Paraconsistent Belief Revision

Based on a Formal Consistency Operator

PhD Thesis

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Preface

Dear Reader,

It is with immense pleasure that I present to you the English translation of my doctoral thesis, a work that stands as a significant milestone in my academic journey. Since my doctoral studies, I have had the opportunity to share the results of the research with the international community through various congresses and publications, notably the article "AGM-like Paraconsistent Belief Revision," published in the Logic Journal of the IGPL, which featured the majority of the technical results from this thesis. However, despite these presentations, several details of the thesis, such as the conceptual discussion on the distinction between coherence, consistency, and non-contradiction, and the overarching theme as presented in the original work, have not been fully conveyed to an English-speaking audience. This translation aims to bridge that gap.

Faced with the personal and professional advancements I have accumulated since then, there was a strong temptation to refine and update the ideas presented. However, I have chosen to present to you the thesis as it was originally conceived and written, preserving the essence of my work and thought at that stage of my academic career. To this end, the only modifications made to the original text are the inclusion of "Translation Notes" to reference recent works that present the evolution of the results of my thesis. These works have also been added to the bibliography, allowing readers to access these advancements and understand the ongoing impact of the research.

The decision to translate this work was further motivated by the positive reception it received within the fields of Paraconsistency and Belief Revision. In terms of Paraconsistency, the thesis work is referenced as innovative for considering the phenomenon from the perspective of epistemic agents, particularly in relation to Logics of Formal Inconsistency, clarifying the meaning of the formal consistency operator. In Belief Revision, the developed systems are seen as relevant extensions and refinements of the AGM theory. These results have been cited in key reference works in these areas, including *Paraconsistent Logic: Consistency, Contradiction and Negation* by Carnielli and Coniglio, and *Belief Change: Introduction and Overview* by Fermé and Hansson.

When I originally defended the thesis nine years ago, Brazilian universities were just beginning to recognize the value of bilingual theses, a trend that has grown significantly since then. Driven by the desire to make the research accessible to a broader audience and contribute to ongoing dialogues in these fields, I undertook the challenge of offering the thesis in English.

I am in the process of composing a manuscript that details the evolution of the research and contributions to the field of study. However, before unveiling these new reflections, I believe it is essential that the original version of my thesis be available to those not familiar with the Portuguese language.

This translation is an invitation for you to delve into the pages of my scholarly endeavor, and I hope it serves as a valuable tool for the advancement of knowledge and inspiration for future research.

Acknowledgements

As I present this English translation of my doctoral thesis, my heart is filled with gratitude for the multitude of professionals whose paths have crossed mine over the nearly ten years since its defense. These individuals have intellectually intersected with my journey, challenging and teaching me in profound ways. While I am deeply thankful to all of them, I choose to specifically name those who have played a significant role in the research presented in this work.

First and foremost, I would like to thank Marcelo Coniglio, my doctoral advisor, whose guidance was pivotal in the conception and development of this thesis. His wisdom and support have been unwavering throughout my academic pursuits.

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Finally, I would be remiss if I did not express my profound gratitude for the financial support I received during my doctoral studies and postdoctoral fellowships, which played a crucial role in enabling this English translation of my thesis. The support from The Brazilian National Council for Scientific and Technological Development (CNPq) during my doctoral studies was instrumental in laying the foundation for my research.

During my postdoctoral fellowships, the funding I received further facilitated my academic endeavors. I was a Postdoctoral Research Fellow funded by FAPERJ (2021-2022) at the Institute of Philosophy and Social Sciences of the Federal University of Rio de Janeiro (IFCS-UFRJ), under the grant FAPERJ 202.367/2021. This opportunity not only advanced my research but also played a vital role in the translation process.

Additionally, I was fortunate to be a Postdoctoral Research Fellow funded by FAPESP (2017-2018) at the Faculty of Exact Sciences and Engineering, University of Madeira (FEE-UMa) in Funchal, Portugal, supported by the grant FAPESP 2017/10836-0. This international experience was invaluable for my academic growth and the subsequent translation of my thesis.

Furthermore, my stage as a Postdoctoral Research Fellow funded by FAPESP (2015-2017) at the Centre for Logic, Epistemology, and the History of Science at the University of Campinas (CLE-Unicamp), Brazil, under the grant FAPESP 2014/22119-2, significantly contributed to the depth and breadth of my research.

For all this support, I am immensely grateful.

Sincerely,
Rafael R. Testa

Jundiaí, November 27, 2023

Abstract

Belief Revision studies how rational agents change their beliefs upon receiving new information. The **AGM** system, the most influential work in this area, investigated by Alchourrón, Gärdenfors, and Makinson, postulates rationality criteria for different types of belief changes and provides explicit constructions for them. The equivalence between the postulates and operations is called the *representation theorem*. Recent studies show how the **AGM** paradigm can be compliant with different non-classical logics, which is referred to as AGM-compliance. This is the case for the paraconsistent logics family we analyze in this thesis, known as the Logics of Formal Inconsistency (**LFIs**).

Despite the AGM-compliance, when a new logic is considered, its underlying rationality must be understood, and its language should be used. In this way, new constructions are proposed, which accurately capture the intuition of **LFIs** – what we call the **AGM**$\circ$ system. Thus, we provide a new interpretation for these logics, more aligned with formal epistemology. Alternatively, by considering AGM-compliance, we demonstrate how the **AGM** results can be directly applied to **LFIs**, resulting in the **AGMp** system. In both approaches, we prove the corresponding *representation theorems* where needed.

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List of Acronyms and Notations

Systems

AGM "Classical" Belief Revision System

AGM$\circ$ Paraconsistent Belief Revision System based on $\circ$

AGMp Paraconsistent Belief Revision System based on AGM-compliance

Logics, languages and logical consequence

CPL Classical Propositional Logic

CPL+ Positive Classical Propositional Logic (fragment of **CPL**)

LFIs Logics of Formal Inconsistency

mbC the smallest **LFI**

L an arbitrary **LFI** (extension of **mbC**)

$\mathbb{L}$ object language

Cn closure (logical consequence operator)

$\vdash$ logical consequence relation

$\nvdash$ non-consequence

Belief sets

K epistemic state closed over Cn

K_f trivial epistemic state

$A, C, D...$ set of sentences

B belief base

$\perp$ remainder set

$\perp\!\!\!\perp$ kernel set

γ selection function

σ incision function

$\leq$ epistemic entrenchment relation

Set Theory

$\{...\}$ set

$\in$ inclusion

$\notin$ non-inclusion

$\subseteq$ subset

$\nsubseteq$ non-subset

$\subset$ proper subset

$\not\subset$ proper non-subset

$\cup$ union

$\cap$ intersection

$\setminus$ difference

$\emptyset$ empty set

$\mathcal{P}$ power set

$\langle, \rangle$ ordered pair

Logical Language

$\alpha, \beta, ...$ sentence

$p, q, ...$ atomic sentence

$\Gamma, \Delta, ...$ set of sentences

$\top$ *top* (tautology)

$\mathbf{f}$ $falsum$ (contradiction)

$\wedge$ conjunction

$\vee$ disjunction

$\to$ implication

$\leftrightarrow$ bi-implication

$\neg$ negation (classical in **LPC** and paraconsistent in the **LFIs**)

$\sim$ classical negation (trivializing) in the **LFIs**

$\circ$ consistency (**LFIs**)

$\bullet$ inconsistency (**LFIs**)

General Introduction

The main topic in this work is the dynamics of theories, that is, the information change in a belief system (known as epistemic states) and how this change can be considered rational. Roughly, the epistemic changes in focus are **belief revisions**,[1] which occur when agents receive new information often incompatible with those present in their current epistemic states or, analogously, when some theory comes to accept (incorporate) a new assertion.[2] We will define in clearer terms, along the thesis, what we understand by such terms – for didactic purposes, we can assume that an agent is any entity capable of perceiving the world and acting on it; belief, in turn, is determined by the relation between an agent and a proposition (in a certain language) and epistemic state would be, by these definitions, the beliefs that can be attributed to an agent in a given moment.[3]

[1] Other names can be found in the literature, such as *database updating*, *theory change*, *theory revision*, *theory dynamics*, *belief change* and *belief dynamics*, among others. Despite some criticism relative to the name "Belief revision", we keep it due to a practical reason – it is the most used name in the field's literature.

[2] Theories are, specifically, logically closed sets of sentences in a given formal language.

[3] Note that we follow the terminology used in works in the field of Formal Epistemology. In this case, we should understand "belief" in a wider sense, namely, as strictly formal. We return to this point in Chapter 1.

Belief Revision

Belief revision is the area that studies the rationality of theory change, that is, the formal study of how agents change their beliefs upon receiving new information (not necessarily incompatible with their previously accepted informations). **When considering an agent with a given epistemic state, how would the agent change their beliefs when confronting new information?** This question is the most general formulation of the problem dealt with by the formal systems considered in this research. An agent can be a human, a computer program, or any system to which beliefs can be ascribed and whose behavior can be expected to be rational.

According to Sven Ove Hansson [41], this research field was recognized as a study subject since the mid 80s and has developed from two convergent research traditions: computer science and philosophy. Relative to computation, database update procedures have been developed since the advent of programming and, with the development of Artificial Intelligence (AI), more sophisticated models to study and create rational agents where proposed.[4]

Relative to philosophy, since the latter half of the 20th century, various philosophers have discussed, for example, the mechanisms by which scientific theories develop, and since then, criteria for rationality have been proposed. According to Hansson [41], the works of Isaac Levi [56, 57] can be cited as early incursions in this field of study, especially regarding proposals for criteria for rational belief change, as well as the work of William Harper [47]. The most influential work in this perspective is known as the **AGM** system, named after its creators Carlos Alchourrón, Peter Gärdenfors, and David Makinson, and is presented mainly in [1].

The intuition to be captured is that beliefs are not static but evolve over time. This change can be due to various situations:

[4] The term *agent* in the specific context of AI can be understood, according to Stuart Russell and Peter Norvig [89], as a computational system that uses knowledge to exhibit intelligent behavior, that is, which is capable of receiving and providing information to the external world. An agent is regarded as rational "if it does *the right thing* given what it knows."

new or previously unknown information that becomes known to the agent; a new observation or experiment revealing a new fact; or a change in the very domain of interest, for example, in the facts of the world known to the agent. In all these cases, the accepted beliefs must be adapted to the new information, or the latter can be ignored and not incorporated into the previous information set.

These situations are valid in any structure that deals with information (beliefs, facts, rules, data, etc.) related to a domain of interest. Therefore, its application is possible in diverse areas such as artificial intelligence (Nebel [71]), software engineering and market research (Williams [108]), ontology and web semantics (Flouris [19]), learning (Kelly [53]), epistemology (Hendricks [49]), rational choice theory (Arlo-Costa and Pedersen [4]), philosophy of science (Hansson [43]), and others. Let us consider some quick examples to illustrate the roles belief revision can play:

In robotics and AI The robot *Curiosity* has a map of the Mars environment in which it must move automatically. In this map there are no obstacles in its way, and therefore it can proceed forward. However, its sensors indicate the presence of a large object in its front. Should the robot doubt its sensors and try to continue moving forward? Should it trust its sensors and doubt the map with which it has been programmed? Should it reach out to its human controller or programmer to solve the matter?

In databases In the database which contains information about a library's users there is an entry for Jorge Luis, whose birthdate is 24th of August of 1999. The librarian receives a new request, in which the date of birth of Jorge Luis is 24th of August of 1989. He cannot add another birthdate and it cannot be changed over time. The librarian must decide what to do: keep the old piece of information? Substitute it for the new? Or is it some other Jorge Luis, which must be added to the database?

In diagonists I believe that if you press the correct button in the espresso machine, loaded with coffee beans, I will have a cup full of the beverage. Suppose that I have pressed the correct button in such machine, but the cup remains empty. Should I suppose that I have not pressed the correct button? That the machine is not loaded with beans? Or should I abandon the information that the machine is working?

In everyday life I believed that it always rained in São Paulo. One morning I wake up in São Paulo and find that the weather is mild, without rain. I remove my beliefs, therefore, that it always rains in São Paulo.

In some cases, the new piece of information is seen as something to be directly incorporated. However, in other cases, the new information represents something that is incompatible with the previous knowledge corpus, and some pieces of information must be retracted.

Let us consider the following examples to help illustrate the different possible belief changes.

Example 0.1. *When Joseph Black learned the results of Lavoisier's new experiments, he abandoned his previous beliefs about* phlogiston *theory of combustion, and accepted Lavoisier's oxygen theory.*

Joseph Black, therefore, had to revise his beliefs because both theories were mutually incompatible, that is, they led to a contradiction in case they were jointly considered as accepted. This example illustrates a *belief revision* caused by new information – in this case, Lavoisier's experiments – that contradict beliefs previously accepted by a given agent (in this case, a human agent, Joseph Black). Before accepting the new information, Joseph Black was convinced that *phlogiston* theory was correct, and took it as a concrete fact, not merely as a probability. Even so, by discarding it, we can say that his attitude was *rational*. In the same way, it would be rationally possible to Joseph Black to avoid the contradiction by not accepting Lavoisier's experiments, if he could give arguments good enough to reject the new information received.

The belief change which occurs in the mentioned example is different from the one which occurs when an agent comes to accept something compatible with its previous beliefs.

Example 0.2. *I did not know how much it rained in Lima. When I was told that in most years it does not rain, I revised my beliefs to append such information.*

In this example, no beliefs needed to be removed to avoid the incoherence of a possible contradiction. We have, in this case, a simple *expansion*. An expansion is the simplest operation and consists in adding the new piece of information to the previously accepted set.

Inversely, we could remove one piece of information without necessarily adding another - *contraction*.

Example 0.3. *I believed that Plato had written* Hippias Major*. However, I was told that the dialogue's authenticity as Plato's work is contested among researchers. I abandoned, therefore, my belief that Plato had written* Hippias Major *(without coming to accept the belief's negation).*

As observes David Makinson [61], the literature contains basically two distinct approaches to describe the aforementioned belief revision operations: via *postulates* or via *explicit construction*. From the perspective of postulates, a set of formal conditions is formulated, which the operations must adhere to; in other words, the postulates constrain the behavior and, consequently, the outcomes of these operations. In contrast, under the construction approach, explicit algorithms are provided to represent the different contexts.

The two approaches are not in opposition but are, in fact, complementary. In developing an explicit construction, it's possible to identify the desired and expected results, leading to the determination of applicability conditions and subsequent postulate formulation. Conversely, in the development of postulates, the results of operations are often checked against some explicit formal construction. This helps in determining the applicability and rationality of the proposed postulate set, leading to their refinement.

As such, demonstrating the equivalence between a particular construction and its corresponding postulates is a central result in Belief Revision systems – we say that a construction is *characterized* by a set of postulates in case it satisfies all postulates and, on the other hand, any operation that satisfies these postulates can be obtained from such construction. The result that demonstrates such a characterization is called *representation theorem*.

The AGM System

In the **AGM** system, the authors define a set of rationality postulates for each of the main operations on epistemic states described earlier, namely:

Contraction. When one wishes to remove a belief from the current epistemic state. It may be necessary to remove some other beliefs to ensure the operation's success.

Expansion. When one wishes to add a belief that is compatible with the current epistemic state.

Revision. When one wishes to add a belief that is incompatible with the current epistemic state.[5]

The importance of the **AGM** system and therefore of its choice as a theoretical basis for our research is due to the important results achieved by that system – different explicit constructions, intuitively simple and interesting, are shown to be equivalent, that is, they produce exactly the same class of operations that satisfy the **AGM** postulates for contraction and revision. We highlight the *partial meet selection function*, presented by Alchourrón [1], *epistemic entrenchment*, presented by Gärdenfors [29], *safe contraction*,

[5]The term "Belief Revision" for the theory is named after the homonymous operation. Publications usually refer to all related operations broadly as "revisions" – we will follow this convention, believing that the context will sufficiently clarify when we are specifically referring to contraction, expansion, or revision in the strict sense.

presented by Alchourrón and Makinson [3] and *systems of Grove's spheres*, presented by Adam Grove [34].

Many works in the literature use the **AGM** postulates to deal with different logical concepts and intuitive notions of notorious formal and philosophical interest: Gärdenfors [27] and Rott [87], for example, have laid out the relations between **AGM** theory and the concepts of non-monotonicity and of non-monotonic logics; Witte [109] addressed the connection of the **AGM** postulates with the fuzzy set theory proposed by Zadeh [110] and with fuzzy logics in general; Martin and Osherson [67] and several other authors relate the **AGM** concepts to Bayesian epistemology, as well as Stalnaker [97] to game theory; among many other examples that are abundantly present in the literature, which we have used to motivate the intuitions in this thesis.

In addition, alternative formulations of the postulates have been presented, and their properties and effects are still being studied. Many works have criticized the **AGM** postulates and alternative formalizations have been presented, among which we highlight doxastic logics and dynamic modal logics, as suggested by Segerberg [90] and Rijke [85]. The fact is that many subsequent works have utilized, critiqued, and reformulated the **AGM** model, establishing it as the most influential in the field of belief revision. Consequently, our selection of this model enables us to engage with a diverse range of works present in the literature, positioning us at the heart of the discussion on this topic. This also opens up our work to be utilized and potentially critiqued.

Several philosophical and practical questions related to belief revision and particularly to the **AGM** system can be identified. Underlying the formal constructions necessary to encompass revision operations, we can highlight, among so many other logical-philosophical questions, some which are pertinent to our research: What makes a revision *rational*? What logical and non-logical rules govern rational belief revisions? Would the concept of rationality be intrinsic to the underlying logic of each agent? Do such agents obey, or should them obey, the same logic? Which logic?

AGM Rationality

We can observe the role of non-contradiction in some of the answers to these questions: an epistemic state in which a sentence and its negation coexist is called logically contradictory. The notion that contradiction is undesirable and even impossible forms one of the pillars of classical (and even intuitionistic) logic and underlies the concept of rationality adopted by theories of belief revision. Notably, the requirement for an epistemic state to be free of contradictions is one of the main criteria of rationality in the **AGM** system.

Principle of Non-Contradiction

Many systems focus on eradicating contradictions from belief sets, while others address them by working around these contradictions, either by isolating or locally suppressing them, or by incorporating concepts such as temporality and alethic modalities (notions of possibility and necessity). Nevertheless, all of these systems seem to agree that a belief set containing a contradiction is problematic and must be resolved in some way. We believe this viewpoint is overly simplistic, as it fails to satisfactorily capture, for example, the everyday reasoning of non-ideal agents. It also assumes that a contradiction inherently violates the definition of rationality. Furthermore, the outright rejection of contradictions does not leverage their presence and potential informative value – quite the contrary.[6]

Now, the fact is that the presence of contradictory information in belief systems seems inevitable, often being the norm. Contradictory sentences, say α and $\neg\alpha$, are perfectly acceptable when presented together, and the system does not necessarily need to resolve this situation. In some cases, the joint presence of α and $\neg\alpha$ can be understood as an internal trigger for the system to take logical action. Some theorists, such as Kevin Kelly [53], utilize the informative power of contradictions, considering them necessary and useful for guiding reasoning and for encouraging the acquisition of

[6] Translation note: such a concept was developed by Testa [100].

new information into the belief set. Thus, a belief system capable of classifying different aspects of consistency and dealing satisfactorily with reasoning in the presence of contradictions proves to be, if possible, quite intriguing

Principle of Minimal Change

Another important principle related to the implementation of belief change is the postulate of minimality (or minimal change). This postulate asserts that the epistemic state resulting from a revision should be as close as possible to the original belief set. In other words, among all possible epistemic states that satisfy the other postulates for a revision, one should choose, whenever possible, the state that retains the most previously accepted information. As we will explore in this thesis, this principle is closely related to the principles of informational economy and Occam's razor: information is costly, so unnecessary losses and incorporations should be avoided.

Despite being a consensus among authors in the field, the validity of this principle depends on the exact formal formulation of *minimal change* – which is far from being a consensus. Several heuristics are used to measure information loss, and these have been applied in various ways, as seen in the distinct proposals presented in the literature by authors such as Alchourr'on and Makinson [3], Fuhrmann [21], Gärdenfors and Makinson [29], Grove [34], and Makinson [61]. Moreover, different postulates capture the intuition of information loss in various ways, and there are ongoing debates about which approach is most appropriate, as discussed by Gärdenfors [28], Hansson [38], Makinson [62], among others.

The main reason for this debate is the fact that the logical form of operations is not sufficient to express what should be abandoned in a belief change. Therefore, extra-logical information is necessary, as observed by Gärdenfors [28]. Thus, the way this extra-logical information is structured and used in the system determines the interpretation of the principle of minimal change. This also influences the connection between belief revision and other areas such as counterfactual conditionals, defeasible inference, and more, as suggested by Makinson [63].

Principle of deductive closure

The fact is that there are several questions concerning the underlying principles of rationality and, in general, there is no unique way to answer them because such answers depend on the intended application. For example, one of these questions is related to the choice of representing the epistemic state as a logically closed belief set (AGM approach) or as a finite subset of the language, not closed by logical consequence (referred to as a belief base). In the first case, it is necessary, among other things, that rationality postulates include the requirement that the results of revision operations are also logically closed sets.

When the focus is solely on the representation of the epistemic state, both of the above cases are more or less equivalent – it is possible to calculate all logical consequences of the belief base whenever necessary. However, when dealing with the dynamics of epistemic states, this equivalence is lost.

Changes made to a belief base necessitate temporarily ignoring its logical consequences, creating a clear distinction between explicitly accepted beliefs (present in the base) and implicitly accepted beliefs (logical consequences of explicit beliefs, which cannot be directly changed but are indirectly affected by changes in the base). In contrast, in logically closed sets, there is no distinction between explicit and implicit beliefs, hence the options for change are not confined to the base.

From a computational standpoint, the belief base approach proves to be more expressive and intriguing, as it necessitates dealing with a finite belief set due to the obvious limitations in the memory and computational capacity of the agent. Conversely, the need to formalize theories, where there is a doxastic commitment to accepting and dealing with the logical consequences of beliefs, renders the AGM approach also interesting from a logical and philosophical perspective.[7]

It is worth noting that if the **AGM** model presupposes on principle that epistemic states are closed by logical consequence, then it inherently assumes the existence of an underlying logic. Much of the belief revision literature assumes that this logic satisfies certain properties, known as **AGM** assumptions – it is presumed that the language is closed under all conventional logical connectives and that it satisfies *tarskianicity*, *compactness*, *deduction*, and *supraclassicality*.

Notably, one of the consequences of the **AGM** assumptions is the so-called *explosion principle* – which posits that a single set of beliefs contradictory to all sentences in the language exists. That is, given a contradiction that generates an incoherent epistemic state, the state becomes trivial.

Thus, according to this principle, contradictory epistemic states are not informative and blatantly violate *minimalism*, and therefore must be avoided – precisely what the *principle of non-contradiction* demands. In summary, using the **AGM** system to satisfactorily deal with reasoning amidst contradictions, as we suggested earlier, seems implausible when considering the *principle of deductive closure*.

Our proposal

As we mentioned earlier, the central idea of the thesis is to develop a system (based on the **AGM** model) capable of modeling belief revision in the context of contradictory epistemic states. Our motivations can be listed as follows (the references specific to each motivation will be properly provided throughout the thesis):

(i) The presence of contradictions in belief sets should not be seen as something to be avoided at all costs. Contradictory information is quite common, especially in the daily lives of human agents, and it is often preferable, for various reasons,

[7] Philosophically, there are still authors who advocate the interesting distinction between explicit and implicit beliefs, such as Gilbert Harman [46], and those who defend the contrary, like Robert Stalnaker [96].

to maintain information that is notably incompatible. Despite the status of contradictory theories, they can be quite informative; therefore, it is desirable to establish well-founded reasoning from them. Our system should allow for the possibility of an operation that addresses these facts.

(ii) Even when it is strictly necessary to maintain theories free of contradictions (such as in sets of normative sentences), it is possible, and often necessary, to accept the presence of contradictions at least temporarily, that is, in an intermediate state of reasoning. This idea will be central to our system.

(iii) The learning process, for instance, can be seen as guided by contradictions. Often a belief can be understood as a hypothesis to be tested. The contradiction generated by an incorporation (guided by an observation, for example) can be seen as a stimulus for seeking new information, rather than an incentive for the exclusion of previously accepted information.

(iv) From a theoretical-deductive and argumentative standpoint, the importance of contradiction as a tool for demonstrating theorems is well-known. Classically, if we assume the presence of the negation of a formula and encounter a contradiction, we have demonstrated that the formula is valid. Our system is intended to be versatile enough to also capture this concept of demonstration and apagogic argument.

(v) Argumentative discourse also seems to be guided by contradictions. Dialogic logic, for example, works with the idea of two distinct agents engaging in dialogue with each other, where contradiction is a phenomenon to be sought, precisely because it demonstrates the possible error of the interlocutor. We aim for our system to be compatible with this rationality.

(vi) The principle of informational economy should be part of our system's rationality. Excluding a belief solely because a new piece of information incorporated into the set contradicts it seems incompatible with such a principle, since the rejection of contradiction is not logically necessary for the system. The important relationship between minimality and non-contradiction, therefore, must be explored.

(vii) Human agents intuitively attribute greater weight to certain information than to others. This concept, central to the application of the informational economy principle in the **AGM** belief revision system, must be present and satisfactorily explored in our new model.

The central idea, therefore, is not to simply discuss whether contradictory theories exist, but to construct a system that satisfactorily deals with them. Paraconsistent logics are based on the study of contradictory but non-trivial theories, which is the theoretical framework we use in our system. As Walter Carnielli, João Marcos, and Marcelo Coniglio [9] emphasize, the meaning of paraconsistency as a philosophical program that dares to go beyond consistency is based on the possibility (formal, epistemological, and mathematical) of benefiting from the distinction between asserting, in a formal or natural language, opposite and incompatible things, and ensuring the non-triviality of a theory, whether formal or not. The greatest challenge of paraconsistency, thus, is to weaken deductive closure enough to prevent trivial contradictory theories, while still maintaining a language strong enough to shape a significantly expressive logic.

Paraconsistency[8]

It was at the beginning of the 20th century, due to prevailing conjunctural factors, that authors such as Łukasiewicz and Vasiliev proposed a new approach to non-contradiction, with this period being considered the dawn of contemporary non-classical logics.[9] It

[8] Translation note: An introduction to paraconsistency and paraconsistent logics can be found in Testa [101] (preprint available at https://philpapers.org/rec/TESPBK).

[9] One of the factors that contributed to this dawn derived from the mathematical environment of the late 19th century and the theoretical enterprise that

was between the 1940s and 1960s that the first systems of paraconsistent logic surfaced, such as the works of Stanisław Jaśkowski [51], David Nelson [74] and Newton da Costa [13].

It was during this period that the role of negation was reconsidered in the philosophy of science as well. According to Carnielli, Marcos and Coniglio [9], Popper's falsificationism [80] presented the idea that falsifying a proposition, as an epistemological step to refute it, is not the same as assuming it to be false. This led Popper to think about an apparently paraconsistent logic (Popper [78]), dual to intuitionism, which was later rejected for being too weak to be useful (Popper [79]) – his disciple David Miller later argued for the need of a paraconsistent character to deal with falsificationism (Miller [70]).[10]

The intuition that the consistency of a formula should not be the only sufficient requirement to ensure its explosiveness was present in Da Costa's first system, which he called "good behavior". In his habilitation thesis for full professorship (*livre docência*, in Portuguese), Da Costa [13] summarized in a list the characteristics that a system, if paraconsistent, must satisfy:[11]

- The principle of non-contradiction should not be generally valid.
- From two contradictory sentences, it should not be possible,

followed the crisis of the foundations of mathematics, which fostered different theoretical projects such as logicism, formalism and intuitionism, thus allowing a purification and analysis of the foundations of logic (for an interesting introduction to the history of paraconsistent logic, we suggest the work of Evandro Gomes and Itala D'Ottaviano [32]).

[10] We highlight the proximity of our system with these ideas and the possibility of formally approaching Popperian falsificationism with our belief revision system. It is worth pointing out that we do not intend to formalize that theory, and we recognize that the term "belief" denotes distinct concepts in both cases – as we have already emphasized, that term in Belief Revision is something quite general and formal. Thus, falsificationism should be understood only as the theory on which we base our formal apparatus, but such apparatus does not intend to explain or even justify it.

[11] It is worth saying that the term *paraconsistency* would only be introduced in 1975, by the Peruvian philosopher Miró Quesada during the III SLALM, in Campinas.

in general, to derive all others.

- The system should contain most of the schemes and rules of classical logic, as long as these do not interfere with paraconsistency.

- The extension of these calculations to quantified systems should be immediate.

Contemporarily, Logics of Formal Inconsistency (LFIs), developed by Marcos and Carnielli and explored in [9], introduce consistency as a primitive notion – in fact, the **LFIs** are paraconsistent logics that internalize the notions of consistency and inconsistency in the object language. Due to their high expressive power and recent development, we use this family of logics to construct the paraconsistent belief revision system presented in this work.

The Logics of Formal Inconsistency

Traditionally, the presence of contradictions in a body of knowledge (or theory) and the fact that such theories are trivial – respectively, contradictoriness and triviality – are considered inseparable, that is,

$$\text{Contradiction} = \text{Triviality}$$

A consequence of this is that the concepts of consistency and non-contradiction are equated. Notably, paraconsistent logics challenge this fact. Moreover, by internalizing the concept of consistency in the language, it is possible to make explicit the relation

$$\text{Contradiction} + \text{consistency} = \text{Triviality}$$

The main idea of the **LFIs** is to consider a new consistency operator $\circ$, primitive or not, such that $\circ\alpha$ denotes that α is consistent, in a way that (for any **LFIs** denoted by the consequence operator $\vdash$):

$$(1)\ \alpha, \neg\alpha \nvdash \beta$$

in general, but it is always the case that

$$(2)\ \alpha, \neg\alpha, \circ\alpha \vdash \beta$$

Consistency, non-contradiction and coherence

Considering the important distinctions made by the **LFIs**, contradiction is not equivalent to inconsistency, and conversely, consistency is not equivalent to non-contradiction. It is certainly possible to assert that if a sentence is consistent, then it does not entail contradiction, and if it entails contradiction, then the sentence is inconsistent. However, these are theorems within the **LFIs** and will be demonstrated at appropriate moments.

It is also important to observe that contradictory theories or epistemic states are those in which there is at least one contradiction. In a classical paradigm, such states are usually termed *inconsistent* epistemic states.[12] What we refer to as coherent epistemic states, in this thesis, are those that are not contradictory or, if they are, the sentence involved in each contradiction is not consistent – therefore, the epistemic state is not trivial, as elucidated by situations (1) and (2) mentioned above.[13]

[12]To adapt notation and avoid misinterpretation, whenever the context requires (and permits), we will interchange the terms "consistent" and "inconsistent", referring to sentences and theories, with "non-contradictory" and "contradictory", respectively. This is always done seeking to respect and preserve the original idea of the referenced works; in quotations, we prefer to emphasize the correct interpretation (that is, appropriate to the new, more perceptive terminology) in brackets. Sometimes the context in which the terms "consistent" and "inconsistent" are used is sufficient to specify their meaning – classical or relative to the interpretation of **LFIs** – and such alteration is not necessary. Moreover, it is worth noting that the terms "incoherent" and "trivial" are often used synonymously by us. We also use the term "incompatible", but in an informal sense, that is, in its usual sense.

[13]The goal is to revisit coherentist justification, to be presented in 1.2.2, and to represent the idea that an agent can be coherent even while retaining contradictory beliefs.

Belief revision with consistency operator

The **AGM**$\circ$ Paraconsistent Belief Revision System we propose heavily relies on the formal consistency operator $\circ$. This implies that the constructions, and consequently the postulates, consider this operator central. In a static paradigm (i.e., when focusing on the logical consequence relation), this is already evident. By assuming the consistency of a sentence involved in a contradiction, we face a trivialization (as elucidated in case (2), mentioned above) – which, in a way, captures and articulates the intuition inherent in the expansion process.

Hence, the idea is to also integrate the notion of consistency into the contraction process. In this research, we interpret that a belief being consistent means it cannot be removed from the relevant set of beliefs. In contrast, we introduce a system more aligned with **AGM**, which does not internalize this notion – the **AGMp** system. We will present the technical details, along with the core logical intuitions and constructions, in dedicated chapters.

At this point, what interests us is the overarching fact that a Paraconsistent Belief Revision System facilitates the logical and rational justification of contradictory epistemic states, enabling sensible reasoning about them, beyond mere trivialization. We propose, among several other points, that this capability can be seen as a potential solution, or at least a novel approach, to address certain problems inherent in the **AGM** Belief Revision System, particularly those associated with contradictory sets.

Levi's identity and external revision

One of the issues with the **AGM** model, as exposed by Sven Ove Hansson [40],[14] is that it cannot represent certain interesting operations. This limitation becomes clearer when adopting Levi's principle [56], which suggests that complex belief changes can be reduced to simpler operations:

[14] The following observations follow the considerations raised in the mentioned article.

Principle of decomposition (Fuhrmann [21]) Every legitimate belief change is decomposable into a sequence of contractions and expansions.

It's important to clarify that this principle should not be interpreted as requiring belief changes to be actually carried out as iterations of these operations, alternating between contraction and expansion. The principle's purpose is to ensure that the results of complex operations are equivalent to what would be achieved through such iterative sub-operations.

As previously outlined, the goal of a revision operation is to integrate a new belief into the initial set while avoiding contradictions. According to the principle of decomposition, the first step involves expanding the epistemic state with the new sentence, followed by a prior contraction of its negation. Therefore, a formal construction of a revision can be delineated as follows:

(1) Contraction by the negation of the sentence

(2) Expansion by the sentence itself

This formalization is known as Levi's identity, first introduced by the author who proposed the principle of decomposition. However, in the **AGM** framework, executing these two operations in reverse is problematic. Expanding an epistemic state with a sentence up for revision may lead to contradiction and, in a classical language, a contradictory epistemic state is trivial (as previously illustrated).

If we relax the requirement in AGM's classical rationality principles that epistemic states be theories (logically closed sets), then revisions could be executed in both the order presented in Levi's identity and its reverse, namely:

(1) Expansion by the sentence

(2) Contraction by the negation of the sentence

These two approaches to revision differ both intuitively and in their logical properties, as Hansson [42] notes. Intuitively, these operations correspond to two different scenarios: the contraction-expansion sequence (Levi's identity) respects non-contradiction at each step, leading to a temporary non-committal state where neither the sentence nor its negation is accepted. Conversely, the expansion-contraction sequence (inverse Levi's identity) allows for a potentially contradictory state where both the sentence and its negation are temporarily accepted.

Hansson suggests that expansion-contraction is more plausible when it's clear that new information should be accepted but less clear which prior belief to abandon for successful integration. In contrast, when there is hesitation, neither accepting the new belief nor its negation, contraction-expansion seems more fitting.

However, this distinction is lost in a rationality framework where epistemic states are theories (deductively closed by a classical, non-paraconsistent operator), rendering the expansion-contraction operation undefinable. Allowing for such a distinction, while still adhering to classical closure principles, would necessitate moving away from the concept of logically closed sets of beliefs, which is undesirable when the focus is on dealing with theories, as Hansson [42] indicates. Remarkably, if we define a system capable of handling contradictory theories, we can combine the expressive power of both revisions described above with the rational principles of the **AGM** theory. This integration is precisely one of the goals of the system presented in this thesis.

Non-prioritized revisions

Another significant issue in the **AGM** model of belief revision, which we aim to address (or at least propose a potential solution for) with our new system, is also related to decomposition. There is a type of operation, interesting both from an intuitive and a logical standpoint, that is not definable in the **AGM** model unless we move away from dealing with theories. This operation, known as semi-revision, involves temporarily accepting new information and, if it generates a contradiction, restoring coherence to the resultant epistemic state by retracting either the newly added sentence or one of the previously accepted ones.

In the **AGM** model, the sentence to be incorporated is always accepted in the new epistemic state; it has a priority character (also referred to as the principle of the primacy of new information). However, in semi-revision, this is not necessarily the case (thus, this operation falls into the category of non-prioritized revisions). The operation of semi-revision can be defined as follows:

(1) Expansion by the sentence

(2) Consolidation of the resulting epistemic state

Consolidation, in this context, refers to the operation that restores non-contradictoriness to the epistemic state and involves contracting out the contradictions from the set - which may include removing the newly incorporated sentence. A variant of this, local consolidation, developed by Renata Wassermann [106], retracts only some contradictory sentences, thereby consolidating only a portion of the resulting epistemic state. Consequently, the final belief set might still be contradictory. Notably, for reasons previously mentioned, these operations are not definable within the **AGM** model and are only applicable to belief bases.

However, it should be noted that the **AGM** system can accommodate non-prioritized revisions if they are formulated such that no contradictory intermediate state arises. Broadly, these operations can be characterized by the following steps:

(1) Decision whether the sentence should be accepted or not

(2) If it is accepted, the set is revised by it

An operation exemplifying these steps is *screened revision*, developed by Makinson. This approach considers a set of beliefs, termed the core, which are immune to revision. The belief set is revised by the sentence to be incorporated only if it does not contradict the set's intersection with the core. The revision is then carried out with the stipulation that no element of this intersection is removed.

