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Extreme Science: Mathematics as the Science of Relations as such

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The question of what mathematics is has never received a satisfactory answer, we feel, although "mathematics is the science of patterns" may come close. Devlin's chapter (which takes that as its definition) discusses briefly some answers that have been tried. This chapter by Robert S.D. Thomas is a new contribution to the question, and, we feel, one worth serious consideration. It certainly helps with questions such as the relationship between mathematics and the (other) sciences, and has something to say about the applicability of mathematics.

As this chapter is exploring where mathematics fits into our overall understanding of the world, it is not likely to specifically influence how to teach mathematics. However, both in our courses for mathematics majors, and in our service courses (for students who will use mathematics in the service of their majors, and for students taking mathematics to enhance their general education), there is some value to reflecting *a bit* on the nature of what we are teaching. Students often appreciate this reflection on where the whole enterprise is going and how it might fit into their world-view. This chapter may be of use as you consider how to talk with your students about mathematics and its role.

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Among his philosophical articles that are likely to be of interest to readers of this volume are "Mathematicians and mathematical objects," in *One Hundred Years of Russell's Paradox. Papers from the 2001 Munich Russell Conference*, Godehard Link, ed. (2004), which discusses similar issues as this chapter; "Idea Analysis of Groups," *Philosophical Psychology* (2002) (a response to George Lakoff and Rafael Núñez's, *Where Mathematics Comes From: How the Embodied Mind Brings Mathematics into Being*); "Proto-Mathematics and/or Real Mathematics," *For the learning of mathematics* (1996), and "Meanings in ordinary language and in mathematics," *Philosophia Mathematica* (1991). Also likely to be of interest are a series of articles comparing mathematics to literature: "Mathematics and Narrative," *The Mathematical Intelligencer* (2002), "Mathematics and Fiction I: Identification," *Logique et Analyse* (2000), "Mathematics and Fiction II: Analogy," *Logique et Analyse* (2002), and "The comparison of mathematics with narrative," in *Perspectives on mathematical practices: Bringing together philosophy of mathematics, sociology of mathematics, and mathematics education*, Bart Van Kerkhove and J. P. Van Bendegem, eds. (2006).

1 Introduction

Consideration of any mathematical model, whether from science or operations research, can lead to consideration of the effectiveness of mathematical models for understanding and prediction of the non-mathematical world. This effectiveness was famously called 'unreasonable' by Eugene Wigner [1960] but 'reasonable' by Saunders Mac Lane [1990]. Whether reasonable or unreasonable, its effectiveness does require explanation—actually two explanations. One explanation is of why the *world* is the way it is that allows our rationality to function dependably. This explanation is probably religious even when it does not set out to be so (see [McGrath 2004]). Another explanation is required of how or why *mathematics* is the way it is as a successful vehicle for our rationality. This explanation is probably philosophical, and it is a virtue of the view of mathematics presented here that it makes the mathematical side of the effectiveness seem natural.

Mathematics works so well in the sciences, I say, because it is one of the sciences but not in the simple-minded way of being empirical (based on pattern observation) that is associated with John Stuart Mill. Having lived with this idea for a long time, I find it obvious, obvious but not necessarily correct. Certainly it is not the only way to look at mathematics. Another way to see mathematics is as an art, and I have nothing whatever to say against that view. Mathematics is a complex business, and it would be remarkable if there were only one informative way to see it. The science view has a certain simple plausibility that ought, it seems to me, to require at least a wave of dismissal for those presenting other views, especially those that make mathematics *sui* *generis*.[1] While it has its own character, as both an art and science, it is not all on its own. But even a wave of dismissal of the science view is often not forthcoming. With the exception of Saunders Mac Lane, whose very similar view led him to regard its effectiveness in the world as reasonable, the view is hardly even available to be dismissed. The intention of this essay is to present a sketch of the view in this context so that it is available to be dismissed, argued against, or even further developed.

Seeing mathematics as a science (though not exclusively) does not solve a lot of philosophical problems. In presenting a context of other sciences for mathematics, philosophy of mathematics is set into philosophy of science, leaving most of its problems intact. The effect on a couple of philosophical problems will be mentioned at the end. Mostly what the remainder of this essay will do is context setting.[2] In order to see mathematics as a science, one needs to see it in two of its contexts: historical and then scientific. Accordingly, I begin in the next section with the historical beginnings of mathematics. Next, I describe the sciences in a way that allows mathematics to fit into them. Finally, having fitted mathematics into a picture of the sciences, I say something about a couple of philosophical problems.

