A Merton Model of Credit Risk with Jumps

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Abstract: In this note we consider a Merton model for default risk, where the firm’s value is driven by a Brownian motion and a compound Poisson process.

Keywords: Merton model, default risk, default probability, processes with jumps

1 Introduction

Various models of Merton’s type for credit risk have been studied so far (refer [1] to [8]). This paper aims to our recent results, where a model driven by a jump process is studied in [9] and another model governed by a jumps-diffusion is investigated in [10]. Suppose that the asset value $V_t$ of a company, under a risk neutral measure, is given by the following differential equation

$$dV_t = (r - \beta \lambda)V_t dt + \sigma V_t dW_t + V_t dQ_t,$$

(1.1)

where $W_t$ is a standard Brownian motion, $Q(t) = \sum_{i=1}^{N(t)} Y_i$ is a compound Poisson process, $N(t)$ is a Poisson process with intensity $\lambda > 0$, $Y_i$’s are independent and identically distributed random variables with $E(Y_i) = \beta$. All of these processes are supposed to be considered under the risk neutral measure. In (1.1), $r$ is the interest rate, $\sigma > 0$ is a constant and $N_t$ expresses the number of jumps of $Q_t$ while $Y_i$ is the $i$-th jump size of $Q(t)$.

The model (1.1) reflects a fact that, the firm’s value can change randomly not only in a continuous way but also in a cumulatively discrete fashion.

We will study on the probability of default of the company when its value $V_t$ is less than some debts.

2 Case of one debt $L$

A bankruptcy situation will occur at some time $t$ when the company asset value is less than a debt $L$. And the problem is how to calculate the default probability $P(V_t < L)$.

It is known that the solution of (1.1) is given by (see [7])

$$V_t = V_0 \exp[\sigma W_t + (r - \beta \lambda - \frac{\sigma^2}{2})t] \prod_{i=1}^{N_t} (Y_i + 1).$$

(2.1)

We see that

$$\ln V_t = \ln V_0 + \sigma W_t + (r - \beta \lambda - \frac{\sigma^2}{2})t + \sum_{i=1}^{N_t} \ln(Y_i + 1).$$

And the event $\{V_t < L\}$ or $\{\ln V_t < \ln L\}$ means that

$$\sigma W_t + Z_t < x_t,$$

(2.2)

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We calculate first the characteristic function $$\Psi_{Z_t}(s)$$ of $$Z_t$$. Indeed, according to the Taylor expansion for characteristic function

$$\Psi_{Z_t}(s) = \mathbb{E}(e^{isZ_t}) = \sum_{j=0}^{\infty} \mathbb{E}(e^{isZ_t} | N_t = j) P(N_t = j)$$

$$= \sum_{j=0}^{\infty} \mathbb{E}(e^{is(U_1+...+U_j)}) P(N_t = j)$$

$$= \sum_{j=0}^{\infty} (e^{isU_1}\cdots e^{isU_j}) P(N_t = j)$$

$$= \sum_{j=0}^{\infty} (\psi_U(s))^j \frac{(-\lambda t)^j}{j!} e^{-\lambda t} = \exp[\lambda t (\psi_U(s) - 1)]$$

(2.5)

where $$\psi_U(s)$$ is the common characteristic function of $$U_i$$'s.

It is known also that, for a compound Poisson process as $$Z_t$$ we have $$\mu(t) = EZ_t = \lambda t E(U_i) = \lambda t E\ln(1+Y_i) = \lambda tm$$; $$\sigma^2(t) = \text{Var}Z_t = \lambda t E(U_i^2) = \lambda t E[\ln(1+Y_i)]^2 = \lambda t r^2$$, where $$E[\ln(1+Y_i)] = m$$ and $$E[(\ln(1+Y_i))^2] = \tau^2$$.

Denote by $$Z_t$$ the normalization of $$Z_t$$

$$Z_t = \frac{Z_t - \mu(t)}{\sigma(t)}$$.

And we will show that $$Z_t$$ has an approximately normal distribution.

Indeed, according to the Taylor expansion for characteristic function

$$\psi_U(s) = \sum_{k=0}^{\infty} (is)^k k! |\mathbb{E}[U]^k|$$

we can write

$$\psi_U(s) = 1 + ism - \frac{s^2}{2} + o(s^2).$$

(2.6)

Now we compute the characteristic function of $$Z_t$$

$$Z_t = \frac{1}{\sigma(t)} Z_t - \frac{\mu(t)}{\sigma(t)}$$,

$$\Psi_{Z_t}(s) = e^{-\frac{s^2}{2}} \Psi_{Z_t}(s/\sigma(t))$$.