Other non-prioritized revisions defined within the **AGM** framework include *credibility-limited revision* and *selective revision*. Broadly, these approaches involve considering the possibility of incorporating some beliefs while rejecting others. The accepted beliefs form a set of credible sentences. If a sentence belongs to this set, the epistemic state is revised by it; otherwise, the initial belief set remains unchanged.

Semi-revision represents an intriguing generalization. In typical non-prioritized revisions, new information is either fully accepted or rejected. In semi-revision, however, there is the possibility of accepting only a part of the new information, aligning more closely with our intuitive understanding of belief incorporation.

In summary, the operations in belief bases and sets can be highlighted as follows:

	Belief sets	Belief bases
Contraction-Expansion	Internal revision	Internal revision
Expansion-Contraction	**External revision (only in Paraconsistent Belief Revision systems)**	External revision

Figure 1: Prioritized revisions

	Belief sets	Belief bases
Decision-Revision	Screened Revision, Credibility-limited Revision, Selective revision, among others	
Integrated Choice	Non-prioritized models based on epistemic entrenchment and non-prioritized Grove spheres, among others	
Expansion-Consolidation	**Semi-Revision (only in Paraconsistent Belief Revision systems)**	Semi-Revision

Figure 2: Non-prioritized revisions

The tables presented (adapted from Hansson [40]) highlight four aspects: (i) studies about the decision-revision scheme have only been conducted for belief sets; (ii) studies about the expansion-consolidation scheme have solely focused on bases; (iii) the same applies to external revision, which can be seen as a specific case of semi-revision where consolidation is necessarily achieved by contracting the negation of the incorporated sentence; and (iv) the paraconsistent revision systems developed in this thesis facilitate revisions and semi-revisions under the expansion-contraction and expansion-consolidation schemes, respectively.

The fact that (i) the decision-revision scheme has only been applied to belief sets is understandable when considering that this scheme is essential for avoiding a contradictory intermediate epistemic state, a concern that arises only when dealing with sets. Adapting the constructions of these models to also encompass belief sets is not only unnecessary but also relatively straightforward. The true challenge, however, lies in the reverse process — defining in belief sets the constructions currently exclusive to bases (like external revision and semi-revision, as illustrated by points (ii) and (iii)). This requires not just an adaptation of the formal constructions but also a significant shift in their underlying justifications and logical principles — which is precisely what we aim to accomplish.

Paraconsistent belief revision: a brief genealogy[15]

Finally, it is worth noting that there are several works in the literature that define different systems of Paraconsistent Belief Revision, which assume philosophical and practical presuppositions distinct from those presented in this thesis. For comparison purposes, we identify at least three possible approaches to the development of such systems, enumerated below in relation to their proximity to the classical **AGM** system:

(i) Assume and utilize the same formal constructions as the **AGM**

[15] We use the term *paraconsistent theories* to allude to a theory whose underlying logic is paraconsistent (understood as a logic in line with the previously outlined criteria suggested by da Costa).

Belief Revision System. The primary difference from the classical paradigm is the assumption of a paraconsistent closure and the interpretation of its logical consequences in line with the desired practical and philosophical justifications.

(ii) Start with the formal constructions of the **AGM** system and extend the range of possible results, that is, define new operations over paraconsistent theories based on **AGM**, which are undefinable in theories under classical closure.

(iii) Redefine and reinterpret the formal constructions of the **AGM** system in the context of paraconsistent theories and extend the range of possible results.

The first approach, while highly relevant and interesting from a philosophical and logical-philosophical standpoint, is less compelling from a logical-formal perspective when considering the strong results of recent AGM-compliance. From this perspective, the definability of the **AGM** postulates (and the validity of representation theorems for existing constructions) in certain logics is direct – the main challenge lies in interpreting the **AGM** results within the concepts addressed in their respective logics. This is exemplified by various paraconsistent logics, particularly the Logics of Formal Inconsistency (LFIs).

Depending on one's perspective on paraconsistency, defining a system that follows this approach may be the only viable path, as emphasized in Section A.3. For instance, if one assumes that the principle of non-contradiction should be completely abandoned, it becomes reasonable to reduce any revision to a simple expansion, even if the result is contradictory (provided it is not trivial).

Conversely, if one assumes that paraconsistency enables sensible reasoning about contradictory theories, without necessarily maintaining or seeking such contradiction, then the first approach falls short in capturing the formal constructions necessary for these justifications.

Thus, complementing the first approach, starting from AGM-compliance allows for extending the system by defining operations previously undefinable. The logical-formal interest in this approach lies in the emergence of new constructions and the challenge of formulating them in a way that allows for the proof of representation theorems. It is noteworthy that our alternative system of Paraconsistent Belief Revision, **AGMp**, adopts this perspective.

The key advantage of a system following this strategy is to extend the **AGM** results in a manner necessary to address or illuminate some issues within the **AGM** system, while remaining sufficiently close to engage in meaningful dialogue with it – and to be interpreted as a potential solution to open issues in **AGM**, rather than as a mere alternative system. The work presented by the author (Testa [104]) is considered a precursor to this approach, where the construction of an external revision via paraconsistency led to the **AGMp** system introduced in this thesis.

Our main system, **AGM**$\circ$, enhances the use of the underlying logic by incorporating a new language into its constructions, both in the postulates for different operations and in the explicit construction of the contraction operation. We thus adhere to the third approach mentioned – the significant logical-philosophical interest of this approach is self-evident (cf. Section 4.3.2).

The proposition of Paraconsistent Belief Revision systems is not a novel concept. We can refer to the observations of Priest and Tanaka [81] in this context (particularly concerning the first approach).

Thesis roadmap[16]

Chapter 1 serves as an introduction to the topics covered, introducing some new ideas. In this chapter, we present the theory of Belief Revision in a general way, primarily based on the definitions provided by Gärdenfors.

Chapters 2 and 3 discuss the **AGM** systems and belief bases found in existing literature, and preview some results related to AGM-compliance, with a focus on the work of Flouris and subsequent developments by Ribeiro and Wassermann.

Chapters 4 and 5 form the core of the thesis, where we introduce our **AGM**$\circ$ system of Paraconsistent Belief Revision and explore semi-revisions within this framework.

Chapter 6 is another crucial part of the thesis, in which we present the alternative **AGMp** system, based on the concept of AGM-compliance.

Finally, in **Chapter 7**, we conclude by summarizing our contributions and suggesting open questions for future research.

Publications

Translation note: For updated list of publications, please check `https://rafaeltesta.com/outputs/`

[16] Some of the technical results of this thesis are presented in Testa, Coniglio and Ribeiro [102, 103], although it omits certain conceptual discussions. This translation aims to address that gap

Chapter 1

Generic belief systems

In this chapter, we define a generic belief system, the elements of which are the main tools for the formal investigation of the theory described in this work, namely, the study of belief change and dynamics[1]. According to Gärdenfors [25], the factors that form the core of such systems are four: (i) a representation of the epistemic state, that is, of what is altered in a revision, (ii) the classification of epistemic attitudes that describe the status of beliefs, (iii) triggers external to the agent that motivate belief changes, called epistemic inputs and, finally, (iv) a classification of these changes.

[1] It is worth noting that the term belief is very broad, generally used by contemporary analytical philosophers and formal epistemologists basically to refer to the attitude one has when something is assumed to be true. As with the various works in the area of belief revision present in the literature, we use "belief" in a very broad sense, which may in some cases, depending on the application of the system at hand, be taken as "knowledge" – but our formal system does not assume the view that knowledge is a type of belief, we only point to the formalization's generality and its possible applications to formal epistemology. Given such generality, we use *belief*, *information*, *sentence*, among others, as synonyms, and we believe that the context is sufficient to notice such uses.

1.1 Epistemic states in general

The *epistemic state*[2] of an agent is the formal representation of all beliefs held at a given moment. According to Gärdenfors [25], this representation is not a psychological entity, but rather an idealized representation of an agent's cognitive state at a particular moment, shaped by certain criteria of rationality that underlie the belief system in question.

The AGM theory of belief revision, which we discuss in Chapter 2, presupposes that epistemic states are non-contradictory and deductively closed. These criteria, advocated by Levi [57][3], recognize the limitations of human agents – their inability to infer and deduce all consequences of their beliefs.

These human limitations give rise to the 'problem of logical omniscience' in the literature – the unrealistic expectation that an agent is aware of all logical consequences of their beliefs, including all logical tautologies. The various problems associated with logical omniscience can be briefly summarized as follows, according to K. M. Sim [92]:

Deductive Closure Problem The unrealistic expectation that an agent understands all logical consequences of their beliefs.

Irrelevant Beliefs The requirement for an agent to acknowledge all tautologies.

[2] Many authors argue that the term 'epistemic state,' as used by Gärdenfors, alludes to methodologically constructed knowledge, in contrast to individual opinions and beliefs. The Greek word $\grave{\epsilon}\pi\iota\sigma\tau\eta\mu\eta$ (episteme) is often interpreted as true, rational, and scientific knowledge. Consequently, some prefer the term 'doxastic state', given that the Greek $\delta o\xi\alpha$ (doxa) signifies common belief or opinion, and does not necessarily imply knowledge. We, however, retain the use of 'epistemic', as it is the more commonly used term in the literature. This emphasizes our reference to beliefs that are rationally justified by certain criteria, acknowledging the nuances highlighted in the previous note.

[3] In his seminal work, considered a cornerstone of belief revision theories, Levi posits that an epistemic state (termed a 'corpus') should be consistent and deductively closed. He argues that an inconsistent corpus inherently contains errors and is, therefore, an untenable assumption. Moreover, without deductive closure, such a corpus would fail to recognize all truths it implies.

Inconsistent Beliefs If the belief set is inconsistent (i.e., contradictory), the agent comes to believe everything (trivialization).

Computational Intractability Practical limitations in time and memory that prevent agents from explicitly representing every belief.

To align his theory with these human constraints, Levi interprets the epistemic state as a set of sentences that an agent is committed to believing, whether they are aware of them or not[4]. Gärdenfors concurs, noting that these rationality criteria are often breached, leading to the concept of an epistemic state as a state of equilibrium: a set of beliefs must be adjusted by a rational agent to achieve coherence when faced with incoherence[5]

In addition to the model in which the epistemic state is coherent and closed under logical consequence (AGM theory), several works in the literature advocate an alternative approach, in which they are characterized as *belief bases* – sets of sentences not necessarily closed under logical consequence, presented mainly by Fuhrmann [21], Hansson [38] and Nebel [73], among others already mentioned (the Theory of Belief Bases, which we present in Chapter 3). From a computational point of view, one could say that such an approach proves itself more interesting when taking into account, for example, the aforementioned computational intractability problem.

One of the main contributions of our research is to explore and justify a new approach, in which the epistemic state is deductively closed but the criterion of non-contradiction is challenged (Chapter 4).

In addition to the difference between deductively closed epistemic states and belief bases, it is possible to formally represent the concept of belief in various ways, which generates, unavoidably, distinct characterizations of epistemic states.

[4]This idealized notion of an epistemic state is seen as a 'doxastic commitment' to all logical consequences of one's existing beliefs (see Levi [58], page 8).

[5]This concept of equilibrium traces back to Rawls [82], reflecting the deontological aspect of rationality criteria in epistemic states.

1.1.1 Formal representation of belief

Several logicians and formal epistemologists often characterize belief as a propositional attitude[6], which is the attitude of forming an opinion about the truth or falsehood of a proposition or about the state of affairs that makes this proposition true. Such an opinion about the truth of a proposition can be of a **qualitative** form, as for example when I believe that my book is on the table, and of a **quantitative** form, when my degree of belief that the book is on the table is at least twice as high as my degree of belief that the book on that table is open on page 30, for instance.

Both forms can be related in at least two ways – it is possible to assert that an agent believes in a proposition if and only if their degree of belief that it is true is greater than their degree of belief that it is false and, according to a second proposal, an agent should believe in a proposition if and only if their degree of belief in this proposition is above a certain threshold.

In this research we specifically address epistemic states modeled as sets of logical sentences in a propositional language; we will not address, therefore, Bayesian belief state models used in decision theory, for example, in which beliefs are represented by a probability measure defined over a given object language or event space, as presented by J. Pearl [76], for example, neither will we address models in which epistemic states are sets of possible worlds.

Such non-basic models certainly add greater realism to belief change but, as Gärdenfors [25] points out, they also make the model more complex. The research strategy, the author concludes, is to approach a simpler model, from which it is possible to expand to something more complex – according to its suitability and to the phenomena to be addressed[7]. Such a strategy is possible because,

[6]In general, propositions are usually understood as that which is expressed by assertions, that is, by declarative sentences with meaning. Thus, if two assertions mean the same thing then they express the same proposition.

[7]This view follows Nebel [71], according to whom there are at least two adequacy criteria that a formalization must satisfy: epistemic adequacy – a formalization must be able to express everything that is necessary to solve the problem to which it is directed; and heuristic adequacy – a formalization must

although the constructions presented here are based on a propositional epistemic state model, it can be said that such models are non-linguistic in the sense that, in general, the description of their components is not dependent on the object language in which beliefs are represented[8].

Indeed, the results presented and obtained here can be easily extended to different models, with language being basically a tool to express the contents of epistemic states and not something on which such concepts are built and, therefore, depend. Furthermore, if necessary, the results of **AGM** belief revision theory can be inserted into the object language, as demonstrated by Rijke [86] and Segerberg [91][9], but in these approaches the dynamic character of the belief system is lost, and the motivations for the foundations, characterized by the rationality criteria for the belief revision models, are diluted in the axioms – which makes the theory dependent on a language and uninteresting from the standpoint of generality and the capacity to be taken as a heuristic starting point for the formulation of distinct models.

1.2 Epistemic attitudes

Given a model for beliefs it is possible for an agent to have various *epistemic attitudes* towards each of the elements in this model

be appropriate to be used by a system.

[8] In models based on possible worlds, for example, we have the seminal work of Hintikka [50] in which epistemic logic was developed as a branch of intensional logic, in which the object language is augmented by epistemic operators, which makes the whole theory specific to an object language. As Gärdenfors states, "in contrast to this, my strategy is to "epistemize" the whole semantics, in the sense that I locate the epistemological machinery in the belief systems rather than in the object language. This does not mean that I have any aversion to epistemic logic – on the contrary. However, because I believe that the study of epistemic operators in a formal or natural language is not of primary concern for understanding the dynamics of knowledge and belief, I have chosen to keep the object language as simple as possible." (Gärdenfors [25], p.29).

[9] Rijke has demonstrated how the postulates of **AGM** theory can be translated into the object language of a modal logic. Segerberg demonstrated how the entire **AGM** theory can be translated into a modal logic called doxastic dynamic logic.

– the agent may accept or reject a particular fact as true, or even accept or reject a sentence with a certain probability, in the case of probabilistic models, for example. If the epistemic state is modeled as a belief set, the agent can have at least three epistemic attitudes towards the sentences of this model: the sentence is accepted, rejected or indeterminate[10].

On the other hand, starting from an epistemic state modeled as a belief base, four attitudes are possible: the sentence is *explicitly* accepted, *implicitly* accepted, rejected or indeterminate. The characterization of epistemic states, therefore, brings to light the distinction between implicit and explicit belief.

1.2.1 Implicit and explicit beliefs

The distinction between implicit and explicit beliefs was first addressed directly in theory dynamics by Harman [46], who defines explicit belief as one that involves an explicit mental representation of the belief's content, and implicit belief as one that is derivable from explicit beliefs. For example, by believing that the planet Earth has exactly one moon, one can easily deduce that the Earth does not have two moons, does not have three moons, and so on. All these other things are implicitly assumed to be true.

In a belief set, the distinction is lost – the epistemic state, being represented as closed by logical consequence, does not distinguish between the agent's explicit and implicit knowledge.[11] On the other hand, the elements of a belief base are divided into *basic* and *derived*, i.e., explicit and implicit respectively. In this context, belief changes are made in the bases, that is, derived beliefs are altered only as a consequence of changes in the base.

Let us consider the following example presented by Hansson [41]:

[10] Certainly, the new Paraconsistent Belief Revision System developed by us encompasses other epistemic attitudes, a direct consequence of the enhancement of its underlying language.

[11] Formally, we have that $K = Cn(K)$.

Example 1.1. *I believe that Paris is the capital of France (α). Furthermore, I believe that there is milk in the fridge (β). Therefore, I believe that Paris is the capital of France if and only if there is milk in the fridge ($\alpha \leftrightarrow \beta$). I open the fridge and find that I need to replace my belief in β with $\neg\beta$. I cannot, therefore, at the price of consistency, maintain my belief in both α and $\alpha \leftrightarrow \beta$ at the same time.*

Belief-set approach. *Both α and $\alpha \leftrightarrow \beta$ are elements of the belief set. When I open the fridge and find that there is no milk, I need to choose between retaining my belief in α or in $\alpha \leftrightarrow \beta$. The removal of $\alpha \leftrightarrow \beta$ does not automatically follow, and needs to be ensured by a selection mechanism.*

Belief-base approach. *Since β is a basic and explicit belief, $\alpha \leftrightarrow \beta$ is merely a belief derived from it. When β is removed, $\alpha \leftrightarrow \beta$ disappears automatically. The option to retain it is not even considered.*

We can notice that in the belief-base approach, the agent's implicit beliefs are automatically lost when one removes those explicitly accepted which logically supported them[12]. The underlying idea is that implicit (derived) beliefs cannot be retained in the belief base by themselves, i.e., even if we are committed to believing in the logical consequences of our basic beliefs, such consequences are subject to changes in basic beliefs. In this same sense, explicit beliefs should be seen as self-sustaining, worthy of being retained in the belief base.

This idea that explicit beliefs are basic emphasizes the comparison often made in the literature that belief bases are related to the foundationalist point of view in the theory of knowledge, while belief sets represent a coherentist point of view, as suggested by Alchourrón [1].

[12] Martins and Shapiro [68] call this process *disbelief propagation*.

1.2.2 Coherentism and foundationalism

Roughly speaking, foundationalist theory states that an agent must keep justifications for a belief, i.e., propositions that are not justified should not be accepted. The epistemic state, in this sense, has a structure in which some beliefs serve as justification for others, that is, a belief is logically justified by one or several others, but it is not justified by itself. In this way, some beliefs need to be self-justified so that there is no infinite regress.

On the other hand, coherentist theory asserts that it is not necessary to consider justifications, since the focus is on the logical structure of beliefs, that is, what matters is how the beliefs of an epistemic state cohere with each other.

The definition of rational change, in this way, is distinct in both viewpoints. While in foundationalism a revision consists of abandoning all beliefs not satisfactorily justified and, then, adding new appropriately justified beliefs, in coherentism the focus is on maintaining the consistency (that is, non-contradiction) of the revised epistemic state and, through minimal changes, ensuring a coherent epistemic state.

The distinction between coherentism and foundationalism is often illustrated through two metaphors, that of the pyramid and that of the boat, expressed by Ernest Sosa [94] as follows:

> "For the foundationalist every piece of knowledge stands at the apex of a pyramid that rests on stable and secure foundations whose stability and security does not derive from the upper stories or sections. For the coherentist a body of knowledge is a free-floating raft every plank of which helps directly or indirectly to keep all the others in place, and no plank of which would retain its status with no help from the others."

This boat metaphor is derived from another one used by Otto Neurath [75] to express the fact that it is not possible, nor desirable, to start from scratch when developing a language for scientific discourse:

> "We are like sailors who on the open sea must reconstruct their ship but are never able to start afresh from the bottom."

According to John Pollock [77], this last example illustrates a point of view called negative coherence – according to the author, coherentist theories can be divided into four types, which can be grouped into two groups, presented below:

1a – Positive coherence. The agent must have reasons to keep a belief, that is, each belief must have a "positive support".

1b – Negative coherence. It is justifiable for the agent to keep a belief while there are no reasons to think otherwise, that is, "all beliefs are innocent until proven guilty".

2a – Linear coherence. The agent adopts a foundationalist point of view with respect to reasons even if this leads to an infinite sequence of reasons or to a certain circularity in the structure of reasons.

2a – Holistic coherence. It is justifiable for the agent to hold a certain belief due to its relationship with all other beliefs.

It is worth noting that a coherentist theory may have more than one of the aspects described above and, as Gärdenfors states, this is the case with the AGM belief revision theory, which is "coherentist by nature" (cf. [28]).

Just like the distinction between explicit and implicit beliefs, the split between two distinct belief revision theories, foundationalist theory and coherentist theory, was originally proposed by Harman [46]. Although partially true, we do not share the idea presented by many authors that belief bases are necessarily directly related to the foundationalist point of view in the theory of knowledge, while belief sets represent a coherentist point of view, as Alchourrón argues [1]. A more detailed analysis of what Harman asserts in his work can help us render this fact explicit. Regarding foundationalism, he makes the following assertion:

> "The foundations theory holds that some of one's beliefs "depend on" others for their current justification; these other

> beliefs may depend on still others, until one gets to foundational beliefs that do not depend on any further beliefs for their justification. In this view reasoning or belief revision should consist. First, in subtracting any of one's beliefs that do not now have a satisfactory justification and, second, in adding new beliefs that either need no justification or are justified on the basis of other justified beliefs one has." (Harman [46] p.29)

On the other hand, in relation to coherence theory, Harman asserts that:

> "It is not true that one's ongoing beliefs have or ought to have the sort of justificational structure required by the foundations theory. In this view ongoing beliefs do not usually require any justification. Justification is taken to be required only if one has a special reason to doubt a particular belief. Such a reason might consist in a conflicting belief or in the observation that one's beliefs could be made more "coherent", that is, more organized or simpler or less ad hoc, if the given belief were abandoned (and perhaps if certain other changes were made). According to the coherence theory, belief revision should involve minimal changes in one's beliefs in a way that sufficiently increases overall coherence." (Harman [46] p.30)

Although committed to a coherentist point of view, we can notice the distinction made by him between fully accepted beliefs and those not fully accepted – considered as hypotheses. The fact is that an epistemic state, as considered by Harman, is something a little more complex than simply considering (or not) logically closed sets of beliefs.

Taking into account the nuances pointed out by Harman, it is possible to perceive the need to develop specialized systems – whose focus is to formalize a specific aspect of a belief system.

1.2.3 Specialized belief systems: the example of agents with limited resources

The idea of agents with limited resources was explored by Renata Wassermann [105] in her doctoral thesis. In that work, the author considers agents with memory and logical ability limitations – that is, non-idealized agents. The system proposed by Wassermann follows Harman – in fact, that system can be understood as a formalization of the ideas presented by Harman [46], together with the theory of minimal agents proposed by Cherniak [10].

The theory presented by Cherniak defines minimal agents as those who possess the minimum abilities required to be called rational. The paradigmatic point, according to the author, is that every feasible agent must necessarily be finite, with limitations in their cognitive ability – limitations of time, memory or even in their deductive abilities. His central hypothesis is that the definition of rationality universally assumed in philosophy is so idealized that it does not apply to human agents. He defines, thus, a hierarchy at whose top figure ideal agents, with epistemic states closed by logical consequence, and right below its lower bound are agents incapable of making any logical inference – which cannot, therefore, be called rational.

According to the author, any rational agent (even those in the middle of the hierarchy) must satisfy what he calls *minimum conditions of general rationality*, from which he derives the *minimum condition of inference* – the requirement that, given a set of beliefs, agents must make some, but not necessarily all, apparently appropriate inferences. In this way, apparently non-rational behaviors can be justified by the agent's limitations.

Another important point is the distinction between *active* beliefs, or those taken into account, and *inactive* beliefs. A simple inference can become more difficult in case not all premises are active. We do not intend to expose all of this author's theory, but only to emphasize that it attacks the usually accepted idea that any agent must reason according to a correct and complete logic, as well as to outline the elements used in the system developed by Wassermann [105]. For example, one should not expect an agent to check all the facts that could contradict an assertion, but at least some of the relevant possibilities should be considered – the process of eliminating counter-examples is as limited as the available resources.

One of the main contributions of the aforementioned thesis is precisely formalizing the ideas present in the cited authors – Cherniak and Harman. Roughly, in such a theory, agents are considered as entities, natural or artificial, with limitations of memory and logical abilities. Some of the agent's beliefs are explicitly represented – the number of explicit beliefs is finite, but large enough not to have practical limits.

Each agent has an associated inference operation, which provides all possible inferences at each step of the revision process. In addition, the set of implicit beliefs is given by the closure of the explicit beliefs, and all reasoning occurs in a small part of the agent's epistemic state represented by the active beliefs (beliefs are activated, for example, when recently acquired, when relevant to the reasoning at hand, or even when recently inferred). In such a set there are also conjectures (or hypotheses) which the agent has not yet accepted fully, but whose acceptance is considered by the agent. Let us observe the following diagram (taken from [105]).

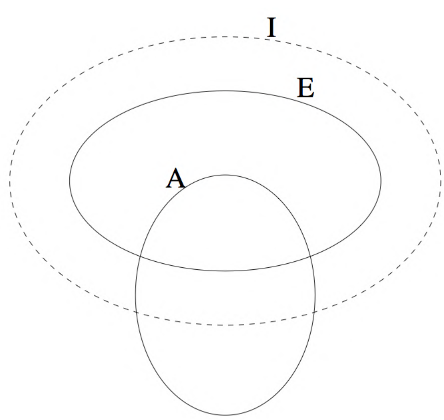

Figure 1.1: Agents with limited resources – Belief structure

Explicit beliefs (active and inactive) are organized as a network in which connections (links) denote some type of relevance. Beliefs are then organized into compartments, and beliefs in the same compartment tend to be activated together.

Roughly speaking, upon receiving new information the agent provisionally incorporates it into its set of active beliefs and, then, puts it into question – deciding whether or not to incorporate it into their set of explicit beliefs. In this way, it is possible to define several operations over an epistemic state, in which beliefs:

- which are explicit become active;
- which are active become rejected;
- are provisionally assumed as active;
- change from active to explicit;
- are inferred from active beliefs;
- are provisionally removed from the set of explicit beliefs.

Obviously, such basic operations can be combined to allow all possible changes that one would expect from an epistemic state.

It is important to note, at this point, that the approach adopted in the system of agents with limited resources is interestingly related to the system developed in this thesis – we highlight, for example, the fact that it is possible for the agent to hold contradictory beliefs (at a certain point during the revision process) without it being considered non-rational. We will discuss this relationship further at the appropriate times.

We would like to emphasize that, certainly, models with more complex epistemic states, of specialized Belief Revision, such as the one suggested by Wassermann, better approximate the different specific phenomena to be formalized. We emphasize, therefore, that we do not disregard these approaches when suggesting a simpler system, based on less complex epistemic states – quite the contrary: we reiterate that the main advantage of simpler models is precisely to use them as starting points, heuristic and formal (material), to the construction of other distinct and more complex systems[13], as well

[13]We intend to use our **AGM** system in the development of more specialized and complex models – with a more specific application – in future work.

as relating them more directly to different logical concepts (philosophical and material), as we have mentioned in the Introduction.

1.3 Epistemic input and operations

The third factor to be considered in a belief system are the motivations that lead the agent to change its beliefs, called by Gärdenfors as *epistemic entries*. The form of these entries is irrelevant, being defined in an abstract way considering its effects on epistemic states, that is, the resulting changes – which comprise the fourth factor, central to the representation of a belief system. Such changes in epistemic states, induced by some epistemic input, are usually called operations. Both in belief sets and in belief bases, we consider three types of operations:

Expansion. Leads the agent to accept a new proposition;

Revision. Leads the agent to accept a new proposition in a non-contradictory way;

Contraction. Leads the agent to abandon a proposition, that is, to make it indeterminate.

One of the main challenges in Belief Revision is to define the rationality criteria for these changes. What is expected of a rational agent when it changes its beliefs? Some authors, such as Harman [46], suggest that the agent should avoid being contradictory and, when changing its beliefs, should obey some minimality criterion.

1.3.1 Rationality criteria

The postulates of rationality specify the constraints that the aforementioned operations must satisfy. To define the postulates of the different operations, the **AGM** model follows the rationality criteria below, presented by Gärdenfors and Rott [30]:

1. Whenever possible, the epistemic states must remain consistent (non-contradictory);

2. Any sentence that is a logical consequence of an epistemic state must belong to the set;

3. When modifying epistemic states, information loss must be minimal;

4. Beliefs considered stronger must be kept at the expense of those considered weaker.

The third criterion, addressed as the *Principle of Informational Economy* by Grove [25] and called the principle of *Minimum Change* by Harman [46], can be understood as a variant of the logical principle called *Occam's Razor*, but applied to the removal of information. This principle is often designated by the Latin expression *Lex Parsimoniae* (Law of Parsimony) enunciated as "*entia non sunt multiplicanda praeter necessitatem*", that is, "entities should not be multiplied beyond necessity", and allows choosing, among several hypotheses to be verified, the one that contains the fewest unproved assertions, which facilitates the verification of the theory and constitutes one of the pillars of reductionism in the scientific method. This economical heuristic is central to belief revision – information is generally not free, so unnecessary losses should be avoided. When we change our beliefs, we should retain as much of our old beliefs as possible.

The second principle defines the conception of epistemic state as a set of beliefs, as adopted by the **AGM** revision model presented in the following chapter. This criterion, along with the principle of informational economy, forces us to demand non-contradiction (the first criterion cited) – if a new belief contradicts some previous one, a revision becomes necessary to maintain coherence and, thus, "beliefs are not multiplied beyond necessity", by the classical principle of explosion.

1.4 Partial considerations

In summary, in this chapter we defined a belief system as suggested by Gärdenfors [25], characterized by the following:

(i) The *epistemic state* of an agent represents, ideally, the set of everything that the agent believes and how these beliefs relate to each other at a given moment. In Chapters 2 and 3 we present, respectively, two distinct ways of representing epistemic states – as logically closed sets of sentences (beliefs sets) and as arbitrary sets of sentences (belief bases). Although the focus is on the former (notably, the systems we developed in Chapters 4 and 5 are defined over sets), our interest in belief bases is due to operations which, classically, are possible only in such epistemic states – but which are, whoever, definable in a paraconsistent paradigm.

(ii) Given a belief state, the agent may have a series of *epistemic attitudes* towards each element of this model. Notably, different models of epistemic states lead to different epistemic attitudes.

(iii) The belief changes we have studied are due to new information, and what matters in Belief Revision systems is only the effect of this information on the epistemic state.

(iv) The different types of change are precisely the effect of the presence of new information.

Chapter 2

AGM System

We present in this chapter the main concepts and results of the **AGM** Belief Revision theory – a system in which the epistemic state of an agent is represented by a logically closed set of sentences. One of the main challenges in the area of belief revision is to define rationality criteria for the different operations (expansion, contraction and revision), and to answer the question of what to expect from a rational agent when it changes its beliefs. In section 1.3.1, we point out such criteria, from which postulates are defined that characterize the different operations. In light of these postulates, it is possible to construct operations that satisfy them and, furthermore, to demonstrate that such constructions are fully characterized by the postulates. This equivalence between rationality postulates and a particular construction, the so-called *Representation Theorem*, is central to belief revision – once such a theorem is proved, it is possible to approach the operation solely through its postulates, allowing for greater abstraction.

2.1 Formal preliminaries

In the original presentation by Alchourrón, Gärdenfors and Makinson [1], operations are constructed over a language $\mathbb{L}$ governed by a logic identified by its consequence operator Cn. We define this logic as the pair $\langle \mathbb{L}, Cn \rangle$. We assume that the reader is familiar with formal logic, at least propositional logic (an introduction to the main concepts is presented in B.1, page 175).

The language of $\mathbb{L}$ is propositional, whose atomic sentences are represented by the lowercase letters $p, q, r, \ldots$, while the lowercase Greek letters $\alpha, \beta, \gamma \ldots$ represent arbitrary sentences.

Uppercase letters represent sets of sentences – in particular, K represents a set of beliefs (logically closed). Such sets are subject to the usual operations of set theory.

Notation 2.1. *We adopt the following common notations:*

$\alpha \in A$	*α is an element of A*
$A \subseteq B$	*A is a subset of B*
$A \subset B$	*A is a proper subset of B*
$A \cup B$	*union of A and B*
$A \cap B$	*intersection of A and B*
$\emptyset$	*empty set*

The negation of the first three symbols is represented by a crossbar – $\alpha \notin A$ represents that α is not an element of A, just as $\nsubseteq$ negates $\subseteq$ and $\not\subset$ negates $\subset$.

2.1.1 Logical closure and AGM assumptions

The **AGM** model is quite general, as very little is assumed about $\langle \mathbb{L}, Cn \rangle$. It is usually required only that $\mathbb{L}$ be closed under the usual classical connectives ($\wedge, \vee, \rightarrow$ and $\neg$) and that $\langle \mathbb{L}, Cn \rangle$ satisfies the following properties, called the **AGM** assumptions:

Definition 2.2 (**AGM** Assumptions)**.** *In the **AGM** theory, $\langle \mathbb{L}, Cn \rangle$ satisfies:*

(Tarskianicity) *the logic is monotonic, idempotent and satisfies inclusion (cf. B.1 for details on these properties).*

(Compactness) *if* $\alpha \in CnA$ *then there exists a finite* $A' \subseteq A$ *such that* $\alpha \in Cn(A')$.

(Deduction) $\alpha \in Cn(A)$ *iff* $\beta \rightarrow \alpha \in Cn(A)$.

(Supraclassicality) *every consequence of classical propositional logic is a consequence of* $\langle \mathbb{L}, Cn \rangle$.

Notably, a logic that satisfies these assumptions is classical propositional logic (**CPL**) – Classical Propositional Logic. A consequence of the **AGM** assumptions is the property known as the *principle of explosion*:

Observation 2.3 (**Principle of explosion**). *If* A *is contradictory then for any* $\beta \in \mathbb{L}$, *we have that* $\beta \in Cn(A)$.

It is worth noting that if logical closure is defined by a paraconsistent logic, this principle ceases to be generally valid. The main contribution of this thesis is to abandon classical assumptions and to explore the consequences of assuming, as underlying the constructions of belief revision and consequently the agent, a non-classical rationality and logic – notably, of a paraconsistent nature. In this way, different epistemic attitudes are assumed and some of the properties described above are challenged.

2.1.2 Belief set

As already laid out, the epistemic states of the **AGM** model are logically closed sets. Formally, we have the following:

Definition 2.4 (**Belief Set**). *An epistemic state* K *is a logically closed set if and only if* $K = Cn(K)$

It is not required that the belief set be finite, but it is convenient to refer to finite examples of logically closed sets. A simple way to do this is to restrict the language to the atomic sentences contained in it. Thus, for example, the elements of $K = p \wedge q$ can be described as follows:

Example 2.5. *The elements of* $K = p \wedge q = Cn(p \wedge q)$ *in a language consisting of* p, q *can be described as follows (the arrows denote logical implication):*

$$
\begin{tikzcd}
 & \top & \\
q \to p \arrow[ur] & p \vee q \arrow[u] & p \to q \arrow[ul] \\
p \arrow[u] \arrow[ur] & p \leftrightarrow q \arrow[ul] \arrow[ur] & q \arrow[ul] \arrow[u] \\
 & p \wedge q \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}
$$

2.1.3 Epistemic attitudes in AGM

Three types of epistemic attitudes are considered in relation to a sentence α in $\mathbb{L}$ (where K represents the agent's belief set):

Accepted if $\alpha \in K$

Rejected if $\neg\alpha \in K$

Undetermined if $\alpha \notin K$ and $\neg\alpha \notin K$

It is worth noting that such epistemic attitudes are defined having in mind classical negation – the dichotomy between accepting and rejecting a sentence reflects the classical definition of a negation, in which a proposition is true if and only if its negation is false. In fact, many authors define epistemic atitudes in the following way:

Accepted if $\alpha \in K$

Rejected if α is inconsistent relative to (contradictory with) K

Undetermined if $\alpha \notin K$ and α is consistent relative to (non-contradictory with) K

Both presentations are, according to classical logic, equivalent: $\neg\alpha \in K$ if and only if α is inconsistent with K, and $\neg\alpha \notin K$ if and only if α is consistent with K.

2.1.4 Epistemic inputs and AGM operations

Epistemic inputs are represented by an atomic sentence of the object language. As we have already emphasized, more complex representations can be found in the literature, from which we highlight those presented by Hansson and Fuhrmann [23] and Spohn [95]. An operation can be understood as the prescription of the way in which a belief state should be altered in view of an epistemic input. The **AGM** model admits three different types of operations, or changes, in belief sets, whose notations are as follows:

Expansion. $(K + \alpha)$ Incorporation of a new sentence α about K without the removal of any previous sentences in K

Contraction. $(K - \alpha)$ Removal of a sentence α from K without the introduction of any new sentence.

Revision. $(K * \alpha)$ Incorporation of a new sentence α about K, with a possible removal of a previous sentence in K to maintain consistency (non-contradiction).

Formally, an operation is a function that takes a set K and an epistemic input α and generates a new set of beliefs. Such operations are described, in the literature, in two ways – by rationality postulates and by different constructions. The postulates, then, are related to the constructions via *representation theorems*.

2.2 Expansion

In this section, we define the expansion operation on belief sets, which models the process of changing a belief set K to include a new sentence. Gärdenfors [25] asserts that expansion is a simple way to model an epistemic change that occurs when one *learns* something, through observation or through some new piece of information provided by someone (epistemic input), and presents a series of postulates to define it; moreover, he shows via a representation theorem that these postulates are equivalent to the following construction, introduced by Levi.

