks of reality.

One further instance of this attitude can be seen at work in Cantor's conception of the *continuum*. This latter, he thought, could be understood only through a process of logical simplification, which could provide us with solid conceptual knowledge of its internal components. This process he thought he had carried out with his theory of derived point-sets and related cardinal powers. But Kant's notion of continuity, like that of Aristotle and of the Scholastic philosophers, he suggests, is dependent upon the *a priori* intuition of *space* and *time*, which, in turn, cannot provide us with any conceptual knowledge of the phenomenal world.

> [...]the continuum is thought to be an unanalysable concept or, as others express themselves, a pure *a priori* intuition which is scarcely susceptible to a determination through concepts. Every arithmetical attempt at determination of this *mysterium* is looked on as a forbidden encroachment and repulsed with due vigour. (Cantor (1883), p. 903).

Time and space, he continues, are only *syncategorematic*, that is, *relational* concepts, and they have failed to produce, via Kant's conception, any tangible progress in our knowledge. He continues:

> Such a thing as *objective* or *absolute time* never occurs in nature, and therefore *time* cannot be regarded as the measure of *motion*; far rather motion as the measure of *time* - were it not that *time*, even in the modest role of a *subjective necessary a priori* form of intuition [Anschauungsform] has not been able to produce any fruitful, incontestable success, although since Kant the time for this has not been lacking. (Cantor (1883), p. 904).

Gödel expressed the same dissatisfaction with Kant's "subjectivism." For a Platonist, and a conceptual realist like him, this is hardly surprising. However, his relationship with Kant's doctrines is more articulated. The main point of friction with Kant is Gödel's notion of *intuition*.

While intuition in Kant's system is related to the work of the intellect, which "elaborates" the representations of our senses, Gödel's intuition is something stronger and deeper: it should give us knowledge of *objects* (and *concepts*) which are seemingly formed by us, but, in fact, already *exist* within us. Intuition (*Anschauung*), in Kant's view, serves the purpose of "constructing" concepts; in Gödel's, that of "seeing" or "perceiving" objects (and concepts thereof).[21] However, this latter process also needs some form of elaboration. Gödel crucially explains:

> It should be noted that mathematical intuition need not be conceived as a faculty giving an *immediate* knowledge of the objects concerned. Rather it seems that, as in the case of physical experience, we *form* our ideas also of t'ose objects on the basis of something else which *is* immediately given. [...] Evidently the "given" underlying mathematics is closely related to the abstract elements contained in our empirical ideas. It by no means follows, however, that the data of this second kind, because they cannot be associated with actions of certaines things upon our sense organs, are something purely subjective, as Kant asserted. (Gödel (1964), in (1990), p. 268).

21 See footnote 12 above.

Gödel seems to envisage a role for intuition which is analogous to that envisaged by Kant in his account, that is, that of a faculty which operates on something *given* in order to "derive" something else, but the two conceptions differ to a substantial extent. While, for Kant, intuition acts on sensory data to derive conceptual information, Gödel's intuition acts on some given conceptual content, to produce other forms of conceptual content (such as *mathematical objects*). However, for Gödel, such an intellectual operation has the character of objectivity, insofar as it provides us with knowledge of *ideal forms*. Such move is seen by most commentators as connected to Gödel's adhering to Husserl's *transcendentalism*. Husserl's transcendental intuition, unlike Kant's, has a fully objective character, as it is directed at "the things themselves," and operates through the process called *eidetic reduction*.[22] Thus, Gödel seems to re-use Kant's original conception to produce something like a more powerful version of it.[23]

As already said, Gödel's attitude to Kant is more articulated and varied than Cantor's. His interpretation of Kant's notion of "phenomenon" is, in this respect, revealing. In his Gödel (1949), Gödel makes the somewhat baffling claim that Einstein's relativity theory, by showing the deceitfulness of the notion of *temporal simultaneity*, has confirmed

> […] the view of t'ose philosophers who, like Parmenides, Kant, and the modern idealists, deny the objectivity of change and consider change as an illusion or an appearance due to our special mode of perception. (Gödel (1949), in (1990), p. 202).

In this passage, Kant is associated with such philosophers as Parmenides, who had explicitly denied the reality of motion. In Gödel's perspective Kant is more the philosopher who has revealed the deceitfulness of the phenomenal world, rather than, as in Cantor, the strong advocate of subjectivism. However, we have seen that, elsewhere, Gödel, not unlike Cantor, had judged Kant's subjectivism a major shortcoming.

5. Set-theoretic Platonism

Gödel's ontological conceptions gradually evolved towards a form of thoroughgoing Platonism, more specifically, Platonism about *sets*. In published work, his platonistic leanings are declared first in his Russell paper (1944) and, afterwards, re-asserted in the Cantor paper (1947). In 1944, he wrote:

---

22 The bulk of Husserl's phenomenological ideas can be found in the 3 volumes of the *Ideen*, Husserl (1980), (1982), and (1989). See also Husserl (1988). A quick review of the main concepts of phenomenology can be found in one of the articles/books I mentioned above, footnote 16, or, for instance, in Christian Beyer's entry "Husserl" in the *Stanford Philosophical Encyclopedia*, 2013, which also includes an up-to-date bibliography.

23 See, in particular, Parsons, (1995), p. 56-70, concerning the difficulties with Gödel's notion of intuition. Parsons' interpretation, especially of Gödel's quotations from Gödel (1964), seems inclined to explain away the presence of phenomenological elements in Gödel's thought. For instance, with regard to the notion of "immediately given," he says: "The picture resembles Kant's, for whom knowledge of objects has as "components" a priori intuition and concepts. It is un-Kantian to think of pure concepts as given, immediately or otherwise. But Gödel's picture seems clearly to be that our conceptions of physical objects have to be constructed from elements, call them primitives, that are given, and that some of them (whether or not they are much like Kant's categories) must be abstract and conceptual." (p. 68).

> It seems to me that the assumption of such objects is quite as legitimate as the assumption of physical bodies and there is quite as much reason to believe in their existence. (Gödel (1944), in (1990), p. 128).

In the Cantor paper, the *existence* of a set-theoretic reality is even more unequivocally and firmly asserted. Gödel's argument, expounded there, aims to show that (CH) necessarily has a determinate truth-value, if one believes that set theory describes a *well-determined reality*. As a consequence, it makes sense to search for its truth-value, even after one has shown that such truth-value cannot be determined by the ZFC axioms.

> It is to be noted, however, that on the basis of the point of view here adopted, a proof of the undecidability of Cantor's conjecture from the accepted axioms of set theory (in contradistinction, e.g., to the proof of the transcendency of) would by no means solve the problem. For if the meanings of the primitive terms of set theory [...] are accepted as sound, it follows that the set-theoretical concepts and theorems describe some well-determined reality, in which Cantor's conjecture must be either true or false. Hence its undecidability from the axioms being assumed today can only mean that these axioms do not contain a complete description of that reality. (Gödel (1964), in (1990), p. 260).

Later, in his unpublished Gibbs lecture (Gödel (1951), Gödel sketched some further arguments in favour of Platonism. I will not deal with those arguments here. Rather, I wish to examine whether and to what extent Gödel's set-theoretic Platonism is indebted to Cantor's set-theoretic Platonism.

In section 2, I have briefly discussed the notion of *intra-subjective* existence in Cantor's *Grundlagen* and shown how it is connected to a peculiar form of conceptual realism. In the same footnote in which he gives an account of the notion of *idea vera*, as constituting the historical and conceptual ground for his notion of *immanent existence*, Cantor also claims that his conception of "set" fits perfectly into Plato's conception of a *third gender* of being which is defined, in the *Philebus*, as μικτόν [*miktón*].