Taking account of (2.5) and (2.6) we have

$$\Psi_{Z_t}(s) = e^{-is\frac{\mu(t)}{\sigma(t)}} \exp[\lambda t (\psi_U(s/\sigma(t)) - 1)]$$

$$= e^{-is\frac{\mu(t)}{\sigma(t)}} \exp[i\lambda tm - \frac{s^2}{2} \frac{\lambda t r^2}{\sigma^2(t)} + o(s^2\frac{2}{t})]$$

$$= e^{-is\frac{\mu(t)}{\sigma(t)}} \exp[is\frac{\mu(t)}{\sigma(t)} - \frac{s^2}{2} \frac{\lambda t r^2}{\sigma^2(t)} + o(s^2\frac{2}{t})]$$

$$= \exp[-\frac{s^2}{2} + o(s^2\frac{2}{t})], \text{ as } t \to \infty.$$
Then $Z_t \sim \mathcal{N}(0, 1)$ or $Z_t \sim \mathcal{N}(\mu(t), \sigma^2(t))$, where $\mu(t) = \lambda t E \ln(1 + Y_i)$, $\sigma(t) = \sqrt{\lambda t E [\ln(1 + Y_i)]^2} = \sqrt{\lambda t} \gamma$.

Now we can consider $\sigma W_t + Z_t$ as a sum of two independent normal random variables for each $t$ large enough, so it has also a normal distribution with mean

$$\mu^*(t) = \mu(t) = \lambda t E \ln(1 + Y_i)$$

and variance

$$\sigma^*(t) = \sigma^2 + \sigma^2(t) = \sigma^2 + \lambda t E [\ln(1 + Y_i)]^2,$$

where $\sigma > 0$ is a known constant as in (1.1).

And

$$P(\sigma W_t + Z_t < x_t) \approx \Phi\left(\frac{x_t - \mu^*(t)}{\sigma^*(t)}\right),$$

(2.7)

where $\Phi(x)$ is the standard normal distribution function.

We are now in the position to state the following theorem.

**Theorem 2.1** The default probability can be approximated by

$$P_{\text{default}}(0, T) \approx \frac{1}{\sigma^*(t) \sqrt{2\pi}} \int_{-\infty}^{x_t} e^{-(u - \mu^*(t))^2/2\sigma^2(t)} du,$$

(2.8)

where

$$x_t = \ln L - (r - \beta \lambda - \sigma^2/2)t - \ln V_0$$

$$\mu^*(t) = \lambda t E \ln(1 + Y_i), \quad \sigma^*(t) = \lambda t E [\ln(1 + Y_i)]^2.$$


3 Case of many liabilities $L_1, L_2, \ldots, L_m$

Now we consider the case where the company faces up numerous debts $L_1, L_2, \ldots, L_m$ that should be paid at times $t_1, t_2, \ldots, t_m$ respectively, with $t_1 < t_2 < \ldots < t_m = T$.

The company will jump into default position before the time $T$ if and only if at one of time $t_i$ ($i = 1, 2, \ldots, m$), it happens that $V_i < L_i$.

So the probability of default before $T$ is

$$P_{\text{default}}(0, T) = 1 - P(V_i > L_i, \forall i).$$

Denote $L = \max\{L_1, \ldots, L_m\}$ It is easy to see that for all $t_i (i = 1, \ldots, m)$ we have

$$(V_i > L_i) \supset (V_i > L).$$

Then

$$P_{\text{default}}(0, T) \leq 1 - P(V_i > L, \forall i).$$

(3.1)

Put $X_i = \sigma W_t + Z_t$, where, as before $Z_t = \sum_{i=1}^{N_t} U_i, U_i = \ln(1 + Y_i)$. The inequality $V_i > L$ is equivalent to

$$X_i = \sigma W_t + Z_t > \ln L - \ln V_0 - (r - \beta \lambda - \sigma^2/2)t_i := x_i.$$

Consider the event

$$A = \{V_i > L, \forall i\} = \bigcap_{i=1}^{m} \{X_i > x_i\}.$$

(3.2)

Then

$$P_{\text{default}}(0, T) \leq 1 - P(A).$$

It is known that a compound Poisson process is a process of independent increments. The processes $(W_t)$ and $(Z_t)$ are independent and both are of independent increments, so is the process $X_t = \sigma W_t + Z_t$.