Definition 2.6 (**Expansion – Levi [56]**). *Let K be a set of beliefs and α a sentence. $K + \alpha$ is defined as:*

$$K + \alpha = Cn(K \cup \alpha)$$

2.2.1 Postulates for expansion

The expansion of a set K by an epistemic input α is denoted $K + \alpha$. Formally, it is assumed that $+$ is a function that takes pairs of belief sets and sentences to belief sets. This property can be expressed by the following postulate:

(Closure) For any sentence α and set K, $K + \alpha$ is a belief set.

The next postulate ensures that the operation preserves, in the epistemic state resulting from the expansion, the added sentence (epistemic input). As previously stated, there is no need to define the form of epistemic inputs – these can be identified with the change caused, that is, it can be described as the requirement that α be accepted in the expanded set, that is:

(Success) $\alpha \in K + \alpha$

The effect of the Principle of Informational Economy, described earlier, can be perceived in the following postulate, which ensures that no belief is unnecessarily removed in an expansion:

(Inclusion) $K \subseteq K + \alpha$

The following postulate represents a limit case, in which nothing needs to be done if the epistemic input is already accepted in K:

(Vacuity) if $\alpha \in K$ then $K + \alpha = K$

Monotonicity guarantees that if an epistemic state is contained in another, then the expansion of both by the same sentence will preserve this relationship:

(Monotonicity) If $K \subseteq K'$ then $K + \alpha = K' + \alpha$

Another effect of minimum change can be described as follows:

(Minimality) For any set K and sentence α, $K + \alpha$ is the smallest set that satisfies the previous postulates.

In summary, the postulates that characterize expansion in belief sets are the following:

Definition 2.7 (**Postulates for Expansion**). *Expansion satisfies the following:*

(Closure) *For any sentence α and set K, $K + \alpha$ is a set of beliefs.*

(Success) $\alpha \in K + \alpha$

(Inclusion) $K \subseteq K + \alpha$

(Vacuity) *If $\alpha \in K$, then $K + \alpha = K$*

(Monotonicity) *If $K' \subseteq K$ then $K' + \alpha = K + \alpha$*

(Minimality) *For every set K and sentence α, $K+\alpha$ is the smallest set that satisfies the previous postulates.*

Theorem 2.8 (**Representation of Expansion** [25]). *The expansion function $+$ satisfies the postulates of the definition 2.7, if and only if $K + \alpha = Cn(K \cup \alpha)$*

To sum up, the above postulates only determine that the expansion of K by α is the set of all logical consequences of K joined with α. Henceforth, keeping in mind this equivalence, we will follow most of the works in the area and present expansion as an operation defined via set theory, that is, as a simple union. As we will see later, it is not possible to provide definitions in the same way for revisions and contractions.

2.3 AGM contraction

A contraction occurs when a belief is removed but none is added. According to Gärdenfors [25], such a change can occur, for example, in a debate in which one of the interlocutors believe in α and the other believes in $\neg\alpha$, that is, when α is present in the belief set K while $\neg\alpha$ is present in the belief set K'. To avoid the conflicting belief, both can temporarily exclude α and $\neg\alpha$, respectively, from K and K', as well as the beliefs that imply them. From the resulting belief set, both can continue the debate, in which the intention is precisely to find arguments to respectively support α and $\neg\alpha$ without begging the question (*petitio principii*).

Example 2.9 (Hansson [41])**.** *You do not share my belief that there will be an economic recession next year. For the sake of argument, let us assume that neither of us knows whether this belief is true or false...*

The result of a contraction by α, then, is always a set of beliefs in which α is no longer accepted (unless it is a tautology). In this way, this operation excludes a previous sentence but no other is added. However, it is quite difficult to find examples of a *pure contraction*, that is, in which there is truly no belief that is also incorporated (thus characterizing a revision in the strict sense). As Hansson [41] states, the usual reason for abandoning a belief is the incorporation of something that forces this fact. Let us remember example 0.3, presented in the Introduction:

Example 2.10. *I believed that Plato had written* Hippias Major. *However, I was told that the authenticity of this dialogue as Plato's is contested among scholars in the field. Therefore, I abandoned my belief that Plato wrote* Hippias Major *(without coming to believe in the negation of this statement).*

If we interpret this example literally, it is not an example of *pure contraction* because new information had to be received, namely, that the authorship of *Hippias Major* is uncertain. According to

Hansson [43], it is common to interpret such situations as contractions – the new piece of information that generated the retraction is conveniently neglected and is not included in the new set of beliefs. This convention, the author adds, is imprecise but convenient for finding examples of contractions that, in light of the previous observations, can be considered pure.

For this reason, Hansson states, it is common for some authors to use hypothetical contractions – as we presented at the beginning of this section. These hypothetical contractions, or *contractions for the sake of argument*, can be considered as pure contractions without major problems, even though they are questionable for not effecting an actual change in the agent's belief set, as Fuhrmann [21] emphasizes.[1]

Unlike expansion, the definition of contraction requires extra-logical factors because, when removing a belief α from K, there may be other beliefs in K that imply α, and therefore criteria must be established for such beliefs to be removed as well. For example, if α is in K only because it is a logical consequence of ψ and γ present in K, then the contraction must also exclude ψ or γ, or both. The extra-logical factor, therefore, lies in choosing which belief should be abandoned to effect the contraction.

Nevertheless, it is possible to define the rationality postulates that contraction must obey, regardless of the chosen contraction function, which we will show next.

Consider the following example:

Example 2.11. *I believe that it is not safe to invest all my money in the stock market ($\neg s$) and that, because it is not safe, I should sell my stocks immediately ($\neg s \rightarrow v$). When meeting a friend, they try to convince me not to sell my stocks immediately. For the sake of argument, I may want to contract v from my beliefs.*

[1] This observation should be understood only as an argument in favor of the fact that contraction is, usually, an intermediate step necessary for revision, with pure contraction being more natural in databases (where there are good reasons to instruct a computer to remove an item from its database) than in the dynamics of human agents (cf. Hansson [43], p. 51).

Through the postulates described in the next section we can characterize the operation of contraction and, in this way, highlight the possible results of the contraction $K = \neg s, \neg s \rightarrow v = Cn(\neg(s \vee \neg v))$ by v from the example above.

2.3.1 Postulates for contraction

The contraction of a set K by an epistemic input α is denoted by $K - \alpha$. Contraction can be used along with expansion to perform revision, as we will see later. Formally, it is assumed that $-$ is a function that takes pairs of belief sets and sentences to belief sets, and therefore the result of contraction is a belief set. This property can be expressed by the following postulate:

(Closure) $K - \alpha = Cn(K - \alpha)$

Closure is a direct consequence of the second constraint presented in section 1.3.1, which is that any sentence that is a logical consequence of an epistemic state must belong to the set. Let's see an example:

Example 2.12. *Recall the contraction from example 2.11, $K = \neg s$, $\neg s \rightarrow v = Cn(\neg(s \vee \neg v))$ by v. Figure 2.1 shows all possible belief sets of the language restricted to s, v – represented by each node.* Closure *guarantees us that the result of contraction is one of the nodes in the diagram.*

In the classic **AGM** system, contraction is always successful (that is, $\alpha \notin K - \alpha$), unless the sentence to be removed is a tautology. The following postulate guarantees this fact, that is, the contraction of K by α should be a set of beliefs that does not imply α:

$\alpha \notin K - \alpha$

However, in case α is a tautology, it is not possible to satisfy such a requirement since $\alpha \in K - \alpha$. In particular, if α is a tautology, $\alpha \in Cn(A)$ for all A. Therefore, the only situation in which it is not possible to remove a sentence is when it is a tautology. Thus the success of contraction is defined by the following postulate:

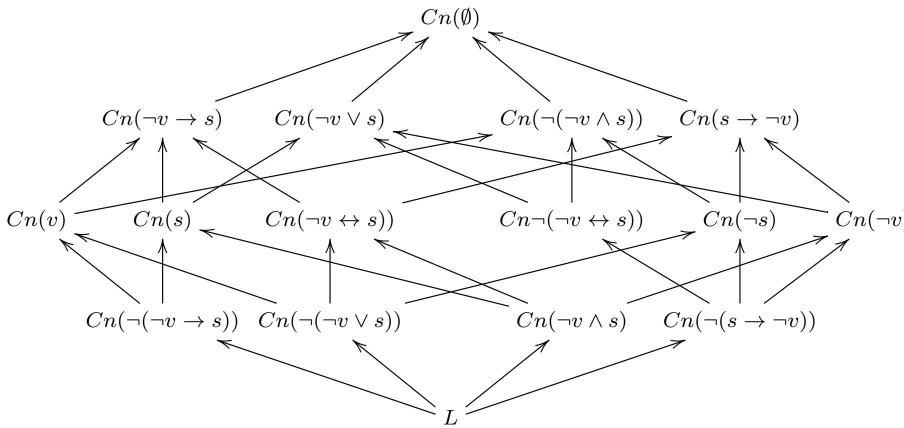

Figure 2.1: Diagram for the example 2.12

(Success) If $\alpha \notin Cn(\emptyset)$ then $\alpha \notin K - \alpha$

As we emphasized in the previous chapter, the constructions presented here should not depend on the language, that is, the description of theory's components is not dependent on the object language in which beliefs are represented. The following postulate guarantees, in a sense, that contraction takes into account only the content of the sentences involved and not their form:

(Extensionality) If $Cn(\alpha) = Cn(\beta)$ then $K - \alpha = K - \beta$

Example 2.13. *Let us return to our diagram:* Success *jointly with* Extensionality *guarantee that the result of contraction will not be one of the belief sets that contain* $Cn\{v\}$ *– figure 2.2*

The next three postulates aim to guarantee that, when performing a contraction, the loss of information should be the minimum necessary to ensure success. Unlike expansion, however, these postulates are not sufficient to ensure the minimality and uniqueness of the operation, requiring extra-logical factors for that purpose – to be presented in the formal construction of this operation. Let us return to the postulates.

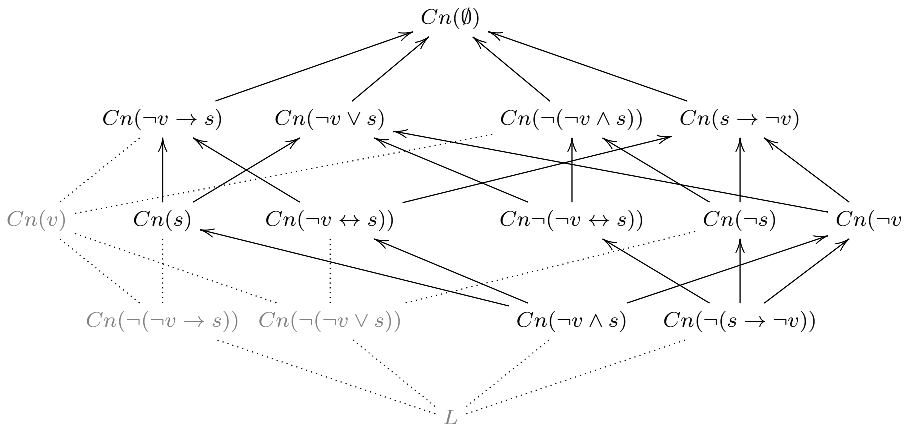

Figure 2.2: Diagram for the example 2.13

First, when removing a sentence from belief set, new sentences should not be unnecessarily added. Thus, we have the following:

(Inclusion) $K - \alpha \subseteq K$

The role of *inclusion* can be illustrated by the following diagram.

Example 2.14. Inclusion *requires that the result of contraction is contained in* $Cn(\neg(s \vee \neg v))$, *that is, the epistemic state resulting from contraction must be above* $Cn(\neg(s \vee \neg v))$ *in the diagram of Figure 2.3 – therefore we must, regarding the previous diagram, eliminate* $Cn(\neg v \wedge s)$, $Cn(\neg(s \to \neg v))$, $Cn(s)$, $Cn\neg(\neg v \leftrightarrow s))$, $Cn(\neg v)$ *and* $Cn(\neg v \vee s)$ *as possible epistemic states of our contraction.*

In addition, the operation of removing a sentence that was not previously in the belief set is vacuous, that is, the original set remains unchanged:

(Vacuity) If $\alpha \notin K$ then $K - \alpha = K$

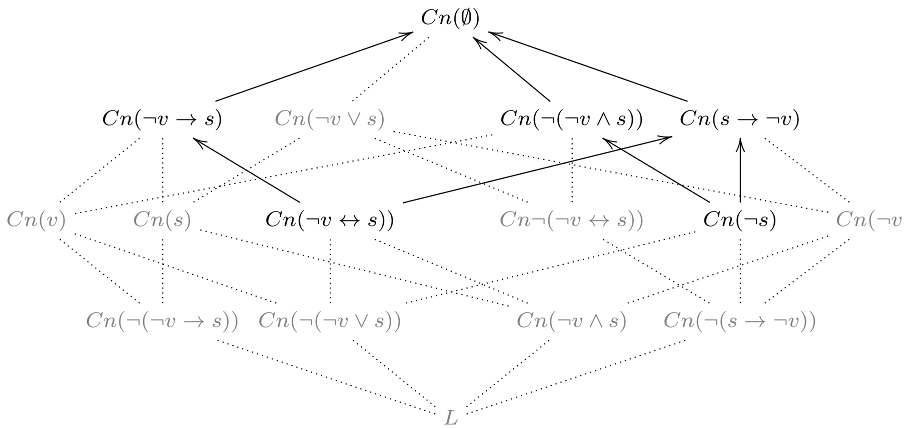

Figure 2.3: Diagram for the example 2.14

The last, most important and controversial postulate that aims to guarantee the minimality of a contraction is *recovery*, which ensures the contraction to be small enough so that the re-addition of α to $K - \alpha$ recovers the entire set K:

(Recovery) $(K - \alpha) + \alpha = K$

Example 2.15. *Recovery guarantees that* $(Cn(\neg(s \lor \neg v) - v) + v = Cn(\neg(s \lor \neg v)$*, that is,* $\neg(s \lor \neg v) - v, v \models \neg(s \lor \neg v)$*. By* deduction, *we have that* $\neg(s \lor \neg v) - v \models v \to (\neg(s \lor \neg v))$*. Moreover, since* $v \to (\neg(s \lor \neg v)) \equiv s \to \neg v$*, we have that* $\neg(s \lor \neg v) - v \models s \to \neg v$*. Thus, in the diagram of figure 2.4, the possible resulting epistemic states, if they satisfy recovery, must be under* $s \to \neg v$*.*

Indeed, as Hansson [41] explains, it is reasonable to think that one of the simplest sequences of belief change, namely, the removal of a sentence followed by its re-addition leaves the agent's epistemic state unchanged, that is, the expansion by α *recovers* what was lost in contraction. Gärdenfors [24] also states that "it is reasonable to require that we get all of the beliefs [in a set] back again after first contracting and then expanding with respect to the same

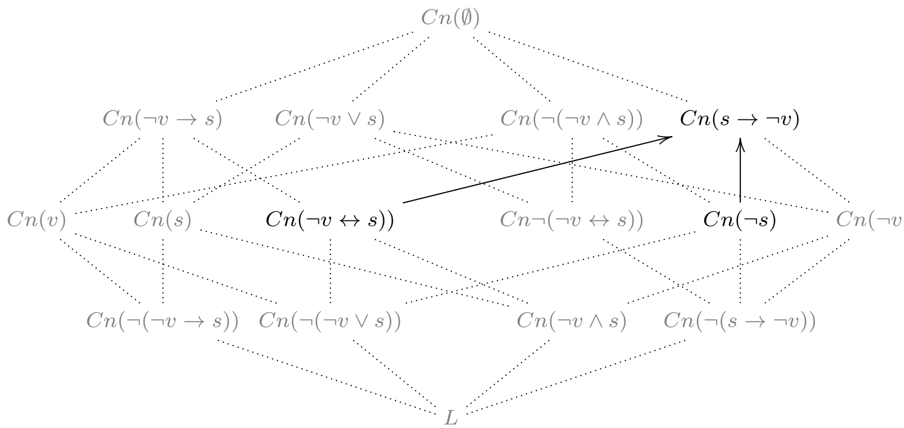

Figure 2.4: Diagram for the example 2.15

belief." On the other hand, many authors criticize that postulate, such as Fuhrmann [21], Hansson [37], Levi [58], and Lindström and Rabinowicz [59], while others, such as Makinson [62], believe that it is "open to query" and propose the definition of *withdraw* contraction – which satisfies all the postulates described below, except for recovery.

Definition 2.16 (***Withdraw* contraction – Makinson [62]).** *The operation $-$ on a set of beliefs is* withdraw *if and only if it satisfies the closure, inclusion, vacuity, success, and extensionality postulates.*

Hansson, despite arguing against the recovery postulate, points out that a *withdraw* contraction violates the principle of informational economy. As an example, he defines the following function $-$ for K:

$$K - \alpha = \begin{cases} K & \text{if } \alpha \notin Cn(K) \\ Cn(\emptyset) & \text{otherwise} \end{cases}$$

Although it satisfies the postulates of *withdrawal* contraction, this function is far from satisfying minimality – when a sentence is removed, all non-tautological beliefs are also removed.

The fact is that *recovery* is not a consensus among the various authors, and remains a controversial point. We will return to discuss recovery but, since it is not the focus at the moment, we will accept such postulate due to its widespread use in the literature, despite the criticisms.

The AGM trio [1] also presents two more postulates, referred to as auxiliary, to specifically characterize contraction by conjunction. These will be addressed in section 2.5, where it will be presented a generalization of the aforementioned postulates that covers the features of contractions by conjunction without the need for specific postulates.

In summary, we can characterize contraction with the six postulates presented in this section. We have, therefore, the following definition of contraction.

Definition 2.17 (**Postulates for contraction** [**1**]). *Contraction satisfies the following postulates:*

(Closure) $K - \alpha = Cn(K - \alpha)$

(Success) $\alpha \notin Cn(\emptyset)$ *then* $\alpha \notin K - \alpha$

(Inclusion) $K - \alpha \subseteq K$

(Vacuity) *if* $\alpha \notin K$ *then* $K - \alpha = K$

(Recovery) $(K - \alpha) + \alpha = K$

(Extensionality) *If* $Cn(\alpha) = Cn(\beta)$ *then* $K - \alpha = K - \beta$

Example 2.18. *Let's recall example 2.11. We have that the possible epistemic states resulting from the contraction of* $K = \neg s, \neg s \to v$ *by* v *are* $K^1 = Cn(\neg v \leftrightarrow s)$*,* $K^2 = Cn(\neg s)$ *and* $K^3 = Cn(s \to \neg v)$*, that is:*

K^1 *I don't sell my stocks if and only if investing in the stock market is safe.*

K^2 *Investing in the stock market is not safe. In this case, I had to eliminate the implication "if it is not safe to invest then I must sell immediately" – well, it's possible to sell later, or simply to stop buying.*

K^3 *If it is safe to invest, then I don't sell my stocks. This way I eliminate the belief that investing is not safe – but I don't assert that it is safe (i.e., I have as a result the intersection of K^1 and K^2).*

2.3.2 Constructions for AGM contraction

Given the conditions that contraction must satisfy, we must analyze how the operator that satisfies such postulates can be constructed. The **AGM** model presents four main constructions – selection functions over subsets of K, Grove spheres systems, epistemic entrenchment, and safe contraction. We will focus our attention on partial meet contraction, built from selection functions.

Selection Functions

One way to construct the contraction of K by α is to focus on the largest possible subsets of K that do not imply α and consider, in this way, the principle of minimal change. Such a set can be defined as follows:

Definition 2.19 (**Remainder set**). *A set K' of beliefs is a maximal subset of K that does not imply α if and only if:*

(i) $K' \subseteq K$

(ii) $\alpha \notin K'$

(iii) If $K'' \subset K' \subseteq K$, *then* $\alpha \in Cn(K')$

The set of all belief sets that are maximal subsets of K that do not imply α is called the remainder set, denoted by $K \perp \alpha$.

Roughly speaking, $K \perp \alpha$ contains more than one maximal subset. The main idea in constructing a contraction function is to apply a selection function γ to choose one of the elements of $K \perp \alpha$.[2]

Definition 2.20 (**Maxichoice contraction**).

$$K - \alpha = \begin{cases} \gamma(K \perp \alpha) & \textit{whenever } K \perp \alpha \neq \emptyset \\ K & \textit{otherwise} \end{cases}$$

This function satisfies the postulates for contraction over K.

Theorem 2.21 (**Representation of maxichoice contraction**). *A maxichoice contraction operation over K satisfies the postulates for contraction of definition 2.17 if and only if there exists a selection function γ such that $K - \alpha = \gamma(K \perp \alpha)$.*

However, assuming such a function to construct the contraction operation leads to the following undesirable result.

Theorem 2.22. *Let K be a belief set and $\alpha \in \mathbb{L}$. If $\alpha \in K$ and $K - \alpha$ is defined by a maxichoice contraction function, then for any proposition β, either $\alpha \vee \beta \in K - \alpha$ or $\alpha \wedge \beta \in K - \alpha$.*

In other words, maxichoice contraction retains a lot of information. This fact can be better illustrated by defining the revision operation using this contraction (via the Levi identity to be presented on page 66).

Corollary 2.23. *Let $-$ be a maxichoice contraction over K. If a revision function $*$ is defined from $-$ by the Levi identity, then, for any α such that $\neg\alpha \in K$, $K * \alpha$ is a complete theory.*

By retaining unnecessary information in a contraction, the agent ends up having, in the resulting revision, an opinion on the truth or falsity of every proposition in the language, which does not match

[2] Intuitively, γ selects the sets that contain the beliefs that the agent believes most strongly (epistemically entrenched), which follows the fourth principle presented in section 1.3.1 – beliefs considered stronger must be maintained over those considered weaker. This qualitative distinction between beliefs is the extra-logical factor mentioned earlier.

our intuitive expectations of this operation, nor does it obey the minimality principle – which is precisely the main objective of this operation. For this reason, it is natural to consider another function, namely, the one that returns all elements of $K \perp \alpha$ – *full meet selection function*, which defines the following function:

Definition 2.24 (**Full meet contraction**).

$$K - \varphi = \begin{cases} \cap(K \perp \alpha) & \textit{whenever } K \perp \alpha \neq \emptyset \\ K & \textit{otherwise} \end{cases}$$

As expected, full meet contraction satisfies the basic postulates for contraction.

Theorem 2.25 (**Representation for full meet contraction**). *An operation $-$ of full meet contraction on K satisfies the postulates for contraction in definition 2.17 if and only if there exists a selection function γ such that $K - \alpha = \cap(K \perp \alpha)$.*

However, we again have the following undesired result:

Theorem 2.26. *Let K be a set of beliefs and $\alpha \in \mathbb{L}$. If $\alpha \in K$ and $K - \alpha$ is defined by a full meet function, then for any proposition β, $\beta \in K - \alpha$ if and only if $\beta \in K$ and $\neg\alpha \vdash \beta$.*

It is notable that *maxichoice* contraction retains a lot of information. This fact can again be better illustrated by defining the revision operation by this contraction (via the Levi identity to be presented).

Corollary 2.27. *Let $-$ be a full meet contraction on K. If a revision function $*$ is defined from $-$ by the Levi identity, then for any α such that $\neg\alpha \in K$, $K * \alpha = Cn(\alpha)$.*

Once again the minimality principle is contradicted and too much information are removed in the resulting revision. The solution is to take an intermediate stance with respect to the two extremes presented, and define a function γ that returns a subset of $K \perp \alpha$ – partial meet contraction. Such an operation, the most widely accepted in the literature, is the intersection of the sets chosen by the selection function.

Definition 2.28 (***Partial meet* contraction).**

$$K - \alpha = \begin{cases} \cap\gamma(K \perp \alpha) & \textit{whenever } K \perp \alpha \neq \emptyset \\ K & \textit{otherwise} \end{cases}$$

Example 2.29. *Let $-$ be a partial meet contraction. In 2.11, in which $K = \{\neg s, \neg s \to v\} = Cn(\neg(s \vee \neg c))$ we have that*

$$K \perp v = \{Cn(\neg v \leftrightarrow s, Cn(\neg s)\}$$

Let $\gamma(K \perp v) = (K \perp v)$. In this case,

$$K - v = Cn(\neg v \leftrightarrow s) \cap Cn(\neg s) = Cn(\neg v \to s)$$

In the case that $\gamma(K \perp v) = Cn(\neg v \leftrightarrow s)$, we have that

$$K - v = \bigcap Cn(\neg v \leftrightarrow s) = Cn(\neg v \leftrightarrow s)$$

In the case that $\gamma(K \perp v) = Cn(\neg s)$, we have that

$$K - v = \bigcap Cn(\neg v) = Cn(\neg v)$$

As the above example illustrates, the postulates of contraction precisely characterize partial meet contraction – the possible resulting epistemic states coincide with those presented in 2.18.

Theorem 2.30 (Representation for partial meet contraction [1])**.** *An operation $-$ of partial meet contraction over K satisfies the postulates for contraction in definition 2.17 if and only if there exists a selection function γ such that $K - \alpha = \cap\gamma(K \perp \alpha)$.*

Transitively relational contraction and supplementary postulates for contraction

A selection function for the belief set K must, for all elements of the remainder set, choose those considered most entrenched. One possible refinement of this construction is to require that the selection function choose beliefs according to a pre-defined preference relation.

Definition 2.31 (**Transitively relational function**). *A function γ for the set K is transitively relational if and only if there exists a relation $\boldsymbol{R}$ such that for all sentences α, if $K\perp\alpha$ is non-empty, then $\gamma(K\perp\alpha) = \{X \in K\perp\alpha | X'\boldsymbol{R}X$ for all $X' \in K\perp\alpha\}$ and moreover $\boldsymbol{R}$ is transitive.*

This function leads to the following:

Definition 2.32 (**Transitively relational partial meet contraction**). *A partial meet contraction is transitively relational if and only if it can be determined by a transitively relational function.*

To characterize this contraction two new postulates related to conjunctions are necessary. The first requires that, in order to abandon a belief of the form $(\alpha \wedge \beta)$, the agent must abandon one of the propositions that constitute it, or even both.

Conversely, another principle to be added asserts that whenever a contraction by α and β is possible, then it must also be possible to carry out the contraction by the belief constituted by the conjunction of both propositions $(\alpha \wedge \beta)$. We have, therefore, the following:

Definition 2.33 (**Supplementary postulates for contraction** [1]). *In addition to the postulates in definition 2.17, a contraction satisfies the following:*

(Conjunctive inclusion) *If $\alpha \notin K-(\alpha\wedge\beta)$ then $K-(\alpha\wedge\beta) \subseteq K-\alpha$*

(Conjunctive intersection) $(K-\alpha) \cap (K-\beta) \subseteq K-(\alpha \wedge \beta)$

Theorem 2.34 (Representation theorem for transitively relational partial meet contraction [1]). *Let K be a belief set. A contraction is transitively relational partial meet on K if and only if it jointly satisfies the contraction postulates in definitions 2.17 and 2.33*

Epistemic entrenchment

When presenting the selection function for partial meet contraction we emphasized the importance of the intuitive idea that, when forced to abandon prior beliefs, the agent should abandon those beliefs that are less deeply rooted in its epistemic state. This idea is exactly what the contraction by *epistemic entrenchment* formalizes – given two beliefs α and β of an epistemic state, to say that "β is more entrenched than α" is to say that β is more useful in deliberation, or even more strongly believed, that is, has a stronger epistemic status than α.

In a contraction, the less entrenched beliefs should be the ones that are most easily abandoned – that is, if we have in mind the construction of the selection function, it chooses the remainder sets that have the most entrenched beliefs in the epistemic state.

Given an epistemic state K, Gärdenfors and Makinson [29] proposed five postulates that an epistemic entrenchment should satisfy ($\alpha \leq \beta$ should be read as "β is at least as entrenched as α in K"):

Definition 2.35 (Postulates for epistemic entrenchment[29])**.** .

(Transitivity) *If* $\alpha \leq \beta$ *and* $\beta \leq \alpha$ *then* $\alpha \leq \alpha$

(Dominance) *If* $\alpha \vdash \beta$ *then* $\alpha \leq \beta$

(Conjunctivity) $\alpha \leq (\alpha \wedge \beta)$ *or* $\beta \leq (\alpha \wedge \beta)$

(Minimality) *If the set of beliefs* K *is non-trivial, then* $\alpha \notin K$ *if and only if* $\alpha \leq \beta$ *for all* β

(Maximality) *If* $\beta \leq \alpha$ *for all* β*, then* $\alpha \in Cn(\emptyset)$

An epistemic entrenchment relation defines a contraction according to the following definition:

Definition 2.36 (**Epistemic entrenchment contraction** [29])**.**

$$\beta \in K - \alpha \textit{ if and only if } \beta \in K \textit{ and } \alpha < (\alpha \vee \beta) \textit{ or } \alpha \in Cn(\emptyset).$$

The epistemic entrenchment contraction coincides exactly with the transitively relational partial meet contraction, and thus we have the following:

Theorem 2.37 (Representation for transitively relational partial meet contraction [29]). *Let K be a belief set. $-$ is an epistemic entrenchment contraction on K if and only if it jointly satisfies the postulates for contraction from definitions 2.17 and 2.33.*

2.4 AGM revision

The revision of a set K by a belief α is denoted by $K*\alpha$. Revision is particularly important when α is incompatible with K and the agent wants to incorporate it in such a way that the resulting set of beliefs remains non-contradictory, that is, some of the previous beliefs must be removed to prevent the presence of contradictions.

The criterion of informational economy, again, plays a central role and requires that the smallest amount of beliefs be removed so that the revision operation is, in a certain sense, a minimal change.

Ideally, as in contraction, minimality is a consensus – but this unanimity dissipates when considering the postulate that guarantees it.

2.4.1 Postulates for revision

The revision of a set K by an epistemic input α is denoted by $K * \alpha$. As in contraction, it is assumed that $*$ is a function that takes pairs of belief sets and sentences to belief sets.

(Fecho) $K - \alpha = Cn(K - \alpha)$

Success guarantees that the new piece of information is in the revised epistemic state.

(Success) $\alpha \in K * \alpha$

Inclusion ensures that the revision of K by the epistemic input α is a subset of the expansion of K by α. Notably, this is trivial when the negation of the epistemic input is present in K (as a consequence of the principle of explosion).

(Inclusion) $K * \alpha \subseteq K + \alpha$

The next postulate complements the previous one and states that if the negation of the epistemic input is not present in K, then revision is equal to expansion.

(Vacuity) If $K + \alpha$ is consistent (non-contradictory), then $K * \alpha = K + \alpha$

The next postulate, although present in the original work of the AGM trio [1], has not been used in recent works on Belief Revision – it is only presented, as suggested by Makinson [62], as a way of defining contraction from revision.

(Harper's Identity) $K * \neg\alpha \cap K = K - \alpha$ for some contraction $-$

Thus, we have the following postulates for revision:

Definition 2.38 (Postulates for Revision [25])**.** *An operation $*$ is a* revision operator *if it satisfies the following postulates:*

(Closure) $K * \alpha = Cn(K * \alpha)$

(Success) $\alpha \in K * \alpha$

(Inclusion) $K * \alpha \subseteq K + \alpha$

(Vacuity) *If $K + \alpha$ is consistent then $K * \alpha = K + \alpha$*

(consistency) *If α is consistent then $K * \alpha$ is consistent.*

(Extensionality) *If $Cn(\alpha) = Cn(\beta)$ then $K * \alpha = K * \beta$*

2.4.2 Constructions for revision

We know that the two main tasks of revision are to add a new belief α to the theory and to ensure that the resulting theory is not contradictory, unless α is, itself, contradictory (self-contradictory). The first task can be guaranteed by expanding the theory by α; the second by the prior contraction of $\neg\alpha$. The composition of these two sub-operations generates the following definition for revision, called the *Levi identity*.

Definition 2.39 (Levi Identity). *Suppose $-$ is a contraction for K that satisfies the postulates of definition 2.17. A revision $*$ for K is constructed as*

$$K * \alpha = (K - \neg\alpha) + \alpha$$

Thus, if $-$ is a partial meet contraction, the operator $*$ defined by the Levi Identity is a partial meet revision. In particular, the revision postulates of definition 2.38 characterize precisely this operation and, as expected, any partial meet revision satisfies these postulates – we have, therefore, the following representation theorem:

Theorem 2.40 (Representation theorem for partial meet revision [1]). *An operation $*$ of partial meet revision on K satisfies the postulates of definition 2.38 if and only if there exists a selection function γ such that $K * \alpha = (K -_{\gamma} \neg\alpha) \cup \alpha$.*

2.4.3 Supplementary postulates for revision

Just like in contraction, supplementary postulates to the basic ones for revision have been proposed. We highlight the following:

Definition 2.41 (Supplementary postulates for revision [1]). *In addition to the basic postulates, we have the following:*

(Superexpansion) $K * (\alpha \wedge \beta) \subseteq (K * \alpha) + \beta$

(Subexpansion) *If* $\neg\beta \notin Cn(K * \alpha)$ *then* $(K * \alpha) + \beta \subseteq K * (\alpha \wedge \beta)$

2.5 Generalized postulates for contraction and revision

There are generalizations of the basic postulates in the literature that cover operations in K whose input is a set A – rendering the supplementary postulates superfluous. It is worth noting that a set $\alpha_1, \alpha_2, ..., \alpha_n$ is equivalent to a sentence $\alpha_1 \wedge \alpha_2 \wedge ... \wedge \alpha_n$, that is, $Cn(\alpha_1, \alpha_2, ..., \alpha_n) = Cn(\alpha_1 \wedge \alpha_2 \wedge ... \wedge \alpha_n)$.

One of these generalizations is the following:

Definition 2.42 (Generalized postulates for contraction [23]). *An operation $-$ is a* contraction operator *in* **L** *if it satisfies the following postulates:*

(Closure) $K - A = Cn(K - A)$

(Success) *If* $A \not\subseteq Cn(\emptyset)$ *then* $A \not\subseteq K - A$

(Inclusion) $K - A \subseteq K$

(Vacuity) *If* $A \notin K$ *then* $K - A = K$

(Recovery) $(K - A) + A = K$

(Extensionality) *If* $Cn(A) = Cn(B)$, *then* $K - A = K - B$

Definition 2.43 (Generalized postulates for revision [23]). *An operation $*$: is a revision if it satisfies the following postulates:*

(Closure) $K * A = Cn(K * A)$

(Success) $A \subseteq K * A$

(Inclusion) $K * A \subseteq K + A$

(Vacuity) *If* $K + A$ *is consistent (non-contradictory) then* $K * A = K + A$

(consistency) *If* A *is consistent (non-contradictory), then* $K * A$ *is consistent*

(Extensionality) *If* $Cn(A) = Cn(B)$, *then* $K * A = K * B$

In addition to allowing the operation's input to be a sentence set, the generalized postulates are quite useful since they allow, as Flouris argues [19], for a greater applicability of the **AGM** system in different logics – notably different non-classical logics.

2.6 AGM-compliance

The main concern of Flouris' work [19] is precisely to elucidate the applicability of the **AGM** system in different non-classical logics – which he calls AGM-compliance.

Definition 2.44 (AGM-compliance [19])**.** *A logic is AGM-compliant if and only if for every belief set* K *there exists at least one operation* $-$ *on* K *that satisfies the (generalized) postulates for contraction.*

Notably, having ensured AGM-compliance, we have, by construction, the validity of revision operations. This interesting result is central to the application of the **AGM** system in epistemic states modeled on a logic distinct from classical logic, and ensures the applicability of the postulates and the validity of representation theorems in these logics. Given this importance, this result was satisfactorily generalized by Márcio Ribeiro and Renata Wassermann – mainly addressed by Ribeiro [85, 84].

Theorem 2.45. *A logic is AGM-compliant if and only if it satisfies the* ***AGM*** *suppositions.*

This result, although on the one hand positive in the sense that it ensures AGM-compliance in different logics, on the other hand has a negative counterpart in the sense that it asserts the non-applicability of the **AGM** system in several groups of logics of notable logical-philosophical and computational interest (such as Horn logic, intuitionistic logic, distributive logic, linear temporal logic, and description logics, for example) – in these cases, the strategy presented by Ribeiro [84] is to modify the postulates sufficiently to allow such applicability. [3]

According to the author, replacing in contraction the recovery postulate with relevance proves to be an interesting strategy – this postulate was suggested by Hansson [37] to capture the notion of minimality without the counterintuitive consequences of the recovery postulate.

(Relevance) If $\beta \in K \backslash K - A$ then there exists K' such that $K - A \subseteq K' \subseteq K, A \neg \subseteq K'$ but $A \subseteq K' + \beta$

The relevance postulate ensures the operation's minimality because it prevents irrelevant sentences from being removed by imposing that no element β can be removed from K unless β contributes to logically entail A, that is, for some K' such that $K - A \subseteq K' \subseteq K$, the set $K' \cup \{\beta\}$ proves A. The fact is that in logics that satisfy the **AGM** assumptions, relevance and recovery are equivalent in the presence of the other **AGM** postulates, as demonstrated by Hansson [37].