> In general, by a "manifold" or "set" I understand every multiplicity [jedes Viele] which can be thought of as one, i.e. every aggregate [Inbegriff] of determinate elements which can be united into a whole by some law. I believe that I am defining something akin to the Platonic εἶδος or ἰδέα as well as to that which Plato called μικτόν in his dialogue "Philebus or the Supreme Good". He contrasts this to the ἄπειρον (i.e. the unbounded, undetermined, which I call the improper infinite) as well as to the πέρας, i.e. the boundary; and he explains it as an ordered "jumble" of both. Plato himself indicates that these concepts are of Pythagorean origin. (Cantor (1883), in p. 916).

This scanty remark affords us three important pieces of information about Cantor's conception of sets:

1. Sets arise from putting together elements of a multitude, by using a specified 'uniting' law.
2. The notion of "set" can be successfully compared to Plato's notion of ἰδέα, that is, "intelligible form". Such is the ontological status also of the μικτόν, that is, mixed entity, that Cantor is referring to here as corresponding to his notion of set.
3. Via Plato's conceptualisation, the notion of "set" is, in turn, related to some Pythagorean conception of "set."

The ideas underlying (1) and, partly (3), have been extensively explored by Hallett in his Hallet (1984), and here I will use his interpretation of Cantor's set-theoretic conception as based on a theory of "ones".[24]

Sets would be constituted of *irreducible unities*, which are, afterwards, transformed into *new unities*, which are the sets themselves. Such conception is used by Cantor to define numbers. Hence, it is an essential ingredient of Cantor's strategy to reduce all numbers, finite and transfinite, to *sets*. In other passages, cardinal numbers are also defined as being *abstracted* from certain particular collections (which are, in turn, collections of *ordinals*). Cantor's set-theoretic ontology thus collapses to well-ordered collections of given number-sets consisting of "unities", the *ordinals*, from which *cardinal numbers* are subsequently abstracted. On this picture, sets are already (ideal) numbers. This is clearly shown by the reference to platonic ἰδέαι [*idéai*] in (2).

As we have seen, Cantor's definition of number-sets uses a complicated theory appearing in Plato's *Philebus*, whereby the generation of numbers is seen as the outcome of a dialectical process involving the interaction of the ἄπειρον [*ápeiron*], the "Unlimited", with the πέρας [*péras*], the "Limit."

Although we are not very sure what the nature of the ἄπειρον to which Plato refers in the *Philebus* is, it is most likely that the concept can be interpreted as corresponding to the *potential infinite*. The πέρας, thus, operates on (merges with) the potential infinite in a *process of determination,* which yields (ideal) numerical entities (this also explains why such entities are called μικτά [*miktá*], "mixed": because they participate of both concepts).[25]

It should be noted that Cantor needs to construe the platonic process in a different way: μικτά have now become determinations of the *actual*, not the *potential* infinite. But determinations of the actual infinite are nothing other than transfinite numbers. It is no surprise, then, that in another passage from Cantor (1885), Cantor describes *transfinite ordinals* as being ἀριθμοὶ νοητοί [*arithmoì noetoí*] or εἰδητικοί [*eidetikoí*], using a terminology that is reminiscent of that used by Plato.[26]

In published work, Gödel never gave any further details about his set-theoretic Platonism. However, in conversations with Wang, we find quite a few remarks about the nature of "sets," which seem to align his position with, or even clearly echo, Cantor's conceptions. For instance, he says:

24 See, in particular, Hallett, (1984), p. 128-142.

25 In Cantor (1887-1888), reprinted in (1932), p. 380, Cantor also uses the Greek word μονάς [*monás*] to refer to number-sets, a term which is borrowed from a definition in Euclid's *Elements* he mentions in that work. He says (my translation): "Cardinal numbers as well as order-types are *simple* conceptual formations; each of them is a *true Unity* (μονάς), as in them a plurality and multiplicity of *Ones* is unitarily bound together [Die Kardinalzahlen sowohl, wie die Ordinungstypen sind *einfache* Begriffsbildungen; jede von ihnen ist eine *wahre Einheit* (μονάς), weil in ihr eine Vielheit und Mannigfaltigkeit von *Einsen einheitlich* verbunden ist]." He also reports instances of the notion of μονάς as can be found in Nicomachus' *Institutio Arithmetica* and Leibniz. Nicomachus' neo-Pythagorean view about numbers also implied that they are συστήματα μονάδων [*systémata monádon*], that is, *aggregations of unities (monads)*. In Cantor's quoted passage from Leibniz's *De arte combinatoria*, Leibniz says: "Abstractum autem ab uno est unitas, *ipsumque totum abstractum ex unitatibus, seu totalitas, dicitur numerus*."

26 Cantor (1885), reprinted in (1932), p. 372.

> 8.2.1 [...] It is a primitive idea of our thinking to think of many objects as one object. We have such ones in our mind and combine them to form new ones.
>
> 8.2.2 [...] Sets are multitudes which are also unities.
>
> 8.2.3 This [fact]—that sets exist—is the main objective fact of mathematics which we have not made in some sense: it is only the evolution of mathematics which has led us to see this important fact. [...], there must be something objective in the forming of unities. (Wang (1996), p. 254).

Gödel uses a language, which overlaps Cantor's: he talks of "unities", "multitudes, ""many objects as one". Such language aims to convey the idea, very similar to Cantor's, that sets are *new unities*, arising out of multitudes. This is particularly relevant, insofar as, as we shall see in section 7, Gödel may have thought to embed Cantor's conception of sets, qua μονάδες [*monádes*], into Leibniz's conception of *monads*.

In what he says above, a second fundamental point is Gödel's idea that the process of "uniting" objects into such "unities" is proof of the objectivity of mathematics. One should be wary of seeing any "constructive" overtone in evoking a process of "unification." If the process itself is possible, it is only because such unifications can be carried out successfully *a priori*.

I will describe one more, fundamental feature of this form of set-theoretic Platonism in the next section.

6. The absolute infinite and the universe of sets

In his Cantor paper, Gödel famously proposes the extension of the system of axioms of set theory to settle open problems in set theory and mathematics. His argument views *strong* axioms of infinity, that is, axioms positing the existence of *large cardinals*, as the most suitable axiom candidates for extending ZFC. At that time, Gödel thought that they might, in particular, have a significant impact on such problems as the Continuum Problem.[27]

Gödel's argument in favour of the acceptance of such large cardinal hypotheses is an argument from "intrinsic necessity," that is, it is based on considerations related to the features of the *iterative concept of set*. The case for the extension of ZFC is introduced in the following way:

> First of all the axioms of set theory by no means form a system closed in itself, but, quite on the contrary, the very concept of set on which they are based suggests their extension by new axioms which assert the existence of still further iterations of the operation "set of." (Gödel (1964), in (1990), p. 260)

In footnote 18, he explains:

> Similarly the concept "property of set" (the second of the primitive terms of set theory) suggests continued extensions of the axioms referring to it. Furthermore, concepts of "property of property of set" etc. can be introduced. (Gödel (1964), in (1990), p. 260).

27 At the time of composition of Gödel (1947), and of its revision in Gödel (1964), it was not known that large cardinal axioms do not fix the power of the continuum, as Gödel had conjectured that they might. This was first shown by Solovay and Lévy in Lévy and Solovay (1967) using measurables, but the result generalises to all known large cardinals. An analogous result for smaller large cardinals had already been proved by Cohen in Cohen (1964). See Kanamori, (2003), p. 126.