2 Historical Context

One needs to push back a long way in order to include the whole development of mathematics, since mathematical records go back a very long way indeed even if you don't count notches on sticks found in stone-age sites, and I don't. We have tablets several thousand years old—how many hardly matters—recording the solution of mathematical problems. If we think about the learned landscape of such a period, we note that most current scholarly disciplines had not been invented. History was a long way from birth. Of the sciences, astronomy, botany, and zoology were beginning their pre-scientific period of data collection. The social sciences and humanities, the latter based on what we call the classics, did not exist. Our classics had not yet been written! All there was on the learned landscape was myth, of which we still have records. Is the mathematics the same as or different from these myths? Obviously different in both manner and subject matter. Or are the manners so different? In previous writing [2002], in which I have compared mathematical proof to narrative, I have dismissed algorithms as being so obviously narrative in form as to require no comment. I thought that the interesting comparison was of the things that were not so obviously similar. But what of the things that *are* obviously similar? What we have from the most distant past that is mathematical is algorithmic rather than theoretical. Quite differently from contemporary presentation of algorithms, specific problems are solved using their peculiar data because, lacking algebraic notation with which to represent either general numbers or arithmetic operations, verbal description of how to solve a particular problem was all that could be written down. More could be learned, however. The apparent intention was that, by learning a few examples, the algorithm could be mastered and applied elsewhere. This is, as

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[1] I have in mind those, like the realists of Charles Chihara's essay in this volume, that give mathematics a subject matter and way of knowing it totally different from other subjects.

[2] What is offered is, while not necessarily the 'big picture' of Charles Chihara's essay, a bigger picture than a portrait of mathematics alone.

one easily recognizes, a learning technique still used by students—with the same attendant pitfall that the range of usefulness of the algorithm is not learned. I draw attention to this because it is somewhat similar to how myths work.

A narrative can perform a variety of functions. Fiction informs us of possibilities, and history informs us about what happened in the past. A function of myth, according to a recent theory [Peterson 1999] is that myth informs us how things ought to be. Myth is one of the few ways in which value is communicated. Mythical stories are applied to present reality allegorically to indicate the distinction between good outcomes and bad outcomes. An example of how a story can suggest how things ought to be is the stories we heard of from the path of the 2004 Boxing Day tsunami. Some societies had stories that motivated them to seek high ground when the earth shook. Whether those stories were history or something like *Chicken Licken,*[3] they worked, and those that had no such stories suffered the avoidable consequence of lacking an appropriate myth. The same process is applied whenever we apply a proverb to assure ourselves that what seems to be happening or what has happened is in accordance with the expectations we ought to have had, even if we did not have them. The process of applying a numerical example to another numerical example by way of an unspecified algorithm is surprisingly similar to the application of a story allegorically to a situation. It appears then that myths, which existed long before history or literary fiction, are both important and compose the literary context in which the earliest serious mathematics was written down. And the modes of interpretation of a mythical narrative and a special-case algorithm are surprisingly parallel.

The way of communicating that I am attributing to ancient mathematicians involves writing about something without worrying what that something is going to turn out to be in the situation of the reader. This is like knowing that interpretation will be allegorical. Particulars stand in for other particulars. In the recounting on a clay tablet of how to solve a particular mathematical problem by an algorithm that could not itself be written down, a whole class of solutions for a whole class of inputs were being recorded in the only way available.

Long before the invention of what we recognize as natural science at the Renaissance, the mathematics from which ours descended had become theoretical—quite different from telling little numerical stories. Recipe mathematics can be extremely useful; ask most engineers. It was recipe mathematics that helped build the pyramids. By Euclid's time, whether in Indian or Hellenistic civilization, mathematics had ceased to be primarily algorithmic, a transformation that did not occur in China. Another shift took place along with that from recipes to theorems. The language used shifted from the particulars that were used to represent the ancient algorithms to attempts at generality. The ubiquitous triangle PQR in Euclid's *Elements* is just any (non-degenerate) triangle for the reader. Syntactically, it is a particular, but functionally it stands for any triangle at all. Any choice of a particular number, used to represent an arbitrary number in an algorithm, still has its own special properties, perhaps divisible by various factors and perhaps prime, but the generic triangle used by Euclid has no particular features. It can be right or isosceles or both, and so on. This indefiniteness became much more explicit when algebra was invented and a symbol not a number was specifically set to have the value of an unknown. Talking about things without knowing *which* things (if any) seems to be an easily observed fundamental

[3] Also known as *The sky is falling*.

mathematical technique. I am suggesting that it predates algebra and is more pervasive than is often thought.