The equivalence between relevance and recovery, however, disappears in various non-classical logics – making them completely distinct postulates. Therefore in such logics, Ribeiro [84] points out, it is possible to choose whether to prioritize the recoverability guaranteed by the recovery postulate or the guaranteed minimality ensured by the relevance postulate.

[3] The strategy of modifying the postulates (and hence the explicit constructions), besides allowing the applicability of the **AGM** system in non-AGM-compliant logics, as demonstrated by Ribeiro [84], can also be used in AGM-compliant logics, as argued in this thesis. The justification for this is logically simple, but philosophically significant – the system's postulates, although not, on the one hand, language-dependent (as already stated earlier), on the other hand reflect only the rationality of the language (and hence of the logic) initially assumed as underlying the system by the AGM trio. Thus other different logics, although AGM-compliant, do not have their expressive power satisfactorily explored by the classic postulates and constructions and these, therefore, need to be modified sufficiently to incorporate the specificities captured by the distinct logics to be addressed. Such modification, it is worth noting, is quite simple from a formal point of view, since these do, indeed, depend on the language. The biggest challenge, then, is to find the best way to capture the intuition to be formalized by the new language.

Furthermore, the representation of relevant contraction (partial meet contraction with the relevance postulate instead of recovery) holds in any compact logic, while the representation of partial meet contraction with recovery holds, as already explained, only in logics that satisfy the **AGM** assumptions – which makes the former also more interesting, since it deals with a larger class of logics, in addition to the original reasons advocated by Hansson.

The paraconsistent logics used in our Paraconsistent Belief Revision system are compact and satisfy the **AGM** assumptions, allowing us to choose which minimalism postulate to use.

2.7 Partial considerations

In addition to introducing the **AGM** system of belief revision, the aim of this chapter is to raise some pertinent questions:

(i) The minimality criterion is central to revision operations and the selection heuristic present in these operations is not easy to formally construct. This fact can be noted by the difficulty of defining a selection function for contraction – and our interest in presenting different constructions allows us to highlight this difficulty.

(ii) We can extract from the presentation of the **AGM** system a roadmap to be followed in presenting our new paraconsistent revision system, which is:

1. To define an epistemic state;
2. To present the different epistemic attitudes;
3. To define the possible operations:

3.1. To define the rationality postulates (and highlight the criteria assumed for this);

3.2. To define an explicit construction for each operation;

3.3. To prove the respective representation theorems.

(iii) The representation theorem for contraction in belief sets is valid for logics that satisfy the **AGM** assumptions, and when this is not the case, it is possible to modify the postulates and explicit constructions in a necessary and sufficient manner for such satisfaction.

(iv) AGM-compliance guarantees the direct applicability of **AGM** results, but the modification mentioned in (iii) is also an interesting strategy when one wishes to better capture the intuitions present in a potential new language.

Chapter 3

Belief Bases

In this chapter we briefly present the belief base system, in which epistemic states are represented by arbitrary sets of sentences and, unlike **AGM** theory, distinguishes what the agent explicitly believes from what they believe as a logical consequence of their explicit beliefs. Although the focus of this thesis is on closed systems, what is logically relevant is the fact that belief bases support different revision operations in which intermediary contradictory epistemic states are allowed, such as external revision, as well as operations in which the task of accepting or rejecting a particular sentence is delegated to the selection function – semi-revision.[1] Notably, such operations are not possible in **AGM** theory, despite their interesting justifications in the literature.

Similar to the previous chapter, we present the postulates that characterize the different operations and their formal constructions, as well as their respective *representation theorems*.

[1] We will define these operations for epistemic states closed over a paraconsistent logic, taking as theoretical justification, among other things, the underlying intuitions behind these operations.

3.1 Belief bases and epistemic attitudes

In the theory of belief bases, epistemic states are arbitrary sets B of sentences. However, the operations are also built on a language $\mathbb{L}$, governed by a logic identified by its consequence operator Cn, defined as the pair $\langle \mathbb{L}, Cn \rangle$ (*cf.* section 2.1, on page 44).

Four epistemic attitudes are admitted regarding a sentence α in this belief system, namely:

Explicitly accepted If $\alpha \in B$

Implicitly accepted If $\alpha \in Cn(B) \setminus B$

Rejected If α is inconsistent with B

Undetermined If $\alpha \notin Cn(B)$ and α is consistent with B

3.2 Expansion

Expansion is the operation that leads a sentence to be accepted by the agent, defined exactly as in **AGM** simply by the following:

Definition 3.1 (Expansion – Levi [56]). *Let B be a belief base and α a sentence. $B + \alpha$ is defined as:*

$$B + \alpha = B \cup \{\alpha\}$$

3.3 Contraction in belief bases

Just like in belief sets, contraction in belief bases leads a sentence that was initially accepted (either explicitly or implicitly) to become indeterminate. Again, following the principle of informational economy, this operation must make minimal changes. Most of the postulates for belief bases are the same as for belief sets – obviously, closure is not valid for belief bases and therefore the *success* condition needs to be adapted to ensure that the removed sentence does not belong to the base's closure:

(Success) $\alpha \notin Cn(\emptyset)$ then $\alpha \notin Cn(B - \alpha)$

Inclusion and *vacuity* remain unchanged:

(Inclusion) $B - \alpha \subseteq B$

(Vacuity) If $\alpha \notin B$ then $B - \alpha = B$

The postulate of *extensionality*, in belief bases, does not guarantee that equivalent sentences are contracted from the same set of beliefs in an equivalent way – something a little stronger is needed:

(Uniformity) If for every subset B' of B, $\alpha \in Cn(B')$ if and only if $\beta \in Cn(B')$, then $B - \alpha = B - \beta$

As Hansson [41] stresses, the postulate of *recovery* is not valid in belief bases, so another postulate that ensures the minimality of the contraction operation is necessary. In the literature, there are two postulates suggested for this purpose, namely:

(Relevance) If $\beta \notin B - \alpha$ then there exists a B' such that $B - \alpha \subseteq B' \subseteq B$ and $\alpha \notin Cn(B')$, but $\alpha \in Cn(B' \cup \{\beta\})$

(Core-retainment) If $\beta \notin B - \alpha$ then there exists a B' such that $B' \subseteq B$ and $\alpha \notin Cn(B')$ but $\alpha \in Cn(B' \cup \{\beta\})$

The role of these postulates is to ensure that when removing α from B, another sentence β is also removed from B only if, somehow, it helps to logically derive α. It is worth noting that the way they are stated, *relevance* is stronger than *core-retainment*, and that both, in the presence of *inclusion*, make *vacuity* redundant.

We can, therefore, characterize contraction with two distinct sets of postulates – each of which corresponds to one of the contraction operations described below.

3.3.1 Partial meet contraction in bases

Partial meet contraction in bases is defined identically to partial meet contraction in sets – given the remainder set $B \perp \alpha$ (constructed as defined in Definition 2.19, on page 58) and a selection function γ, we have the following:

Definition 3.2 (**Partial meet contraction in bases**).

$$B -_{\gamma} \alpha = \bigcap \gamma(B \perp \alpha)$$

This construction is fully characterized by the following:

Definition 3.3 (Postulates for partial meet contraction). *An operation $-$ is a partial meet contraction if it satisfies the following postulates:*

(Success) *If $\alpha \notin Cn(\emptyset)$ then $\alpha \notin Cn(B - \alpha)$*

(Inclusion) $B - \alpha \subseteq B$

(Uniformity) *If for every subset B' of B, $\alpha \in Cn(B')$ iff $\beta \in Cn(B')$, then $B - \alpha = B - \beta$*

(Relevance) *If $\beta \notin B - \alpha$ then there exists a B' such that $B - \alpha \subseteq B' \subseteq B$ and $\alpha \notin Cn(B')$, but $\alpha \in Cn(B' \cup \beta)$*

Theorem 3.4 (Representation [45]). *Let $\langle \mathbb{L}, Cn \rangle$ be a monotonic compact logic. The operation $-$ is a partial meet contraction for B iff $-$ satisfies the postulates in definition 3.3.*

3.3.2 Kernel contraction

In kernel contraction, instead of considering the largest subsets of B that do not imply α (remainder set), we consider the smallest subsets of B that imply α (kernel set). Notably, this is not possible in belief systems closed under logical consequence. Let's see:

Definition 3.5 (Kernel set). *Let B be a belief base and α a sentence of the language. The kernel set with respect to α, represented by $B \perp\!\!\!\perp \alpha$, is a set such that $X \in B \perp\!\!\!\perp \alpha$ if and only if:*

(i) $X \subseteq B$

(ii) $\alpha \in Cn(B)$

(iii) *If* $X' \subset X$ *then* $\alpha \notin Cn(X')$

That is, the kernel set is constituted by all subsets of B that derive α from B and are minimal. The kernel contraction is defined by a function σ that chooses at least one sentence from each kernel set to be removed and, in this way, prevents the sentence to be contracted from being derived from the base.[2] Such a function is called an incision, and it is defined as follows:

Definition 3.6 (Incision function)**.** *Let* B *be a belief base and* $\alpha \in B$*.* σ *is an incision function such that for every* α*:*

(i) $\sigma(B \perp\!\!\!\perp \alpha) \subseteq \bigcup B \perp\!\!\!\perp \alpha$ *and*

(ii) *If* $\emptyset \neq X \in B \perp\!\!\!\perp \alpha$ *then* $X \cap \sigma(B \perp\!\!\!\perp \alpha) \neq \emptyset$

Definition 3.7 (**Kernel contraction**)**.** *Let* σ *be an incision function in* B *and* $\alpha \in B$*.*

$$B - \alpha = B \setminus \sigma(B \perp\!\!\!\perp \alpha)$$

This construction is fully characterized by the following:

Definition 3.8 (Postulates for kernel contraction)**.** *An operation* $-$ *is a kernel contraction if it satisfies the following postulates:*

(Success) *If* $\alpha \notin Cn(\emptyset)$*, then* $\alpha \notin Cn(B - \alpha)$

(Inclusion) $B - \alpha \subseteq B$

(Uniformity) *If for every subset* B' *of* B*,* $\alpha \in Cn(B')$ *if and only if* $\beta \in Cn(B')$*, then* $B - \alpha = B - \beta$

(Core-retainment) *If* $\beta \notin B - \alpha$*, then there exists a* B' *such that* $B' \subseteq B$ *and* $\alpha \notin Cn(B')$*, but* $\alpha \in Cn(B' \cup \{\beta\})$

[2] Intuitively, σ selects the agent's beliefs that are the least deeply rooted and, again, follows the fourth principle presented in section 1.3.1 – stronger beliefs should be kept at the expense of weaker ones.

Theorem 3.9 (Representation [45]). *Let $\langle \mathbb{L}, Cn \rangle$ be a monotonic compact logic. The operation $-$ is a kernel contraction for B if and only if it satisfies the postulates of definition 3.8.*

3.4 Belief base revision

Just like in **AGM** theory, revision is defined as the operation that makes a sentence accepted by the agent (in this case, explicitly) in a coherent way. Definition 2.39, on page 66, characterizes such operation on logically closed sets based on a contraction and an expansion. Revision in belief bases is defined in the same way:

$$(B - \neg\alpha) + \alpha$$

However, in belief bases it is possible to invert the two sub-operations involved in the above Levi identity:

$$(B + \alpha) - \neg\alpha$$

As we have seen before, this construction for revision in belief bases is called the inverse Levi identity. Notably, that operation is not viable in belief sets – it is possible for $K + \alpha$ to be contradictory and, therefore, $K + \alpha = K_f$, that is, we lose all the information originally present in K. The same does not occur when dealing with belief systems defined on bases.

The two identities above define, respectively, *internal revision* and *external revision*.

Both revisions satisfy the following postulates:

(Success) $\alpha \in B * \alpha$

(Inclusion) $B * \alpha \subseteq B + \alpha$

Moreover, revisions must satisfy some postulate that guarantees the operation's minimality. Depending on the contraction used to define revision, *kernel* or *partial meet*, we consider respectively the postulates *core-retainment* or *relevance*:

(Core-retainment) If $\beta \in B \setminus B * \alpha$ then there exists a $B' \subseteq B \cup \{\alpha\}$ such that $B' \cup \{\alpha\}$ is consistent (non-contradictory), but $B' \cup \{\alpha, \beta\}$ is not.

(Relevance) If $\beta \in B \setminus B * \alpha$ then there exists a B' such that $B * \alpha \subseteq B' \subseteq B \cup \{\alpha\}$ and $B' \cup \{\alpha\}$ is consistent (non-contradictory), but $B' \cup \{\alpha, \beta\}$ is not.

Internal revision, in specific, satisfies, in addition to above postulates, the following:

(Uniformity) If for every $B' \subseteq B$, $B' \cup \alpha$ is consistent if and only if $B' \cup \beta$ is also consistent, then $B \cap (B * \alpha) = B \cap (B * \beta)$.

On the other hand, the external revision satisfies a weaker version of this postulate:

(Uniformity) If, for every $B' \subseteq B$, $B' \cup \{\alpha\}$ is consistent (non-contradictory) if and only if $B' \cup \{\beta\}$ is also consistent, then $B \cap (B * \alpha) = B \cap (B * \beta)$

On the other hand, external revision satisfies a weaker version of this postulate:

(Weak Uniformity) If $\alpha, \beta \in B$ and, for every $B' \subseteq B$, $B' \cup \{\alpha\}$ is consistent (non-contradictory) if and only if $B' \cup \{\beta\}$ is also consistent, then $B \cap (B * \alpha) = B \cap (B * \beta)$

Moreover, external revision satisfies, as expected, the postulate of pre-expansion:

(Pre-expansion) $B + \alpha * \alpha = B * \alpha$

Therefore, it is possible to define the following distinct revisions in belief bases.

3.4.1 Internal revision

Definition 3.10 (**Partial meet internal revision**).

$$B *_\gamma \alpha = (\bigcap \gamma(B \perp \neg\alpha)) + \alpha$$

Theorem 3.11 (Representation [45]). *The operation $*$ is an internal partial meet revision for B iff it satisfies the success, inclusion, uniformity, and relevance postulates.*

Definition 3.12 (**Kernel internal revision**).

$$B *_\sigma \alpha = (B \setminus \sigma(B \perp\!\!\!\perp \neg\alpha)) + \alpha$$

Theorem 3.13 (Representation [45]). *The operation $*$ is a kernel internal revision for B iff it satisfies the success, inclusion, uniformity, and core-retainment postulates.*

3.4.2 External revision

Definition 3.14 (**External partial meet revision**).

$$B *_\gamma \alpha = \bigcap \gamma((B + \alpha) \perp \alpha$$

Theorem 3.15 (Representation [45]). *The operation $*$ is an external partial meet revision for B iff it satisfies the success, inclusion, weak uniformity, pre-expansion, and relevance postulates.*

Definition 3.16 (**Kernel external revision**).

$$B *_\sigma \alpha = (B + \alpha) \setminus \sigma((B + \alpha) \perp\!\!\!\perp \neg\alpha)$$

Theorem 3.17 (Representation [45]). *The operation $*$ is a kernel external revision for B iff it satisfies the success, inclusion, weak uniformity, pre-expansion, and core-retainment postulates.*

3.5 Semi-revision

Unlike the operations presented so far, semi-revision does not assume that the agent must necessarily accept the new sentence α

to be incorporated. The semi-revision presented by Hansson [36] delegates the task of accepting or rejecting α to the selection mechanism (selection or incision function) – which challenges the success postulate.

The construction of semi-revision is defined as the expansion of B by the sentence α to be incorporated, followed by the contradiction's removal – defined as a contraction of the falsum particle **f**.

Naturally, it is possible to define this operation via kernel contraction or partial meet.

Definition 3.18 (Kernel semi-revision).

$$B?_\sigma\alpha = (B + \alpha) \setminus \sigma((B + \alpha)\perp\!\!\!\perp \boldsymbol{f}$$

The only difference from kernel external revision is the absence of the success postulate, as well as the replacement of weak uniformity by the following:

(Internal exchange) If $\alpha, \beta \in B$ then $B?\alpha = B?\beta$

Theorem 3.19 (Representation [105]). *Let $\langle \mathbb{L}, Cn\rangle$ be a monotonic compact logic. The operation $?$ is a kernel semi-revision for B iff $?$ satisfies the inclusion, core-retainment, pre-expansion, and internal exchange postulates.*

Definition 3.20 (Partial meet semi-revision).

$$B?_\gamma\alpha = \bigcap \gamma((B + \alpha)\perp \boldsymbol{f}$$

Theorem 3.21 (Representation [105]). *Let $\langle \mathbb{L}, Cn\rangle$ be a monotonic compact logic. The operation $?$ is a partial meet semi-revision for B iff $?$ satisfies the inclusion, relevance, pre-expansion, and internal exchange postulates.*

3.6 Partial considerations

As we highlighted in the introduction, the main objective of addressing the belief bases system in this thesis, besides to provide a relevant literature review, is to explore the constructions definable in this system that are not present in **AGM**, namely:

(i) External revision In which there exists an intermediary epistemic state that is contradictory, captured by the pre-expansion postulate.

(ii) Semi-revision Which can be considered a generalization of external revision, in which the success postulate is not assumed, because the acceptance or rejection of the belief to be incorporated is delegated to the selection mechanism – and is not, therefore, assumed *a priori*.

Notably, one of the objectives of this thesis is precisely to define them on a paraconsistent system, in which the existence of an intermediate contradictory state can be satisfactorily captured – which is what we will do in the pages that follow.

Chapter 4

Paraconsistent Belief Revision: AGM∘ System

In this chapter, we present our paraconsistent belief revision system, in which the epistemic state is represented by a deductively closed set over a Logic of Formal Inconsistency (**LFI**). This is any logic **L** that extends **mbC** – the simplest of the **LFIs** to be considered. Despite these logics being AGM-compliant, the intuitive idea we aim to capture with such a system is better represented by substantially altering the rationality postulates of belief revision operations[1], and therefore also its explicit constructions. This satisfactorily interprets the very notion of paraconsistency underlying the **LFIs** – whose strategy is the internalization of the concept of consistency (or inconsistency) within the object language, thus granting it greater expressive power that we aim to explore.

[1] In an alternative approach presented in the next chapter, we introduce a Paraconsistent Belief Revision system that assumes AGM-compliance as a starting point. This allows for the preservation of all AGM constructions and postulates. This system aligns with the second approach to Paraconsistent Revision presented on page 22. Although it may seem logically uninteresting due to the nullification of the consistency operator in a certain sense, the logical interest of this approach lies in its capacity to approach classical **AGM** results (complementing them, in a sense) and to start from a more general definition of paraconsistency, not necessarily linked to the consistency operator.

4.1 Motivations

As presented in the introduction of this work, it seems plausible that agents may hold contradictory beliefs. From a formal standpoint, we can consider the paradigmatic example of external revision in belief bases, where there is an intermediate epistemic state that is contradictory yet coherent: non-trivial and rationally justified. As Hansson [40] points out, external revision becomes plausible when the acceptance of new information is evident, but it is less obvious which previous belief should be abandoned to satisfactorily incorporate this new information.

Semi-revision is a significant generalization of external revision. It is an operation in which a contradictory intermediary epistemic state is also required, but the decision of which previous belief to retract is delegated to the selection function (or another choice mechanism employed by the system), thereby better capturing the aforementioned intuition.

However, in the **AGM** system, if a belief set contains a contradiction, the resulting epistemic state is rendered both incoherent and trivial. Even if this problem can be overcome by modifying the underlying logic, the justification for the possibility of contradictory epistemic states, or rather, the criteria that support them, remains necessary. Does this pose a challenge for the coherentist theory of epistemic justification? Can contradictory epistemic states be considered coherent? Consequently, it becomes essential to investigate rationality postulates that justify the coherence of contradictory belief sets. The formal consistency operator, as presented by the **LFIs**, offers ideal theoretical support for this.

In the following section, we introduce the Formal Inconsistency Logics, which we henceforth assume underlie our model of paraconsistent belief revision—the **AGM**∘ system.

4.2 Logics of Formal Inconsistency

As we have previously discussed, the fundamental concept underlying the **LFIs** involves the introduction of a new consistency operator, $\circ$, whether primitive or derived. In this framework, $\circ\alpha$ signifies the consistency of α. Thus, for any **LFI** expressed by the consequence operator $\vdash$:

$$\alpha, \neg\alpha \nvdash \beta \text{ in general, but it is always holds that } \alpha, \neg\alpha, \circ\alpha \vdash \beta$$

With this, the Logics of Formal Inconsistency, by internalizing the concept of consistency in the language, balance the relation

$$\text{Contradiction} + \text{consistency} = \text{Triviality}$$

in which consistency is explicitly denoted. Formally, we have the following:

Definition 4.1 (**LFI**). *Let $\boldsymbol{L}$ be a logic with a negation $\neg$. The logic $\boldsymbol{L}$ is a Logic of Formal Inconsistency if there exists a non-empty set $\bigcirc(p)$ of formulas in the language of $\boldsymbol{L}$ that depend exclusively on the propositional variable p, such that:*

(i) *There exist sentences α and β such that $\neg\alpha, \alpha \nvdash_{\boldsymbol{L}} \beta$*

(ii) *There exist sentences α and β such that:*

(a) $\bigcirc(\alpha), \alpha \nvdash_{\boldsymbol{L}} \beta$

(b) $\bigcirc(\alpha), \neg\alpha \nvdash_{\boldsymbol{L}} \beta$

(iv) *For every sentence α and β: $\bigcirc(\alpha), \neg\alpha, \alpha \vdash_{\boldsymbol{L}} \beta$*

For each formula α, the set $\bigcirc(\alpha)$ aims to express, in a specific sense, the consistency of α relative to the logic **L**. When this set is unitary, we denote by $\circ\alpha$ the unique element of $\bigcirc(\alpha)$ and, in this case, $\circ$ defines a formal operator (connective) of consistency. It is worth recalling that $\circ$ is not, necessarily, a primitive operator of the signature of **L**.

The most basic of the **LFIs** considered is the propositional logic **mbC**, developed by Carnielli, Coniglio, and Marcos [9].

Definition 4.2 (The logic **mbC**). *The smallest Logic of Formal Inconsistency in the family under review is consituted by the following:*
Axioms:

(A1) $\alpha \to (\beta \to \alpha)$

(A2) $(\alpha \to \beta) \to ((\alpha \to (\beta \to \delta)) \to (\alpha \to \delta))$

(A3) $\alpha \to (\beta \to (\alpha \wedge \beta))$

(A4) $(\alpha \wedge \beta) \to \alpha$

(A5) $(\alpha \wedge \beta) \to \beta$

(A6) $\alpha \to (\alpha \vee \beta)$

(A7) $\beta \to (\alpha \vee \beta)$

(A8) $(\alpha \to \delta) \to ((\beta \to \delta) \to ((\alpha \vee \beta) \to \delta))$

(A9) $\alpha \vee (\alpha \to \beta)$

(A10) $\alpha \vee \neg\alpha$

(bc1) $\circ\alpha \to (\alpha \to (\neg\alpha \to \beta))$

Inference rule:

(Modus Ponens) $\alpha, \alpha \to \beta \vdash \beta$

It is worth noting that (A1)-(A9) plus *Modus Ponens* constitutes an axiomatization for the positive logic **LPC+**.

The following theorems about **LFIs** are important in our system, and therefore we present them in this section. Their proofs can be found in the references, but when they are also important for understanding our system we will present them here at least in a schematic and summarized form.

Theorem 4.3 ([9]). *In **mbC** there are no theorems of the form* $\circ\delta$

Proof. It suffices to use classical truth tables over $0, 1$ for the usual operators $(\wedge, \vee, \to,$ and $\neg)$ and to define a truth table for $\circ$ with a constant value of 0. □

The importance of this theorem is to establish the fact that, in the **AGM**$\circ$ system to be defined, the agent accepts $\circ\alpha$ for some belief α only when it is deliberately incorporated into its epistemic state, because, as this theorem asserts, it is not possible to derive it logically from other previously accepted sentences, not even from the belief α itself.

The following fact is important for the proofs of the *representation theorems*, in which it is necessary to safeguard the possibility of proofs by cases. Moreover, we will use the Deduction Metatheorem to prove other important facts regarding **mbC** and other **LFIs**.

Theorem 4.4. *In **mbC** the following holds:*

(i) *If $\Gamma, \alpha \vdash_{mbC} \delta$ and $\Gamma, \beta \vdash_{mbC} \delta$ then $\Gamma, \alpha \vee \beta \vdash_{mbC} \delta$. In particular, it is possible to carry out proofs by cases in mbC, that is, if $\Gamma, \alpha \vdash_{mbC} \delta$ and $\Gamma, \neg\alpha \vdash_{mbC} \delta$ then $\Gamma \vdash_{mbC} \delta$.*

(ii) *$\Gamma, \alpha \vdash_{mbC} \beta$ iff $\Gamma \vdash_{mbC} \alpha \to \beta$, that is, the Deduction Metatheorem holds in mbC.*

The next theorem can be understood as an instance of what is described in theorem 4.9 – classical rules can be recovered by assuming the consistency of certain formulas.

Theorem 4.5. *The following contraposition rules hold in **mbC**:*

(i) $\circ\beta, (\alpha \to \beta) \vdash_{mbC} (\neg\beta \to \neg\alpha)$

(ii) $\circ\beta, (\alpha \to \neg\beta) \vdash_{mbC} (\beta \to \neg\alpha)$

(iii) $\circ\beta, (\neg\alpha \to \beta) \vdash_{mbC} (\neg\beta \to \alpha)$

(iv) $\circ\beta, (\neg\alpha \to \neg\beta) \vdash_{mbC} (\beta \to \alpha)$

Proof. All cases are demonstrated in a way analogous to what follows: in (i), note that, by the Deduction Metatheorem, it suffices to prove that $\circ\beta, (\alpha \to \beta), \neg\beta \vdash_{mbC} \neg\alpha$. This follows immediately if we use proof by cases on α and $\neg\alpha$, along with *Modus Ponens* and (bc1). $\square$

The following results are extremely useful for understanding **LFIs** and the rationality criteria of the belief revision system based on them.

Theorem 4.6. *In mbC we have the following:*

(i) *mBC distinguishes between consistency and non-contradiction:*

$$\circ\alpha \vdash_{mbC} \neg(\neg\alpha \wedge \alpha)$$

but the converse does not hold.

(ii) *mbC distinguishes between inconsistency and contradiction:*

$$\alpha \wedge \neg\alpha \vdash_{mbC} \neg\circ\alpha$$

but the converse does not hold.

Next, we can see that the *falsum* particle is definable in the language and, with that, a trivializing negation such as the classical one is definable. We will see later that this negation, denoted by $\sim$, has a central role in our system – as much as the consistency operator itself. The fact is that in the smallest **LFIs** of the family we explore in the **AGM∘** system, the interdefinability of $\sim$ from $\neg$ and $\circ$ together is not present and, therefore, the appearance of the negation operator $\sim$ in the initial definitions is necessary, which forces an intuitive interpretation for it and requires a specific epistemic attitude. It is worth noting that, given the ideas present in the system, such interpretation is natural.

Other important results are the following:

Theorem 4.7. *In* ***mbC*** *it holds that:*

(i) *Let δ be a formula. Then $\mathbf{f} =_{def} \delta \wedge \neg\delta \wedge \circ\delta$ is a* falsum *particle in* ***mbC****, that is, $\mathbf{f} \vdash_{mbC} \beta$ for all β.*

(ii) *The formula $\sim\alpha =_{def} (\alpha \to \mathbf{f})$ defines a classical negation in* ***mbC****, that is, $\vdash_{mbC} \alpha \vee \sim\alpha$, and $\alpha, \sim\alpha \vdash_{mbC} \beta$ for all β.*

Demonstration:

(i) is an immediate consequence of (bc1).

(ii) is a direct consequence of the fact that, by axiom (A9), $\alpha \vee \sim\alpha$ is a theorem.

■

The following theorem, at first glance, could seem counter-intuitive or even problematic to our system.

Theorem 4.8. *In **mbC**:*

(i) $(\alpha \wedge \beta) \dashv\vdash_{mbC} (\beta \wedge \alpha)$ *is valid,*
however $\neg(\alpha \wedge \beta) \dashv\vdash_{mbC} \neg(\beta \wedge \alpha)$ *does not hold.*

(ii) $(\alpha \vee \beta) \dashv\vdash_{mbC} (\beta \vee \alpha)$ *is valid,*
however $\neg(\alpha \vee \beta) \dashv\vdash_{mbC} \neg(\beta \vee \alpha)$ *does not hold.*

(iii) $(\alpha \wedge \neg\alpha) \dashv\vdash_{mbC} (\neg\alpha \wedge \alpha)$ *is valid,*
however $\neg(\alpha \wedge \neg\alpha) \dashv\vdash_{mbC} \neg(\neg\alpha \wedge \alpha)$ *does not hold.*

We define the system **AGM**$\circ$ with this peculiarity of the **LFIs** in mind, so that the explicit constructions work like the intuitive ideas that we intend to encompass. This peculiarity illustrates the failure, in general, of the replacement property. With regard to Belief Revision, two beliefs are logically equivalent when the result of revising an epistemic state by each of these beliefs has exactly the same result, as Goldblatt asserts [31]. Therefore, the theorem above does not interfere with any result of our system. The question, then, is merely to perceive the fact that certain sentences that are logically equivalent in a classical paradigm are not so in **LFIs**.

Recovering Classical Logic

The following Derivability Adjustment Theorem (DAT) can be proven:

Theorem 4.9. *Let* $\Gamma \cup \{\alpha\}$ *be a set of formulas in **LPC**. Then* $\Gamma \vdash_{LPC} \alpha$ *iff there exists some* Δ *such that* $\bigcirc(\Delta), \Gamma \vdash_{mbC} \alpha$

The fact is that by incorporating certain beliefs of the form $\circ\alpha$ for their respective sentences α, we obtain a certain symmetry with respect to classical behavior within the **AGM∘** system regarding expansion. Notably, by accepting $\circ\alpha$ for every α in the language, the system no longer accepts epistemic states that are contradictory without this being inconsistent and trivializing.

It is worth noting that the aforementioned symmetry does not hold for contractions (and therefore, revisions) – the reason for this is that we interpret the role of consistency in the success of such operation, as we will see in the following pages, which makes it differ from the classical **AGM** system and **AGMp**.

Extensions of mbC

As we asserted before, different **LFIs** entail different logical consequences and, therefore, reflect different rationalities.

Definition 4.10 (Extensions of **mbC** [8])**.** *Let us consider the following axioms:*

(ciw) $\circ\alpha \vee (\alpha \wedge \neg\alpha)$

(ci) $\neg \circ \alpha \to (\alpha \wedge \neg\alpha)$

(cl) $\neg(\alpha \wedge \neg\alpha) \to \circ\alpha$

(cf) $\neg\neg\alpha \to \alpha$

The extensions of ***mbC*** *we have considered are the following:*

mbCciw $=$ ***mbC***+*ciw*

mbCci $=$ ***mbC***+*ci*

bC $=$ ***mbC***+*cf*

Ci $=$ ***mbC***+*ci*+*cf* $=$ ***mbCi***+*cf*

mbCcl $=$ ***mbCci***+*cf*+*cl*

Cil $=$ ***mbC***+*ci*+*cf*+*cl* $=$ ***mbCci***+*cf*+*cl* $=$ ***mbCcl*** + *cf* $=$ ***Ci***+*cl*

Let us see some interesting results, to be further explored ahead. As we stated before, the relationship between $\sim$, $\circ$, and $\neg$ depends on the **LFI** under consideration – in **mbC**, $\circ\alpha, \neg\alpha \vdash_{mbC} \sim \alpha$, but the converse is not true.

Let us observe the following table, which illustrates that fact.

α	$\neg\alpha$	$\circ\alpha$	$\sim \alpha$
1	1	0	0
	0	1	0
		0	0
0	1	1	1
		0	1

Figure 4.1: Strong negation in **mbC**

On the other hand, such equivalence is the case on **mbCciw**.

α	$\neg\alpha$	$\circ\alpha$	$\sim \alpha$
1	1	0	0
	0	1	0
0	1	1	1

Figure 4.2: Strong negation in **mbCciw**

Theorem 4.11. ***mbcCi*** *includes the following restricted forms of contraposition:*

(i) $(\alpha \to \circ\beta) \vdash_{mbcCi} (\neg\circ\beta \to \neg\alpha)$

(ii) $(\alpha \to \neg \circ \beta) \vdash_{mbcCi} (\circ\beta \to \neg\alpha)$

(iii) $(\neg\alpha \to \circ\beta) \vdash_{mbcCi} (\neg\circ\beta \to \alpha)$

(iv) $(\neg\alpha \to \neg \circ \beta) \vdash_{mbcCi} (\circ\beta \to \alpha)$

Demonstration: The proof is a consequence of the fact that $\vdash_{mbCci} \circ\circ\beta$ (*cf.* Carnielli and Coniglio [8]). ■

Theorem 4.12. *In* ***Cil****, consistency and non-contradiction are equivalent:*

$$\circ\alpha \equiv \neg(\alpha \wedge \neg\alpha)$$

Theorem 4.13. *In* ***Ci****, it holds that*

$$\circ\alpha \vdash_{Ci} \circ\neg\alpha$$

4.3 The AGM∘ system

The **AGM∘** system studies changes of beliefs in epistemic states modeled as logically closed sets of sentences. This closure, however, has distinct characteristics relative to the **AGM** model – notably, we assume a paraconsistent logic **L** with a consistency operator incorporated into the language.

4.3.1 Formal preliminaries

In what follows, we assume some given **LFI**, denoted by **L**, such that it extends **mbC**. The deductively closed theories of **L** are called *belief sets* over **L** and denoted by K. As usual, Cn will represent the deductive closure operator in logic **L** (in this case, such closure obeys the properties presented in 4.2).[2] The language $\mathbb{L}$ of **L** is generated by the connectives $\wedge$, $\vee$, $\rightarrow$, $\neg$, $\circ$, and the constant $\mathbf{f}$. Classical or *strong* negation is defined, as usual, by the abbreviation $\sim\alpha =_{def} (\alpha \rightarrow \mathbf{f})$, whereas $(\alpha \leftrightarrow \beta)$ is an abbreviation for $(\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$.

Among the properties of the underlying logic **L**, we highlight the following:

Lemma 4.14. *Let* $X \cup \alpha \subseteq \mathbb{L}$ *such that* $X, \alpha \vdash \neg\alpha$*. Then* $X \vdash \neg\alpha$*.*

Proof. Suppose that $X, \alpha \vdash \neg\alpha$. It always holds that $X, \neg\alpha \vdash \neg\alpha$, hence $X, \alpha \vee \neg\alpha \vdash \neg\alpha$, since we are assuming that **L** (being an extension of **mbC**) has a classical disjunction $\vee$ (cf. section 4.2). But, since $\vdash \alpha \vee \neg\alpha$ (since this is valid in **mbC**), then $X \vdash \neg\alpha$. $\square$

[2]For reasons of clarity and notational simplicity, we prefer to maintain the classic notation K and Cn to denote, respectively, belief sets and the deductive closure of **L**. The same will be done when denoting the usual **AGM** operations (expansion, contraction, and revision). We believe that the context is sufficient to make the distinction between the different logics explicit.

Other properties of the **LFIs**, concerning the logical consequence relation in general, are extremely important but not strictly necessary for understanding our system, and are presented in the Appendix. We suggest referring to it whenever necessary.

4.3.2 Revisited epistemic attitudes

Due to the linguistic richness of the logics of formal inconsistency, we distinguish three groups of epistemic attitudes:

I Propositional Pertaining to the acceptance of a belief in the epistemic state.

II Quasi-Modal (or auxiliary modal)[3] Pertaining to the entrenchment of a belief.

III Modal Pertaining to the way in which a belief is accepted in the epistemic state.[4]

Let us examine in detail each of these aforementioned attitudes.

I. Propositional epistemic attitudes

Four propositional epistemic attitudes are considered with respect to a sentence $\alpha \in \mathbb{L}$. Let K be the agent's belief set, a sentence α can be:

Underdetermined (or indeterminate) if $\alpha \notin K$ and $\neg\alpha \notin K$, that is, neither α nor $\neg\alpha$ are accepted in K

Rejected if $\neg\alpha \in K$, that is, $\neg\alpha$ is accepted in K

[3]These terms relate to the auxiliary verb that expresses modality only when in conjunction with other verbs. The idea is precisely to capture the fact that the epistemic attitude in question does not constitute modality itself – but it is when in conjunction with a propositional attitude.

[4]The relationship between modality and paraconsistency was first proposed by Béziau [6] and widely studied by J. Marcos [65, 66]. We believe that our system does justice to this relationship and contributes with a new interpretation of some of its points, mainly regarding the modal and quasi-modal epistemic attitudes we propose. We will come back to these issues at appropriate moments.