In his conversations with Wang, we find further details about how new axioms should reflect the iterative concept of set. Some remarks contain mention of such properties of the universe of sets as *reflection*, *uniformity*, *closure*, etc., all of which could orient the selection of new axioms.[28]

The principle I am mostly interested in here is *reflection*. This is how Gödel comments on it:

> 8.7.3 *Reflection principle*. The universe of sets is structurally indefinable. One possible way to make this statement precise is the following: The universe of sets cannot be uniquely characterized (i.e., distinguished from all its initial segments) by any internal structural property of the membership relation [...]. This principle may be considered a generalization of the closure property. Further generalizations and refinements are in the making in recent literature. The totality of all sets is, in some sense, indescribable. (Wang (1996), p. 280-281).

In this account of the principle, the notion of the *indescribability* of the universe of sets plays a crucial role. But where did Gödel get it? And why did he see it as essential for his purposes? In order to answer these questions, we have to do some careful interpretive work.

In another revealing passage reported by Wang, Gödel talks about the so-called von Neumann axiom, and describes it in the following way:

> 8.3.7 As has been shown by von Neumann, a multitude is a set if and only if it is smaller than the universe of all sets. This is understandable from the objective viewpoint, since one object in the whole universe must be small compared with the universe and small multitudes of objects should form unities because being small is an intrinsic property of such multitudes. (Wang (1996), p. 262).

As Wang explains, the "axiom" Gödel attributes to von Neumann, is, in fact, due to Cantor,[29] and made his first appearance in a letter that this latter sent to Dedekind in 1899.[30] In that letter, responding to Dedekind's concerns that Cantor's notion of "sethood" might be unclear and paradox-laden, Cantor draws a distinction between *consistent* and *inconsistent* multiplicities. The latter are what would become to be described subsequently by von Neumann as "classes," whereas the former are "sets" in the proper sense. Classes are collections that are too "big" to be considered sets.[31]

This conception lies at the roots of the so-called *limitation of size* doctrine. It implies, in particular, that classes do not have a transfinite size. To be more precise, the size of inconsistent multiplicities is that of the *absolute infinite*. In turn, the absolute infinite is the infinite of God.[32]

28 Wang discusses them in both Wang (1974) and (1996), but this latter contains a more detailed account.

29 Wang (1996), p. 261: "[...] Cantor called multitudes "like" V *inconsistent multitudes*, and introduced a general principle to distinguish them from sets."

30 English translation in (Ewald (1996), p. 931-935.

31 However, as we have seen (footnote 4), Gödel was fully aware of the fact that Cantor, not von Neumann, had first introduced the distinction between sets and classes. On this point, see also van Atten (2009).

32 See Hallett (1984), in particular, p. 164-194. With regard to von Neumann's re-statement of Cantor's principle (what Gödel calls "von Neumann axiom"), see, in particular, p. 286-298.

In Cantor (1885), a summary of the different forms of *actual infinite* provides the following definition of the absolute infinite:

> One can question the Actual Infinite in three main forms: first, insofar as it is in *Deo extramundano aeterno omnipotenti sive natura naturante*, that is the *Absolute*, second, insofar as it can be found *in concreto seu in natura naturata*, and I call it *Transfinitum*, and third, it can be questioned *in abstracto*, that is, insofar as it can be understood by human beings in the form of *actual infinite* or, as I have called them, of *transfinite numbers*, or in the more general form of *transfinite order-types* (ἀριθμοί or εἰδητικοί) [Man kann nämlich das A.-U. in *drei Hauptbeziehungen* in Frage stellen: *erstens, sofern es in Deo extramundano aeterno omnipotenti sive natura naturante*, wo es das *Absolute* heißt, *zweitens* sofern es *in concreto seu in natura naturata* vorkommt, wo ich es *Transfinitum* nenne und *drittens* kann das A.-U. *in abstracto* in Frage gezogen werden, d.h. sofern es von der menschlichen Erkenntnis in Form von *aktual-unendlichen*, oder wie ich sie gennant habe, von *Transfiniten Zahlen* in der noch allgemeineren Form der *Transfiniten Ordnungstypen* (ἀριθμοί oder ειδητικοί) aufgefaßt werden könne] (Cantor (1885), in (1932), p. 372, my translation).

Some years before, in the *Grundlagen*, after expounding the point of view of various authors, he had referred to the absolute infinite in the following way:

> However different the theories of these writers may be, in their judgement of the finite and infinite they essentially agree that finiteness is part of the concept of number and that the true infinite or Absolute, which is in God, permits no determination whatsoever. As to the latter point I fully agree, and cannot do otherwise; the proposition: "omnis determinatio est negatio" is for me entirely beyond question. (Cantor (1883), in p. 890-891).

In footnote, he adds:

> The absolute can only be acknowledged [anerkannt] but never known [erkannt]—and not even approximately known. [...] As Albrecht von Haller says of eternity: 'I attain to the enormous number, but you, O eternity, lie always ahead of me.' (Cantor (1883), p. 916).

In that work, Cantor uses theological tones, whereas in his letter to Dedekind, the *absolute infinite* is viewed as an eminently mathematical phenomenon, relating to collections too "big" to be measured (such as the class of all ordinals or the class of all cardinals). Such oscillation (and maybe tension) has been acknowledged and examined by the secondary literature.[33] It seems to me that, in view of all these interpretive efforts, one can, at least, say that Cantor's absolute infinite plays a dual role in his work:

1. To provide a general justification for his *limitation of size* doctrine, whereby one should distinguish between *consistent* and *inconsistent* multiplicities.

33 See, in particular, Hallett, (1984), p. 41-48 and 165-176. Wang discusses Cantor's conception in connection with Gödel's criteria for introducing new axioms especially in Wang, (1996), p. 188-190. Jané addresses Cantor's conception in full in Jané (1995). Jané lays strong emphasis on the tension between the idea that the *Absolute* cannot be measured (and determined), and the fact that it can still be seen as a sort of "quantitative maximum" for the actual infinite, a tension which was perceived and addressed by Cantor in different ways over his career. Jané thinks that, in the end, God's *absoluteness* and *mathematical absoluteness* fell apart, as Cantor was forced to accept, mathematically, that the *absolute infinite* is not a form of the *actual infinite*.

2. To rebuke successfully the objection that measuring the infinite might lead to some kind of rational *pantheism*. God's infinity is still clearly and determinately distinguished from that of the transfinite, through attributing to him an absolute infinity.

As we have seen, the absolute infinite characterises itself for being, essentially, indescribable, indefinable. However, in a sense, although indefinable, it can still be seen as endowed with some properties, and be thought of as a sort of *aggregate*. However, these properties cannot be ascribed to it directly (in view of its *indefinability*), but to rank initial segments of it, which would, thus, in a sense, reflect it. This form of *reflection* Gödel sees at work in one further set-theoretic principle, that he defines as the "basic axiom of set theory," Ackermann's axiom. This axiom does not mention the reflection principle directly, but can be seen as a consequence of it.

> (A) *Ackermann's axiom*. Let y and z be in V and $F(x, y, z)$ be an open sentence not containing V, such that, for all x, if $F(x, y, z)$, then x is in V. There is then some u in V, such that, for all x, $F(x, y, z)$ if and only if x belongs to u. (Wang (1996), p. 282).

Ackermann's axiom can be glossed in the following way: if there exists a set-theoretic property, whose formulation does not mention V, then there must be some set *in* V which satisfies such property. It is crucial to mention that the set under consideration is in V, that is, it is crucial to "reflect" V onto a set, which has that property.

It is Gödel himself who acknowledges the connection between Ackermann's axiom and Cantor's absolute. He says in 8.7.9: "All the principles for setting up the axioms of set theory should be reducible to a form of Ackermann's principle: The Absolute is unknowable." (Wang (1996), p. 283).