Interpretation is so obviously a feature of accounts of the past that I cannot imagine that historical works have ever been straightforwardly and commonly accepted as telling it like it is in the way that mathematics does. Mathematics is the very paradigm of dependable knowledge and I think has been since it became theoretical with the axiom-proof format.[4] So used are we in the present day to accepting accounts as accurate that we forget that financial accounts, like historical accounts, are made to convince someone of something, even if the something is not quite true. People may believe what they read in newspapers because they believed what they read in textbooks at school. And television is believable because seeing is believing. In the world of two thousand years ago, in which deductive geometry flourished, one of the most important studies was rhetoric. Convincing folks may even be more important now, but we are less frank about needing to do it. Mathematics, in contrast, is mercifully free of that sort of thing. Logic, not rhetoric, rules. Mathematics is unusual in this. It is not faulty mathematics that is used to convince shareholders of Enron that their shares will increase in value; it is assumptions on which calculations are based. I mention this to emphasize that, in being logical and dependable, mathematics has for two thousand years—a long time in human affairs—been what thinking has aspired to if it is to be regarded as above reproach.

What I have been concerned to indicate with this sketch of mathematics before and after the invention of the theorem-proof way of expression is that, as written documents, it began similar to myths in form and interpretation and was transformed from narrative (always about particulars) to theoretical form with language intended to be general, assuming a unique place as what any serious intellectual enterprise would be if it could.

3 Scientific Context

Plainly there are many ways to look at mathematics. As my title indicates, I am putting one forward. Some would call it a view of the nature of mathematics.[5] The main point I want to make in this essay is that one way to see mathematics is as sitting at the extreme of a spectrum of sciences. Since such boundaries don't matter, I don't see it as important whether it is just beyond the extreme end of the spectrum and so not a science or just inside the end and so is the most abstract science. I do see it as sufficiently important to want to make two subsidiary points: The sciences do compose a spectrum, and something important can be learned about mathematics by seeing it in its place on (or beyond) that spectrum, with one end being chemistry, physics, mathematics and with the other end containing the subject matters that are interesting in their own right like psychology. Whether one finds mathematics, physics, chemistry, sociology, or psychology interesting is not what I am concerned with. That is a question about prior interests, modes of presentation, and inclinations of various sorts. I am drawing attention to the gradual difference in what these subjects *study*—what is there before the study begins. With psychology there are folks with their varied minds. With sociology there are whole groups of folks with their

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[4] Does it matter that not all of mathematics is axiomatized? Not at all; axioms are just a remarkably effective coping strategy, as they are in science.

[5] I am sceptical about mathematics' having a nature. It has been pointed out to me by Carlo Cellucci that I suggest variously fictionalism, modal structuralism, and deductivism. All three have things to say worth hearing.

common mind and differences. But with chemistry there are just reagents; even a chemist would have a hard time working up interest in a jar of Glauber's salts. With physics there is anything at all, so long as it is flying through the air, sliding along a surface, flowing in a channel, or doing any of the other physical things that objects and substances do, but such contemplation abstracts from all of the aspects of the objects that make them interesting in themselves. The psychologist's person with a mental life has become simply a rigid body or a point mass with friction. The subject matters of chemistry and physics, near the one extreme, are sufficiently undifferentiated not to be of intrinsic interest, since chemistry considers relations among all substances that interact chemically and physics considers relations among all things that are physical. As I am going to spell out in greater detail in the next section, a physicist in considering a person as a point mass with friction is not negating personality but simply considering what physics considers, physical relations. Physics is about how things interact physically, chemistry about how they interact, as we say, chemically, biology about the new relations added by being alive, and psychology about the new relations added by thinking. Even a thinking thing has merely physical relations.

The contradictory view of what scientific subjects are about, which I consider myself to be combatting because it misrepresents mathematics, is that the different sciences simply have different stuffs as subject matters, and that the stuff of mathematics is the things philosophers call mathematical objects or even what a self-confessedly ill-informed poster to the POMSIGMAA listserve has too often said, numbers. I do not see that widening the focus from numbers to more than just numbers improves this view. At its narrowest, it is plain wrong, but even broadened I find it misleadingly uninformative. The argument against mathematics' having a stuff has been carried through some way by Charles Chihara in his essay in this volume.

I am putting forward a picture, the frame of which I have now described. In order to see mathematics as I see it, one needs to see chemistry and physics—and for that matter psychology—as I see them. I take the defining feature of science as we understand it now, as distinct from natural philosophy before Galileo, to be its study of relations rather than of the things themselves. The sciences are ways of understanding not minds, chemicals, and things in general but the ways minds interact with one another and the world, the way chemicals relate—chiefly react, and the way things in general behave physically: statics, kinematics, dynamics, thermodynamics, fluid mechanics. Psychologists do not pontificate on the nature of mind. You will not find a chemist say anything about the essence of antimony. And even when they profess to be thinking the thoughts of God, physicists do not tell us what gravity *is* any more than Newton did. Physicists tell us the magnitude and direction of gravitational acceleration, whether Newtonian or relativistic, but that is as far as they go. And clearly the science that opposed Galileo, based on Aristotelian common sense and observation, was essentialist to the core. It is an interesting historical question how conscious Galileo was of modelling his new science on mathematics in anything like the way I have suggested. But whether it was conscious or not, that is what he did. It is a possible interpretation of his famous statement that the laws of physics are written in mathematical language.