Accepted if $\alpha \in K$

Overdetermined (or contradictory) if $\alpha \in K$ and $\neg\alpha \in K$, that is, both α and $\neg\alpha$ are accepted in K

$$\begin{tikzcd}
& \alpha, \neg\alpha \in K \arrow[dl, dash] \arrow[dr, dash] & \\
\alpha \in K \arrow[dr, dash] & & \neg\alpha \in K \arrow[dl, dash] \\
& \alpha, \neg\alpha \notin K &
\end{tikzcd}$$

Figure 4.3: Propositional epistemic attitudes of **AGM∘**

We base our attribution of these four propositional epistemic attitudes on Belnap's seminal work [5], in which the author suggests a significant interpretation of the 4-valued system in the context of databases stored in a computer – hence the name of his work: "How a computer should think". The main contexts of application of this multi-valued system are research in relevant logic and computer applications – in both cases, the interpretation of this multi-valued system is as follows (where the set of truth-values is taken as $W = \{\emptyset, \{\bot\}, \{\top\}, \{\bot, \top\}\}$ with respect to a particular state of affairs):

$\emptyset$ there is no information about this state of affairs;

$\{\bot\}$ information indicating that the state of affairs is faulty;

$\{\top\}$ information indicating that the state of affairs is the case;

$\{\bot, \top\}$ conflicting information asserting that the state of affairs is the case and is faulty.

It is worth noting that, in this context, an agent accepting and rejecting a sentence is possible – that is, it is not incoherent and does not generate trivialization.

Let us consider the following example:

Example 4.15. *I believe in the existence of Poseidon ($p \in K$). I will also accept, for the sake of argument, your idea that Poseidon does not exist ($\neg p \in K$) to reflect on it further.*

Thus, the coherent possibility of accepting a contradiction, previously impossible to perform without trivialization (in the classical paradigm), allows us to create a new epistemic attitude specific for it.

It is interesting to note that the example 4.15 presented captures the same type of reasoning formalized in *dialogical logic*, briefly presented by Keiff [52], in which, roughly speaking, argumentation is a type of game between two interlocutors – the agent temporarily accepts the interlocutor's belief (as well as its logical consequences) to compare it to their own. We believe that this fact illustrates a type of *dialectical reasoning*, in which the agent can end up accepting an intermediate belief: something between their previous belief and that of their interlocutor – notably, a part of the logical consequences of the contradiction. We will return to this point when describing **AGM**∘ semi-revision, in Chapter 5.

Furthermore, we believe that the incorporation illustrated in example 4.15 is analogous to the examples of pure contraction described earlier, called *contraction for the sake of argument* – and therefore we call this type of incorporation *expansion for the sake of argument*. Through this analogy, we intend to argue that just as it is possible to claim that there are no truly pure contractions[5] – since these must be understood as an intermediate step in revision – then incorporations that generate an overdetermination (contradiction) can be understood as a necessary intermediate step in (external) revision and, mainly, in semi-revision. The following example highlights this fact.

Example 4.16. *The investigator in a theft case believes that only A or B could have committed the crime ($a \in K$ or $b \in K$), and that they are not accomplices ($(a \rightarrow \neg b) \in K$) and ($(b \rightarrow \neg a) \in K$). His working hypothesis requires him to investigate the possibility that*

[5] As claims Hansson [43], for example.

both A and B committed the theft and to incorporate both a and b, i.e., ($a \in K$) and ($b \in K$).

In this example, at the end of the investigation, the investigator can retain in their epistemic state one of the conflicting pieces of information, or even part of their (contradictory) conjunction – that is, that both A and B committed the crime and, therefore, despite not being accomplices, acted as such in this specific theft (we have again an example of **AGM∘** semi-revision to be presented in chapter 5).

The attitude of neither accepting nor rejecting a sentence, although already possible in the classical **AGM** system, deserves a prominent role in our system because of its duality with overdetermination. Let us see:

Example 4.17. *I do not accept the existence of Poseidon ($p \notin K$). However, I also do not reject it ($\neg p \notin K$)*

One of the central points of this work is to show that, just as it is natural to accept epistemic states with underbdetermined sentences, as in the agnostic example 4.17, there are cases in which overdetermined epistemic states are also perfectly acceptable – let us remember external revision, which we presented in section 3.4.2, page 80, in which it is perfectly acceptable (and rational) to have a contradictory intermediate state. We intend to argue that in paraconsistent belief sets this is also possible and, we emphasize, necessary for the minimality principle to be respected.

In addition to these four epistemic attitudes, we define in our system three others – with the latter two, which we call modals, being defined based on this one we present now.

II. Quasi-modal epistemic attitude

Only one quasi-modal epistemic attitude is considered in relation to a sentence $\alpha \in \mathbb{L}$. Let K be the agent's belief set, a sentence α can be:

Consistent if $\circ\alpha \in K$, i.e., $\circ\alpha$ is accepted in K (regardless of the acceptance or rejection of α).

$$\begin{tikzcd}
 & \alpha, \neg\alpha \in K \arrow[dl, dash, dotted] \arrow[dr, dash, dotted] & \\
\alpha \in K \arrow[dr, dash, dotted] & \circ\alpha \in K & \neg\alpha \in K \arrow[dl, dash, dotted] \\
 & \alpha, \neg\alpha \notin K &
\end{tikzcd}$$

Figure 4.4: Quasi-modal epistemic attitudes of **AGM∘**

A sentence being consistent in K means that any epistemic attitude towards it is irrefutable (unfalsifiable[6]) – if the agent accepts or rejects such a sentence, they will do so in a way that K is not revisable, respectively, by $\neg\alpha$ and α and, furthermore, the accepted sentence will be so deeply rooted in the epistemic state that excluding the former from the latter is not a possibility.[7]

Such entrechnment can be due to different factors such as, for example, preferences in previous beliefs or even due to the hierarchy deliberately fixed by a programmer in a database or even in a normative set, in which certain norms are considered as impossible to retract from the system. Moreover, consistency may also indicate that the belief in question is not susceptible to refutation since the agent simply believes that there are no arguments for refutation.

In short, a sentence α being consistent in K means that:

[6] The term alludes to an important concept in the philosophy of science, coined by Karl Popper [80]. According to the philosopher, for an assertion to be considered refutable or falsifiable in principle, it must be possible to make an observation or experiment that tries to show that the assertion is false. Conversely, an irrefutable assertion is impossible to demonstrate as false. Notably, our system of Paraconsistent Belief Revision satisfactorily interprets Popper's ideas of falsifiability, but we do not intend to effectively formalize his theory, but only serve as one of the possible formal approximations to it.

[7] It is worth noting that the quasimodal epistemic attitude captures, in the object language, part of the intuition presented by epistemic entrenchment and partly by the selection function. We intend to explore this relationship in future work.

(i) Contracting K by α is not possible because, if α is accepted in K, it is so deeply rooted in the epistemic state that it is not possible to remove it (notably, this is also the case when α is a theorem[8]). In this case, we say that α is irrefutable in K. This epistemic attitude can be understood as a deliberate act of the agent to mark those sentences that he or she is not willing to abandon.

(ii) Revising K by $\neg\alpha$ is only possible if α is rejected or indeterminate in K. Let us remember that assuming a formula to be consistent does not necessarily imply that it is accepted (nor rejected) in the epistemic state.

The following example helps us describe this fact:

Example 4.18. *I believe that it is not rationally possible to refute the existence of Poseidon ($\circ p \in K$) due to the metaphysical nature of the question.*

This example reflects a deliberate attitude of the agent to mark a sentence that he is not willing to abandon – in this case, about the existence of Poseidon – not because of a personal preference for it but because he believes that it exceeds any rational argumentation that would allow for refutation. It is worth noting that the agent in the example can still accept, reject, or even not determine the existence of Poseidon.

One might expect that the agent in this example, by considering the existence of Poseidon ($\circ p \in K$) as irrefutable, would also consider its non-existence irrefutable (that is, $\circ\neg p \in K$) for exactly the same reasons cited. However, this is not the case in **mbC**, but this claim is valid from **Ci** (cf. theorem 4.13, page 92). The fact is that different extensions of **mbC**, or rather, different **LFIs** reflect different rationalities – and deal with the propagation of consistency in different ways. Thus, taking these peculiarities into account, if one wishes to illustrate the above situation, the previous example could

[8]In these cases, as expected, $K - \alpha = K$. Trivially, this also holds when α is not in K.

be better described as follows (in order to capture the intuitive idea in the different **LFIs**).

Example 4.19. *I believe that it is not rationally possible to refute any opinion about the existence of Poseidon ($\circ p, \circ\neg p \in K$) due to the metaphysical nature of the issue.*

We will return to discussing the propagation of consistency and the peculiarities of each **LFI** at an appropriate moment. For now, it is important to note that accepting the consistency of a sentence (and therefore the irrefutability of the epistemic attitude towards it) does not mean accepting that its truth-value has been conclusively established and that such a sentence can be “elevated to the status” of knowledge, as it may seem at first glance. On the contrary – assuming consistency for any assertion we call a belief is a behavior diametrically opposed to what could be called knowledge.

Furthermore, interpreting belief (in general) as a genus of which knowledge is a species is a mistake. By accepting the consistency of a sentence, the agent excludes the rational possibility of arguing in favor of it (via hypothetical reasoning that, as we have already explained, presupposes the prior contraction of the belief) and prevents it from being corroborated by other incorporations without begging the question.

It is also interesting to note, at this point, that it is possible to define the attitude of accepting the non-consistency of a sentence, namely, $\neg\circ\alpha \in K$. This can be defined in language as the acceptance of inconsistency $\bullet\alpha \in K$. In a way dual to consistency, accepting inconsistency can be understood as a deliberate act by the agent to mark those sentences which the agent is rationally willing to abandon – by assuming that some of their previous beliefs (possibly but not necessarily all), no matter how strongly confirmed and coherent with their body of knowledge, must always be understood as types of hypotheses that future incorporations caused by new ideas, information, and experiences may refute.

Example 4.20. *I reject your opinion that Poseidon exists ($\neg p \in K$) but I am open to discussing it ($\bullet p \in K$).*

In summary, a sentence α being inconsistent in K means that:

(i) Contraction of K by α is possible – in this case we say that α is refutable in K.

(ii) Revising or even simply expanding K by $\neg\alpha$ is not incoherent, even if α is previously accepted in K (and, conversely, revising or even expanding K by α is not incoherent even if $\neg\alpha$ is previously accepted in K).

It is worth noting that assuming the inconsistency of a formula does not necessarily entail that it is contradictory in the epistemic state (but the converse is true). Furthermore, the relationship between $\circ$ and $\bullet$ is not as straightforward as it may seem at first – we will address these nuances in 4.3.3, but this will not be the focus of our exposition.

The aim of this research is to present a belief revision system based on the consistency operator (**AGM∘**). As such, working out the possibility of formalizing its dual (**AGM•**), as well as exploring some of its characteristics, is done only for didactic purposes. In addition to contributing to a better understanding of the specificities of the Logics of Formal Inconsistency, we can better understand (by comparison) the idea to be captured by the consistency operator. Let us return to the epistemic attitudes of **AGM∘**.

The acceptance of the consistency of a sentence (and therefore of its irrefutability), when combined with the other epistemic attitudes of the system, defines the following modes of acceptance and rejection of α in K, as follows.

III. Modal epistemic attitudes

Let K be the agent's belief set, a sentence α can be:

Strongly accepted if $\alpha \in K$ and $\circ\alpha \in K$, that is, if both α and $\circ\alpha$ are accepted in K

Strongly rejected if $\sim \alpha \in K$

It is worth noting that in **mbC**, a sentence being rejected and consistent entails that it is strongly rejected, but the converse is not necessarily true – let us remember tables 4.1 and 4.2.

Therefore, we consider the following modal attitudes, shown in the diagram in figure 4.5.

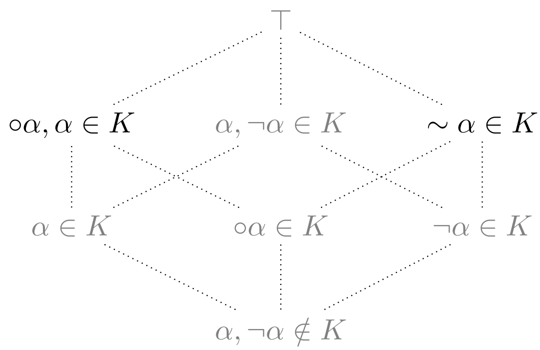

Figure 4.5: Modal epistemic attitudes of **AGM∘**

A sentence α being strongly accepted in K means that α is accepted in K and this set is not susceptible to contraction by α, and moreover, K is not revisable by $\neg\alpha$. Let us see:

Example 4.21. *I strongly believe in the existence of Poseidon ($\circ p, p \in K$). Therefore, at the cost of coherence, I cannot accept your idea that Poseidon does not exist ($\neg p \notin K$), not even for the sake of argument.*

This example is substantially different from 4.18 on page 98. Conversely, a sentence α being strongly rejected means that this set is not revisable by α (due to the presence of $\sim \alpha$ – incorporated either directly or as a consequence of the joint presence of $\circ\alpha$ and $\neg\alpha$).

Example 4.22. *I strongly reject the existence of Poseidon ($\sim p \in K$). Therefore, at the cost of coherence, I cannot accept your idea that Poseidon exists ($p \notin K$), not even for the sake of argument.*

It seems to us that modal epistemic attitudes can be used to capture the characteristic of human agents that Hansson calls the "stubbornness of human belief", that is, their stubbornness (or tenacity) in accepting certain beliefs in such a way as to not want to retract them from their epistemic state.

Example 4.23 (Hansson [41], p.236)**.** *Let us consider the following examples:*

1. *Alice is a fundamentalist. Nothing can make her believe that anything in the Bible is wrong.*

2. *Bernard is an atheist. Nothing can make him believe that God exists.*

3. *Cynthia is convinced in her heart of hearts that John loves her. Nothing can make her abandon this conviction.*

According to the author, it is not possible to capture these attitudes in Belief Revision Systems based solely on revisions and contractions – notably, Hansson had in mind only the **AGM** system and belief bases. He states:

> "Alice's fundamentalism is lost if her belief set is revised by any sentence $\neg\alpha$ such that α is a consequence of something in the Bible. Bernard, in turn, becomes a theist when he revises his set by 'God exists', and Cynthia can easily contract the sentence 'John loves me' from her set of beliefs."

In our case, such examples are formalized as simple epistemic attitudes, without the need to introduce a modal metalanguage as suggested by Hansson [41], p.236 – this fact reinforces the modal character of these epistemic attitudes.

Example 4.24. *Consider the examples from 4.23 mentioned above:*

1. *Let b be any proposition present in the Bible. We have that $b, \circ b \in K$, where K is Alice's epistemic state.*

2. *Let d be the proposition that God exists. In this case, $\sim d \in K$, where K is Bernard's epistemic state.*

3. *Let j be the proposition that John loves Cynthia. We have that $j, \circ j \in K$, where K is Cynthia's epistemic state.*

In **1**, Alice strongly accepts any proposition from the Bible and therefore her epistemic state cannot be revised by sentences that contradict it. In **2**, Bernard strongly rejects the existence of God and therefore revising his set of beliefs to include the existence of God is not rationally possible. In **3**, Cynthia strongly accepts the belief of John's love for her – therefore this information is irrefutable in her epistemic state (and any new information to be incorporated by her will be properly filtered to be coherent with that prior belief).

In summary, the seven epistemic attitudes defined in **AGM∘** are as follows.

Definition 4.25 (Epistemic attitudes in **AGM∘**, see Figure 4.6)**.** *Let K be the agent's belief set, a sentence α can be:*

Accepted *if* $\alpha \in K$

Rejected *if* $\neg\alpha \in K$

Undetermined *(or indeterminate) if* $\alpha \notin K$ *and* $\neg\alpha \notin K$

Overdetermined *(or contradictory) if* $\alpha \in K$ *and* $\neg\alpha \in K$

Consistent *if* $\circ\alpha \in K$

Strongly accepted *if* $\alpha \in K$ *and* $\circ\alpha \in K$

Strongly rejected *if* $\sim \alpha \in K$

We can see that the consistency operator is central to the dynamics of revision, but in a static paradigm (in which the focus is on the consequence operator of **L** and not on the dynamics of sentences), this role cannot be fully expressed, although it is satisfactorily perceived.

The very theorems of the **LFIs** that deal with the $\circ$ operator express, for example, the fact that a contradictory theory is not

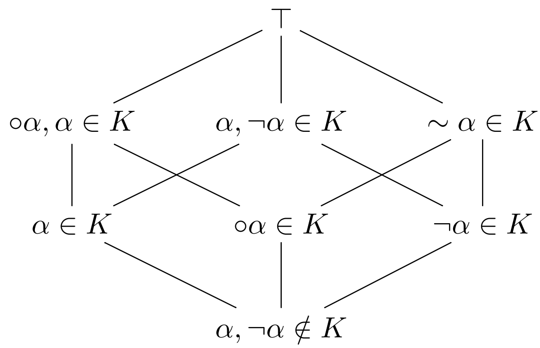

Figure 4.6: Epistemic attitudes in **AGM∘**

trivial unless one of the sentences involved in the contradiction is considered or comes to be perceived as consistent, or conversely, if a theory jointly accepts a certain sentence and its consistency, then accepting its negation leads to the trivialization of the set.

In short, while the consistency operator is central to the dynamics of **AGM** theories, on the other hand, this fact helps us to realize that the idea underlying this operator in a static paradigm (in the **LFIs**) is precisely to express, in a certain sense, such dynamics.

It is worth noting that although the **AGM** model of Belief Revision is not language-dependent, as we stated in the initial chapter of this thesis, the above observations reflect the fact that it is indeed possible to enrich the model to express different phenomena previously not captured.

4.3.3 On the inconsistency operator and the AGM• system

Let us recall that the definition of the inconsistency operator • is as follows:

$$\bullet\alpha =_{def} \neg\circ\alpha$$

As already laid out, assuming the inconsistency of α highlights the possibility of contracting K by α (i.e., α is refutable in K) and therefore revising K by $\neg\alpha$ is not incoherent. It is worth noting that assuming the inconsistency of a belief does not necessarily entail (or rather, does not entail in **mbC**) that it is contradictory in the epistemic state, but its converse is true:

$$(\alpha \wedge \neg\alpha) \vdash_{mbC} \bullet\alpha$$

It can be said that in **mbC**, in a certain sense, the inconsistency operator is innocuous in relation to propositional attitudes, that is, all accepted sentences are considered inconsistent (but not necessarily contradictory), or rather, they are not considered consistent until asserted otherwise.

On the other hand, a significant behavior of the inconsistency operator (and therefore of its use as a primitive operator of the language)[9] can be captured from **Ci**, defined over **mbC** precisely by adding the axiom that deals with inconsistency,

$$\textbf{(ci)}\ \neg\circ\alpha \rightarrow (\alpha \wedge \neg\alpha)$$

together with **(cf)** $\neg\neg\alpha \rightarrow \alpha$ (*cf.* Carnielli and Coniglio [8]).

In this case, although all sentences of the language are already considered not consistent (until asserted otherwise), incorporating the inconsistency of a sentence is not innocuous because it is equivalent to affirming that it is overdetermined. Thus, it is possible to equate inconsistency and contradiction, that is, accepting the inconsistency of a sentence entails that it is contradictory, that is:

Theorem 4.26. *In* ***Ci*** *we have that* $\bullet\alpha \dashv\vdash (\alpha \wedge \neg\alpha)$

In this case, example 4.20 becomes uninteresting since it is trivially valid and does not capture our intuitive idea of consistency (and inconsistency), which is not the case when assuming a weaker **LFI** – namely, **mbC**. For this reason, assuming $\bullet\alpha$ as an epistemic attitude is interesting in our **AGM**$\circ$ system only in the restricted case of **mbC**. In the remaining cases (that is, from **Ci**), the definitions become trivially valid, and assuming inconsistency is equivalent to accepting a contradiction (which is still an epistemic attitude, but does not use the expressive power of $\bullet$).

[9] Since $\bullet\alpha \equiv \neg\circ\alpha$ and also $\neg\bullet\alpha \equiv \circ\alpha$.

Thus, defining an **AGM$\bullet$** system in which the operator $\bullet$ is taken into account in the initial definitions (and constructions) is necessary to explore such operator (as we do with consistency). However, we will not do this in the present research.

4.3.4 The rationality criteria of the AGM$\circ$ system

As in the **AGM** model, the rationality postulates specify the constraints that revision operations must satisfy. In order to define the postulates for the different operations, our **AGM$\circ$** model follows almost the same criteria presented by Gärdenfors and Rott [30] for **AGM**, with some obvious adaptations that deserve some clarification (we name the criteria according to our system, to facilitate future references):

(1) Non-contradiction Whenever possible, epistemic states should remain non-contradictory;

> **(1.1) Coherence** In the case of contradiction, the epistemic state should be coherent – the sentence involved in the contradiction should not be strongly accepted or strongly rejected;

(2) Deductive closure Any sentence that is a logical consequence of an epistemic state should belong to the set;

(3) Minimal change When modifying epistemic states, the loss of information should be minimal;

(4) Epistemic entrenchment Beliefs considered stronger should be maintained at the expense of those considered weaker;

> **(4.1) Non-revisability** Consistent beliefs are not subject to removal from the epistemic state.

The first criterion, the *principle of non-contradiction*, requires that epistemic states should, whenever possible, remain non-contradictory. If they are contradictory, they should at least be coherent – to avoid trivialization at all costs. It is worth noting that clause

(1.1) distinguishes this criterion from its classical version, since, in the paraconsistent system, contradiction is logically possible.

It is interesting to note that despite the possibility of contradictory epistemic states, the same argument that supports the fact that the **AGM** system is coherentist can be applied to the **AGM**∘ system, given the separation of the concepts of non-contradiction and coherence. We do not adhere to this classification, nor do we defend it (at least not in this work), but we emphasize that this is a possibility.[10]

A question that could be raised, at this point, is about external revision: if we accept the criterion of *non-contradiction* then, even if logically possible, external revision should not be rationally possible because it violates this principle due to the intermediate contradictory state – if the contradiction can be avoided by prior contraction, then according to **(1)** the agent should perform it.

At this point, the criterion of *minimality* comes into play – if it has priority over the first criterion, then external revision prevails over internal revision, because as can be seen (and as will be formally laid out in section 4.6.1), it is the previous contraction of a belief in internal revision that is no longer necessary, and therefore it is this that constitutes an unnecessary loss of information. On the other hand, if the first criterion is prioritized, then internal revision is the only one that satisfies the rationality criteria described above.

In this way, it is essential to define both revisions, but it is important to make it clear that they are competing: internal revision prioritizes criterion **(1)**, while external revision prioritizes **(3)**. However, both, as a final result, must obey **(1)** and thus have a non-contradictory epistemic state as a result whenever possible – there are situations, however, in which contradiction is unavoidable due to criterion **(4.1)**.

We can see the importance of economical heuristics in belief revision – as we have already stated, information is generally not free,

[10] We firmly believe that the coherentist theory of epistemic justification merits exploration in relation to the Logics of Formal Inconsistency, and our system can be seen as an initial step towards this objective.

so unnecessary losses should be avoided. When we change our beliefs, we should retain as much as possible of our old beliefs but, in a paraconsistent paradigm, in which contradictions are logically possible, this criterion clashes head-on with the principle of non-contradiction. In this sense, we can say that revision is a game of balance between both criteria, and different priorities characterize different rationalities to be followed[11] – this statement can be better understood later when characterizing the internal and external revisions of **AGM**$\circ$.

The other criteria remain unchanged in relation to **AGM**, only with the introduction of clause **(4.1)** concerning consistent beliefs – these must remain in the agent's epistemic state at all costs, even at the expense of coherence and possible trivialization. Despite being expensive, trivialization is the price to be paid for accepting the non-refutability of certain beliefs without due care to verify the possibility of accepting other beliefs that contradict them.

4.3.5 Epistemic attitudes and the underlying rationality of different LFIs

In addition to the criteria described in the previous section, when taking into account what is asserted in **(2)**, that is, that any sentence that is a logical consequence of an epistemic state must belong to the set, then distinct new criteria are brought about by the different logics we assume as underlying the **AGM**$\circ$ system, presented in 4.2.

We do not intend to exhaust all the different theorems of the **LFIs** that we have addressed. The central point of this section is to understand how the paraconsistent belief revision system developed in this thesis can be seen as a pertinent and expressive interpretation of the **LFIs**, intuitively (and formally) explicating some of their results that are quite natural under our system.

[11] This game cannot be perceived in a system in which contradiction is logically impossible, such as **AGM**.

On accepting inconsistency, consistency, and its propagation

Let us remember Theorem 4.13, in which we have that

$$\circ\alpha \vdash_{Ci} \circ\neg\alpha$$

Certainly, in this case, an agent whose underlying logic is **Ci** incorporates, as a ratioinality criterion, the fact that accepting the consistency of a belief entails that the negation of such a belief is also consistent. Examples 4.18 and 4.19 seem to illustrate this fact quite naturally.

Moreover, Theorem 4.12 states the following:

$$\circ\alpha \equiv \neg(\alpha \wedge \neg\alpha)$$

That is, an agent whose underlying logic is **Ci** equates non-contradiction with consistency.

Furthermore, we have in **mbCci** that

$$\vdash_{mbCci} \circ\circ\alpha$$

This case is quite significant: the agent whose underlying logic is **mbCci**, upon accepting the consistency of a belief, such consistency becomes irrefutable in their epistemic state – since $\circ\circ\alpha \in K$ for every α. Therefore, it is not possible for such an agent to remove the belief $\circ\alpha$ from its epistemic state.

Moreover, we can see that an agent (whose underlying logic is any **LFI**, extension of **mbC**), upon accepting a contradiction, that is, upon overdetermining some sentence, also incorporates the fact that such a sentence is inconsistent (or better, non-consistent). Let us look at Table 4.7, which illustrates this fact (and that its converse is not true).

In this case, upon performing an external revision, for instance, in which there exists an intermediate contradictory state, such agent marks, in its epistemic state, the revisability of the sentence in question. Thus, incorporating its consistency becomes incoherent – that is, once a sentence is considered as overdetermined, the agent stores

<table>
<tr><th>α</th><th>$\neg\alpha$</th><th>$\circ\alpha$</th><th>$\neg\circ\alpha$</th></tr>
<tr><td rowspan="4">1</td><td>1</td><td>0</td><td>1</td></tr>
<tr><td rowspan="3">0</td><td rowspan="2">1</td><td>1</td></tr>
<tr><td>0</td></tr>
<tr><td>0</td><td>1</td></tr>
<tr><td rowspan="3">0</td><td rowspan="3">1</td><td rowspan="2">1</td><td>1</td></tr>
<tr><td>0</td></tr>
<tr><td>0</td><td>1</td></tr>
</table>

Figure 4.7: Non-consistency in mbC

the information that it is inconsistent, and it is not rationally possible to come to accept its consistency, unless such agent deliberately contracts its epistemic state by $\neg\circ\alpha$.

4.3.6 Epistemic inputs and AGM∘ operations

An interesting consequence the language's enrichment is the operations' very definition. The **AGM∘** system allows for the three main types of changes, or operations, on belief sets:

Expansion $(K + \alpha)$ Incorporation of a new belief α into K without removing any previous beliefs in K.

This operation remains the same as in the classical model and is, therefore, trivializing if the resulting epistemic state is incoherent. It is worth noting that the resulting set may be contradictory, but this is not a problem as long as coherence is maintained (the distinction between non-contradiction, consistency, and coherence was explained in the introduction and can be better understood formally in Section 4.2, where we present the main definitions and theorems of the **LFIs**). The same is not true for contraction:

Contraction $(K - \alpha)$ Possible removal of a belief α from K without the introduction of any new beliefs.

The main distinction relative to the classical **AGM** system is the fact that this operation may fail, which captures the idea that some beliefs are so deeply rooted in the agent's epistemic state that excluding them is not a possibility. As we have defined, such beliefs are those strongly held by the agent.

Revision $(K * \alpha)$ Incorporation of a new belief α on K, with the possible removal of a previous belief in K in an attempt to preserve non-contradiction.

We use the term *possible removal* in two distinct senses:

(i) The removal of a previously accepted belief may not be necessary since, if the belief to be incorporated does not generate a contradiction, the incorporation can be done directly and, in this case, revision is equivalent to expansion.

(ii) The removal of a previously accepted belief may not be possible since, even if the belief to be incorporated generates a contradiction, the former is strongly accepted in the initial epistemic state. In this case, revision also equates to expansion.

It is in this sense that revision *attempts to preserve non-contradiction*: in the case expressed in (ii) above, we will certainly have a contradictory set of beliefs. In both cases, the belief to be incorporated is accepted – in the first case with minimal change, in the second (less interesting) case, trivially.

Classical and paraconsistent revision operations assume that the agent always accepts the new sentence α to be incorporated, which can be perceived through the *success* postulate. As previously laid out, Hansson [36] describes, in belief bases, a generalization of revision called *semi-revision* – an operation that delegates to the selection mechanism the task of choosing the sentence to be retracted to avoid contradiction, which allows retracting the newly added sentence, thus violating *success*. Revision, he argues, can only be applied after the agent has decided to accept α, which does not capture certain intuitive ideas about belief incorporation. The **AGM**∘ system allows this operation to be defined as well.

Roughly speaking, semi-revision is constructed by adding α to the set of beliefs followed by the removal of the possible contradiction generated by the incorporation (consolidation operation) – which may or may not remove the newly added sentence α:

Semi-revision ($K?\alpha$) Possible incorporation of a belief α, depending on the initial epistemic state and the agent's belief status (entrenchment).

Consolidation ($K!$) Removal of contradictions from the epistemic state.

From the aforementioned operations it is possible to define several others – as particular cases of them and their iterations. At the end of this and the next chapters we will outline some operations that we consider significant. However, before doing so, we need to better understand the classical operations, that is, those already present in the literature on belief sets and belief bases in the **AGM** system, but in their paraconsistent versions or, better yet, in their formalized consistency versions.

4.4 Expansion

Expansion is the operation that simply incorporates a sentence α into the epistemic state.

Definition 4.27 (**Expansion**). *Let K be a set of beliefs and α a sentence. $K + \alpha$ is defined as:*

$$K + \alpha = Cn(K \cup \alpha)$$

We can note that, as in the classical **AGM** model and as would be expected, if $\alpha \in K$ then $K+\alpha$ is equivalent to K. Incorporating a belief already present in the epistemic state is a redundant operation that does not bring about any change. However, unlike the classical model, performing the operation $K+\alpha$ on a belief set where $\neg\alpha \in K$ does not necessarily result in a trivial set of beliefs, despite being contradictory.

If we accept the idea that the non-triviality of a belief set is sufficient to justify the rationality criteria for the operation that has it as a resulting set, then, even if contradictory, we could argue that simple expansion can be understood as a revision operation. Therefore, removing any sentence to retrieve an epistemic state free from contradictions is not necessary given the possibility of overdetermining a belief without this causing the epistemic state's trivialization.

By committing oneself, for example, to a dialetheist[12] position regarding belief revision, it becomes necessary to accept non-triviality as sufficient to justify the rationality of the resulting set and, therefore, to accept expansion as a revision operation – given *success*, even a self-contradictory sentence must be present in the resulting epistemic state. In this case, contracting any sentence to ensure the absence of contradictions in the belief set (as proposed by the revision operation) is impossible unless the recently expanded sentence is itself excluded.

Our idea of revision, on the other hand, is broader: it is an incorporation operation in which the resulting epistemic state is, whenever possible, non-contradictory. The only case in which contradiction and trivialization itself persist is when the negation of the belief to be incorporated is unfalsifiable in the set – due to the presence of consistency.

We can better understand the possible situations generated by the consistency of a sentence using the following schemata:

$$\circ\alpha \in K \text{ and } \begin{cases} \alpha \in K \text{ and } & \begin{cases} \neg\alpha \in K & \text{then } Cn(\{\alpha, \sim\alpha\}) \subseteq K = K_f \quad (1) \\ \neg\alpha \notin K & \text{then } Cn(\{\circ\alpha, \alpha\}) \subseteq K \quad (2) \end{cases} \\ \alpha \notin K \text{ and } & \begin{cases} \neg\alpha \in K & \text{then } Cn(\{\sim\alpha\}) \subseteq K \quad (3) \\ \neg\alpha \notin K & \text{then } Cn(\{\circ\alpha\}) \subseteq K \text{ and } \alpha \text{ is indeterminate} \quad (4) \end{cases} \end{cases}$$

If both α and $\neg\alpha$ are accepted in K, expanding the belief set with $\circ\alpha$ is incoherent and, as expected, the resulting set is trivial. This occurs because the presence of $\circ\alpha$ causes α to be strongly rejected (if $\neg\alpha \in K$). Thus, as situation **(2)** above describes, there is no problem in strongly accepting a belief unless, as we see in **(1)**, its negation is also present.

Situation **(3)** is analogous – there is no problem in strongly rejecting a belief unless it is incorporated into the set (which leads to

[12] A very short description of dialetheism can be found in the Appendix.

the undesired case **(1)**). Case **(4)** is interesting – the agent has a position regarding the consistency of α but such belief is indeterminate, that is, the agent neither accepts nor rejects it but, if come to do either, it will do so strongly.

The next scheme illustrates the situations in which $\circ\alpha \notin K$, that is, the agent has no defined position regarding the consistency of α and, therefore, it can be overdetermined without being incoherent and generating trivialization.

$$\circ\alpha \notin K \text{ and} \begin{cases} \alpha \in K \text{ and} & \begin{cases} \neg\alpha \in K & \text{then } Cn(\{\alpha, \neg\alpha\}) \subseteq K & (5) \\ \neg\alpha \notin K & \text{then } Cn(\{\alpha\}) \subseteq K & (6) \end{cases} \\ \alpha \notin K \text{ and} & \begin{cases} \neg\alpha \in K & \text{then } Cn(\{\neg\alpha\}) \subseteq K & (7) \\ \neg\alpha \notin K & \text{then } \alpha \text{ is indeterminate} & (8) \end{cases} \end{cases}$$

In situations **(6)** and **(7)**, the agent can incorporate $\neg\alpha$ and α, respectively, without this being incoherent since, as can be noted in **(5)**, the contradiction does not lead to trivialization. Situation **(8)** illustrates the case in which both α and $\neg\alpha$ are not accepted by the agent, who also has no opinion on the consistency of the belief in question.

4.5 AGM∘ contraction

As we assume in our system, a contraction represents the action of removing a previously held belief from an epistemic state, which may occur, for example, in an argument or hypothetical reasoning. Let us consider the following example adapted from Ribeiro [85]:

Example 4.28. *I believe that butter is unhealthy ($\neg s$) and therefore I should not eat much of it ($\neg s \rightarrow \neg c$). Upon reading recent studies that claim the opposite, I may want to, for the sake of argumentation, contract $\neg c$.*

4.5.1 Postulates for AGM∘ contraction

The contraction of a set K by a belief α is denoted by $K - \alpha$. As in the classical **AGM** system, we assume that $-$ is a function that takes pairs of belief sets and sentences to belief sets:

(closure) $K - \alpha = Cn(K - \alpha)$

Example 4.29. *Recall from the contraction in previous example,* $K = \{\neg s, \neg s \to \neg c\} = Cn(\neg(s \vee \neg c))$ *by* $\neg c$*. Each node in the following diagram shows all possible belief sets of the language restricted to* $\{s, c\}$*. As in classical* ***AGM****,* closure *ensures that the result is one of these nodes in Figure 4.8*

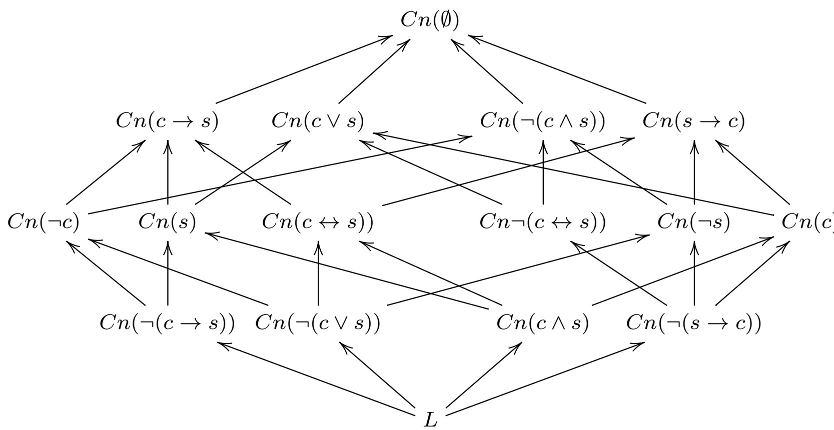

Figure 4.8: Diagram for example 4.29

Furthermore, contraction is an operation that removes a sentence from the belief set and, in this way, it should be the case that

$\alpha \notin K - \alpha.$

However, if α is a tautology, $\alpha \in Cn(\emptyset)$ and thus violating closure would be necessary. In this case, success must be defined as

If $\alpha \notin Cn(\emptyset)$ then $\alpha \notin K - \alpha$.

This is precisely the statement of the success postulate in the classical **AGM** system. However, in the **AGM**∘ system, we need to take into account the possibility of α being consistent in K, that is, $\circ\alpha \in K$ – in this case, the epistemic status of α is such that it cannot be refuted and removed from the belief set (when previously present)[13]. Now, in this case, success needs to be violated, unless the postulate itself incorporates the information that success fails if one tries to contract the set by a sentence that is strongly held in K – and that is exactly what we do.

(Success) If $\alpha \notin Cn(\emptyset)$ e $\circ\alpha \notin K$ then $\alpha \notin K - \alpha$.