So, we have finally made the following picture available to ourselves. The doctrine of the Absolute, first expounded by Cantor, became a basic principle of his *limitation of size* doctrine. This latter, in turn, was made into an axiom by von Neumann. Such an axiom has consequences on the *definability* of the universe of sets, and, thus, encourages the discovery of such properties as reflection. The reflection principle, in the version given by Gödel, is, in turn, encapsulated by one single set-theoretic principle: Ackermann's axiom, that Gödel believed should be considered the most basic axiom of set theory. Thus, Gödel's doctrine concerning the extension of ZFC, insofar as it is essentially based on the *reflection principle*, is largely indebted to one single conception, Cantor's conception of the *absolute infinite*.[34]

The use of Cantor's conception might also be viewed as instrumental for Gödel's own parallel belief of the *inexhaustibility of mathematics*. Gödel addresses the notion of *inexhaustibility* in his Gibbs lecture, but there is no connection there between this latter and Cantor's absolute infinite. However, the connection can be reconstructed

34 However, this form of reflection principle does not justify *very large* cardinal hypotheses. Gödel was maybe already aware of this fact, when, in a footnote added in 1966 to Gödel (1964), referring to the axiom asserting the existence of a *measurable cardinal*, he stated (p. 260-61): "That these axioms are implied by the general concept of set in the same sense as Mahlo's has not been made clear yet [...]", whereas he is aware of the fact that small large cardinals such as Mahlo's can be connected to *reflection* successfully: "Mahlo's axioms have been derived from a general principle about the totality of sets which was first introduced by Levy (1960). It gives rise to a hierarchy of precise formulations." For details about this hierarchy, see Kanamori, (2003), p. 57-59.

indirectly. In the text we possess, Gödel says that the collection of all mathematical *truths* represents "objective" mathematics, as opposed to "subjective" or "mechanised" mathematics, which consists of all *demonstrable* propositions. The first incompleteness theorem shows that the collection of mathematical truths is *larger* than that of demonstrable truths.[35]

Thus, Gödel construes his incompleteness theorem as proof that the realm of "objective mathematics" is *larger* than that of "mechanical" (or "subjective") mathematics.[36] The gap between the two cannot be filled, because our grasp of objective mathematics is incomplete. This fact is seen, Gödel continues, in relation to the goal of providing something like a "definitive" axiomatisation of set theory. The task is impossible, for the following reasons:

> [...]if one attacks this problem, the result is quite different from what one would have expected. Instead of ending up with a finite number of axioms, as in geometry, one is faced with an infinite series of axioms, which can be extended further and further, without any end being visible and, apparently, without any possibility of comprising all these axioms in a finite rule producing them. (Gödel (1951), in (1995), p. 306).

In turn, the process of formation of ever new axioms never ceases, since, as Gödel explains, further ordinals can always be formed. This leads him to conclude that:

> You will realize, I think, that we are still not at an end, nor can there ever be an end to *this* procedure of forming the axioms, because the very formulation of the axioms up to a certains stages gives rise to the next axiom. (Gödel (195) in (1995), p. 307).

To summarise, Cantor's conception of the absolute infinite has two bearings on Gödel's set-theoretic Platonism. One can see it as connected, essentially, to two doctrines, which Gödel, at some point, held:

1. The universe of sets is *indescribable*, yet is, in a sense, characterisable. Since it is not describable, though, all of its characterising properties reflect onto rank initial segments (*reflection principle*).
2. There is no endpoint in the series of axioms expressing ways to generate sets. This is also the reason for the *incompletability* of mathematics.

7. Trans-subjective ("transient") existence

Alongside the "immanent" one, Cantor also mentions one further form of "existence" of mathematical objects:

> But then, reality can also be ascribed to numbers to the extent that they must be taken as an expression or copy of the events and relationships in the external world which confronts the intellect, or to the extent that, for instance, the various number-classes (I), (II), (III), etc. are representatives of powers that actually occur in physical and

35 More precisely, Gödel's First Incompleteness Theorem shows that any theory of arithmetic of the same strength as PA (Peano Arithmetic) is *incomplete*, namely, that it does not prove all *arithmetical truths*.

36 However, it could actually turn out that already "subjective" mathematics is larger than "mechanical" mathematics, should human minds prove to be *stronger* than machines, something Gödel had already cast as a conjecture in the Gibbs lecture and which is again reported by Wang in Wang (1996), p. 186. I am indebted to Gabriella Crocco for pointing me to the subtle difference between "subjective" and "mechanical" in Gödel's formulations.

> mental nature. I call this second kind of reality the *transsubjective* or the *transient reality* of the integers. (Cantor (1883), p. 895-896).

Cantor's mention of a *trans-subjective* form of reality of mathematical entities and, in particular, his idea that number-classes (that is, the ℵs) are, somehow, instantiated in the physical reality have baffled many commentators.

In order to make sense of such statements and see their connections with Gödel's analogous claims, we have to expound some further Cantorian conceptions relating to the nature of physical reality.

As I have pointed out many times, Cantor's views are essentially metaphysical. For instance, concerning the status of *analytic mechanics* and *mathematical physics*, he says:

> These disciplines are, in my opinion, in their foundations as well as in their aims *metaphysical*; if they seek to make themselves free from metaphysics, as has been recently proposed by a celebrated physicist [Gustav Kirchhoff], they degenerate into a "description of nature" in which the fresh breeze of free mathematical thought—as well as the power of *explaining* and *justifying* natural phenomena—must be absent. (Cantor (1883), p. 897).

Always in the *Grundlagen*, he encourages physicists' efforts to develop a different form of physics, which may overcome the limitations of a purely *mechanical explanation* of natural forces, and, instead, be committed to some form of *organicism*, which he saw in essential accordance with the systems of Leibniz and Spinoza. This leads him to assert that:

> For, alongside of (or in place of) the mechanical explanation [Erklärung] of nature (which inside its proper domain has all the aids and advantages of mathematical analysis at its disposal, but whose one-sidedness and insufficiency have been strikingly exposed by Kant) there has until now not been even the start of an *organic* explanation of nature that would attempt to go further and that would be armed with the same mathematical rigour; this organic explanation can, I believe, be initiated only by taking up afresh and continuing the works and endeavours of t'ose thinkers [i.e., Leibniz and Spinoza, *my note*]. (Cantor (1883), p. 892).

But, in just what sense would an organic explanation of the physical world be in accordance with Leibniz's and Spinoza's intuitions? Cantor's use of Spinozian *ideae verae* and *adaequatae* has already been discussed. Now, in Spinoza's conception, ideas are features of reality belonging to both the *material* and the *mental* ontological realms. Cantor's views seem to follow suit. *Transfinite* concepts, insofar as they exist immanently, should have some physical correlates as any other concept. In particular, this justifies Cantor's claim that the ℵs (cardinal powers) will *occur in* or *represent* aspects of the real world.

Some further details of this conception are tied more prominently to Leibniz's doctrines. Such phenomenal aspects of the real world as the *continuum*, *matter* and *corporeal forms* should be reduced to logically simpler entities, that Cantor calls *atoms* in the *Grundlagen*. These atoms should not be construed as the atoms of the *atomistic* tradition, but rather as something comparable to Leibniz's *monads*, infinitely small points without parts. In the *Grundlagen*, he mentions their presence in the physical reality incidentally, in the following passage about Bolzano's view on the actual infinite:

> The proper-infinite, as we find it in, for example, well-defined point-sets or in the construction of bodies from point-atoms [punktuellen Atomen] (I thus do not mean here the chemical-physical (Democritean) atoms, because I cannot hold them for existent, either in thought or reality, although much that is useful has been achieved up to a certain limit by this fiction) has found its most determined defender etc. (Cantor (1883), in p. 894).