Mathematics, in its shift from algorithms to theory, moved from what we could do with numbers to the study of relations among numbers, relations among points, lines, and planes. That shift to the rigorous determination of relational consequences of relations had been consolidated nearly two millennia before Galileo. He had little choice. As I tried to indicate in the previous section, mathematics and its approach were all there was to imitate. Myth was hardly appropriate, history is similar to the natural philosophy he was replacing, and Europe was innocent of serious prose fiction. The systematic study of relations and their consequences, which mathematics had been doing apparently forever, was the right target.[6]

We all know that the definitions of points, lines, etc., at the beginning of Euclid's *Elements* are not used in the sequel. What are used in ruler-and-compasses geometry are the positions of lines and points and planes and the construction of new ones in specific positions and proofs of their relations. What a line *is* does not matter. What a magnitude *is* does not matter. *All that matters is how they are related.* When the *Elements* was written, it was thought to be about relations in physical space. But because the enterprise was to prove what followed from the assumptions, it did not matter whether Euclidean space was indeed an accurate model of physical space. The subject, modulo some implicit assumptions, was coherent in itself, and it showed that such study was possible. When it came time to do heliocentric astronomy, it was not only the conic sections that could be taken over from geometry but also the method of making assumptions about relations and seeing what those relations led to without regard to the obvious enough fact that the relations were among physical objects with their varied inner constitutions. The only thing that was required was that they have mass, and even that is a matter of inertial and gravitational behaviour not of essence. And so essences quietly became irrelevant. They turned out to be no loss, since no one had known what they were anyway. We no longer worry about the difference between magnitudes; they each have their dimensions, but aside from keeping track of those dimensions, we treat all magnitudes as numbers. They all behave identically as to arithmetic operations, and so from our structuralist mathematical point of view there is nothing other than their dimensions to distinguish them.

4 Mathematics as a Science

Evidence for the view that mathematics can be regarded as a science is the recent turn to experimental mathematics. Not just to experimenting to look for counterexamples or patterns that might turn out to be universal, but the search for evidence in the absence of proofs. Those writing about these matters in books [Borwein/Bailey 2004], [Borwein/Bailey/Girgensohn 2004], and the recent papers [Bailey/Borwein 2005], [Borwein 2008], are mathematicians interested in proving what they have evidence for, but it is easy to see that they need not have that interest or even be mathematicians. Moreover, there are the proofs up to a pre-set level of probability advocated by Doron Zeilberger [1993], for instance that a given number is prime with 99.9% probability. An old-fashioned mathematician will probably say that proofs help us to understand the results and do not *just* assure us of their correctness. You cannot say that of a 'proof' to a certain level of probability, no matter how high; it is just limited assurance of a scientific-like fact.

An aspect of this notion of scientific-like facts, of things that happen to be true, is that they do not have the interest of justified facts whose justification is much of why they are interesting. Gregory Chaitin has often said, e.g., [Chaitin 1998], that there are so very many such facts that

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[6] More precisely, Galileo chose *the right way*, with primary properties and no essences, to imitate mathematics, Aristotle having attempted a more superficial and less successful imitation. The difference was noted by Kant in the preface of his [Kant 1992], according to [Cassirer 1923].

have no old-fashioned reason to be true. It remains to be seen whether such facts have any permanent place in mathematics, where, after all, interest is a joint value with correctness.

The reader that has come this far can now see that the case I am making can be put so simply that it looks as though I am saying nothing at all. Mathematics is like the sciences because the sciences have been constructed to imitate mathematics. This is easily said, but to mean anything—or to mean what I mean—some content has to be given to the words.

If one sees science in the way I do, then one can easily see mathematics as the next step, where it simply does not matter—is not taken into account—what the things are that are being talked about so long as they behave in the way that we are interested in studying. Mathematics is the extreme of the sciences because they are themselves approximations to varying degrees to its method and matter. If one calls the characterless objects of mathematics point masses, then suddenly one is doing physics, but if one does not, one is doing mathematics—perhaps, depending on motivation, applied mathematics.