Denoting by $A_i$ the event $\{X_i > x_i\}, i = 1, 2, \ldots, m$ we can see that

$$A_1 = \{X_1 > x_1\} = \{X_1 - X_0 > x_1\},$$

$$A_2 = \{X_2 > x_2\} = \{X_2 - X_1 > x_2\},$$

$$\vdots$$

$$A_m = \{X_m > x_m\} = \{X_m - X_{m-1} > x_m\}.$$
\[ A_2 = \{X_{t_2} > x_{t_2}\} = \{X_{t_2} - X_{t_1} > x_{t_2} - x_{t_1}\} \supset \{X_{t_2} - X_{t_1} > x_{t_2} - x_{t_1}\}, \]

if \( A_1 \) occurs.

\[ \ldots \]

\[ A_m = \{X_{t_m} > x_{t_m}\} = \{X_{t_m} - X_{t_{m-1}} > x_{t_m} - x_{t_{m-1}}\} \supset \{X_{t_m} - X_{t_{m-1}} > x_{t_m} - x_{t_{m-1}}\}, \]

if \( A_1, \ldots, A_{m-1} \) occur.

Put \( B_i = \{X_{t_i} - X_{t_{i-1}} > x_{t_i} - x_{t_{i-1}}\} \) for \( i = 1, 2, \ldots, m \) and \( x_0 = 0 \) by convention. It follows that

\[ \bigcap_{i=1}^{m} B_i \subset \bigcap_{i=1}^{m} A_i = A. \]

Because of the independence of increments we have

\[ P(A) \geq P\left(\bigcap_{i=1}^{m} B_i\right) = \prod_{i=1}^{m} P(B_i), \tag{3.3} \]

And by definition of \( B_i \),

\[ P(B_i) = P(X_{t_i} - X_{t_{i-1}} > x_{t_i} - x_{t_{i-1}}) = P(\sigma(W_{t_i} - W_{t_{i-1}}) + (Z_{t_i} - Z_{t_{i-1}}) > x_{t_i} - x_{t_{i-1}}). \tag{3.4} \]

Put \( \overline{X}_i = X_t - X_{t_{i-1}} \), \( \overline{W}_i = \sigma(W_{t_i} - W_{t_{i-1}}) \) and \( Z_i = Z_t - Z_{t_{i-1}} \), where \( Z_i \) is defined as in (2.4). The random variable \( \overline{W}_i \) has normal distribution \( \mathcal{N}(0, \sigma^2(t_i - t_{i-1})) \). The random variable \( Z_i = \sum_{k=N_{i-1}+1}^{N_i} U_k \) has the same distribution with that of \( \sum_{k=1}^{N_i-t_{i-1}} U_k \) since \( U_i \)'s are i.i.d and \( N_i \) is a process of stationary and independent increments.

We can see that the distribution of \( Z_i \) is given by

\[ F_{Z_i}(z) = P(Z_i \leq z) = \sum_{n=0}^{\infty} P(N_{i-1} = n)P(Z_i \leq z/N_{i-1} = n) \]

\[ = \sum_{n=1}^{\infty} \frac{\lambda^n(t_i - t_{i-1})^n}{n!} e^{-\lambda(t_i - t_{i-1})} P(Z_i \leq z/N_{i-1} = n) \]

\[ = \sum_{n=1}^{\infty} \frac{\lambda^n(t_i - t_{i-1})^n}{n!} e^{-\lambda(t_i - t_{i-1})} P\left(\sum_{k=1}^{n} U_k \leq z\right) \]

\[ = \sum_{n=1}^{\infty} \frac{\lambda^n(t_i - t_{i-1})^n}{n!} e^{-\lambda(t_i - t_{i-1})} F_{U_i}^n(z), \tag{3.5} \]

where \( F_{U_i}^n \) is the \( n \) fold convolution of common distribution of \( U_i \)'s.

Suppose now that \( U_i \)'s are continuous random variables, so are \( Z_i \)'s and \( Z_i \)'s. Then the density function of \( \overline{X}_i = \overline{W}_i + Z_i \) is

\[ f_{\overline{X}_i}(x) = f_{\overline{W}_i} \ast f_{Z_i}(x) = \int_{-\infty}^{\infty} f_{\overline{W}_i}(x-z)f_{Z_i}(z)dz \]

\[ = \frac{1}{\sigma\sqrt{2\pi(t_i - t_{i-1})}} \int_{-\infty}^{\infty} \exp\left[-\frac{(x-z)^2}{2\sigma^2(t_i - t_{i-1})}\right] f_{Z_i}(z)dz, \tag{3.6} \]

where \( f_{Z_i}(z) = \frac{dz}{dx} F_{Z_i}(z) \) is the density function of \( Z_i \).