However, in (Cantor (1885b), Cantor's Leibnizian characterisation of "atoms" is more explicit. In the following passage, he also clarifies the connection between monads and transfinite cardinal numbers. He says:

> Following Leibniz, I call the *simple* elements of nature, from whose composition *matter* is, in some sense, constituted, *Monads* or *Unities*. […] it is many years that I have formulated the *hypothesis* that the *power* of *corporeal matter* is what I have called, in my investigations, the *first power* and that, on the contrary, the power of *ethereal matter* is the second class [Ich nenne in Anschluss an Leibniz die *einfachen* Elemente der Natur, aus deren Zusammensetzung in gewißem Sinne die *Materie* hervorgeht, *Monaden* oder *Einheiten* […] …in dieser Beziehung habe ich mir schon vor Jahren die *Hypothese* gebildet, daß die Mächtigkeit der *Körpermaterie* diejenige ist, welche ich in meinem Untersuchungen die *erste* Mächtigkeit nenne, daß dagegen die Mächtigkeit der *Äthermaterie* die *zweite* ist]. (Cantor (1885b), in (1932), p. 275 and 276, my translation)[37].

Mention of monads, of point-atoms, of an *organic*, as opposed to *materialist*, explanation of the physical world, gives us a taste of the broadness of Cantor's conception of the *trans-subjective*.[38]

Is there an analogous mention of a trans-subjective form of existence in Gödel's thought? Although we cannot find any direct connection to Cantor's conception, we find, in some of Gödel's incidental observations, reference to a well-structured metaphysical ontology. Especially on the grounds of what Wang reports in his (1996), it seems reasonable to assert that also Gödel took *monads* to be the essential constituents of the world. Like Cantor, he also thought that *monadology* might provide a different, alternative description of the physical world. My claims are substantiated by what he says in the following remarks:

> 9.1.20 We should *describe* the world by applying these fundamental ideas: the world as consisting of monads, the properties (activities) of the monads, the laws governing them, and the representations (of the world in the monads).
>
> 9.1.21 The simplest substances of the world are the monads.
>
> 9.1.22 Nature is broader than the physical world, which is inanimate. It also contains animal feelings, as well as human beings and consciousness. (Wang (1996), p. 295).

---

37 Using the subsequent $\aleph$-notation, the *powers* of the first and the second class are, respectively, $\aleph_0$ and $\aleph_1$ Under CH, the power of ethereal matter is also **c**. For further details on this claim, and its connection to Cantor's set-theoretic work, see Dauben, (1979), p. 126 and Ferreirós, (2004), in particular, p. 75-77.

38 An articulated review of extra-mathematical themes in Cantor's thought is the aforementioned Ferreirós, (2004). Cantor's ideas related to *organicism* were not altogether foreign to the scientific debate, as Ferreirós shows (see p. 77). On this point, see also Purkert-Ilgauds, (1985), p. 67-68.

However, things are not so simple as they appear. As explained by Tieszen in Tieszen (2012), Gödel may have thought of Leibniz's monads in terms of *transcendental egos*, in the way indicated by Husserl.[39]

But there are other passages in Wang's book, which have been extensively examined by van Atten in van Atten (2009), wherein Gödel would seem to hold a more "standard" view concerning monads. In particular, he would seem to have been inclined to identify *sets* with *monads*.[40]

In connection, again, with Ackermann's axiom, which I have discussed in the preceding section, Gödel also says:

> 8.7.14 There is also a theological approach, according to which V corresponds to the whole physical world, and the closeness aspect to what lies within the monad and in between the monads. According to the principles of rationality, sufficient reason, and pre-established harmony, the property $P(\mathrm{V}, x)$ of a monad x is equivalent to some intrinsic property of x, in which the world does not occur. In other words, when we move from monads to sets, there is some set y to which x bears intrinsically the same relation as it does to V. Hence, there is a property $Q(x)$, not involving V, which is equivalent to $P(\mathrm{V}, x)$. According to medieval ideas, properties containing V or the world would not be in the essence of any set or monad. Wang (1996), p. 284).

And in one further remark, he says:

> 9.1.27 *Objects*. Monads are objects. Sets (of objects) are objects. A set is a unity (or whole) of which the elements are constituents. [...] Sets are the limiting case of spatiotemporal objects and also of wholes. (Wang (1996) p. 296).

In published work, Gödel was more wary of proposing connections between ideal entities and physical reality. In the Cantor paper, he advocates a sharp distinction between the epistemic status of Euclid's fifth postulate and CH. The former is based on our interpretation of the physical reality, whereas the latter can only be settled by purely mathematical considerations. However, Gödel's remarks on this point leave some room for the possibility that, in the future, we could find a trans-subjective meaning for transfinite set theory as well:

> On the other hand, the objects of transfinite set theory, [...], clearly do not belong to the physical world, and even their indirect connection with physical experience is very loose (owing primarily to the fact that set-theoretical concepts play a minor role in the physical theories of today). (Gödel (1964) in(1990) p. 267).

To sum up, I believe that Cantor's and Gödel's conceptions of the trans-subjective can also be successfully compared. At bottom, what they seem to share is a parallel attitude to apply to the physical reality the same attitude they applied to mathematics, which I have defined in section 4 "arithmetical purism," namely a tendency to reduce complex

39 Cf., in particular, Tieszen (2012), p. 38. According to Husserl, "*Monads are transcendental egos in their full concreteness*. Transcendental egos in their full concreteness are not "mere poles of identity," but are rather egos with all the predicates that attach to these poles of identity, so that each monad is distinct from every other monad. We know that Leibniz has a range of different kinds of monads, but Husserl's focus is much narrower. It is on the kind of 'monads' that we are." Tieszen also observes that we do not know to what extent Gödel wanted to use Leibniz's monadological conception in a way which would conform to Husserl's."

40 Van Atten notices (p. 4), that the sole fact that "Leibniz denies the existence of infinite wholes of any kind" would doom Gödel's attempt to failure.

phenomena to simpler elements. This attitude seems to be aptly reflected by their thick speculations on the existence of atomic (monadic) constituents of the reality, whatever these latter might be, either simple substances or vitalistic soul-like principles.

8. Connection of immanent and transient

Cantor distinguishes two forms of "existence." However, he also commits himself to the belief in their fundamental identity. This belief is expressed in the following passage:

> Because of the thoroughly realistic but, at the same time, no less idealistic foundation of my point of view, I have no doubt that these two sorts of reality always occur together in the sense that a concept designated in the first respect as existent always also possesses in certain, even infinitely many, ways a transient reality. (Cantor (1883), p. 896).

Further, in the subsequent paragraph, he adds: "This linking of both realities has its true foundation in the *unity* of the all to which we ourselves belong." (Cantor (1883) p. 896).

In footnote, he provides us with some further details about the nature and the origin of his conception. In particular, he assures that his position is in accordance, as ever, with Plato's, Spinoza's and Leibniz's philosophical systems, and quotes one famous proposition from Spinoza's *Ethics*: "As for Spinoza, I need only mention his statement in *Ethica*, part II, prop. VII: '*ordo et connexio idearum idem est ac ordo et connexio rerum*' [The order and connection of ideas is the same as the order and connection of things (my translation)]" (Cantor (1883), in p. 918).