Reviel Netz claims [1999, p. 197] that the relational view was present even in ancient Greek mathematics, taking his book on it to vindicate such a claim on the part of just one of the mathematicians that has held the view. Newton viewed numbers as relations (ratios) between magnitudes[7] but I don't know about geometry. The relational view is easier to see in pure mathematics than in mathematics before the invention of pure mathematics, perhaps by Riemann but which for our purposes we can date between Newton and Frege. But Newton may possibly have had a clearer idea of what he was doing than the average mathematician. Another clearer than average thinker was Gauss, to whom Bourbakiste Jean Dieudonné [1977] attributed the same view. After the invention of pure mathematics, the view becomes more common. It was Russell who made the claim Netz vindicated, that mathematics considers 'types of relation' [Russell 1956, p. 3]. Poincaré [1902, p. 20] states the view very clearly, 'Mathematicians do not study objects, but the relation between objects.' Hilbert and von Neumann seem to have agreed, although the latter, like Saunders Mac Lane, preferred functions, which are interdefinable with relations in general. Carnap's philosophy of physics was so formalistic that he was accused of turning physics into mathematics.[8]

Another philosopher that seems to have taken this view was Ernst Cassirer, quoting [1953, vol. 3, p. 293] approvingly more of the above quotation from Russell. [9] Gödel at one time at least appears to have embraced the scientific-style basis for axiom choice advocated by Russell ('their justification lies (exactly as in physics) in the fact that they make it possible for [what one wants] to be deduced' [Gödel 1944, p. 121]). Sir Michael Atiyah's presidential address to the Royal Society of London [Atiyah 1995] contained the following statement, speaking of abstraction, which he had been saying was used in science: 'Mathematics takes the process to its ultimate conclusion: the identity of the players is ignored, only their mutual relations are studied. It is this abstraction that makes mathematics such a universal language: it is not tied to any particular interpretation.' Note that this ignoring does not empty mathematical language of all meaning;

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[7] Austin, translator of [Frege 1980], gives the reference *Arithmetica Universalis,* Vol. I, cap. ii, 3 at §19.

[8] [Carnap 1967, §15, p. 27], cited in [H. Wang 1974, p. 40] and n. 2 to that page.

[9] A contemporary philosopher that assimilates mathematics to science and in particular deduction in mathematics to deduction in science is Carlo Cellucci in his book [Cellucci 2002], of which only the introduction is available in English as [Cellucci 2005].

rather it allows its various patches to be filled with a variety of meanings including new meanings. My most up-to-date indication that this is a common view of reflective mathematicians is the recent book [Widdows 2004], in which the mathematician author expresses a similar view both at the beginning and at the end.

I have not been concerned to combat the blinkered view that mathematics is just about numbers, although my examples tend to be geometrical. That is an unnecessary limitation. If one finds something in the world that is an application of the four-group, that is not because the elements of that group bear any resemblance to the elements of Klein's four-group or of the general abstract four-group. It is because their squares are the identity and they multiply one another to give one another. It is group *behaviour* that we find in the world, not the elements of abstract groups. If one models something with a graph in the sense of graph theory, one may be modelling activities by the edges of a graph whose vertices just represent the termination of the tasks, as in the critical path method, or one may be modelling places by vertices and the routes among them by edges, as in the travelling-salesman problem. But places and times are not significantly *like* the vertices, which have no character to be like. And activities and distances are not *like* edges, which are just pairs of vertices. In these and other cases, graphs represent naked relations with any quantitative elements as add-ons. Perhaps my exposure to graph theory as a beginning graduate student is what sensitized me to this aspect of mathematics.

One may reasonably ask what relations are studied in pure mathematics—those that are later applied elsewhere. Not to give arithmetic priority over geometry, one of the earliest relations has to be sameness. In arithmetic sameness is equality, in geometry congruence. I think it is fair to say that there is little to study about these two fundamental relations, but one does need them in order to consider others. Arithmetic appears to be based on the successor relation that allows us to assign both ordinal and cardinal numbers, whichever came first historically. Both kinds of number, that is, the objects invented to carry the successor relation as far as we wish, have long been studied. The relation that connects pairs of numbers to their sums can be thought of as connecting the number of things in the combination of two counted clusters to their counts. This can be thought of as an operation on two numbers to produce their sum or as a function on the cartesian product of the numbers with themselves. The cartesian product itself is based on the relation between two things and the single thing that is the pair of them. The relation expressed by the operation of addition gives rise to the relation expressed by the operation of subtraction. In order to make subtraction work more of the time, we consider what the numbers would have to be that would allow us to subtract always. In this way, repeated *mutatis mutandis* to form fractions, real numbers, and complex numbers, the raw material of the study of the relations is expanded in a way that the raw material of the other sciences cannot be. (And yet, the extension of physics to infrared, ultraviolet, and X-radiation is not utterly divorced from what had happened earlier in mathematics.) The study of addition logically produces two quite different offspring: the further scientific consideration of addition and the technology of addition. It is not enough to be able to add two numbers a and b by counting b numbers beginning with $a + 1$. We need to be able to take a and b, written in the customary notation for numbers and produce algorithmically the customary notation for $a + b$. As notation (to think only of Europe) changed from Greek to Roman to the contemporary decimal system, the technology of addition had to be changed and improved with much thought; the addition of Greek and of Roman numbers is not so easily performed. On the other hand, if one is adding a number of cases of the same number, then one stumbles on a new relation, that expressed by the operation of multiplication—again an operation on two numbers or a function from the cartesian product to the numbers. Multiplication gives rise to the new relation of two numbers to their quotient in much the same way as addition gives rise to subtraction. It is discovered that zero misbehaves seriously in division as it did not in subtraction. Beginning with the integers, one quickly needs a new sort of number to deal with practical examples of division. And again there is the technology of performing divisions and multiplications as well as the coming to understand how they work. The relation of equality is basic to this because one is often looking for the succinct and standard number that is equal to the one for which one has a cumbersome expression: one wants the standard expression for the integer $a + b$ or $a \div b$ rather than the expressions '$a + b$' or '$a \div b$', that is, the standard expression for the integer *equal to* $a + b$ or $a \div b$.