Now we have

\[ P(B_i) = 1 - \int_{-\infty}^{x_i-x_{i-1}} f_{\overline{X}_i}(x)dx, \]

where \( f_{\overline{X}_i}(x) \) is defined by (3.6).

And so, the following assertion is ready to be stated:
Theorem 3.1 If $U_i$’s are continuous random variables then the probability of default before $T$ is estimated by

$$P_{\text{default}}(0,T) \leq 1 - \prod_{i=1}^{m} \left( 1 - \int_{-\infty}^{\gamma_{i}-\eta_{i-1}} \left[ \frac{1}{\sigma \sqrt{2\pi(t_i-t_{i-1})}} \times \right. \right.$$

$$\times \left. \left. \int_{-\infty}^{\gamma_{i}-\eta_{i-1}} \exp \left[ -\frac{(x-z)^2}{2\sigma^2(t_i-t_{i-1})} \right] f_{Z_i}(z) \, dz \right] \, dx \right), \quad \text{(3.8)}$$

where

$$x_i = \ln L - \ln V_0 - (r - \beta \lambda - \frac{\sigma^2}{2}) \tau_i$$

and

$$f_{Z_i}(z) = \sum_{n=0}^{\infty} \frac{d_n \lambda^n(t_i-t_{i-1})^n}{n!} e^{-\lambda(t_i-t_{i-1})} e^{-z^2/2n} \quad \text{(4.1)}$$

From (3.8) and (4.1) we have

$$P_{\text{default}}(0,T) \leq 1 - \prod_{i=1}^{m} \left( 1 - \sum_{n=1}^{\infty} \frac{\lambda^n(t_i-t_{i-1})^n}{n!} e^{-\lambda(t_i-t_{i-1})} \frac{1}{2\pi \sigma \sqrt{n(t_i-t_{i-1})}} \times \right.$$

$$\times \left. \int_{-\infty}^{\gamma_{i}-\eta_{i-1}} \int_{-\infty}^{\gamma_{i}-\eta_{i-1}} \exp \left[ -\frac{(x-z)^2}{2\sigma^2(t_i-t_{i-1})} - \frac{z^2}{2n} \right] dz \, dx \right). \quad \text{(4.2)}$$

4 Particular cases of Theorem 3.1

We consider some particular cases for distribution of $U_i$’s.

4.1 Case of normal random variables

Suppose that $U \sim N(0, 1)$ then we have $\sum_{k=1}^{m} U_k \sim N(0, n)$ with density function $\frac{1}{\sqrt{2\pi n}} e^{-z^2/2n}$ and the density of $Z_i$ is

$$f_{Z_i}(z) = \frac{1}{\sqrt{2\pi n}} \sum_{n=1}^{\infty} \frac{\lambda^n(t_i-t_{i-1})^n}{n!} e^{-\lambda(t_i-t_{i-1})} e^{-z^2/2n} \quad \text{(4.1)}$$

From (3.8) and (4.1) we have

$$P_{\text{default}}(0,T) \leq 1 - \prod_{i=1}^{m} \left( 1 - \sum_{n=1}^{\infty} \frac{\lambda^n(t_i-t_{i-1})^n}{n!} e^{-\lambda(t_i-t_{i-1})} \frac{1}{2\pi \sigma \sqrt{n(t_i-t_{i-1})}} \times \right.$$

$$\times \left. \int_{-\infty}^{\gamma_{i}-\eta_{i-1}} \int_{-\infty}^{\gamma_{i}-\eta_{i-1}} \exp \left[ -\frac{(x-z)^2}{2\sigma^2(t_i-t_{i-1})} - \frac{z^2}{2n} \right] dz \, dx \right). \quad \text{(4.2)}$$

4.2 Case of exponential random variable $U_k$ with parameter $\nu > 0$

We know that if $U_k \sim \exp(\nu)$ then $\sum_{k=1}^{m} U_k \sim \text{Gamma}(n, \mu)$ with the density function

$$\frac{z^{n-1} e^{-z/\nu}}{\nu^n \Gamma(n)},$$

where $\Gamma$ is Gamma function. Then

$$f_{Z_i}(z) = \sum_{n=1}^{\infty} \frac{\lambda^n(t_i-t_{i-1})^n}{n!} e^{-\lambda(t_i-t_{i-1})} \frac{z^{n-1} e^{-z/\nu}}{\nu^n \Gamma(n)}. \quad \text{(4.1)}$$