However, the emphasis on Spinozian "monism," in Cantor's philosophy, is tied to a specific goal, that of emphasising, once more, the objectivity of mathematical *concepts*. In a sense, such monism only serves as a strengthening of conceptual objectivity. It can be paraphrased in the following way: the conceptual constructs we seem to build up (but, in fact, *reminisce*) are related to equally objective constructs, which are given, trans-subjectively, in the physical reality. The emphasis on the connection between the two realms works, thus, essentially as a striking epistemological metaphor: finding mathematical concepts is equivalent to detecting new "forms" in the physical reality.

This leads Cantor to conclude that:

> The mention of this linking has here only one purpose: that of enabling one to derive from it a result which seems to me of very great importance for mathematics, namely, that mathematics, in the development of its ideas, has *only* to take account of the *immanent* reality of its concepts and has *absolutely no* obligation to examine their *transient* reality. (Cantor (1883) p. 896).

It seems reasonable to assume that also Gödel might have conceived of a sort of connection between concepts and objects. An instance of such connection could be the one we have described in the preceding section, that between *sets* (qua also interpretable as concepts) and *monads* (objects).

In Wang's book, we find one observation Gödel makes concerning axioms and models, which resumes the aforementioned Spinozian doctrine that appealed so much to Cantor, that of the connection between *ordo idearum* and *ordo rerum*. He says:

> 4.3.9 The axioms correspond to the concepts, and the models which satisfy them correspond to the objects. The representations give the relation between concepts and objects. For Spinoza the connection of things are connections of ideas. [...] We have here a general proportionality of the membership relation (the concept) and the sets (the objects). The original difference is that concepts are abstract and objects are concrete. In the case of set theory, both the membership relation and the sets are abstract, but sets are more concrete. (Wang (1996) p. 141).

However, whether Gödel here refers to objects as *trans-subjective* constructs is unclear. In any case, even so, we do not know what relevance the reported Spinozian point of view would globally have in Gödel's conceptions. As a matter of fact, we know that Gödel rejected Spinoza's *pantheistic* views and held a *theist* conception.[41] But he could have been inclined to accept some form of *monism* regardless of any commitment to Spinoza's doctrines. An alternative way could be suggested, once more, by monadology. After all, monads have a dual aspect: on the one hand, they are *simple substances* which also interact with the physical reality and, on the other, they are soul-like forms, or, as Husserl would put it, *transcendental egos*. In the first aspect, they bear on the structure of reality, including the physical reality, in the other, they belong to the realm of the "intelligible," the "immaterial," the "conceptual." Thus, in a sense, the two aspects of reality, to which Cantor was referring, would be successfully *unified* in them.

This interpretation might help us to understand the following very famous, but also cryptic, Gödelian statement in the Cantor paper. As we know, Gödel assimilates the "data" resulting from *conscious* (*introspective*) *elaboration* to "data" resulting from *sensory elaboration*. Such data, he says, need not, as Kant thought, be conceived as an "expression" of human subjectivity.

> Rather they, too, may represent an aspect of objective reality, but, as opposed to sensations, their presence in us may be due to another kind of relationship between ourselves and reality. (Gödel (1964), in (1990) p. 268).

The relationship Gödel is referring to here might be the one we would naturally expect to find in a monadological system: that between conceptual, mental, intellectual aspects of monads and their physical instantiation in reality. In this sense, we could say that the connection between "immanent" and "trans-subjective," posited by Cantor, could have also appealed to Gödel.

9. The development of mathematics

In one of her most widely known papers, Maddy (1996), Maddy brings to the fore and stresses the importance of a fundamental aspect of Gödel's philosophical production, that is, his defence of "extrinsicness" as legitimate evidence in favour of the acceptance of the axioms. As we will see, also Cantor fostered the importance of some sort of "extrinsicness" in his *Grundlagen*. In his Cantor paper, Gödel had stated that:

---

41 Cf. Wang (1996) p. 112: "Gödel gave his own religion as "baptized Lutheran" (though not a member of any religious congregation) and noted that his belief was *theistic*, not pantheistic, following Leibniz, rather than Spinoza. [...] Gödel was not satisfied with Spinoza's impersonal God."

> [...]even disregarding the intrinsic necessity of some new axiom, and even in case it has no intrinsic necessity at all, a probable decision about its truth is possible also in another way, namely, inductively, by studying its "success." Success here means fruitfulness in consequences, in particular, in "verifiable" consequences, i.e., in consequences demonstrable without the new axiom [...]. There might exist axioms so abundant in their verifiable consequences, shedding so much light upon a whole field, and yielding such powerful methods for solving problems (and even solving them constructively, as far as that is possible) that, no matter whether or not they are intrinsically necessary, they would have to be accepted at least in the same sense as any well-established physical theory. (Gödel (1964), in (1990) p. 261).

In Maddy's view, this statement is proof that Gödel's "ontological" realism is mitigated by more pragmatic, "naturalistic" concerns, which would coalesce into the claim that a decision concerning the "acceptability" of an axiom should also be tied to considerations stemming from *intra-mathematical* practice. Overall, what comes out of Maddy's reconstruction is a different picture of Gödel's realism, as laying as much emphasis on "intra-mathematical practice" as on "intrinsic (conceptual) necessity" of new set-theoretic axioms. However, she says:

> I don't claim that this second picture of Gödel's views is completely accurate, any more than the first one was, but I do think it provides a useful perspective. Gödel's views are often presented in connection with the first of the two aforementioned attractions of realism: its faithfulness to mathematical experience. This alternative reading highlights the second attraction of realism—an account of the meaningfulness of the independent questions—but its working parts bypass realism altogether; they argue directly for the meaningfulness of those questions on purely mathematical grounds, not via philosophical realism. (Maddy (1996) p. 498).

Maddy has further elaborated upon her interpretation of Gödel's naturalism in Maddy (1997). If her interpretation is correct, then there must necessarily be some tension between these two sets of criteria for the acceptance of the axioms, "intrinsic" and "extrinsic." I will say something on this point later, but for the time being I will try to show that Gödelian naturalism seems to square well with some Cantorian remarks in the *Grundlagen* I am going to quote. While formulating the *generating principles* for the ordinals,[42] Cantor warns that:

> It is not necessary, I believe, to fear, as many do, that these principles present any danger to science. For in the first place the designated conditions, under which alone the freedom to form numbers can be practised, are of such a kind as to allow only the narrowest scope for discretion. Moreover, every mathematical concept carries within itself the necessary corrective: if it is fruitless or unsuited to its purpose, then that appears very soon through its uselessness, and it will be abandoned for lack of success. (Cantor (1883), p. 896).

The first part of this quotation introduces the Cantorian theme of mathematicians' freedom, and argues that such freedom does not imply any form of *arbitrariness*. In a sense, freedom already contains some sort of *necessity* in it, insofar as one is only free to *follow* the "designated conditions" for the regulated *awakening* of a logically

42 Generating principles are the three principles Cantor uses in the *Grundlagen* to "construct" the whole series of transfinite ordinals (see Cantor (1883), p. 907-909), *viz.*, the *successor*, the *limit* and the *restriction* principle, whereby one can build, respectively, successor-ordinals ($\omega + 1$, $\omega + 2$...), limit-ordinales ($\omega + \omega$, ...) and initial ordinals (ω_0, ω_1, ...).

transparent concept. The second part introduces what Maddy would call "naturalistic" concerns, which seem to clash with the "intrinsic" ones. What Cantor says seems to imply that concepts, even those consistent with the older ones, that is, *coherent* with a previously established conceptual network, might nonetheless be abandoned as *useless*, for lack of success.