Example 4.30. *Success ensures that, in case $\circ\alpha \notin K$, the result of contraction will not be among the nodes that contain $Cn(\neg c)$, which can be observed in Figure 4.9*

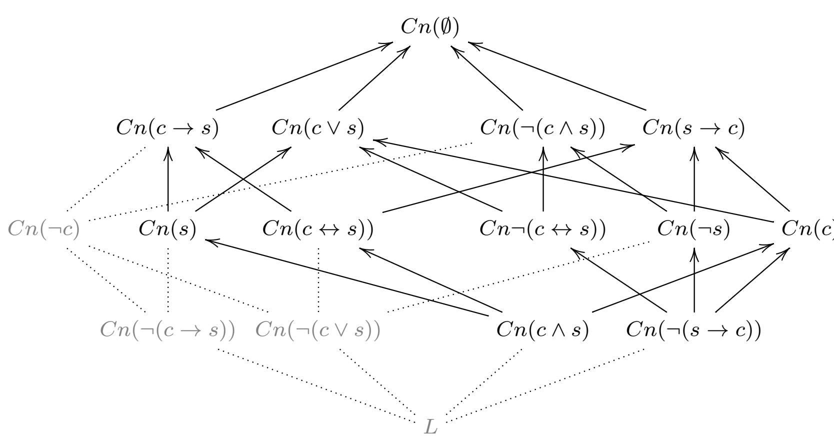

Figure 4.9: Diagram for example 4.30

The postulate of inclusion is exactly the same as in the classical **AGM** system and ensures that, upon removing a sentence α, no other sentence will be incorporated into the belief set:

(Inclusion) $K - \alpha \subseteq K$.

[13]This postulate illustrates the fact that strongly held beliefs behave like tautologies within the epistemic state.

Example 4.31. Inclusion *requires that the result of the contraction is contained in* $Cn(\neg(s \vee c))$*, that is, the resulting epistemic state after contraction must be located on or below* $Cn(\neg(s \vee c))$ *in Figure 4.10. Therefore, we must eliminate* $Cn(c \wedge s)$, $Cn(\neg(s \to c))$, $Cn(s)$, $Cn\neg(c \leftrightarrow s))$, $Cn(c)$ *and* $Cn(c \vee s)$ *as possible epistemic states of our contraction.*

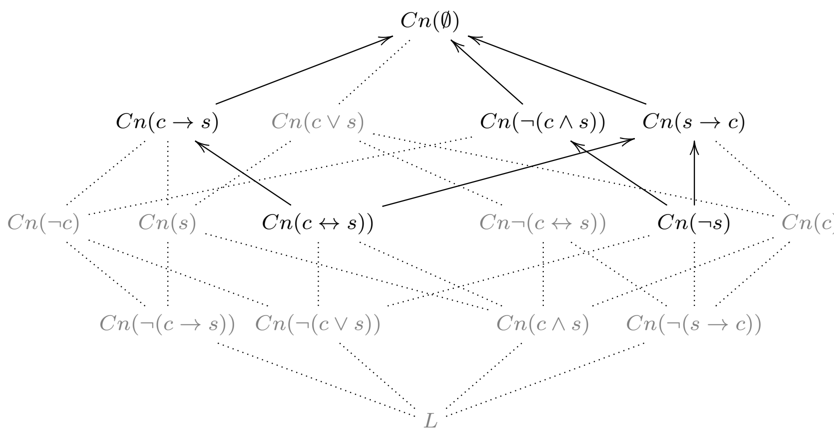

Figure 4.10: Diagram for example 4.31

The next postulate is central to describing contraction in **AGM**∘, and highlights the difference between it and the classical paradigm. *Failure* complements *success* and governs the behavior of contraction when $\circ\alpha$ is accepted in K. The intuitive idea is precisely to capture the fact that attempting to remove a non-falsifiable sentence from K is ineffectual, resulting in the same epistemic state K as before (due to inclusion).

Let us adapt example 4.28 to better understand this paradigmatic case:

Example 4.32. *I believe that butter is unhealthy* ($\neg s$) *and therefore I should not eat too much of it* ($\neg s \to \neg c$)*. Moreover, since I have believed this all my life and trust what most scientists say, I believe that my prior information is irrefutable and, therefore, it is* *consistent that I should not eat butter ($\circ\neg c$). However, upon reading recent studies that contradict this belief, I may want to, for the sake of argument, contract $\neg c$.*

The diagram helps us understand some interesting facts that occur in this example, where the removal of the sentence is not possible (consider $Cn(X) = Cn(X \cup \{\circ\neg c\})$).

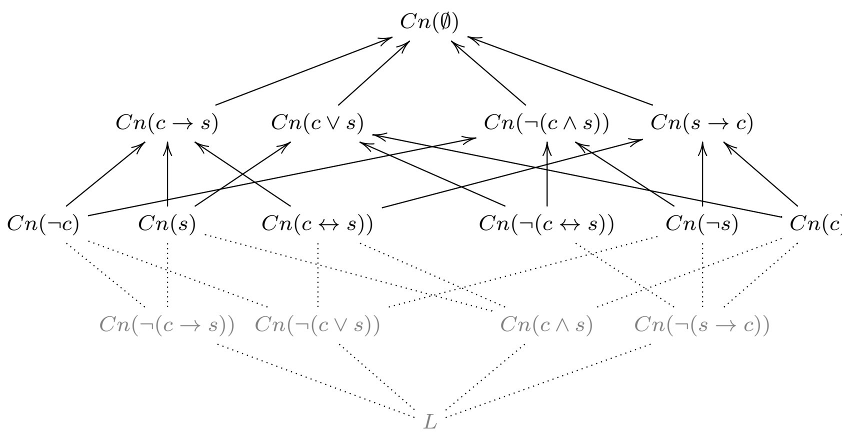

Figure 4.11: Diagram for example 4.32

Figure 4.11 shows the possible $K - \neg c$ that satisfy *closure* (all nodes in the diagram), *success* (since $\circ\neg c \in K$, the diagram does not exclude sets containing $Cn(\neg c)$), and *inclusion* (the diagram excludes sets larger than K). Notably, the remaining sets are just the logical consequences of K – that is, K itself. Thus, when attempting to contract the belief set by a sentence considered consistent, the operation fails, and exactly the same previously accepted beliefs persist in the system – including the very one to be retracted.

This fact is satisfactorily described by the postulate:

(Failure) If $\circ\alpha \in K$, then $K - \alpha = K$.

This postulate reflects an intuitive idea already explained earlier, namely, the irrefutability of certain beliefs in the agent's epistemic state. We will return to the *failure* and its direct relation to the idea of unfalsifiability, formalized by the acceptance of a belief's consistency, when presenting the explicit construction for contraction. Before that, let us consider the last postulate – relevant only in the case where $\circ\alpha \notin K$ in the operation $K - \alpha$ and therefore the contraction is successful, being ineffectual otherwise.

The next postulate, *relevance*, ensures the minimality of the operation and replaces *recovery* as presented in the classical paradigm. As we have seen, Hansson [37] showed that, for logics that satisfy the **AGM** assumptions, both are equivalent in the presence of the other postulates. Furthermore, as Ribeiro [85] asserts, this is also the case for any compact logic. Our choice for such a postulate, therefore, is due to the fact that it is compatible with a larger class of logics. Additionally, we agree with Hansson's argument that the *recovery* postulate is counter-intuitive in various situations, making the *relevance* postulate a more interesting and intuitive option.

The *relevance* postulate ensures the operation's minimality by preventing irrelevant sentences from being removed from the initial set – no element β can be removed from K unless β contributes to proving the sentence α to be removed, that is, for some K' such that $K - \alpha \subseteq K' \subseteq K$, the set $K' \cup \beta$ proves α. Thus, we have the following:

(Relevance) If $\beta \in K \setminus K - \alpha$ then there exists K' such that $K - \alpha \subseteq K' \subseteq K$, $\alpha \notin K'$, and $\alpha \in K' + \beta$.

Therefore, we have the following postulates for contraction:

Definition 4.33 (Postulates for **AGM**∘ contraction)**.** *The operation* $-$ *satisfies the following:*

(Closure) $K - \alpha = Cn(K - \alpha)$.

(Success) *If* $\alpha \notin Cn(\emptyset)$ *and* $\circ\alpha \notin K$ *then* $\alpha \notin K - \alpha$.

(Inclusion) $K - \alpha \subseteq K$.

(Failure) *If* $\circ\alpha \in K$ *then* $K - \alpha = K$.

(Relevance) *If $\beta \in K \backslash K - \alpha$ then there exists K' such that $K - \alpha \subseteq K' \subseteq K$, $\alpha \notin K'$, and $\alpha \in K' + \beta$.*

4.5.2 AGM∘ partial meet contraction

Let us now see the construction of partial meet contraction for **AGM∘**. It is worth noting that we need to incorporate, in its definition, the intuitive idea of non-revisability satisfactorily captured by the *success* and *failure* postulates. We consider again the maximal subsets of K that do not imply α – the set of all these subsets is the well-known remainder set, defined below:

Definition 4.34. *Let K be in $\mathbf{L}$ and $\alpha \in \mathbb{L}$. The set $K \perp \alpha \subseteq \wp(\mathbb{L})$ is such that, for every $X \subseteq \mathbb{L}$, $X \in K \perp \alpha$ if and only if the following clauses are satisfied:*

1. $X \subseteq K$;
2. $\alpha \notin Cn(X)$;
3. *If $X \subset X' \subseteq K$ then $\alpha \in Cn(X')$.*

To ensure the theorem of representation for the different paraconsistent revisions for **AGM∘** which will be presented in future sections, the following lemmas about remainder set are necessary:

Lemma 4.35. *If $X \in K \perp \alpha$, then $X \in \mathbb{L}$.*

Proof. If $\beta \in \mathbb{L}$, then $\alpha \notin Cn(X \cup \{\beta\})$ and, since X is maximal (item 3 of definition 4.34), $\beta \in X$. □

Lemma 4.36 (Upper bound property)**.** *For every belief set K, every $X \subseteq K$, and every set A in a compact logic in which $A \cap Cn(X) = \emptyset$, there exists X' such that $X \subseteq X'$ and $X' \in K \perp A$.*

Proof. First, we enumerate the elements of K in a sequence $\beta_1, \beta_2, \ldots$. Let $X_0 = X$ and, for every $i \geq 1$, we define X_i as follows:

$$X_i = \begin{cases} X_{i-1} & \text{if } A \cap Cn(X_{i-1} \cup \{\beta_i\}) \neq \emptyset \\ X_{i-1} \cup \{\beta_i\} & \text{otherwise.} \end{cases}$$

For every i, we have $A \cap Cn(X_i) = \emptyset$. Let $X' = \bigcap_i X_i$. Verify that $X \subseteq X' \subseteq K$. Furthermore, if $\beta \in K$ and $\beta \notin X'$, then $A \cap (X' \cup \{\beta\}) \neq \emptyset$. If $A \cap Cn(X') \neq \emptyset$, then there exists $\beta \in A \cap Cn(X')$. By compactness, $\beta \in Cn(X'')$ for some finite $X'' \subseteq K$. In this case, $\beta \in Cn(X_i)$ and hence $A \cap Cn(X_i) \neq \emptyset$ for some i, which would be a contradiction. □

Once again we consider a function γ that selects some elements from $K \perp \alpha$ whenever possible and returns K itself otherwise. Intuitively, γ selects those sets that contain the beliefs that the agent believes in most strongly. However, within the paraconsistent paradigm and the language of the **LFIs**, this notion of epistemic entrenchment is also satisfactorily incorporated by the consistency of the belief: if $\circ\alpha \in K$, α is entrenched in K to the point that this belief cannot be removed from K and, in this case, the remainder set will be K itself.

Definition 4.37 (Selection Function)**.** *The* selection function *for K is a function γ such that for every α:*

1. *$\emptyset \neq \gamma(K, \alpha) \subseteq K \perp \alpha$ if $\alpha \notin Cn(\emptyset)$ and $\circ\alpha \notin K$.*
2. *$\gamma(K, \alpha) = \{K\}$ otherwise.*

Thus, we incorporate the idea of non-revisability into the selection function itself. This strategy is quite natural when we consider that, indeed, beliefs considered consistent will not even be chosen by the agent as retractable – even if they are retracted as a last resort, such as the agent's most deeply held beliefs. On the contrary, consistent beliefs remain in the set of beliefs in any situation, unless the agent retracts the very belief that such a sentence is consistent – which, it is worth remembering, is not possible in all **LFIs** due to the propagation of consistency (as we saw in Section 4.3.5).

Another important difference in the selection function presented in this chapter is the fact that we parameterize γ with respect to a specific sentence and belief set, and not to the entire remainder set. Thus, with respect to α and K, for example, we define $\gamma(K, \alpha)$, and not $\gamma(K \bot \alpha)$. It is worth noting that this is a technical difference that does not directly affect the essence of the results obtained and is, therefore, used by us to define the selection function of the **AGM** system itself.[14]

We define, as in **AGM**, partial meet contraction as the intersection of the sets selected by γ.

Definition 4.38. *A contraction $-$ in K is a partial meet contraction iff there exists a selection function γ for K such that, for any α,*

$$K -_{\gamma} \alpha = \bigcap \gamma(K, \alpha)$$

Let us turn back to our previous examples:

Example 4.39. *Let $-$ be a partial meet contraction. In the example 4.28, in which $K = \{\neg s, \neg s \rightarrow \neg c\} = Cn(\neg(s \vee c))$ we have the following as remainder set:*

$$K \bot \neg c = \{Cn(\neg c \leftrightarrow s), Cn(\neg s)\}$$

It is worth noting that the possible resulting subsets correspond exactly to those of the postulates. The selection function γ can choose the following:

Case (i) *If $\gamma(K, \neg c) = K \bot \neg c$, meaning that the selection function chooses all elements of the residual set, we have that*

$$K - \neg c = Cn(c \leftrightarrow s) \cap Cn(\neg s) = Cn(s \rightarrow c)$$

Case (ii) *If $\gamma(K, \neg c) = Cn(\neg c \leftrightarrow s)$, meaning that the selection*

[14] Technically, the parameterization requires that the function γ be defined for a specific sentence and, therefore, logically equivalent sentences may not behave in the same way with respect to a selection function over a remainder set. Thus, the *extensionality* postulate ceases to be significant. In addition to being more specific, this definition follows the **LFIs** because it does not rely on logical equivalence – remember that many logically equivalent sentences in a classical paradigm are not so in the **LFIs**. Furthermore, it is worth noting that many works in the literature of **AGM** belief revision already use this parameterization without major concerns when defining a selection function for partial meet contraction.

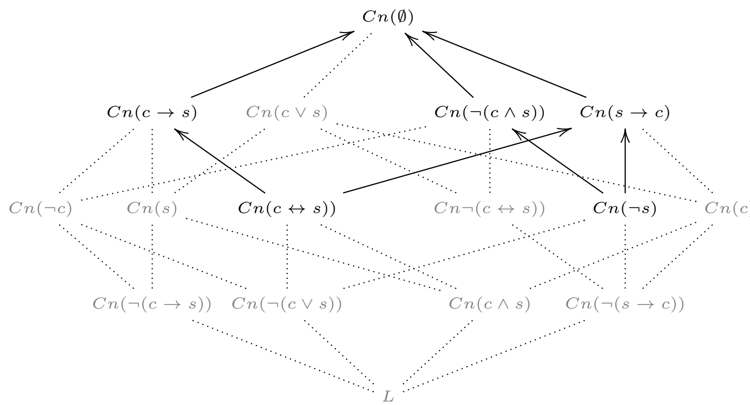

Figure 4.12: Remainder set diagram

function chooses $\neg c \leftrightarrow s$, we have that

$$K - \neg c = \bigcap Cn(c \leftrightarrow s) = Cn(c \leftrightarrow s)$$

Case (iii) *If $\gamma(K, \neg c) = Cn(\neg s)$, meaning that the selection function chooses $\neg s$, we have that*

$$K - \neg c = \bigcap Cn(\neg s) = Cn(\neg s)$$

Example 4.40. *Let $-$ be a partial meet contraction. In example 4.32, in which $K = \{\neg s, \neg s \to \neg c, \circ c\}$, we have that*

$$\gamma(K, \neg c) = K$$

and therefore

$$K - \neg c = \bigcap K = K$$

The postulates of Definition 4.33 precisely characterize the **AGM∘** partial meet contraction, as can be observed in the presented examples. Therefore, we should be able to demonstrate the representation theorem for this operation.

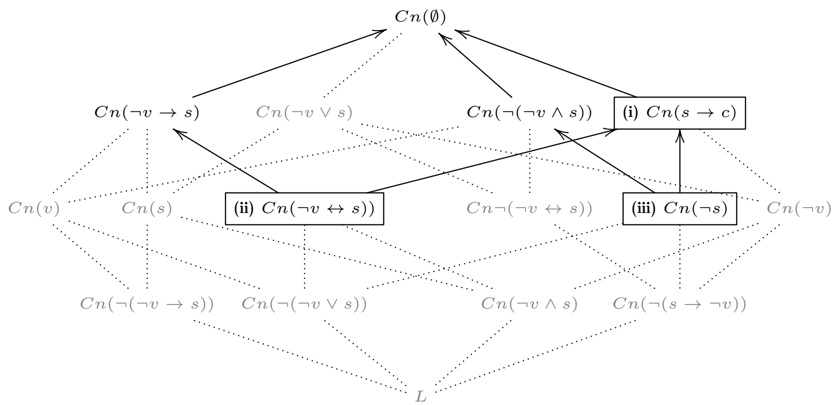

Figure 4.13: Contraction results adduced by the function γ

Theorem 4.41 (representation)**.** *An operation $-$ on K satisfies the postulates of Definition 4.33 for every α iff there exists a selection function γ such that $K - \alpha = \bigcap \gamma(K, \alpha)$.*

Proof. **(construction ⇒ postulates)**

Closure: Let $X \in K \perp \alpha$ and $\beta \in Cn(X)$. Then $\alpha \notin Cn(X \cup \{\beta\})$, and since X is maximal, $\beta \in X$. Hence, for every $X \in K \perp \alpha$, we have $X = Cn(X)$. Therefore, $K -_\gamma \alpha = \bigcap \gamma(K, \alpha)$, where the elements of $\gamma(K, \alpha)$ are closed sets and, since the intersection of closed sets is itself closed, we have that $K -_\gamma \alpha$ is closed.

Success: If $\alpha \notin Cn(\emptyset)$, then by Lemma 4.36, $K \perp \alpha \neq \emptyset$.

Inclusion: Follows from the construction.

Failure: Follows from the construction.

Relevance: If $\beta \in K \setminus K - \alpha$, then there exists $X \in \gamma(K, \alpha)$ such that $\beta \notin X$. By definition, $K -_\gamma \alpha \subseteq X \subseteq K$, $\alpha \notin Cn(X)$, and $\alpha \in Cn(X \cup \{\beta\})$.

(postulates $\Rightarrow$ construction) Let $-$ be an operator satisfying the above postulates and let γ be the following function:

$$\begin{aligned}\gamma(K,\alpha) &= \{X \in K\perp\alpha : K-\alpha \subseteq X\} \text{ se } \alpha \notin Cn(\emptyset) \text{ ou } \circ\,\alpha \notin K \\ &= \{K\} \text{ otherwise .}\end{aligned}$$

We need to prove that 1) γ is a selection function, and 2) $K-\alpha = \bigcap\gamma(K,\alpha)$.

1. $\gamma(K,\alpha) \subseteq K$ follows directly from the construction. If $\alpha \notin Cn(\emptyset)$ and $\circ\alpha \notin K$ then *success* and *inclusion* ensure that $\alpha \notin K-\alpha \subseteq K$. By Lemma 4.36, there exists X such that $K-\alpha \subseteq X \in K\perp\alpha$, and therefore, $\gamma(K,\alpha) \neq \emptyset$.

2. If $\alpha \in Cn(\emptyset)$ then *relevance* and *inclusion* guarantee that $K-\alpha = K$. Similarly, if $\circ\alpha \in K$ then *failure* ensures that $K-\alpha = K$. In these two cases, $\bigcap\gamma(K,\alpha) = K$ because $\gamma(K,\alpha) = \{K\}$.

3. If $\alpha \in Cn(\emptyset)$ then *relevance* and *inclusion* guarantee that $K-\alpha = K$. Similarly, if $\circ\alpha \in K$ then *failure* guarantee that $K-\alpha = K$. In these two cases, $\bigcap\gamma(K,\alpha) = K$, because $\gamma(K,\alpha) = \{K\}$. If $\alpha \notin Cn(\emptyset)$ then $K-\alpha \subseteq K -_\gamma \alpha$ by construction. It remains to prove that $K -_\gamma \alpha \subseteq K-\alpha$. Let $\beta \notin K-\alpha$ and assume that $\beta \in K$ (otherwise, $\beta \notin \bigcap\gamma(K,\alpha)$ trivially). By *relevance*, there exists K' such that $K-\alpha \subseteq K' \subseteq K$, $\alpha \notin Cn(K')$, and $\alpha \in Cn(K' \cup \{\beta\})$. By Lemma 4.36, there exists X such that $K' \subseteq X \in K\perp\alpha$. Since $K' \subseteq X$, $\alpha \in Cn(K'\cup\{\beta\})$ and $\alpha \notin Cn(X)$, we have $\beta \notin X$. Therefore, $\beta \notin \bigcap\gamma(K,\alpha)$.

□

4.6 AGM∘ revision

The existence of contradictory belief sets, without the agent being required to at least attempt to restore an epistemic state free of contradictions, is in a way justified by accepting non-triviality as sufficient for principles of rationality.

Levi's previous statement that the presence of a contradiction is something common should not be understood as the author's acceptance of the legitimacy of contradictory epistemic states. On the contrary, Levi argues that an epistemic state of this kind is "not useful" because it is unfeasible as a source of access to factual possibilities – since truth values collapse into incoherence and indiscriminateness[15] – and useless as a source for reasoning and practical deliberation. Inconsistency, he claims, is an "epistemic hell" from the perspective of a deliberative agent.[16]

The paraconsistent belief revision model notably addresses the first cause of the epistemic hell by restricting the principle of explosion and, in this way, controlling the intractability of a contradictory epistemic state. However, restoring the usefulness of a belief set "as a source for reasoning and practical deliberation" requires demanding that it be free from contradictions and that, therefore, the resulting epistemic state after incorporation is, whenever possible, non-contradictory. In fact, this is precisely the first principle we presented in section 4.3.4 (page: 106):

(1) Non-contradiction Whenever possible, epistemic states should remain non-contradictory.

The two main tasks of a revision, therefore, are to incorporate a new belief α into the belief set and to ensure that the result is contradiction-free whenever possible. Just like in the classical model, we can construct this operation from two sub-operations: expansion by α and contraction by $\neg\alpha$.

However, unlike in **AGM**, our new system allows, similar to belief bases, the definition of the sub-operations in two distinct orders, namely:

[15] Notably as a reflection of the principle of explosion.

[16] The term "epistemic hell" was first used by Peter Gärdenfors [25] when taking a similar viewpoint regarding inconsistent epistemic states.

Internal revision

$$K * \alpha = (K - \neg\alpha) + \alpha$$

External revision

$$K * \alpha = (K + \alpha) - \neg\alpha$$

As we stated earlier, both definitions are competing as each one prioritizes a distinct criterion of rationality. Although both aim to result in a non-contradictory or at least coherent epistemic state, internal revision prioritizes criterion **(1)**, namely, non-contradiction, while external revision prioritizes **(3)**, which is:

(3) Minimality When modifying epistemic states, information loss should be minimal.

External revision is rationally justified when considering the above criterion – if it takes priority, external revision prevails over internal revision as the prior contraction of a belief is no longer logically necessary and, thus, it can be considered unnecessary information loss. Furthermore, as we will see, many of the logical consequences of the intermediate contradictory set are perfectly acceptable and often desirable in the resulting epistemic state after revision.

It is beyond the scope of this thesis, at least at this moment, to advocate for a position about which criterion should prevail over the others, and we will limit ourselves to presenting both possibilities, as well as their motivations, logical consequences, and intuitive implications. Considering that external revision, as we have stated in the introduction of this thesis, is precisely one of the motivations behind the development of our system (notably, an epistemic state that is always free of contradictions does not require a paraconsistent system), we will begin the exposition with this type of revision.

4.6.1 External AGM∘ revision

As mentioned before, the main objective of the criterion of informational economy is that the revision of a belief set is neither smaller nor larger than necessary to accept the new sentence being incorporated. In this section, we present the postulates that outline the notion of minimal change we adopt for this operation, taking into account the fact that the prior contraction of a sentence is logically unnecessary (or even impossible, in the case of strongly accepted sentences) and, therefore, a revision only requires the non-contradiction of the epistemic state at the end of the operation – allowing for an intermediate contradictory state.

Postulates

The first postulate states, as expected, that the result of the revision is a logically closed set of beliefs.

(Closure) $K * \alpha = Cn(K * \alpha)$.

Similar to the classical **AGM** paradigm, the sentence to be incorporated is always accepted into the belief set.

(Success) $\alpha \in K * \alpha$.

Notably, if for some reason α is strongly rejected in K, success is trivially satisfied. This paradigmatic case illustrates the previously stated fact that strongly accepting or rejecting a sentence is equivalent to making the set non-revisable by its negation or by the sentence in question, respectively. In this case, when attempting to force the revision, the resulting inconsistency generates a trivial epistemic state, exactly as expected. The same applies to the following postulates.

(Inclusion) $K * \alpha \subseteq K + \alpha$.

In both cases, if the set is non-revisable by α, the postulates are trivially satisfied, and the revision amounts to expansion. The next postulate asserts that the result of the revision is a non-contradictory set of beliefs, as required by its homonymous principle.

(Non-contradiction) If $\neg\alpha \notin Cn(\emptyset)$ and $\sim \alpha \notin K$, then $\neg\alpha \notin K * \alpha$.

It is worth noting that the postulates being presented are adaptations of the classical **AGM** postulates, or even reinterpretations of them. The following postulate, however, characterizes precisely the core of the **AGM**∘ paraconsistent revision and is specific to this system – the failure postulate. As we stated earlier, if the belief to be incorporated in the revision is strongly rejected in the initial set, one of the clauses in the previous postulate is not satisfied.

(Failure) If $\sim\alpha \in K$, then $K * \alpha = \mathbb{L}$.

The fact is that the intermediate contradictory state is not always coherent, and therefore it is possible for the result of the revision to be a trivial set, equivalent to the language itself.[17] This fact shows that a revision (in this case, external) is logically interesting only when the set is actually revisable by the new belief in question, with the result being trivial otherwise.

We believe that it is not necessary to require such revisability, that is, to demand that the belief set K under revision does not strongly reject the belief to be incorporated – its very failure would be an expression of the irrationality of an agent attempting to perform a revision that they themselves accept as not possible (by strongly accepting or rejecting certain beliefs).

The next postulate expresses the idea of minimality in revision. As mentioned before, this postulate was suggested by Hansson as a replacement for recovery. *Relevance* ensures the operation's minimality by preventing irrelevant sentences from being removed, since it imposes that no element β can be removed from K unless β contributes to proving α, meaning that for some K' such that $K - \alpha \subseteq K' \subseteq K$, the set $K' \cup \{\beta\}$ proves α.

[17] It is possible to attempt to address this issue by potential previous removal of $\sim \alpha$ from K, which would result in the potential removal of $\circ\alpha$ or $\neg\alpha$ in case $\sim \alpha$ follows from their presence in the epistemic state. The latter case corresponds to internal revision, to be presented below. The former is also definable in **AGM**∘ – the fact is that the new framework allows us to define distinct revisions, and doing so depends on different justifications for them.

(Relevance) If $\beta \in K \setminus K * \alpha$ then there exists K' such that $K * \alpha \subseteq K' \subseteq K + \alpha$ and $\neg\alpha \notin K'$, but $\neg\alpha \in K' + \beta$.

Finally, we have the postulate that reflects the existence of the potentially contradictory intermediate state, which substantially distinguishes external revision from internal revision, in which the requirement of prior contraction aims to avoid such contradictory state.

It is worth noting that, if the set is not revisable by the new belief to be incorporated, then prior contraction is not possible, and therefore demanding it is futile. Now, in other cases (where the set is revisable by the new belief), contraction is possible, but in these cases the contradictory intermediate state is coherent and non-trivializing – and demanding prior contraction is not futile, but it is logically unnecessary (as a strategy to avoid trivialization). [18]

(Pre-expansion) $(K + \alpha) * \alpha = K * \alpha$

The postulates for **AGM∘** external revision, therefore, are the following:

Definition 4.42 (Postulates for **AGM∘** external revision)**.** *An **AGM∘** external revision operation satisfies the following postulates:*

(Closure) $K * \alpha = Cn(K * \alpha)$.

(Success) $\alpha \in K * \alpha$.

(Inclusion) $K * \alpha \subseteq K + \alpha$.

(Non-contradiction) *If* $\neg\alpha \notin Cn(\emptyset)$ *and* $\sim\alpha \notin K$ *then* $\neg\alpha \notin K * \alpha$.

(Failure) *If* $\sim\alpha \in K$ *then* $K * \alpha = \mathbb{L}$

(relevance) *If* $\beta \in K \setminus K * \alpha$ *then there exists* K' *such that* $K * \alpha \subseteq K' \subseteq K + \alpha$ *and* $\neg\alpha \notin K'$, *but* $\neg\alpha \in K' + \beta$.

(Pre-expansion) $(K + \alpha) * \alpha = K * \alpha$

[18]We reiterate the assertion that, when considering the requirement that all revision steps be non-contradictory, prior contraction is logically necessary. As we previously stated, this is the core of the non-contradiction *versus* minimality game.

Construction: Inverse Levi Identity

By the Levi identity, we are now able to use the **AGM∘** partial meet contraction to define a construction for external revision. A revision defined in this way is called ***AGM∘** partial meet external revision*, clearly defined over a function γ.

$$K *_\gamma \alpha = (K + \alpha) -_\gamma \neg\alpha = \bigcap \gamma(K + \alpha, \neg\alpha)$$

Any partial meet external revision satisfies the **AGM∘** postulates from definition 4.42 and, furthermore, just like in contraction, the postulates precisely characterize the partial meet external revision, meaning that the representation theorem holds.

Theorem 4.43 (Representation)**.** *An operation $*$ over K satisfies the postulates for **AGM∘** partial meet external revision from definition 4.42 for every α iff there exists a selection function γ such that $K * \alpha = \bigcap \gamma(K + \alpha, \neg\alpha)$.*

Proof. **(Construction ⇒ Postulates)**

Closure: It follows for the same reason stated in the previous theorem.

Success: In cases where $\neg\alpha \in Cn(\emptyset)$ or $\circ\alpha \in K$, by definition we have $K *_\gamma \alpha = K + \alpha$ and success follows trivially. Now, let $X \in (K + \alpha)\bot\neg\alpha$, and suppose for the sake of contradiction that $\alpha \notin X$. Let $X' = X \cup \{\alpha\}$. Since $X \subset X' \subseteq K + \alpha$, we have $\neg\alpha \in Cn(X')$, by the maximality of $\bot$. Therefore, $\neg\alpha \in Cn(X \cup \{\alpha\})$ and, by Lemma 4.14, we have that $\alpha \in Cn(X)$. However, this contradicts the fact that $\neg\alpha \notin Cn(X)$. We conclude that $\alpha \in X$ for every $X \in (K + \alpha)\bot\neg\alpha$. Therefore, $\alpha \in K *_\gamma \alpha$.

Inclusion: It follows directly from the construction.

Non-contradiction: Suppose that $\neg\alpha \in K * \alpha = (K + \alpha) - \neg\alpha$. By the success of contraction, we have $\neg\alpha \in Cn(\emptyset)$ or $\circ\alpha \in K$.

Failure: If $\sim\alpha \in K$ then $K + \alpha = \mathbb{L}$ and, hence, $\circ\alpha \in K + \alpha$. Then, by the failure of contraction, we have $K + \alpha - \neg\alpha = \mathbb{L}$.

Relevance: Let $\beta \in K \setminus ((K + \alpha) - \neg\alpha)$. Then, $(K + \alpha)\bot\neg\alpha \neq \emptyset$ (otherwise, $(K + \alpha) - \neg\alpha = K + \alpha$ and $K \setminus ((K + \alpha) - \neg\alpha) = \emptyset$, which would be a contradiction). Therefore, there exists $X \in \gamma(K + \alpha, \neg\alpha) \subseteq (K + \alpha)\bot\neg\alpha$ such that $\beta \notin X$. By construction, $K * \alpha \subseteq X \subseteq K + \alpha$. Let $X' = X \cup \{\beta\}$. Then $X \subset X' \subseteq K + \alpha$ since $\beta \in K$. By definition, $\neg\alpha \in Cn(X')$, that is, $\neg\alpha \in X + \beta$.

Pre-expansion: $(K + \alpha) * \alpha = ((K + \alpha) + \alpha) - \neg\alpha = (K + \alpha) - \neg\alpha = K * \alpha$.

(Postulates ⇒ Construction) Let $*$ be an operator satisfying the above postulates and let γ be the following function:

$$\begin{aligned}\gamma(K, \neg\alpha) &= \{X \in K\bot\neg\alpha : K * \alpha \subseteq X\} \text{ if } \circ\,\alpha \notin K \text{ and } \neg\alpha \notin Cn(\emptyset)\\ &= \{K\} \text{ otherwise.}\end{aligned}$$

We need to prove that 1) γ is well-defined, 2) γ is a selection function, and 3) $K * \alpha = (K + \alpha) - \neg\alpha = \bigcap\gamma(K + \alpha, \neg\alpha)$.

1. Let $K \neq K'$ such that $K + \alpha = K' + \alpha$. By *pre-expansion*, $K * \alpha = (K + \alpha) * \alpha = (K' + \alpha) * \alpha = K' * \alpha$. Thus, γ is well-defined.

2. It directly follows from the construction that $\gamma(K + \alpha, \neg\alpha) \subseteq (K + \alpha)\bot\neg\alpha$ in the case where $\circ\alpha \notin K$ and $\neg\alpha \notin Cn(\emptyset)$. If $\circ\alpha \in K$ or $\neg\alpha \in Cn(\emptyset)$, then $\gamma(K + \alpha, \neg\alpha) = \{K\}$ by definition. Otherwise, we will show that $\gamma(K + \alpha, \neg\alpha) \neq \emptyset$. By *non-contradiction*, we have $\neg\alpha \notin K * \alpha$. By *closure* and *inclusion*, $\neg\alpha \notin K * \alpha = Cn(K * \alpha) \subseteq K + \alpha$. Therefore, by Lemma 4.36, there exists $X \in (K + \alpha)\bot\neg\alpha$ such that $K * \alpha \subseteq X$. It follows that $X \in \gamma(K + \alpha, \neg\alpha)$ and thus $\gamma(K + \alpha, \neg\alpha) \neq \emptyset$.

3. Let $\circ\alpha \notin K$ and $\neg\alpha \notin Cn(\emptyset)$. In this case, $K * \alpha \subseteq \bigcap\gamma(K + \alpha, \neg\alpha)$, by construction. Let $\beta \notin K * \alpha$. We will show that

there exists $X \in \gamma(K+\alpha, \neg\alpha)$ such that $\beta \notin X$. If $\beta \notin K+\alpha$, then $\beta \notin X$ for every $X \in \gamma(K+\alpha, \neg\alpha)$ (since every $X \in \gamma(K+\alpha, \neg\alpha)$ is in $K+\alpha$). Let $\beta \in K+\alpha$. By *pre-expansion*, $\beta \notin (K+\alpha) * \alpha$, and thus by *relevance*, there exists Z such that $K * \alpha = (K+\alpha) * \alpha \subseteq Z \subseteq (K+\alpha)+\alpha = K+\alpha$, $\neg\alpha \notin Cn(Z)$ and $\neg\alpha \in Z+\beta$. By Lemma 4.36, there exists $X \in (K+\alpha)\perp\neg\alpha$ such that $K * \alpha \subseteq Z \subseteq X$. Therefore, $X \in \gamma(K+\alpha, \neg\alpha)$. Since $\neg\alpha \in Z+\beta$, it follows that $\neg\alpha \in X+\beta$ and, therefore, $\beta \in Cn(X)$ (otherwise, $\neg\alpha \in Cn(X)$). It follows that $\beta \notin X$, and hence, $\beta \notin \bigcap\gamma(K+\alpha, \neg\alpha)$. We conclude that $K * \alpha = \bigcap\gamma(K+\alpha, \neg\alpha)$. Now, if $\circ\alpha \in K$ or $\neg\alpha \in Cn(\emptyset)$, by construction, $\bigcap\gamma(K+\alpha, \neg\alpha) = K+\alpha$. On the other hand, if there exists $\beta \in (K+\alpha) \setminus (K * \alpha)$, then $(K+\alpha)\perp\neg\alpha \neq \emptyset$, which would be a contradiction. We conlude that $K * \alpha = K+\alpha = \bigcap\gamma(K+\alpha, \neg\alpha)$.

□

4.6.2 AGM∘ internal revision

The intuition to be captured by internal revision is exactly the same as in the classic **AGM** system (which, in this case, is necessarily internal), namely, the need to retract the beliefs that contradict the new sentence to be incorporated from the epistemic state.