We can see the estimation in (3.8):

$$P_{\text{default}}(0,T) \leq 1 - \prod_{i=1}^{m} \left( 1 - \int_{-\infty}^{\gamma_{i}-\eta_{i-1}} \frac{1}{\sigma \sqrt{2\pi(t_i-t_{i-1})}} \int_{0}^{\infty} \exp \left[ -\frac{(x-z)^2}{2\sigma^2(t_i-t_{i-1})} \right] \times \right.$$

$$\times \left. \int_{-\infty}^{\gamma_{i}-\eta_{i-1}} \int_{0}^{\infty} \exp \left[ -\frac{(x-z)^2}{2\sigma^2(t_i-t_{i-1})} - \frac{z^2}{2n} \right] dz \, dx \right). \quad \text{(4.3)}$$
5 When \( U = U_k \)'s are general discrete random variables

In this case we have

\[
P(Z_i = z) = P\left( \sum_{k=1}^{N_{t_i-1}} U_k = z \right) = \sum_{n=1}^{\infty} P(N_{t_i-1} = n)P\left( \sum_{k=1}^{N_{t_i-1}} U_k = z/N_{t_i-1} = n \right)
= \sum_{n=1}^{\infty} \lambda^n(t_{i-1})^n e^{-\lambda(t_{i-1})} P\left( \sum_{k=1}^{n} U_k = z \right).
\]

(5.1)

Denote by \( \mathcal{L} \) the set of all possible values of \( Z_i \equiv \sum_{k=1}^{N_{t_i-1}} U_k \). So that

\[
P(\overline{X}_i < x) = P(\sigma \overline{W}_i + Z_i < x) = \sum_{z \in \mathcal{L}} P(\sigma \overline{W}_i < x - z)P(Z_i = z)
= \sum_{z \in \mathcal{L}} \sum_{n=1}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{u^2}{2\sigma^2(t_i-t_{i-1})}\right] \times
\frac{\lambda^n(t_i-t_{i-1})^n}{n!} e^{-\lambda(t_i-t_{i-1})} P\left( \sum_{k=1}^{n} U_k = z \right)du.
\]

(5.2)

The default probability in this case is estimated by

\[
P_{\text{default}}(0,T) \leq 1 - \prod_{i=1}^{m} \left( 1 - \sum_{z=0}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{(z - x)^2}{2\sigma^2(t_i-t_{i-1})}\right] dx \times \frac{\lambda^n(t_i-t_{i-1})^n}{n!} e^{-\lambda(t_i-t_{i-1})} P\left( \sum_{k=1}^{n} U_k = z \right) \right).
\]

(5.3)

6 \( U \) is Poisson random variable with parameter \( \beta > 0 \)

If \( U = U_k \sim \text{Poisson}(\beta) \) then

\[
\sum_{k=1}^{n} U_k \sim \text{Poisson}(n\beta)
\]

with mass probability

\[
p_z = P\left( \sum_{k=1}^{n} U_k = z \right) = e^{-n\beta}(n\beta)^z/z!, \quad z = 0, 1, 2, ...
\]

Then

\[
P_{\text{default}}(0,T) \leq 1 - \prod_{i=1}^{m} \left( 1 - \sum_{z=0}^{\infty} \frac{\lambda^n(t_i-t_{i-1})^n}{n!} e^{-\lambda(t_i-t_{i-1})} \frac{n\beta)^z}{z!} \right) \times \frac{1}{\sigma \sqrt{2\pi(t_i-t_{i-1})}} \times \frac{1}{\sigma \sqrt{2\pi(t_i-t_{i-1})}} \times \frac{1}{\sigma \sqrt{2\pi(t_i-t_{i-1})}} \times
\]

\[
= 1 - \prod_{i=1}^{m} \left( 1 - \sum_{z=0}^{\infty} \frac{\lambda^n(t_i-t_{i-1})^n}{n!} e^{-\lambda(t_i-t_{i-1})} \frac{n\beta)^z}{z!} \right) \times \frac{1}{\sigma \sqrt{2\pi(t_i-t_{i-1})}} \times \frac{1}{\sigma \sqrt{2\pi(t_i-t_{i-1})}} \times \frac{1}{\sigma \sqrt{2\pi(t_i-t_{i-1})}} \times \frac{1}{\sigma \sqrt{2\pi(t_i-t_{i-1})}} \times
\]

\[
\right).
\]

(6.1)

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