As a consequence, there seems to be also some tension in Cantor between a realistic ontology and this kind of "pragmatism" about the introduction of new concepts. It should be noticed that this tension is not eased by the fact that "extrinsic evidence" only counts after one has fully carried out the process of ascertaining whether a concept has "intrinsic necessity." As a matter of fact, what Cantor says above does not seem to exclude that a concept is consistent, *sufficiently determinate* and distinguished from the other ones, in other terms that it complies with the "intrinsic" requirements, but that, at the same time, it is abandoned as *useless*. The importance of such considerations, however, should not be overestimated. We have to keep in mind that Cantor, in the *Grundlagen*, is also concerned with responding successfully to his critics, who had argued that using the actual infinite might turn out to run the risk of bringing inconsistencies into mathematical thought.

Cantor's counterargument has two parts: the first, by emphasising the existence of conceptual constraints within mathematicians' thought, has the effect of dismantling the objection that the introduction of "new numbers" is arbitrary. The second is directed at denying the risk that science might be harmed by novelties, by emphasising what, in his eyes, was natural and even obvious: mathematicians are able to detect the potential inconsistency or unsuitability of a concept and, in that case, they can always retreat on their steps. This second part does not need to be seen as a naturalistic doctrine, although it certainly contains an element of pragmatism.

The case of Gödel's naturalism is, maybe, slightly different, as his "pragmatism" seems more structured. He seems to take "inductive" evidence quite seriously, and the analogy between physics and mathematics is certainly striking. However, he sometimes uses the analogy between physics and mathematics in a different way, namely to prove that mathematical knowledge is dependent upon the internal elaboration of the "perceiving" subject in the same way as physical knowledge is (i.e., to claim that the "immediately given" of both does not merely consist of "data"). Which of the two analogies is, then, more faithful to his conceptions?

Elsewhere, Gödel seems to bring forward some sort of *indispensability argument*, particularly when he says that the assumption of the existence of mathematical objects can be compared to the assumption of the existence of bodies, both assumptions being *indispensable* for obtaining a *satisfactory* theory of mathematics and physics, respectively.

Cantor seems to have used the same argument when, in presenting the transfinite ordinals, he asserted:

> I am so dependent on this extension of the number concept that without it I should be unable to take the smallest step forward in the theory of sets [Mengen]; this circumstance is the justification (or, if need be, the apology) for the fact that I introduce seemingly exotic ideas into my work. (Cantor (1883) p. 882).

To conclude, in both Gödel's and Cantor's conceptions, there is some emphasis on extrinsic evidence. However, the full import of their arguments considered globally, within their respective conceptions, is difficult to judge. What seems to be certain is that Gödel could, again, see Cantor's thought as an antecedent of his views concerning the importance of "extrinsicness."

10. Concluding remarks

It is time to make some final considerations. I have been suggesting, in the introduction, that Gödel's conceptions are strikingly similar to Cantor's and that he may have been consciously or unconsciously influenced by them, and also that he may have used them in some particular circumstances. Now that a more detailed picture is available, I want to reconsider my claim more systematically and add some further remarks.

As said at the beginning, we do not know whether and to what extent Gödel knew Cantor's work, but the comparison of many passages from both authors' works has shown that there is a high number of textual overlappings, conceptual resemblances, even similarities in the use of linguistic expressions. Consequently, one could reasonably conjecture that Gödel drew upon Cantor's thought, or that, at least, he was fully aware that his philosophical positions were in line with Cantor's. However, I believe that the point may not be so crucial. It would be worth exploring such similarities regardless of whether they have been produced intentionally or not. Philosophically, these similarities involve, very frequently, the use of the same sources. Among them, in particular, Plato, Leibniz, Kant, Spinoza. Their philosophical choices reveal a substantial identity of tastes: objectivistic, metaphysical, systematic, theological, "right-wing" conceptions are preferred to subjectivistic, positivist, "left-wing" ones.

Another crucial point of contact is that, although both are essentially concerned with mathematics, their philosophical projects seem, at times, more ambitious and, in fact, all-encompassing, spanning material and ideal objects, the finite and the infinite, numbers, sets, V, God.

As a consequence of these similarities, in comparing Gödel's and Cantor's thought, one can sometimes get a more precise picture of what Gödel is aiming at. For instance, this is what happened to me when I struggled to understand Gödel's reference to a "different relationship between ourselves and reality." The knowledge of Cantor provided some grounds to assert that those words referred to a connection between *conceptual* and *non-conceptual* aspects of reality. Similarly, Gödel's remarks on the nature of the Absolute and his mention of the von Neumann axiom strike the right note in anyone acquainted with Cantor's work and, in turn, become more intelligible because of that.

Sometimes, Gödel's views are just a *transformation* (in a certain sense, a continuation) of Cantor's. Take the example of the notion of "intuition." Both are committed to believing that there exists a mathematical intuition that allows us to "perceive" mathematical objects/concepts. In Gödel's view, such an intuition should, in principle, also allow us to solve all mathematical problems *uniquely* and *determinately*.[43] Cantor was only initiating such conception, which perhaps had some

43 In the Cantor paper, Gödel's optimism is very robust, but elsewhere (the Gibbs lecture and some other remarks in Wang 1996), it is mitigated by the observation that the intuition of mathematical objects may be fallible and, what is more important, *incomplete*, thus leaving it open whether we are able to find solutions to all set-theoretic problems.

influence on the young Husserl, who, in turn, provided Gödel with the theory Cantor was looking for. In this sense, Gödel was continuing investigations or developing intuitions fully in the wake of Cantor's work. Gödel's conceptions can even be seen, sometimes, as improvements on Cantor's.

Some other time, Gödel's thoughts make apparent what is concealed or follows from Cantor's conceptions. Take the idea of inexhaustibility of mathematics. In Gödel's conception, we never cease to produce new axioms, and this is, in a sense, a consequence of the existence of an inexhaustible infinite (Cantor's *absolute infinite*). Cantor used this concept to forestall the paradoxes, and as the logical basis of the distinction between classes and sets, whereas Gödel successfully connected it to its incompleteness theorems. Gödel also understood that the *indescribability* of V could be turned into a positive phenomenon, that is, into a justification of the reflection principle.

The work I am presenting here may be subject to further generalisation. For instance, a natural corollary would be to imagine a new brand of realism, Cantor-Gödel Platonism, based on the conceptual intersections between Gödel's and Cantor's thought I have described, and that I would like to resume briefly:

1. Mathematical entities (essentially, *sets*) exist both as *conceptual* and as *trans-subjective* objects. Their relationships and properties, both at the conceptual and at the trans-subjective level, exist independently of our mind.
2. There is essentially one correct method to develop mathematics, which consists in using "intuition" to grasp concepts and their relations. Such intuition is strong enough to decide, in principle, all problems of set theory (incidentally, this is also the correct procedure to determine whether a concept is consistent).
3. Although there is only one correct method to develop mathematics, in certain circumstances it is useful for mathematicians to look at extrinsic criteria for deciding whether a concept (or an axiom) are legitimate.
4. Set theory and logic are inexhaustible, as the actual infinite is (given the existence of the absolute infinite). A parallel phenomenon is the *indescribability*, in a sense, of the universe of sets, which encourages the re-iteration of processes of formation of ordinals (and the postulation of axioms positing new ordinals).
5. God is absolutely infinite.

I believe that we are not very far from being able to delineate something like Cantor-Gödel Platonism as a determinate and distinguished form of realism. Of course, one may legitimately ask how the identification of such a conception could benefit us. I may suggest one way in which this could happen. The issue of realism in the contemporary philosophy of mathematics is still thriving and widely debated. Gödel was instrumental in revitalising it, and his connection to Cantor should be paid the right attention and given proper emphasis, precisely because of our purpose to evaluate the tenability of realism.

However, if and how Cantor-Gödel Platonism might be thought of as a successful or adequate reconstruction of mathematical thought and of set theory as a whole or whether it could be used for present mathematical purposes is, of course, far from being clear.