Although this contraction is the **AGM∘** contraction itself and is, therefore, minimal, it cannot be stated that revision itself is also minimal in general. The main reason is that in exactly the same cases where revision is non-trivial, both for external and internal revision, external revision allows for a smaller information loss than internal revision and, therefore, we can assert that minimality is best achieved with external revision.

On the other hand, when one desires to ensure that the epistemic state remains free of contradictions whenever possible (at every step of the operation), internal revision remains the best minimal solution for carrying out an incorporation.

Another argument in favor of keeping internal revision as a rational possibility is one already laid out, dealt with by Hansson when justifying the distinction between the two revisions – namely, both represent intuitively distinct scenarios.

Postulates

Most of the postulates are exactly the same as in **AGM**$\circ$ external revision, namely, *closure, success, inclusion, non-contradiction*, and *relevance*. The main difference is that **AGM**$\circ$ internal revision does not have the postulate of *pre-expansion*, as expected – since the potentially contradictory intermediate state resulting from the prior incorporation represented by such expansion is no longer present.

Another important difference is related to the postulate that characterizes **AGM**$\circ$ paraconsistent revision, namely, *failure*. This postulate, in internal revision, can be considered weaker – it asserts only that if the negation of the belief to be incorporated is consistent, revision is equivalent to expansion, because the previous contraction of said negated sentence is not possible unless it can be contracted from K (that is, if $\neg\alpha$ is consistent).

(Failure) If $\circ\neg\alpha \in K$ then $K * \alpha = K + \alpha$.

We have, therefore, the following postulates:

Definition 4.44 (Postulates for **AGM**$\circ$ internal revision)**.** *An* ***AGM***$\circ$ *internal revision operation satisfies the following postulates:*

(Closure) $K * \alpha = Cn(K * \alpha)$.

(Success) $\alpha \in K * \alpha$.

(Inclusion) $K * \alpha \subseteq K + \alpha$.

(Non-contradiction) *If* $\neg\alpha \notin Cn(\emptyset)$ *and* $\circ\neg\alpha \notin K$ *then* $\neg\alpha \notin K * \alpha$.

(Failure) *If* $\circ\neg\alpha \in K$ *then* $K * \alpha = K + \alpha$.

(Relevance) *If* $\beta \in K \setminus K * \alpha$ *then there exists* K' *such that* $K \cap K * \alpha \subseteq K' \subseteq K$ *and* $\neg\alpha \notin K'$*, but* $\neg\alpha \in K' + \beta$.

Construction: Levi's Identity

With Levi's identity, we are now able to use the **AGM∘** partial meet contraction to define a construction for internal revision. A revision defined in this way is called ***AGM∘** partial meet internal revision*, defined over a function γ.

$$K * \gamma\alpha = (K - \gamma\neg\alpha) + \alpha$$

Any internal partial meet revision satisfies the **AGM∘** postulates from definition 4.44 and, as expected, the postulates precisely characterize partial meet internal revision, that is, the representation theorem holds.

Theorem 4.45 (representation)**.** *An operation $*$ on K satisfies the **AGM∘** postulates of Definition 4.44 for every α iff there exists a selection function γ such that $K * \alpha = \left(\bigcap \gamma(K, \neg\alpha)\right) + \alpha$.*

Proof. **(construction ⇒ postulates)** Let γ be a selection function, and define $K * \alpha = \left(\bigcap \gamma(K, \neg\alpha)\right) + \alpha$. We will prove that $*$ satisfies the postulates for internal revision **AGM∘**.

Regarding the postulates *closure, success, inclusion*, and *non-contradiction*, the proof is analogous to the previous theorem.

Relevance: Let $\beta \in K \setminus K * \alpha$. Then $\beta \notin \bigcap \gamma(K, \neg\alpha) + \alpha$, which means there exists X such that $\beta \notin X \in K\perp\neg\alpha$. Moreover, $K \bigcap K_\gamma\alpha \subseteq X + \alpha$. Since $X \in K\perp\neg\alpha$, we have $X \subseteq K$, $\neg\alpha \notin X$, and because $\beta \in K \setminus X$, $\neg\alpha \in X + \beta$.

Failure: If $\circ\neg\alpha \in K$, then $K - \neg\alpha = K$ by the definition of the selection function, and therefore $(K - \neg\alpha) + \alpha$ is equal to $K + \alpha$.

(postulates ⇒ construction) Let $*$ be an operator satisfying the above postulates, and let γ be the following function:

$$\begin{aligned} \gamma(K, \neg\alpha) &= \{X \in K\perp\neg\alpha : K \cap K * \alpha \subseteq X\} \text{ if } K\perp\neg\alpha \neq \emptyset \\ &= K \text{ otherwise.} \end{aligned}$$

Similarly to the previous theorem, γ is well-defined and we will prove that 1) γ is a selection function and 2) $K * \alpha = \bigcap \gamma(K, \neg\alpha) + \alpha$

1. $\gamma(K, \neg\alpha) \subseteq K \bot \neg\alpha$ by definition. If $\neg\alpha \notin Cn(\emptyset)$ and $\circ\neg\alpha \notin K$ then by non-contradiction $\neg\alpha \notin K * \alpha$. By Lemma 4.36, there exists X' such that $K \cap K * \alpha \subseteq X' \in K \bot \neg\alpha$, hence $X' \in \gamma(K, \neg\alpha)$ and therefore $\gamma(K, \neg\alpha) \neq \emptyset$.

2. We will first prove that $K * \alpha \subseteq \bigcap \gamma(K, \neg\alpha) + \alpha$. By construction, $K \cap K * \alpha \subseteq \bigcap \gamma(K \neg\alpha)$. Therefore, $(K \cap K * \alpha) + \alpha \subseteq \bigcap \gamma(K \neg\alpha) + \alpha$ and thus $K + \alpha \cap (K * \alpha + \alpha) \subseteq \bigcap \gamma(K \neg\alpha) + \alpha$ by distributivity. Thus, by success, inclusion, and closure, $K * \alpha \subseteq \bigcap \gamma(K, \neg\alpha) + \alpha$.

To prove the converse, we have two cases:

1. If $\circ\neg\alpha \in K$. In this case, by failure, $K * \alpha = K + \alpha$ and since $\bigcap \gamma(K, \neg\alpha) \subseteq K$, we have, by closure and success, that $\bigcap \gamma(K, \neg\alpha) + \alpha \subseteq K * \alpha$.

2. If $\circ\neg\alpha \notin K$, we have two sub-cases:

1. If $\neg\alpha \in Cn(\emptyset)$. In this case, by relevance, we have $K \subseteq K * \alpha$. Thus, since there cannot exist $\beta \in K \setminus K * \alpha$, we have $\bigcap \gamma(K, \neg\alpha) \subseteq K * \alpha$.

2. If $\neg\alpha \notin Cn(\emptyset)$. In this case, let us assume by contradiction that $\beta \in \bigcap \gamma(K, \neg\alpha) \setminus K * \alpha$. Since $\beta \in \bigcap \gamma(K, \neg\alpha)$, we have $\beta \in K$ and thus $\beta \in K \setminus K * \alpha$. By relevance, there exists K' such that $K \cap K * \alpha \subseteq K'$, $K' \subseteq K$, $\neg\alpha \notin K'$, and $\neg\alpha \in K' + \beta$. By Lemma 4.36, there exists K'' such that $K' \subseteq K'' \in K \bot \neg\alpha$. Since $\circ\neg\alpha \notin K$ and $\neg\alpha \notin Cn(\emptyset)$, we have $\bigcap \gamma(K, \neg\alpha) \subseteq K''$, and therefore $\beta \in K''$. As $\neg\alpha \in K' + \beta$ and $K' \subseteq K''$ then, if $\beta \in K''$, we would have $\neg\alpha \in Cn(K'')$. Hence, $\beta \notin K''$, by the previous cases 1 and 2. Thus, we conclude that $\bigcap \gamma(K, \neg\alpha) \subseteq K * \alpha$.

Now, in both cases, since $\bigcap \gamma(K, \neg\alpha) \subseteq K * \alpha$, $\bigcap \gamma(K, \neg\alpha) + \alpha \subseteq K * \alpha + \alpha$, and by success and closure, $\bigcap \gamma(K, \neg\alpha) + \alpha \subseteq K * \alpha$.

□

In addition to these two revisions present in the **AGM∘** system – reinterpretations of those already present in the **AGM** and belief base literatures (now applied to sets with a paraconsistent closure) – it is possible, by respecting the assumed criteria of rationality and leveraging the use of the new language, to define several other revisions. We will do this as iterations of the already defined operations and therefore will not define axioms or explicitly specify the constructions for them[19].

4.7 Some other revisions in AGM∘

Coherent revision 1 Incorporation of a new belief α with a possible prior removal of $\circ\alpha$ to try to ensure the coherence of all steps of the operation.

$$(K - \circ\alpha) * \alpha$$

Coherent revision 2 Incorporation of a new belief α with a possible prior removal of $\sim \alpha$ to try to ensure the coherence of all steps of the operation.

$$(K- \sim \alpha) * \alpha$$

Coherent revision 3 Incorporation of a new belief α with a possible prior removal of $\circ\neg\alpha$ to try to ensure the coherence of all steps of the operation.

$$(K - \circ\neg\alpha) * \alpha$$

The revisions of the above definitions can be either external or internal, depending on the rationality one wants to capture. The prior contraction of $\circ\alpha$, $\sim \alpha$, and $\circ\neg\alpha$ illustrate an attempt to prepare the epistemic state for the subsequent revision – hence why we call them "coherent revisions". Respectively, we have the following:

[19]Translation note: in Testa et al. [98], various distinct paraconsistent revisions are defined, placing a particular emphasis on advancing the concept of distinct remainder sets.

Coherent revision 1 By contracting K by $\circ\alpha$, the aim is to prevent the new belief from being strongly accepted. Notably, this is not a problem when considering a subsequent internal revision.

Coherent revision 2 By contracting K by $\sim\alpha$, the aim is to retract the fact that the new belief to be incorporated is strongly rejected (thus ensuring that the epistemic state is indeed revisable by α) to avoid the incoherence of a possible contradictory epistemic state. Particularly, if the presence of $\sim\alpha$ is due to the prior joint presence of $\circ\alpha$ and $\neg\alpha$, the previous operation and even internal revision are shown to be a special case of this.

Coherent revision 3 Notably, the concern of this operation is to avoid the failure of internal revision – retracting $\circ\neg\alpha$ ensures the prior contraction by $\neg\alpha$.

It is important to note that, respectively, if $\circ\circ\alpha$, $\circ\sim\alpha$, and $\circ\circ\neg\alpha$ are previously present in K, the inconsistency persists, so that solutions analogous to the "coherent revisions" (regarding themselves) become necessary – and this occurs repeatedly. This fact is not unthinkable because the agent is not rquired to deliberately incorporate such iterations of consistency – these are direct consequences of the propagation of consistency of certain **LFIs**, which we have addressed in section 4.3.5, and it reflects the fact that, in certain logics, assuming a set as non-revisable by a certain belief entails that it will always be so, in a way that it is not possible to perform a revision operation that determines otherwise.

In these cases, the agent needs to abandon all their previous beliefs and reconstruct their epistemic state from scratch (if they indeed want to revise it based on the mentioned belief). This rationality can be perceived, for example, in the hyperbolic doubt presented by Descartes at the beginning of the "Meditations".

4.8 Partial considerations

Regarding the LFIs: The focus of the **AGM**$\circ$ system is to capture, in the process of contraction, the intuitive idea of formal consistency. This implies that a consistent belief cannot be retracted from the epistemic state in question. It is worth noting that expansion already encapsulates the concept of formal consistency, as this operation is compatible with logical consequence (in a static paradigm). However, the dynamic paradigm's complexity necessitates a new interpretation for the consistency operator to represent the behavior of contraction – and, consequently, revision. Thus, we do not claim that our intuitive idea of consistency precisely mirrors the concepts found in the **LFIs**. Nonetheless, the compatibility of our idea with the theorems of these paraconsistent logics is evident. We believe that our system offers an intriguing (formal) philosophical interpretation of those logics (aligned with the epistemological-formal interests of belief revision theories) and leads to promising applications for such logics.

Regarding Belief Revision in general: Our system illuminates issues concerning contradictory epistemic states, particularly in relation to external revision and, as will be discussed in the next chapter, semi-revision. Admittedly, the alteration in the definition of contraction (relative to **AGM**) may seem an impediment to interpreting our system as a resolution to these issues. However, the **AGMp** system, to be introduced, counters this critique by adopting the same operations as **AGM** (based on the AGM-compliance of paraconsistent logics). Moreover, considering a non-classical system of belief revision allows us to satisfactorily address previously unnoticed issues, such as the conflict between non-contradiction and minimality highlighted by our system.

Regarding AGM-compliance: Although AGM-compliance is quite useful, it cannot be applied without restrictions. We contend that the **AGM**$\circ$ system exemplifies the need for a belief system to undergo necessary and sufficient modifications to accommodate a new language and underlying logic when assuming them.

Chapter 5

Semi-revisions in AGM°

We consider one of the most important consequences of defining a Paraconsistent Belief Revision system to be the possibility of modeling semi-revisions. This is a process in which the new information received is weighted, meaning that its epistemic importance is compared to that of the previously accepted beliefs in the initial set, without assigning priority over the others – hence, such a revision is considered non-prioritized. Conversely, in prioritized revisions (particularly those presented so far), any conflict between the information previously present in the set and the new information is resolved by the (prior or subsequent) abandonment of some of the old beliefs. Attempts have been made in the literature to execute non-prioritized revisions under **AGM** (in the decision-revision framework). However, as stated by Hansson [40], in semi-revisions, there exists an intermediate contradictory state that prevents their definition in logically closed sets (in the expansion-consolidation framework) – which, notably, is not a problem in our system.

5.1 Non-prioritized revisions

The belief revisions we have presented and defined in the previous chapters are operations in which the agent receives new information and accepts it – any conflict between this new information and the previously held beliefs in the epistemic state is resolved by the (prior or subsequent) removal of some old belief. Thus, the following criterion, which has not been explicitly stated in our presentation, is implicitly present:

Principle of the primacy of new information The new information must, always (unconditionally), be accepted.

However, once the new information is accepted, the epistemic state becomes as revisable by it as it is by the other previously held beliefs, unless the new information is strongly accepted, which notably can only be the case in the **AGM**$\circ$ system.

> "It follows from the postulates for revision that the system is totally trusting at each stage about the input information; it is willing to give up whatever elements of the background theory must be abandoned to render it consistent [i.e. non-contradictory] with the new information. Once this information has been incorporated, however, it is at once as susceptible to revision as anything else in the current theory (Cross, C. B. and Thomason, R. H. [11], as cited in Hansson [43], p. 235, the bracketed observation is ours)

New information is often not accepted if it contradicts deeply entrenched previously accepted beliefs in the epistemic state. However, this fact is not satisfactorily captured when considering the aforementioned primacy criterion. Non-prioritary revisions challenge exactly this principle.

5.1.1 Semi-revision

In semi-revision, a new sentence α that contradicts previously held beliefs in the epistemic state is accepted only if it has a higher epistemic value than those beliefs. In this case, enough of the previous beliefs are retracted to make the resulting set satisfy the *non-contradiction criterion*. We can decompose the operation into two steps (as already presented in the introduction, page 20):

1. Expansion by α;

2. Restoration of non-contradiction by abandoning some previous belief or even the newly incorporated α.

It is worth noting that a non-prioritary revision does not necessarily need to be interpreted as the iteration of these sub-operations (as argued in relation to Levi's identity), but rather that the result of the operation is as such.

The first sub-operation can be defined as a simple preliminary expansion[1], but it is necessary to formally define the second sub-operation – called consolidation.

5.1.2 Consolidation

For the reasons mentioned earlier, this operation was originally developed for belief bases: if an epistemic state is contradictory, non-contradiction can be restored by removing part of its elements – notably those intuitively considered less entrenched. A plausible way to accomplish such a removal is to contract the contradiction, that is, the constant **f** – since in a classical paradigm, all contradictions are equivalent to this constant. However, in our system, it is necessary to refine this definition.

[1] It should be noted that the impossibility of modeling semi-revisions in **AGM** is precisely a consequence of such (potentially contradictory) preliminary incorporation. Different solutions for defining non-prioritary revisions in **AGM** can be found in the literature, but they all, broadly speaking, start from the decision-revision model (mentioned in the introduction), where the acceptance or rejection of the new belief is first decided and then, if accepted, a revision is carried out.

5.2 AGM∘ consolidation

Instead of equating contradictions with the constant **f**, we formally define them as sets in which, for some sentence of the language, that sentence and its negation are jointly accepted. In other words, we say that K is *contradictory* if $\{\beta, \neg\beta\} \subseteq K$ for some $\beta \in \mathbb{L}$.

Unlike the other operations already defined and explained, consolidation starts from a belief set and results in a subset of the same set – we will follow the usual notation and denote the consolidation of an epistemic state K as $K!$.

5.2.1 Postulates for AGM∘ consolidation

The first postulate is the well-known closure axiom.

(Closure) $K! = Cn(K!)$.

As we stated, the result of consolidation is a subset of the initial epistemic state – as we are only removing beliefs without incorporating anything.

(Inclusion) $K! \subseteq K$.

Obviously, we require the result of consolidation to be a non-contradictory set – precisely the core of this operation.

(Non-contradiction) $K!$ is non-contradictory.

On the other hand, considering that consolidation is a particular case of contraction, the postulate of *failure* for this operation becomes evident. Thus, if one of the beliefs involved in the contradiction is strongly accepted or rejected, it is not possible to perform the contraction using it. In these cases, the initial set is trivial, and therefore the result of the operation is the set itself.

(Failure) If $K = \mathbb{L}$, then $K! = \mathbb{L}$.

Finally, we require that nothing be unnecessarily removed, which is captured by the well-known relevance postulate, with some obvious adaptations.

(Relevance) If $\beta \in K \setminus K!$ then there exists K' such that $K! \subseteq K' \subseteq K$ and K' is non-contradictory, but $K' + \beta$ is contradictory.

We can see that consolidation is a particular case of contraction, so it is natural that many of its postulates, such as the one mentioned above, are present in this operation. In summary, we have the following:

Definition 5.1 (Postulates for **AGM**$\circ$ consolidation)**.** *Consolidation obeys the following postulates:*

(Closure) $K! = Cn(K!)$.

(Inclusion) $K! \subseteq K$.

(Non-contradiction) *If* $K \neq \mathbb{L}$, *then* $K!$ *is non-contradictory.*

(Failure) *If* $K = \mathbb{L}$, *then* $K! = \mathbb{L}$.

(Relevance) *If* $\beta \in K \setminus K!$ *then there exists* K' *such that* $K! \subseteq K' \subseteq K$ *and* K' *is non-contradictory, but* $K' + \beta$ *is contradictory.*

5.2.2 Construction

Similar to contraction, we use a choice function over a remainder set. The particularity of the remainder set definition for consolidation is that it is defined in relation to a belief set – rather than just a sentence in the language.

Definition 5.2 (Remainder for sets)**.** *Let* K *be in* $\mathbb{L}$ *and* $A \subset \mathbb{L}$. *The set* $K \bot_P A \subseteq \wp(\mathbb{L})$ *is such that for every* $X \subseteq \mathbb{L}$, $X \in K \bot_P A$ *iff the following clauses are satisfied:*

1. $X \subseteq K$
2. $A \cap Cn(K) = \emptyset$

3. *If* $X \subset X' \subseteq K$ *then* $A \cap Cn(X') \neq \emptyset$.

Consolidation considers a specific subset A, namely, the one that represents all contradictory sentences in K.

Definition 5.3 (Set of contradictory sentences)**.** *We define* Ω_K *as the set of contradictory sentences in* K*. That is:*

$$\Omega_K = \{\alpha \in K : \textit{ existe } \beta \in \mathbb{L} \textit{ tq } \alpha = \beta \wedge \neg\beta\}$$

We can finally define *consolidation* as follows:

Definition 5.4. *A* consolidation function *for* K *is a function* γ *such that:*

1. *If* $K \neq \mathbb{L}$ *then* $\emptyset \neq \gamma(K) \subseteq K \perp_P \Omega_K$

2. *If* $K = \mathbb{L}$ *then* $\gamma(K) = \{K\}$

$$K!_\gamma = \bigcap \gamma(K)$$

Theorem 5.5 (Representation)**.** *An operation* $!$ *over* K *satisfies the postulates of Definition 5.1 iff there exists a consolidation function* γ *such that* $K! = \bigcap \gamma(K)$.

Proof. **(construction $\Rightarrow$ postulates)**

Closure: Follows for the same reason stated in the first theorem.

Inclusion: Follows directly from the construction.

Non-contradiction By Lemma 4.36, $K \perp_P \Omega_K \neq \emptyset$. Therefore, by definition, $\bigcap \gamma(K) \cap \Omega_K = \emptyset$.

Failure: Follows from the definition of γ.

Relevance: Let $\beta \in K \setminus K!$. There exists $X \in \gamma(K) \subseteq K \perp_P \Omega_K$ such that $\beta \notin X$. By construction, $K! \subseteq X \subseteq K$. Let $X' = X \cup \beta$. Then $X \subset X' \subseteq K$ since $\beta \in K$. By definition, $\Omega_K \cap Cn(X') \neq \emptyset$, that is, $\Omega_K \cap (X + \beta) \neq \emptyset$.

(postulates ⇒ construction) Let's consider the following function:

$$\gamma(K) = \{X \in K\perp_P\Omega_K : K! \subseteq X\} \text{ if } K \neq \mathbb{L}$$

$$\gamma(K) = \{K\} \text{ otherwise}$$

We need to prove that 1) γ is a consolidation function and 2) $K! = \bigcap\gamma(K)$.

1. It follows directly from the construction that $\gamma(K) \subseteq K\perp_P\Omega_K$. We will show that $\gamma(K) \neq \emptyset$. By *non-contradiction*, we have $\Omega_K \cap K! = \emptyset$. By *inclusion*, $K! \subseteq K$. Therefore, by Lemma 4.36, there exists $X \in K\perp_P\Omega_K$ such that $K! \subseteq X$. It follows that $X \in \gamma(K)$ and thus $\gamma(K) \neq \emptyset$.

2. It follows directly from the construction that $K! \subseteq \gamma(K)$. We need to show that $\gamma(K) \subseteq K!$. To do so, we just need to show that there exists $\beta \notin K!$ such that $\beta \notin \bigcap\gamma(K)$. Let $\beta \notin K!$ and suppose $\beta \in K$ (otherwise, $\beta \notin \gamma(K)$ trivially). By *relevance*, there exists K' such that $K! \subseteq K' \subseteq K$, $K' \cap \Omega_K = \emptyset$, but $K' + \beta \cap \Omega_K \neq \emptyset$.

   By Lemma 4.36, there exists $X \in K\perp_P\Omega_K$ such that $K! \subseteq K' \subseteq X$. Therefore, $X \in \gamma(K)$. Since $\Omega_K \cap K' + \beta \neq \emptyset$, we have $\beta \notin Cn(X)$ (otherwise $\Omega_K \cap X \neq \emptyset$). Therefore, $\beta \notin \bigcap\gamma(K)$.

□

5.3 AGM∘ semi-revision

Having defined consolidation, we can use the previously presented identity to define the semi-revision of K by α, denoted as $K?_\gamma\alpha$, as an expansion followed by consolidation (under a function γ):

$$K?_\gamma\alpha = (K + \alpha)!_\gamma$$

This definition, constructed through two other already known operations, is sufficient to characterize semi-revision. However, for didactic reasons, we will present its postulates – which allow us to better compare it with the revisions already presented.

Definition 5.6. ***AGM*○** *semi-revision obeys the following:*

(Closure) $K?\alpha = Cn(K?\alpha)$.

(Inclusion) $K?\alpha \subseteq K + \alpha$.

(Non-contradiction) *If* $K \neq \mathbb{L}$*, then* $K?\alpha$ *is not contradictory.*

(Failure) *If* $\sim \alpha \in K$*, then* $K?\alpha = \mathbb{L}$.

(Relevance) *If* $\beta \in K \setminus K?\alpha$*, then there exists* K' *such that* $K?\alpha \subseteq K' \subseteq K + \alpha$*,* K' *is not contradictory, but* $K' + \beta$ *is.*

It is worth noting that *success* is not required – this absence is precisely the core of this operation because, as stated, the new sentence to be incorporated is not prioritized over the previously accepted ones, and therefore it will not necessarily be present in the resulting epistemic state.

Let us recall the following example:

Example 5.7. *An investigator of a robbery believes that it is possible only for A or for B to have committed the crime* ($a \in K$ *or* $b \in K$)*, and that they are not accomplices* ($(a \to \neg b) \in K$) *and* ($(b \to \neg a) \in K$)*. Their working hypothesis requires them to investigate, at the same time, the possibility that A and B have committed the robbery* ($a \in K$) *and* ($b \in K$)*.*

In the example, the investigator's epistemic state, after incorporation, is $K = Cn(\{a, b, (a \to \neg b), (b \to \neg a)\})$, noticeably contradictory. In this case, the different possible consolidated epistemic states depend on their selection function – which can be understood as the evidence acquired through their investigation.

It is important to notice, in this example, that a possible solution to the investigation would be $K = Cn(\{a, b\})$ – a direct consequence of considering the contradictory information that both A and B committed the robbery.

Another interesting example highlights the fact that **AGM**$\circ$ semi-revision encompasses the definition of *selective revision*, presented by Fuhrmann and Hansson [16]. Let's consider the following example, adapted from that article, which illustrates this fact:

Example 5.8. *A child tells their father that a dinosaur entered the house and broke the vase in the living room.*

The original idea of this example is to show that in many cases the sentence to be incorporated needs to be filtered. The father can choose to accept, after filtering, only the part of the information that the vase is broken since the impossibility of a dinosaur's existence is evident. On the other hand, **AGM**$\circ$ semi-revision automatically performs this filtering, so that it is not necessary to filter the information to be incorporated beforehand. The selection function itself operates on the consequences of the temporary contradictory epistemic state and, during the consolidation, it will certainly reject the information that a dinosaur entered the house but possibly retain the fact that the vase is broken.

5.3.1 Other non-prioritary operations

In this section, we suggest some other possible operations in our system. Similar to the previous chapter, we will define these operations only in terms of the ones already presented, and will not provide explicit constructions for them.

Local consolidation (or α-consolidation) Consolidation over a specific contradiction involving a particular sentence α.

$$K!_\alpha$$

In this case, we simply consider the subset of $\Omega_K = \{\alpha \in K : \text{there exists } \beta \in \mathbb{L} \text{ such that } \alpha = \beta \wedge \neg\beta\}$ corresponding to a specific β, which in this case is the sentence α being carried in the notation. This operation follows exactly the same definition presented by Wassermann and Hansson [106] for belief bases – now also possible in our system.

Local semi-revision Semi-revision defined over local consolidation.

$$(K + \alpha)!_{\alpha}$$

Strong consolidation Incorporate $\circ\delta$ after a δ-consolidation. $K!_{\delta} + \circ\delta$

The aim of this operation is to encapsulate, to a certain degree, the intuitive concept of classical consolidation as it is defined within belief bases.

5.4 Partial considerations

The significance of defining semi-revisions has been satisfactorily explored in the previous chapters, particularly in the introduction where we address some open problems of the **AGM** system as proposed by Hansson. The key aspect to highlight in this chapter is the potential to view **AGM**$\circ$ semi-revision as a generalization of selective revision, wherein only a portion of the new information is integrated. Specifically, in the context of our system, the determination of which part of the information to retain is made subsequently by the selection function.

Chapter 6

AGM-Compliant Paraconsistent Belief Revision

In this chapter, we present alternative results concerning Paraconsistent Belief Revision. This system, akin to **AGM**$\circ$, represents the epistemic state as a logically closed set over a Logic of Formal Inconsistency, encapsulated by any logic **L** extending **mbC**. However, the framework of this system aligns with the AGM-compliance of its underlying logic. This means we employ the same formal constructions as found in the classical **AGM** system. The aim is to facilitate a dialogue with the classical **AGM** system and bring the results obtained here into closer alignment with it – complementing them, to a certain extent, as this system defines some operations previously undefinable in **AGM** (always based, it must be noted, on the operations as defined in **AGM**). Additionally, this system adopts a broader concept of paraconsistency, given that the consistency operator is not central to the definition of contraction. Therefore, we refer to this system as **AGMp**.

6.1 AGM-compliance

The main concern of the aforementioned work by Flouris [19] is to elucidate the applicability of the **AGM** system in different non-classical logics – which he refers to as AGM-compliance. Considering that expansion, in any belief revision system, is assumed through a usual set-theoretic operation – namely, a mere union – and that revision is defined through expansion and contraction, it can be stated that a logic is AGM-compliant if it is possible to fully characterize a contraction operation in that logic using the classical postulates. Formally, we have the following:

Definition 6.1 (AGM-Compliance – Flouris [19])**.** *A logic is AGM-compliant if and only if for every set K of beliefs, there exists at least one operation $-$ over K that satisfies the (generalized) postulates for contraction.*

Such compliance is related to whether the logic is decomposable or not.

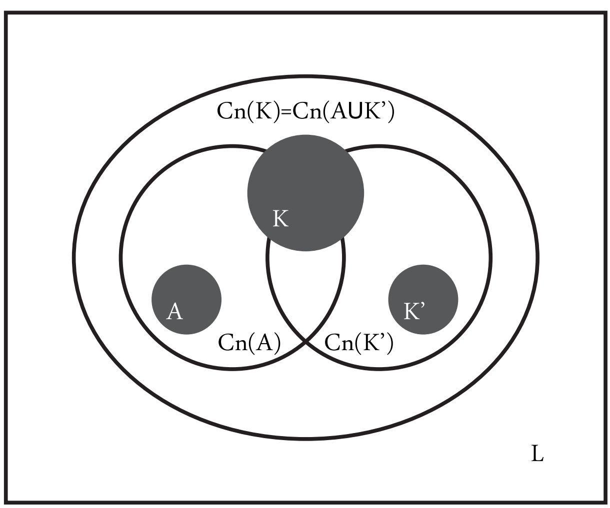

Figure 6.1: Decomposability – informal diagram

The intuition behind decomposability is that the result K' of a contraction $K - A$ should "fill the gap" between K and A. In other words, it should be possible for K to be decomposed with respect to A into two sets, A and K', where both have less information than K when taken separately. That is, K strictly implies each of them, but they have the same informational power as K when combined – they are equivalent to K. As such, the result $K' = K - A$ can be seen as a type of complement to A with respect to K.

As argued by Flouris, complementarity is central to AGM-compliance: the result of an AGM-compliant contraction between two sets of beliefs ($K - A$) should be the complement of A with respect to K. The definition of complement is as follows:

Definition 6.2 (Complementary Sets)**.** *Let $\langle \mathbb{L}, Cn \rangle$ be a Tarskian logic and let $K, A \in 2^{\mathbb{L}}$ be two sets of sentences such that A is finitely representable and $Cn(\emptyset) \subset Cn(A) \subset Cn(K)$. The complement of A with respect to $K(A^-(K))$ is the class of sets $K' \in A \in 2^{\mathbb{L}}$ such that $Cn(K') \subset Cn(K)$ and $Cn(K' \cup A) = Cn(K)$.*

In this way, complementary sets are the subsets of K that, when combined with A, form a set equivalent to K. Decomposability, therefore, is formally defined as follows:

Definition 6.3 (Decomposability)**.** *A logic $\langle \mathbb{L}, Cn \rangle$ is decomposable iff the set of complements of A with respect to K (as defined above) is non-empty.*

Considering the aforementioned definitions, the following theorem asserts which logics are AGM-compliant:

Theorem 6.4 (AGM-compliance – Flouris [19])**.** *A logic $\langle \mathbb{L}, Cn \rangle$ is AGM-compliant if and only if it is decomposable.*

Furthermore, Flouris demonstrated in the mentioned work that Boolean logics are decomposable and, therefore, any logic that satisfies the **AGM** assumptions is AGM-compliant, as expected.

Theorem 6.5. *Let $\langle \mathbb{L}, Cn \rangle$ be a Boolean logic (distributive and closed under negation), then $\langle \mathbb{L}, Cn \rangle$ is decomposable.*

Corollary 6.6. *Let $\langle \mathbb{L}, Cn \rangle$ be a Boolean logic, then $\langle \mathbb{L}, Cn \rangle$ is AGM-compliant.*

Another important result to be presented in this section is regarding the postulate of *recovery*. Let us recall the equivalence of the contraction postulates to what is called relevant contraction, which consists of all the contraction postulates except for *recovery*, replaced by *relevance*. The fact is that we used this postulate in the operations we defined in **AGM**$\circ$, motivated by its intuitive interpretation and by the fact that it characterizes contraction in a broader class of logics.

In order to do the same in **AGMp** without having to demonstrate the Representation Theorems again, we need to safeguard this possibility in general – since we intend to extrapolate the representation theorems from **AGM** constructions to our **AGMp** system.

Indeed, this important result was demonstrated by Ribeiro in his aforementioned doctoral thesis. Therefore, we have the following:

Theorem 6.7 (Equivalence of relevant contraction – Ribeiro [85])**.** *In Boolean logics, the **AGM** postulates are equivalent to the postulates of relevant contraction.*

In finite logics, *distributivity* and *decomposability* imply that *relevance* and *recovery* are equivalent:

Corollary 6.8. *For finite, distributive, and decomposable logics, relevance and recovery are equivalent in the presence of the other **AGM** postulates.*

Lastly, one of the main results presented by Ribeiro [85] generalizes the representation for contraction satisfactorily and establishes the Representation Theorem for relevant contraction, given its equivalence to partial meet contraction.

Theorem 6.9. *Let $\langle \mathbb{L}, Cn \rangle$ be a compact logic and let A be finitely representable, then $K - A$ satisfies relevance and the other **AGM** postulates iff $K - A = \bigcap \gamma(K, A)$ for some selection function γ.*

This result assumes the obvious generalization of the construction of the remainder set, which is as follows:

Definition 6.10 (**Remainder set**). *Let K be a set of beliefs and A be a set of sentences. The remainder set $K \perp A$ is the set such that $X \in K \perp A$ iff:*

(i) $X \subseteq K$ *(subset of K)*

(ii) $A \notin Cn(X)$ *(that does not imply A)*

(iii) If $X \subset X' \subseteq X then then A \subseteq X'$ *(maximal)*

Given the results presented in this section, it is possible to assume the AGM-compliance of the Logics of Formal Inconsistency (since they are compact) by assuming the classical postulates for contraction or even relevant contraction – which is what we do in this chapter.

6.2 Paraconsistent revision – Alternative results (the system AGMp)

We assume once again a given **LFI**, say bf L, such that **L** extends **mbC**. The deductively closed theories of **L** are called *belief sets* over **L** and denoted by K, and Cn represents the deductive closure operator in logic **L**. The language $\mathbb{L}$ of **L** is generated by the connectives $\wedge$, $\vee$, $\rightarrow$, $\neg$, $\circ$, and the constant $\mathbf{f}$. Classical or *strong* negation will also be defined as usual, by the abbreviation $\sim\alpha =_{def} (\alpha \rightarrow \mathbf{f})$, and $(\alpha \leftrightarrow \beta)$ is an abbreviation for $(\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$.

6.2.1 AGMp contraction

As stated in the previous section, the postulates for contraction are as follows:

Definition 6.11 (**Postulates for AGMp contraction**). *Contraction satisfies the following postulates:*[1]

(Closure) $K - \alpha = Cn(K - \alpha)$

[1] We emphasize that, for the reasons already mentioned, we replace *recovery* with *relevance* and omit *extensionality*.

(Success) $\alpha \notin Cn(\emptyset)$ *implies* $\alpha \notin K - \alpha$

(Inclusion) $K - \alpha \subseteq K$

(Vacuity) *If* $\alpha \notin K$ *then* $K - \alpha = K$

(Relevance) *If* $\beta \in K \setminus (K - \alpha)$ *then there exists* K' *such that* $K - \alpha \subseteq K' \subseteq K$, $\alpha \notin K'$, *but* $\alpha \in K' + \beta$

The construction for it is precisely the partial meet contraction, namely,

$$K -_\gamma \alpha = \cap\gamma(K, \alpha)$$

for some selection function γ, and the representation theorem follows directly from the results presented.

6.2.2 AGMp revision

Similar to the **AGM∘** system of paraconsistent belief revision, the **AGMp** system allows for two distinct revisions: internal revision, given by the Levi identity, and external revision, given by the inverse Levi identity.

AGMp internal revision

Just like in contraction, the postulates for internal revision are the same as in **AGM**. It is worth noting the change of name for the *consistency* postulate – which is now called *non-contradiction* for obvious reasons.

Definition 6.12 (**Postulates for AGMp internal revision**). *Internal revision satisfies the following postulates:*

(Closure) $K * \alpha = Cn(K * \alpha)$

(Success) $\alpha \in K * \alpha$

(Inclusion) $K * \alpha \subseteq K + \alpha$

(Vacuity) *If* $K + \alpha$ *is consistent, then* $K * \alpha = K + \alpha$

(Non-contradiction) *If α is not self-contradictory then $K * \alpha$ is not contradictory.*

The construction for it is precisely the Levi identity, which is defined as

$$K * \alpha = (K - \neg\alpha) + \alpha$$

for some selection function γ, and the representation theorem follows directly from the results presented here.