Similar basic notions are involved in geometry. Beyond congruence, there are several basic relations. Coxeter's *Introduction to Geometry* [Coxeter 1961] elaborates an axiom system for the ternary relation 'between' on a line. Euclid does not concern himself with such things, presumably taking their behaviour to be obvious. Collinearity seems to me to be the basic ternary relation in Euclidean geometry. If one has three points, no two identical, then either they are collinear or not. If they are, then fine, the line defined by each of the three pairs is the same. (The line segments may have very different lengths, however—with relations like those in arithmetic.) If they are not collinear, then they define a triangle with sides the line segments defined by each of the pairs. As soon as one has triangles, then one can consider their congruence and then the slightly more sophisticated relations of similarity and equality of area. Congruence will obviously not apply to figures of different shapes, but equality of area may. Differently shaped polygons can be studied for their equality properties, and eventually one reaches the possible relation of areas of polygons and circles and other figures with curved boundaries (eventually leading to integral calculus). But even to say this is to call upon the relations 'boundary of' and 'same number of sides' for polygons. A small increase in the level of sophistication of the relations involved brings one from the geometry well studied in the fourth century B.C.E. to the topology only brought to light in the nineteenth century C.E. This is not to say that the relations are more complicated; they may be simpler. But simple in mathematics does not always mean easy.

I conclude this indication of what sort of relation has been studied mathematically—an indication that is just the start, since all of classical mathematics can be looked at in this way—with a word on what lies beyond arithmetic and geometry. Multiplication and addition give us linear functions, which are just the simplest examples of functions as they were historically viewed. It has taken a long time to change thinking about functions from the process view with which functional relationships originally began to the set-of-ordered-pair view that allows any functional relationship to be specified. No matter how they are specified, functions are a specific sort of relation. Their study developed into analysis with particular attention to those of use in physics. The relation between a function as values and the rates of change of those values was particularly important in this developmental process. Differentiation was not originally thought of as a relation between functions, but that view has become standard. I have already alluded to modern algebra, which is a development of the study of operations (as in addition and multiplication) applied to what are not necessarily numbers. With the development of category theory, the relation view in its function form, which I attributed above to von Neumann and Mac Lane, appears to have triumphed as the practical way to organize mathematical ideas, while the mathematical-object view of set theory still holds centre stage in foundational discussions (a competition hotly disputed).

5 Ontological Consequences

One might reasonably ask what philosophical problem the relational view offers a solution to that the object view does not. My concern is not to solve philosophical problems but rather to have philosophical problems that purport to be about mathematics actually be about what I can recognize as mathematics. Since the relational or scientific view says nothing specific about metaphysics, it does not attempt to solve metaphysical problems, although it does try to avoid unnecessary ones. Ontological arguments concern what exists, often whether mathematical objects exist[10] or even whether abstract objects in general exist with mathematical objects taken as typical abstract objects. It is not clear to me that mathematical objects are typical examples of abstract objects, but I am trying to avoid a concentration on objects.[11] A switch to relations, if taken seriously, would I am sure produce different and more relevant problems. I think that argument about the existence of relations is harder to mount than analogous argument about objects. It is not clear what it means for a relation to exist. It does not seem to require that the relata exist, since a great deal of fictional literature takes its meaning from the fact that relations in fiction are intended to be of the same types as occur in the real world. If that were not so, it would be meaningless to have a fictional child of a fictional parent without explaining what parenthood meant in the world of the fiction. While parenthood *could* be differently defined in a science-fiction world, all such relations, when verbally described, are parasitic on the ordinary relations from which we derive our vocabulary. This view is subversive of ontology as well as trying to avoid it, but it does not seem to favour either answer to the existence question for mathematical objects.