References

CANTOR, Georg (1883) *Grundlagen einer allgemeinen Mannigfaltigkeitslehre. Ein mathematisch-philosophischer Versuch in der Lehre des Unendlichen.* B. G. Teubner, Leipzig.

CANTOR, Georg (1885) “Über die verschiedenen Standpunkte auf das aktuelle Unendlichen.” *Zeitschrift für Philosophie und philosophische Kritik*, 88, pp. 224-33.

CANTOR, Georg (1885b) “Über verschiedene Theoreme aus der Theorie der Punktmengen in einem *n*-fach ausgedehnten stetigen Raume ” (Zweite Mitteilung). *Acta Mathematica,* 7, pp. 105-24.

CANTOR, Georg (1932) *Gesammelte Abhandlungen mathematischen und philosophischen Inhalts.* Berlin, Springer.

CANTOR, Georg (1887-1888) “Mitteilungen zur Lehre vom Transfiniten, I-II.” *Zeitschrift für Philosophie und philosophische Kritik,* I, 91, pp. 81-125; II, 92, pp. 240-65.

COHEN, Paul J. (1964) “The independence of the Continuum Hypothesis”. *Proceedings of the National Academy of Sciences*, pp. 105-110.

CROCCO, Gabriella (2006) “Gödel on Concepts”, *History and Philosophy of Logic*, 27(2), pp. 171-91.

DAUBEN, Warren (1979) *Georg Cantor. His Mathematics and Ph+ilosophy of the Infinite.* Harvard: University Press, Harvard.

DAWSON, Jonh W. (1997) *Logical Dilemmas: the life and work of Kurt Gödel.* A K Peters, Wellesley (MA).

EWALD, William (1996) ed. *From Kant to Hilbert: A Source Book in the Foundations of Mathematics, volume II.* Oxford: Oxford University Press.

FERREIRÓS, José (2004) “The Motives behind Cantor’s Set Theory”. *Science in Context,* 12/1-2, pp. 49-83.

FREGE, Gottlob (1918) “Der Gedanke”. *Beiträge zur Philosophie der deutschen Idealismus*, pp. 58-77.

FREGE, Gottlob (2007) *The Foundations of Arithmetic.* New York: Pearson-Longman.

GÖDEL, Kurt (1933) “The present situation in the foundation of mathematics”. Unpublished.

GÖDEL, Kurt (1944), “Russell’s Mathematical Logic”, *The Philosophy of Bertrand Russell,* ed. P.A. Schilpp, Chicago: Open Court Publishing, pp. 125-153.

GÖDEL, Kurt (1947) “What is Cantor’s Continuum Problem?” *American Mathematical Monthly*, 54, pp. 515-525.

GÖDEL, Kurt (1949) “A Remark on the Relationships between Relativity Theory and Idealistic Philosophy”, *Albert Einstein: Philosopher-Scientist*, ed. P.A. Schilpp, pp. 555-562. Illinois: La Salle, Open Court Publishing.

GÖDEL, Kurt (1951) “Some Basic Theorems on the Foundations of Mathematics and its Applications” (Gibbs Lecture). Unpublished.

GÖDEL, Kurt (1961/?) “The Modern Development of the Foundations of Mathematics”. Unpublished.

GÖDEL, Kurt (1964) “What is Cantor’s Continuum Problem?”, in H., Putnam, and P., Benacerraf, eds. *Philosophy of Mathematics. Selected Readings,* New York: Prentice-Hall, pp. 470-485.

GÖDEL, Kurt (1990) *Collected Works, II: Publications 1938-1974.* Oxford: Oxford University Press.

GÖDEL, Kurt (1995) *Collected Works, III: Unpublished Essays and Lectures.* Oxford: Oxford University Press.

HALLETT, Micheal (1984) *Cantorian Set Theory and Limitation of Size.* Oxford: Clarendon Press.

HAUSER, K. (2006) "Gödel's Program revisited. Part I: The Turn to Phenomenology". *Bulletin of Symbolic Logic,* 12/4, pp. 529-590.

HUSSERL, Edmund (1980) *Ideas Pertaining to a Pure Phenomenology and to a Phenomenological Philosophy. Book 3: Phenomenology and the Foundation of Sciences.* Dordrecht: Kluwer.

HUSSERL, Edmund (1982) *Ideas Pertaining to a Pure Phenomenology and to a Phenomenological Philosophy. Book 1: General Introduction to a Pure Phenomenology.* The Hague: Nijhoff.

HUSSERL, Edmund (1988) *Cartesian Meditations.* Dordrecht: Kluwer.

HUSSERL, Edmund (1989) *Ideas Pertaining to a Pure Phenomenology and to a Phenomenological Philosophy. Book 2: Studies in the Phenomenology of Constitution.* Dordrecht: Kluwer.

JANÉ, Ignacio (1995) "The role of the Absolute Infinite in Cantor's Conception of Set". *Erkenntnis,* vol. 42, pp. 375-402.

KANAMORI, Akihiro (2003) *The Higher Infinite.* Berlin: Springer, Verlag.

LÉVY, Azriel (1967) and SOLOVAY, Robert "Measurable Cardinals and the Continuum Hypothesis". *Israel Journal of Mathematics,* pp. 234-248.

MADDY, Penelope (1996) "Set-theoretic Naturalism". *Bulletin of Symbolic Logic*, 61/2, pp. 490-514.

MADDY, Penelope (1997) *Naturalism in Mathematics.* Oxford: Oxford University Press.

MARTIN, Donald A. (2005) "Gödel's Conceptual Realism". *Bulletin of Symbolic Logic,* 11/2, pp. 207-224.

ORTIZ HILL, C. (1997) "Did Georg Cantor Influence Edmund Husserl?". *Synthèse*, 113, pp. 145-170.

ORTIZ HILL. C. (2004) "Abstraction and Idealization in Edmund Husserl and Georg Cantor". *Poznan Studies in the Philosophy of the Sciences and in the Humanities*, vol. 82/1, pp. 217-244.

PARSONS, Charles (1995) "Platonism and Mathematical Intuition in Kurt Gödel's Thought". *Bulletin of Symbolic Logic*, 1/1, pp. 44-74.

PURKERT, W. and ILGAUDS, H.J. (1985) *Georg Cantor*. Leipzig: Teubner.

RICKETTS, Thomas and POTTER, Michael eds. (2010) *The Cambridge Companion to Frege*. Cambridge: Cambridge, University Press.

TIESZEN, Richard (2005) *Phenomenology, Logic and the Philosophy of Mathematics.* Cambridge: Cambridge, University Press.

TIESZEN, Richard (2011) *After Gödel. Platonism and Rationalism in Mathematics.* Oxford: Oxford University Press.

TIESZEN, Richard (2012) "Monads and Mathematics: Gödel and Husserl". *Axiomathes*, 22, pp. 31-52.

VAN ATTEN, Mark (2009) "Monads and Sets. On Gödel, Leibniz, and the Reflection Principle", *Judgment and Knowledge. Papers in honour of G.B. Sundholm*, London: King's College Publications, pp. 3-33.

VAN ATTEN, Mark and KENNEDY, Juilette eds. (2003) "On the Philosophical Development of Kurt Gödel". *Bulletin of Symbolic Logic,* 9/4, pp. 425-476.

WANG, Hao (1974) *From Mathematics to Philosophy.* London: Routledge & Kegan Paul.

WANG, Hao(1987) *Reflections on Kurt Gödel*. Cambridge: (MA), MIT Press.

WANG, Hao (1996) *A Logical Journey*. Cambridge: (MA), MIT Press.