AGMp external revision

The novelty of this model compared to **AGM** is that, just like in **AGM**$\circ$, it is possible to define revision in reverse order – the inverse Levi identity. We suggest the following new postulates for it:

Definition 6.13 (Postulates for **AGMp** external revision)**.** *An external **AGMp** revision operation satisfies the following postulates:*

(Closure) $K * \alpha = Cn(K * \alpha)$

(Success) $\alpha \in K * \alpha$

(Inclusion) $K * \alpha \subseteq K + \alpha$

(Vacuity) *If* $\neg\alpha \notin K$ *then* $K + \alpha \subseteq K * \alpha$

(Non-contradiction) *If* $\neg\alpha \in K * \alpha$ *then* $\vdash \neg\alpha$

(Relevance) *If* $\beta \in K \setminus (K * \alpha)$ *then there exists* X *such that* $K * \alpha \subseteq X \subseteq K + \alpha,\ \neg\alpha \notin Cn(X)$ *and* $\neg\alpha \in Cn(X) + \beta$

(Pre-expansion) $(K + \alpha) * \alpha = K * \alpha$

Some observations need to be made regarding these postulates. It is possible to notice the absence of the consistency operator in the definitions, and this reflects exactly the idea that we are not giving it a prioritary role in the constructions – so that, in this way, revision is brought closer to the classical **AGM** paradigm. This can be interpreted in three different ways:

1. We consider a weaker paraconsistent logic, in which such an operator is not definable (and therefore, not necessarily an **LFI** as we suggested at the beginning of this chapter). Thus there are no modal and quasi-modal epistemic attitudes, and therefore it is not possible to restore the principle of explosion. All beliefs are susceptible to forming a contradiction without this being trivializing.

2. We consider an **LFI**, but we restrict K to a completely revisable epistemic state, that is, K does not have strongly accepted or rejected beliefs.

3. It is possible that there are strongly accepted or rejected beliefs. Thus, revision is still valid but, in some cases, in a trivial manner. This fact can be captured with a possible definition of the *failure* postulate, similar to external revision in **AGM**$\circ$, to capture this fact.

Regarding **3**, the *failure* postulate has a crucial difference when compared to **AGM**$\circ$: in the latter, the joint presence of $\neg\alpha$ and $\circ\alpha$ (or, directly, $\sim \alpha$) in K, for the sentence α to be incorporated in the revision, implies that it is strongly rejected, and therefore the intermediate contradictory state is trivializing. This trivialization leads to the negation of α is consistent (and thus strongly accepted) and, as such, a subsequent contraction is not possible (since, it should be noted, in that system contraction itself fails in these cases), resulting in an epistemic state equivalent to $\mathbb{L}$.

On the other hand, in **AGMp**, the difference lies in the fact that the subsequent contraction does not fail (since, we should remember, this contraction is the same as in the **AGM** system, in which *failure* is not present because it does not interpret the presence of $\circ\alpha$ as an impediment to the rejection of α). In this case, the resulting epistemic state is $\mathbb{L} - \neg\alpha$, which does not capture any intuitive idea of revision.

The fact is that external revision in **AGMp** proves to be interesting only in cases where K is indeed revisable by the sentence α to be incorporated, an intuition that is better captured by observations **1** and **2** (the same argument applies to semi-revisions).

Theorem 6.14 (Representation)**.** *An operation $*$ over K satisfies the postulates for* ***AGMp*** *external revision from definition 6.13 for every α iff there exists a selection function γ such that $K * \alpha = \bigcap \gamma(K + \alpha, \neg\alpha)$.*

Proof. **(construction $\Rightarrow$ postulates)**

Closure: By definition, $*$ satisfies this property..

Success: Let $X \in (K + \alpha)\bot(\neg\alpha)$, and suppose that $\alpha \notin X$. Consider $X' = X \cup \{\alpha\}$. Given that $X \subset X' \subseteq K + \alpha$, we have that $\neg\alpha \in Cn(X')$, by property 3 in Definition 4.34, that is, $X, \alpha \vdash \neg\alpha$. Thus, $X \vdash \neg\alpha$, by Lemma 4.14. But this contradicts the fact that $\neg\alpha \notin Cn(X)$, by item 2 of Definition 4.34. Therefore, $\alpha \in X$ for all $X \in (K + \alpha)\bot(\neg\alpha)$. Thus, if $(K + \alpha)\bot(\neg\alpha) \neq \emptyset$ then $\alpha \in \bigcap \gamma((K + \alpha)\bot(\neg\alpha)) = K * \alpha$. In the case that $(K + \alpha)\bot(\neg\alpha) = \emptyset$ then it also holds that $\alpha \in \bigcap \gamma((K + \alpha)\bot(\neg\alpha)) = K * \alpha$, because in this case $\gamma((K+\alpha)\bot(\neg\alpha)) = \{K+\alpha\}$,by Definition 4.37 (and obviously $\alpha \in K + \alpha$).

Inclusion: Clearly $K * \alpha = (K + \alpha) - (\neg\alpha) \subseteq K + \alpha$, by the postulates for contraction.

Vacuity: Suppose that $\neg\alpha \notin K$. Thus, $\neg\alpha \notin (K+\alpha)$, by Lemma 4.14. Then $K * \alpha = (K + \alpha) - (\neg\alpha) = (K + \alpha)$, by the postulates for contraction.

Non-contradiction: Suppose that $\neg\alpha \in K * \alpha = (K + \alpha) - (\neg\alpha)$. By the postulates for contraction, $\vdash \neg\alpha$.

Relevance: Let $\beta \in K \setminus ((K+\alpha) - (\neg\alpha))$. Thus, $(K+\alpha)\bot(\neg\alpha) \neq \emptyset$ (otherwise, $(K + \alpha) - (\neg\alpha) = K + \alpha$ and then $K \setminus ((K + \alpha) - (\neg\alpha)) = \emptyset$, a contradiction). Therefore, there exists $X \in \Upsilon(K + \alpha, \neg\alpha) \subseteq (K + \alpha)\bot(\neg\alpha)$ such that $\beta \notin X$. By the definition of $*$, $K * \alpha \subseteq X \subseteq K + \alpha$. Let $X' = X \cup \{\beta\}$.

Thus, $X \subset X' \subseteq K + \alpha$, because $\beta \in K$.By Definition 4.34, $X' \vdash \neg\alpha$, taht is, $X, \beta \vdash \neg\alpha$.

Pre-expansion: $(K + \alpha) * \alpha = ((K + \alpha) + \alpha) - (\neg\alpha) = (K + \alpha) - (\neg\alpha) = K * \alpha$.

(postulates $\Rightarrow$ construction)
Let $*$ be an operator satisfying the above psotulates and let γ be this function:

$$\gamma(K, \neg\alpha) \quad = \quad \{X \in K \perp \neg\alpha : K * \alpha \subseteq X\}$$

We will prove that 1) γ is well-defined, 2) γ is a selection function, and 3) $K * \alpha = \bigcap \gamma(K + \alpha, \neg\alpha)$

1. Suppose that $K \neq K'$ such that $K + \alpha = K' + \alpha$. By *pre-expansion*, we have $K * \alpha = (K + \alpha) * \alpha = (K' + \alpha) * \alpha = K' * \alpha$. Therefore, γ is well-defined.

2. It is clear that $\gamma(K + \alpha, \neg\alpha) \subseteq (K + \alpha) \perp (\neg\alpha)$, if $(K + \alpha) \perp (\neg\alpha) \neq \emptyset$. In order to consider γ as a selection function, we need to prove that $\gamma(K + \alpha, \neg\alpha) \neq \emptyset$ if $(K + \alpha) \perp (\neg\alpha) \neq \emptyset$. Thus, suppose that $(K + \alpha) \perp (\neg\alpha) \neq \emptyset$. Therefore, $\nvdash \neg\alpha$, by item 2 of Definition 4.34. By *non-contradiction*, we have $\neg\alpha \notin K * \alpha$. Using *closure* and *inclusion*, we have $\neg\alpha \notin K * \alpha = Cn(K * \alpha) \subseteq K + \alpha$. Hence, by the upper bound property, there exists $X \in (K + \alpha) \perp (\neg\alpha)$ such that $K * \alpha \subseteq X$. Therefore, $X \in \gamma(K + \alpha, \neg\alpha)$ and hence $\gamma(K + \alpha, \neg\alpha) \neq \emptyset$ if $(K + \alpha) \perp (\neg\alpha) \neq \emptyset$. This shows that γ is a selection function, inducing a contraction operator $-$ in $\mathbf{L}$ as in Definition 4.38.

3. It remains to prove that $K * \alpha = (K + \alpha) - (\neg\alpha) = \bigcap \gamma(K + \alpha, \neg\alpha)$.

3.1 Let us assume first that $(K + \alpha) \perp (\neg\alpha) \neq \emptyset$. Clearly, $K * \alpha \subseteq \bigcap \gamma(K + \alpha, \neg\alpha)$, by the definition of γ. Now, let $\beta \notin K * \alpha$. We want to prove that there exists $X \in \gamma(K + \alpha, \neg\alpha)$ such that $\beta \notin X$. If $\beta \notin K + \alpha$, then $\beta \notin X$ for every $X \in \gamma(K + \alpha, \neg\alpha)$ (since every $X \in$

$\gamma(K+\alpha,\neg\alpha)$ is contained in $K+\alpha$). So, assume that $\beta \in K+\alpha$. By pre-expansion, we have $\beta \notin (K+\alpha)*\alpha$, and thus, by relevance, there exists Z such that $K*\alpha = (K+\alpha)*\alpha \subseteq Z \subseteq (K+\alpha)+\alpha = K+\alpha$, $\neg\alpha \notin Cn(Z)$, and $\neg\alpha \in Cn(Z)+\beta$. By Proposition 4.36, there exists $X \in (K+\alpha)\bot(\neg\alpha)$ such that $K*\alpha \subseteq Z \subseteq X$. Therefore, $X \in \gamma(K+\alpha,\neg\alpha)$. Since $\neg\alpha \in Cn(Z)+\beta$, we have $X,\beta \vdash \neg\alpha$, and hence $X \nvdash \beta$ (otherwise, $X \vdash \neg\alpha$). Thus, $\beta \notin X$, which implies $\beta \notin \bigcap\gamma(K+\alpha,\neg\alpha)$. This proves that $K*\alpha = \bigcap\gamma(K+\alpha,\neg\alpha)$, if $(K+\alpha)\bot(\neg\alpha) \neq \emptyset$.

3.2 Finally, let's assume that $(K+\alpha)\bot(\neg\alpha) = \emptyset$. According to the definition of γ, we have $\bigcap\gamma(K+\alpha,\neg\alpha) = K+\alpha$. On the other hand, if there exists $\beta \in (K+\alpha)\setminus(K*\alpha)$, then, in the same way as proven above, we would have $(K+\alpha)\bot(\neg\alpha) \neq \emptyset$, which is a contradiction.

Therefore, $K*\alpha = K+\alpha = \bigcap\gamma(K+\alpha,\neg\alpha)$.

□

6.3 Semi-revisions in AGMp

Given what has been discussed so far, it is natural to expect that the **AGMp** system allows for the definition of semi-revisions, that is, non-prioritary revisions in the expansion-consolidation scheme. In fact, this is done exactly as we defined in the previous chapter, that is, we construct a consolidation operation and define the semi-revision from an expansion followed by this operation.

$$K?\alpha = (K+\alpha)!$$

The prior expansion is defined in the usual way by set theory, and consolidation is a particular case of contraction – in this case, the same observations regarding *failure* for external revision made on page 158 must be taken into account.

6.4 Partial considerations

The core concept of the **AGMp** system is to preserve the key contributions of the **AGM**$\circ$ system in Belief Revision, as discussed earlier, while also functioning as an extension of the classical **AGM** system. This approach is feasible considering we begin with the AGM-compliance of the underlying logic, thereby retaining the classical results of **AGM**.

Chapter 7

Final Considerations

> *"The truth of a theory can never be proven, for one never knows if future experience will contradict its conclusions."*
>
> Albert Einstein

Despite the extensive formal construction presented and developed in this thesis, our intention is not to definitively answer what it means for an agent to be *rational*. However, our research undeniably contributes to advancing this question and unequivocally clarifies the fundamental role of the underlying logic in attempting to capture such rationality. This necessitates addressing a crucial question: which logic should be used?

The main contributions of this research to the fields of Belief Revision and paraconsistent logics, as well as to the related areas of study, are outlined throughout this thesis to motivate the exposition and development of the tools presented herein. However, there are certain considerations that warrant additional emphasis, which we would like to highlight.

7.1 Perspectives and future work

From a **logical-philosophical** perspective, we propose that this research offers a significant contribution to interpreting Logics of Formal Inconsistency as a potent tool for reasoning about knowledge theory and formal epistemology. The constructions developed here can be seen as an application of **LFIs** in practice – they are utilized as a means to address contradictory yet non-trivial theories.

The expressive power of formal consistency is adeptly captured and integrated within the dynamics of theories. Notably, paraconsistency and the formal consistency operator contribute to several questions in the field of Belief Revision. According to Hansson [41], the relationships among *epistemic value*, *vulnerability to change*, and *probability* require urgent clarification. Significant progress has been made since Hansson's observation, but it is noteworthy that the consistency operator can be perceived as a link among these notions[1]. This provides new insights into the discussion.

Moreover, the distinction between *coherence* and *non-contradiction* in paraconsistent revision systems enhances the debate between *foundationalist* and *coherentist* theories of epistemic justification. While it is debatable whether the **AGM** system formally expresses coherentism, formally presenting a more nuanced definition of coherence is a significant contribution.

The importance of this research in addressing Popper's question, as explored by his disciple Miller, concerning the need for a logic of scientific reasoning akin to paraconsistency, is also paramount. Not only does the paraconsistent nature of our developed system align with this, but the intuitive understanding of formal consistency we capture seems to resonate with scientific reasoning – a scientific theory should not regard any information within its body as consistent, and theories that deem some information consistent (and thus, in our system, unfalsifiable) do not meet the criteria to be considered scientific.

Finally, our contributions to the unresolved questions in **Belief Revision** are noteworthy. We underscore the development of external revisions and semi-revisions in the expansion-contraction and

[1] Recent advancements in **LFIs**, particularly concerning first-order and fuzzy **LFIs** where formal consistency has varying truth values, should be considered when exploring this relationship, aligning well with the concept of relating it to the epistemic weight of a belief.

expansion-consolidation frameworks, respectively, as the pinnacle of this contribution. Additionally, we critique the indiscriminate application of AGM-compliance, without considering the specificities of the new underlying logic. Certain disharmonies in various non-classical Belief Revision systems, including some first-order and modal systems, appear to be a result of this oversight.

In conclusion, considering that our basic system intentionally avoids committing to specific revisions (specialized revision systems), and is deemed a heuristic and formal foundational tool for developing new systems, we anticipate that our research will serve as a foundation for the creation of various distinct systems, each reflecting their particularities and specific needs.

Appendix A

Contradictory Epistemic States

After establishing the logical possibility of contradictory yet coherent epistemic states, a direct consequence of **LFIs** and the systems we have developed, it becomes essential to justify the rational possibility of such states. We could theoretically equate this rational possibility with the logical one, but this approach seems somewhat circular. Hence, we base our justification of contradictory epistemic states on the criteria outlined in section 4.3.4, supported by paradigmatic examples like *external revision* and *semi-revision*. The former illustrates contradiction as a necessary intermediate state of reasoning to maintain *minimality*, while the latter, as a generalization, challenges the *primacy of new information*.

Beyond these examples, the literature in this field presents other scenarios where contradictory epistemic states are not only possible but necessary. Our systems offer a formal foundation for their examination, or alternatively, these examples can be viewed as additional rational justifications for our systems.

A.1 Justified contradictory beliefs: The lottery paradox

Richard Foley [20] presents the possibility of contradictory yet justified beliefs using the lottery and preface paradoxes as examples. In the case of the lottery, an agent may justifiably believe their ticket will lose, especially if the lottery is structured such that the probability of losing is extremely high. However, the same rationale applied to one's own ticket – that it will likely lose –can be applied to every individual ticket in the game. Consequently, one might believe that all tickets will lose, a belief set that contradicts the fact that there will be at least one winner, thereby creating a contradictory epistemic state.

The lottery paradox, originally proposed by Henry Kyburg [55], challenges the notion that rational belief should be based solely on high probability. To illustrate, consider a rational standard that accepts propositions with a probability of at least 0.99 (on a scale from 0 to 1, where 0 denotes complete rejection and 1 denotes complete acceptance of a belief). In a lottery with 100 tickets and only one winner, the probability that the statement "Ticket n will not be the winner" ($\neg g_n$) is true would be high enough for an agent to accept it into their belief set.

In this way:

(1) $\{(\neg g_1, \neg g_2 ... \neg g_{100})\} \subset K$,
by probabilistic acceptance.

(2) $(\neg g_1 \wedge \neg g_2 \wedge ... \wedge \neg g_{100}) \in K$,
by closure in (1).

(3) $(g_1 \vee g_2 \vee ... \vee g_{100}) \in K$,
by the information that at least one ticket will be the winner.

(4) $\neg(\neg g_1 \wedge \neg g_2 \wedge ... \wedge \neg g_{100}) \in K$,
by closure in (3).

(5) $(\neg(\neg g_1 \wedge \neg g_2 \wedge ... \wedge \neg g_{100}) \wedge (\neg g_1 \wedge \neg g_2 \wedge ... \wedge \neg g_{100})) \in K$
by closure in (4) and (2).

Given the evident contradiction in (5), Kyburg proposes to address it by either rejecting the closure of the belief set or abandoning probabilistic acceptance. Kyburg leans towards rejecting closure to circumvent contradiction, a strategy akin to adopting a paraconsistent closure where the principle of explosion is restrained, as advocated in our Belief Revision system. This approach posits that the crux of the issue lies in the consequences of the conjunction of contradictory beliefs, rather than in their mere coexistence.

A.2 More on contradictory beliefs: The preface paradox

The other argument, identified by Foley as akin to the lottery paradox but not involving probability or probabilistic acceptance, is the paradox of the preface, as presented by Makinson [60]. This paradox centers on a common phenomenon in literature: authors often note in the preface of a book that it may contain errors, for which they take full responsibility.

Makinson extrapolates a significant conclusion from this common disclaimer. If we accept each statement in the book as true, which is a reasonable assumption, and also regard the preface's admission of potential errors as true, then we are faced with a contradictory belief set. This arises because the author simultaneously believes in the veracity of each statement in their book while also acknowledging, due to their fallibility, that the aggregate of these statements may include falsehoods. Hence, the author must rationally both believe and disbelieve the entirety of their book's content.

This paradox serves to bolster Henry Kyburg's perspective that a set of contradictory beliefs is feasible. As Sorensen [93] notes, some theorists argue that it's justifiable for an agent to hold incompatible beliefs if their contradictory nature is broadly spread across the agent's overall belief set, as illustrated in the aforementioned examples. However, Knight [54] observes that the discomfort of contradiction intensifies as the belief set becomes more concise, which is particularly evident in self-referential paradoxes.

A.3 The liar paradox: A dialetheist justification?

A sentence is self-referential when it refers to itself or its own referent. The most notable example is the so-called Liar sentence, "This sentence is false," which lies at the core of its homonymous paradox.[1]

The Liar sentence leads to a contradiction when attempting to determine its truth value – if we assume the sentence is true, then it must be false as it claims; conversely, if we assume it is false, then it must be true, as that is what the sentence states. In both cases, we arrive at a contradiction: the sentence appears to be both true and false simultaneously.[2]

The intriguing aspect of this example is that it involves a single premise leading to the negation of itself. Therefore, acknowledging the rational possibility of a contradictory belief set due to such a sentence implies accepting the rationality of the contradictory sentence itself. This acceptance hinges on whether one endorses or rejects self-contradictory sentences.

Graham Priest and other logicians [81] propose that the Liar sentence should be considered both self-contradictory and true (and rationally justifiable). This stance, known as dialetheism, posits that true contradictions, or *dialetheias*, exist. According to dialetheists, paradoxes like the Liar provide evidence that some contradictions are true, emerging naturally from the use of ordinary language and our thought processes.

While dialetheism offers a rational basis for accepting contradictory epistemic states caused by self-contradictory sentences, it also demands an ontological commitment to the actual existence of dialetheias, influencing any belief revision theory based on this concept.

Accepting the existence of contradictory epistemic states, as ex-

[1]For an introduction to this topic, see Bolander [7].

[2]It is worth noting that some authors, as suggested by Feferman [15], argue the Liar sentence cannot be definitively classified as true or false, challenging the *bivalence principle* which posits that every sentence must be either true or false.

emplified by the lottery, preface, and Liar paradoxes (and specifically the true contradictions of dialetheism), also entails acknowledging their implications – probabilistic acceptance and agent fallibility in the lottery and preface paradoxes, mitigated by justifying contradictions dispersed across the belief set; and the existence and dialetheist acceptance of true contradictions. Thus, the necessity of a belief revision model addressing such contradictory states appears closely linked to these concepts and theories, but it's not an absolute requirement.

Although our paraconsistent theory of belief revision accommodates and effectively models these examples, it does not strictly adhere to the justifications outlined above. As previously mentioned, we recognize the existence of contradictory belief sets for various reasons and aim to manage them sensibly.

A.4 A simpler example: Contradictory expansion

The reality is that examples of contradictory epistemic states are not hard to find. Levi [58] points out that contradictions resulting from simple expansions are quite common. An agent can believe in a proposition and, through simple expansion, come to believe in its negation, whether due to direct observation or reliance on another agent's statement. Levi notes that it is typical for agents to acquire contradictory beliefs in this manner.

Echoing this sentiment, Hansson [41] remarks that such scenarios are far from unrealistic. Many individuals harbor contradictory beliefs yet manage to act rationally[3]. Moreover, Hansson suggests there is nothing inherently problematic about a computer accepting two contradictory statements, provided this does not cause incoherence to spread throughout the database.

This perspective aligns with Harman's view, which acknowledges exceptions to the requirement for an agent's beliefs to be

[3] This observation underscores Hansson's preference for epistemic states modeled by belief bases not closed under logical consequence, as traditionally the closure of a contradictory set implies belief in all sentences of the language.

non-contradictory. For instance, an agent may recognize they hold incompatible beliefs but remain uncertain about how to revise them without substantial loss. Notably, Harman emphasizes the *principle of minimality* over the *principle of non-contradiction* in his argument – accepting contradiction is preferable if resolving it would lead to significant loss. In such instances, Harman concludes, the optimal approach might be to preserve the incoherence and avoid inferences that depend on the contradictory sentences.

A.5 Another justification: Compartmentalized epistemic state

The aforementioned fact can be better understood with Stalnaker's idea that a person's cognitive state is better described by multiple belief systems rather than just one (Stalnaker [96]):

> "A person may be disposed to behave, in one type of context or with respect to one type of action, in a way that is best explained by one epistemic state, and at the same time be disposed to behave, in a different type of context or with respect to another type of action, in ways that are best justified by a different, distinct epistemic state. This need not be understood as a sort of jumping from one state to another or vacillating between two states: the agent can be, at the same time, in two stable epistemic states, in two distinct dispositional states that may be manifest in different types of situations."

Stalnaker interprets this phenomenon as indicative of the imperfections in human agents' cognitive abilities. In an idealized scenario, an agent's beliefs should coexist within a single coherent system, yet they may still be potentially contradictory considering the aforementioned arguments.

Wassermann [105] proposed a system that effectively encapsulates this concept of compartmentalization, albeit from a different angle – focusing on agents with limited resources. This approach was briefly outlined in Section 1.2.3.

A.6 Conclusion

Numerous examples of contradictory epistemic states are documented in the literature, each underpinned by varying justifications and criteria. The reality is that our Paraconsistent Belief Revision system serves as a formal tool to delineate such examples. This enables a comparative analysis of their justifications with those employed in our system, thereby encouraging continual re-evaluation and exploration of paraconsistency phenomena. Specifically, it aids in examining formalized consistency and its connections to diverse logico-philosophical questions.

Appendix B

Formal Preliminaries: Abstract Logic

B.1 Logical consequence – General overview

Following primarily Hansson [41] and Ribeiro [85], in this section we present the formal framework that we adopt in the body of the thesis – we provide a rather general definition of the consequence operator Cn, as we consider different logics throughout the thesis. After presenting a brief list of logical properties, we provide some important proofs concerning the relationship between these properties.

We define a logic as a pair $\langle \mathbb{L}, Cn \rangle$, where $\mathbb{L}$ is an enumerable set called the *language* whose elements are *sentences* and the consequence operator is a function $Cn : 2^{\mathbb{L}} \longrightarrow 2^{\mathbb{L}}$. We represent sets of sentences (subsets of $\mathbb{L}$) by the uppercase letters $A, B, C, ...$. The sentences of the language are represented by the lowercase Greek letters $\alpha, \beta, ...$. Let $\alpha \in \mathbb{L}$ and $A, B \in 2^{\mathbb{L}}$:

1. α is a consequence of A iff $\alpha \in Cn(A)$;

2. A is a consequence of B iff every element of A is a consequence of B, that is, $A \subseteq Cn(B)$;

3. A and B are equivalent iff $Cn(A) = Cn(B)$;

4. A is trivial iff $Cn(A) = \mathbb{L}$.

Furthermore, a set A is not a consequence of B iff $A \not\subseteq Cn(B)$, which means that there exists at least one sentence $\alpha \in A$ that is not a consequence of B, and thus $B \cap Cn(A) = \emptyset$.

Some logical properties assumed for Cn are:

Definition B.1 (Alfred Tarski)**.** *A consequence operator on $\mathbb{L}$ is a function Cn that takes each subset of $\mathbb{L}$ to another subset of $\mathbb{L}$ such that:*

(Inclusion) $A \subseteq Cn(A)$

(Monotonicity) *if* $A \subseteq B$ *then* $Cn(A) \subseteq Cn(B)$

(Idempotence) $Cn(A) = Cn(Cn(A))$

An operator Cn that satisfies these properties is called "Tarskian". By metonymy, we refer to a logic whose operator satisfies these properties as a "Tarskian logic" – and the same applies to any property.

Notably, since our scope of interest encompasses non-classical logics, it is important to emphasize that Tarskian logics do not encompass all the logics present in the literature (such as the aforementioned non-monotonic logics). However, Tarskianity includes a reasonable number of logics, such as the **LFIs** considered in this thesis. Therefore, throughout the thesis and the following pages, we simply refer to a "Tarskian logic" as a "logic".

In the course of the thesis, particularly in the proofs of the main results presented, we indiscriminately use (without explicit reference) the following results about Tarskian logics, which deserve further clarification.

Observation B.2. $Cn(Cn(A) \cup Cn(B)) = Cn(A \cup B)$

Proof. $A \subseteq Cn(A)$ and $B \subseteq Cn(B)$ by *inclusion*. Therefore, $A \cup B \subseteq Cn(A) \cup Cn(B)$ and, by *monotonicity*, $Cn(A \cup B) \subseteq Cn(Cn(A) \cup Cn(B))$.

$Cn(A), Cn(B) \subseteq Cn(A \cup B)$ by *monotonicity*. Thus, $Cn(A) \cup Cn(B) \subseteq Cn(A \cup B)$ and, by *idempotence*, $Cn(Cn(A) \cup Cn(B)) \subseteq Cn(A \cup B)$. □

Observation B.3. $Cn(Cn(A) \cap Cn(B)) = Cn(A) \cap Cn(B)$

Proof. $Cn(A) \cap Cn(B) \subseteq Cn(A)$ and, by *idempotence*, $Cn(Cn(A) \cap Cn(B)) \subseteq Cn(A)$. Similarly, $Cn(Cn(A) \cap Cn(B)) \subseteq Cn(B)$. It follows that $Cn(Cn(A) \cap Cn(B)) \subseteq Cn(A) \cap Cn(B)$ and $Cn(A) \cap Cn(B) \subseteq Cn(Cn(A) \cap Cn(B))$ by *inclusion*. □

Observation B.4. *If A and B are equivalent, then $A \subseteq Cn(K)$ iff $B \subseteq Cn(K)$.*

Proof. By *monotonicity* and *idempotence*, $Cn(A) \subseteq Cn(K)$. By hypothesis, $Cn(A) = Cn(B)$ and, by *inclusion*, $B \subseteq Cn(K)$. The converse is analogous. □

We define that $K \subseteq \mathbb{L}$ is closed under Cn if and only if $K = Cn(K)$. Furthermore, we have the following.

Definition B.5. Inclusion *is a partial order on the class of all sets K in the language, and therefore satisfies the following:*

(Transitivity) *If $K_1 \subseteq K_2$ and $K_2 \subseteq K_3$ then $K_1 \subseteq K_3$;*

(Reflexivity) $K_2 \subseteq K_1$*;*

(Antisymmetry) *If $K_1 \subseteq K_2$ and $K_2 \subseteq K_1$ then $K_1 = K_2$*

In addition to the aforementioned basic properties, we refer to a few others throughout the thesis. It is worth noting that we do not assume them to be universally valid – in each case, we explicitly refer to them, such as when describing the **AGM** assumptions.

Definition B.6 (Properties of the consequence operator)**.** *The list of properties we consider throughout the thesis is as follows:*

(Compactness) *If $\alpha \in Cn(A)$, then there exists a finite subset $A' \subseteq A$ such that $\alpha \in Cn(A')$;*

(Complementarity) *The logic is complemented iff every finitely representable set $A \subseteq \mathbb{L}$ has a complement $A' \subseteq \mathbb{L}$;*

The complement *of a set $A \subseteq \mathbb{L}$, if it exists, is a set $A' \subseteq \mathbb{L}$ such that:*

$$Cn(A \cup A') = \mathbb{L}$$

$$Cn(A) \cap Cn(A') = Cn(\emptyset)$$

A set is finitely representable *iff there exists a finite A' such that $Cn(A) = Cn(A')$*

(Distributivity) $Cn(A \cup B) \cap Cn(A \cup C) \subseteq Cn(A \cup (Cn(B) \cap Cn(C)))$

(Decomposability) *A logic $\langle \mathbb{L}, Cn \rangle$ is decomposable iff for every $K, A \in 2^{\mathbb{L}}$ such that $K = Cn(K)$, $Cn(\emptyset) \subset Cn(A) \subset K$, and A is finitely representable, there exists $K' \in \mathbb{L}$ such that $Cn(K') \subset Cn(K)$ and $Cn(K' \cup A) = Cn(K)$.*

Definition B.7 (Boolean logic)**.** *A logic is Boolean iff it is distribute and complementary.*

B.1.1 Logical consequence and language

In this section, we assume a standard language for propositional logics $\mathbb{L} = \{\wedge, \vee, \rightarrow, \neg\}$ and present some important properties of this language. The language $\mathbb{L}$ is called closed under an n-ary connective $\#$ iff, for every $\alpha_1, \ldots, \alpha_n \in \mathbb{L}$, we have $\#(\alpha_1, \ldots, \alpha_n) \in \mathbb{L}$.

We say that a negation $\neg$ of a language $\mathbb{L}$ is classical (or strong) iff $\langle \mathbb{L}, Cn \rangle$ satisfies the following conditions for every $\alpha \in \mathbb{L}$:

1. $Cn(\alpha) \cap Cn(\neg\alpha) = Cn(\emptyset)$

2. $Cn(\{\alpha, \neg\alpha\}) = \mathbb{L}$

We denote such a negation, for the **LFIs**, as $\sim$, so that in such logics $\neg$ represents a weak (paraconsistent) negation – one that does not, in general, satisfy the above conditions 1 and 2.

Definition B.8 (Properties of the consequence operator)**.** *Other properties that we consider in this thesis, which allude to a standard language, are:*

(Deduction) *A logic* $\langle \mathbb{L}, Cn \rangle$ *closed under the* $\rightarrow$ *operator satisfies deduction iff, for every* $\alpha \in \mathbb{L}$ *and every* $A \subseteq \mathbb{L}$*,* $\alpha \in Cn(A \cup \{\beta\})$ *iff* $\beta \rightarrow \alpha \in Cn(A)$*.*

(Supraclassicality) *A logic* $\langle \mathbb{L}, Cn \rangle$ *is supraclassical iff, for every* $\alpha \in \mathbb{L}$ *and every* $A \subseteq \mathbb{L}$*, if* $\alpha \in Cn_{LPC}(A)$ *then* $\alpha \in Cn_{LPC}(A)$*, meaning that every consequence of classical propositional logic is a consequence of* $\langle \mathbb{L}, Cn \rangle$*.*

Other important results used in the thesis without explicit reference are as follows.

Observation B.9. $Cn(\{\alpha \vee \beta\}) = Cn(\{\alpha\}) \cap Cn(\{\beta\})$

Proof. We will prove that (1) $Cn(\{\alpha \vee \beta\}) \subseteq Cn(\{\alpha\}) \cap Cn(\{\beta\})$ and (2) $Cn(\{\alpha\}) \cap Cn(\{\beta\}) \subseteq Cn(\{\alpha \vee \beta\})$.

(1) Let $\delta \in Cn(\{\alpha \vee \beta\})$. By *deduction*, $\alpha \vee \beta \rightarrow \delta \in Cn(\emptyset)$, and therefore $\{\alpha \vee \beta \rightarrow \delta\} \subseteq Cn(\emptyset)$. Assuming *supraclassicality*, $\alpha \rightarrow \delta \in Cn(\{\alpha \vee \beta \rightarrow \delta\})$. By *monotonicity*, from $\{\alpha \vee \beta \rightarrow \delta\} \subseteq Cn(\emptyset)$, we have $Cn(\{\alpha \vee \beta \rightarrow \delta\}) \subseteq Cn(Cn(\emptyset))$, and by *idempotence*, $Cn(\{\alpha \vee \beta \rightarrow \delta\}) \subseteq Cn(\emptyset)$. Assuming that $\alpha \rightarrow \delta \in Cn(\{\alpha \vee \beta \rightarrow \delta\})$, we have $\alpha \rightarrow \delta \in Cn(\emptyset)$. Therefore, by *deduction*, $\delta \in Cn(\{\alpha\})$.

Similarly, we have $\delta \in Cn(\{\beta\})$ and, therefore $Cn(\{\alpha \vee \beta\}) \subseteq Cn(\{\alpha\}) \cap Cn(\{\beta\})$.

(2) Now, let $\delta \in Cn(\{\alpha\}) \cap Cn(\{\beta\})$. By *deduction*, $\alpha \rightarrow \delta \in Cn(\emptyset)$. Similarly, we have $\beta \rightarrow \delta \in Cn(\emptyset)$. Hence, $\{\alpha \rightarrow \delta, \beta \rightarrow \delta\} \subseteq Cn(\emptyset)$. By *monotonicity* and *idempotence*, $Cn(\{\alpha \rightarrow \delta, \beta \rightarrow \delta\}) \subseteq Cn(\emptyset)$. By *supraclassicality*, $\alpha \vee \beta \rightarrow \delta \in Cn(\emptyset)$, and therefore, by *deduction*, $\delta \in Cn(\{\alpha \vee \beta\})$.

□

Observation B.10. $Cn(A \cup \{\alpha, \beta\}) = Cn(A \cup \{\alpha \wedge \beta\}$

Proof. We will prove that (1) $Cn(A \cup \{\alpha, \beta\}) \subseteq Cn(A \cup \{\alpha \wedge \beta\}$ and (2) $Cn(A \cup \{\alpha \wedge \beta\} \subseteq Cn(A \cup \{\alpha, \beta\})$.

(1) By *supraclassicality*, $\{\alpha, \beta\} \in Cn(A \cup \{\alpha \wedge \beta\}$. By *inclusion* and *monotonicity*, $A \subseteq Cn(A \cup \{\alpha \wedge \beta\}$. Thus, $A \cup \{\alpha, \beta\} \in Cn(A \cup \{\alpha \wedge \beta\}$. By *monotonicity* and *idempotence*, $Cn(A \cup \{\alpha, \beta\}) \subseteq Cn(A \cup \{\alpha \wedge \beta\}$.

(2) By *supraclassicality*, $\alpha \wedge \beta \in Cn(A \cup \{\alpha, \beta\})$ and thus $\{\alpha \wedge \beta\} \subseteq Cn(A \cup \{\alpha, \beta\})$. By *inclusion* and *monotonicity*, $A \subseteq Cn(A \cup \{\alpha, \beta\})$ and thus $A \cup \{\alpha \wedge \beta\} \subseteq Cn(A \cup \{\alpha, \beta\})$. By *monotonicity* and *idempotence*, $Cn(A \cup \{\alpha \wedge \beta\} \subseteq Cn(A \cup \{\alpha, \beta\})$.

□

Observation B.11. $Cn(\{\alpha_1, \alpha_2, ..., \alpha_n\}) = Cn(\{\alpha_1 \wedge \alpha_2 \wedge ... \wedge \alpha_n\})$

Proof. Follows direct from iterating the observation B.10. □

The following notation will be very useful.

Notation B.12. *For every set A and sentence α:*

$A \vdash \alpha$ *iff* $\alpha \in Cn(A)$

$\alpha \vdash \beta$ *abbreviates* $\{\alpha\} \vdash \beta$

$\vdash \beta$ *abbreviates* $\emptyset \vdash \beta$

$\nvdash \alpha$ *denotes that* $A \vdash \alpha$ *is not the case.*

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