While we cannot easily and convincingly say what we mean by a relation's existing or not, something we can say about some relations is that they *cannot* relate anything; they are impossible. The obvious example is the relation describable as being not self-identical, sometimes used to define the empty set as consisting of those things that are not self-identical. The empty set, by the way, is the only mathematical object that I am almost sure exists, in some sense of that slippery word. While there are some persons that find contradiction (logical contradiction, not argumentative contradiction) interesting, mostly mathematicians prefer to avoid it at almost any cost. Russell's notorious ridicule of Meinong was based on the latter's ontological espousal of impossible objects, not just non-existent objects. Just what espousal consisted of we need not go into, my point being that mathematicians have no time for them when doing mathematics—however amusing they may find Escher's impossible drawings. Given that we wish to avoid

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[10] Such arguments are discussed by Stewart Shapiro and others in this volume.

[11] Jeremy Gray captures the connection to objects briefly in discussing implicit definitions by axioms, 'There was no attempt to show that the new, implicit, definitions somehow captured the essence of the real object, because the real object was only incidentally what it was about.' [Gray 2006, p. 390] Jean Dieudonné puts it more clearly as follows. To solve eighteenth-century problems in the nineteenth century, it was necessary to abandon 'the semi-"concrete" character of classical mathematical objects; it has to be understood that what is essential about these objects is not the particular features which they seem to have but the *relations* between them. These relations are often the same for objects which appear very different, and therefore they must be expressed in ways which do not take these appearances into account; for example; if we wish to specify a relation which can be defined either between numbers or between functions, it can only be done by introducing objects which are neither numbers nor functions, but which can be *specialized* at will as either numbers or functions, or indeed other kinds of mathematical objects. It is these "abstract" objects which are studied in what have come to be called mathematical *structures* . . . ' [Dieudonné 1992, pp. 2 f.].

impossible relations in particular and contradictions in general, what does the relational view of mathematics suggest to us? It seems to me that it suggests only that, in our study of relations by attributing them to objects made up for the purpose, we need to be careful to avoid the possibility of deducing any logical contradiction. We would like not only to avoid logical contradictions, which we might do by being careful, but also to avoid the possibility of them.[12] And to operate in the realm of no logical contradictions is to operate in the realm of the logically possible. To say that mathematics studies logical possibilities is, while certainly true for most of mathematics, no more informative than to say that we want our deductions to be logically valid because, without the relational subject matter to which I say the subject is devoted, that description would describe logic or logicism. Mathematicians explore, in a scientific spirit, relations that interest us to see how those relations are related to one another regardless of the objects that grammatically are their subject matter, any choice of which, as I have said, is an extra-mathematical enterprise called interpretation or application. A philosopher that studies the consequences of approximately this point of view is Geoffrey Hellman, who calls it [Hellman 1989] modal structuralism, a structuralism without structures. He genuinely attempts to avoid the basis in things that I regard as artificial and misleading. A different attempt to avoid things and structures is made by Charles Chihara [2008].

I have mentioned structuralism, both modal and otherwise, because non-modal structuralism is as close as most philosophy of mathematics gets to the relational view of the subject. Ordinary structuralism, which one can easily attribute to Bourbaki, has been philosophically elaborated chiefly by Michael Resnik [1997] and Stewart Shapiro [1997]. In order, it seems to me, to have objects to talk about, structuralism considers usually sets of objects and their relations as forming structures, which are then said to be the subject matter of mathematics. Mathematics is then about those structures rather than about the somehow lesser objects that compose them; how the objects are inferior to the structures other than by inclusion is unclear. Because structures are objects themselves, the usual discussion of their definition and existence is easy to launch. Structuralism does appear to be inspired by mathematicians' interest in structure rather than in a uniquely mathematical subject matter, whether objects or structures. It is close but, as is widely thought, not quite right. In criticizing structuralism, Fraser MacBride [2005] chooses as its weak link what is called the incompleteness of mathematical objects. Note the return to the objects that philosophers are happy talking about from the structures or positions or relations that structuralism tries to replace them with. I have not found in MacBride's article what seems to me the obvious criticism, namely that it is not positions in structures that have the relations we want but the hypothetical holders of those positions that have them. A set of eggs is what goes into an egg carton, not the egg carton. I do not see talk of the positions instead of their contents as satisfactory. And both of the main advocates of positions structuralism, Michael Resnik and Stewart Shapiro, use essentially this approach. I am concerned, however, to discuss very briefly the actual criticism made of both of them, which is that, as things having only the relations assumed or deduced, these structures are incomplete in the way that Hamlet is incomplete because we do not know the length of his nose or many other things about him that Shakespeare did not tell us. This incompleteness

[12] So far as I know, how to do this dependably is not known. Considering relations rooted in physical relations is one attempt. Intuitionism is another.

of fictional characters is an inevitable result of the way in which they are specified. In the next of a series of novels, the author is free to specify some feature of, say, a serial detective, that was not previously specified—even in the case of Dr Watson to specify features previously *differently* specified.[13] The freedom of fiction does not carry over to mathematics. The incompleteness of mathematical objects, the fact that we do not know everything about them, is I think a direct and harmless consequence of their role in mathematics.

When we use the definition of bachelor as unmarried adult human male, and then deduce that whatever is a bachelor has no husband, say, or wife, we are not using an incomplete object denoted by 'whatever.' We are using a pronoun, the non-fictional uses of which are all objects of whatever kind they may be, none of them incomplete. If we apply the term whatever/bachelor to Hamlet, then that bachelor is incomplete but not because we used the word we used. The incompleteness is not in the pronoun or term but in the antecedent. If we apply the pronoun or term to a real person, say, Sam Smith, then there is no incompleteness. Similarly, because our mathematical expressions are all ultimately of the form 'whatever satisfies the axioms of our system and the hypotheses of our theorem satisfies also the conclusion of our theorem', it is too hasty to claim that our whatevers are incomplete. The time for incompleteness claims is in application not in hypothetical pure mathematics. This incompleteness, by the way, is entirely distinct from the incompleteness (of mathematical systems not objects) of Gödel's famous theorems.

To summarize, the relational view does not settle any ontological problems, but it does suggest two conjectures. Ontology is less important to mathematics than contemporary philosophers often think. (I have in mind those that require an existent subject matter for worthwhile talk or even reference; one can only denote what is not blessed with existence, not refer to it.) Ontology that considers only objects and ignores relations (or regards them as non-existent or unimportant) is too simple-minded to cope with mathematics.

6 Epistemological Consequences

The philosophical aspect that the relation view is particularly useful for is epistemological—for explaining how it is that we can gain and apply mathematical knowledge. Object views have notorious difficulty with gaining mathematical knowledge because they make the subject matter of mathematics even more remote than the mathematical knowledge itself. Since we do not interact with mathematical objects, which are supposed to exist on an altogether different plane from us, we have no way to get information about them—no way to form reasonable hypotheses, and no way to disprove them if they are wrong—not even any way to see that our proofs about them are relevant to them. This is referred to as the problem of access because we have no access to timeless objects outside space. Aristotle's solution to this platonic problem was to locate mathematical objects, likewise mysteriously, 'in' physical objects. So the problem has been recognized as a problem for over two thousand years. It seems to me that the way in which the problem is to be solved is to notice what we mean by access in this context. Obviously it cannot mean sensory access to objects that are not sensible. The relational view allows a solution like Aristotle's but without the metaphorical 'in.' We have access to a relation whenever we can

[13] Conan Doyle is inconsistent in what he says about Watson.

consider objectively what stands in that relation, where 'in' is not metaphorical but is just the standard way in which we speak of x related to y by relation R. We say that x stands *in* relation R to y. Mathematically, we say xRy if the relation happens to be binary, as it need not be. And so, to consider relation R, all we need to be able to do is to talk objectively about any things at all that stand in relation R. To begin to consider marriage, all that we need is some persons that are married, not access to all married couples, past, present, and future. Because marriage is a relation among real persons, it will be difficult to say anything definitive about it, but it is not difficult to see examples of it. Its study would have to be empirical. Because mathematical relations are among mathematical objects, we are free to define them for ourselves—by consensus if we are going to communicate successfully.

We have access to the relation of successor, as used in arithmetic, both as Brouwer claimed with the passage of time and in many physical arrangements of one thing after another, where 'after' is not temporal but can be physical. To have access to this relation it is not necessary that the set of things standing in the relation all be accessible or on the other hand that the set be infinite even though in mathematics we extend the domain of the relation to infinity. Our puny physical access to this relation inspires us to imagine an infinite domain on which it is defined—by us. This is handy even though our physical means cannot even represent what is in such a domain, either on paper or electronically. In the natural sciences we likewise have access only to limited examples of the relations that are studied, some of which we assume to have infinite domains. To the infinite domain of arithmetic we have no access at all, and so our examples and counterexamples in arithmetic are often finite where we can specify and understand them. Once we have this ideal domain and become comfortable with it, we can consider relations within it other than the relation of successor. I have indicated above how we define ternary relations like sum and product, difference and quotient. Once we have these working well, in order to make them work better we further idealize to negative numbers and to rational numbers. Debit balances and portions of a pizza help us out with the *relations* involved, but they do not give us access to the negative numbers and fractions of which any mathematical discussion of such things makes use. Because we have access to ordinary things that stand in the relations among the mathematical objects, even if only approximately, we do not need access to the mathematical objects themselves. They are idealizations or reifications that it is extremely convenient to talk about, but we can get along perfectly well without access to them and—a nominalist will say—without their being in any sense real. But their reality is more irrelevant than relevantly false. Most mathematical relations arise among mathematical objects themselves. In physics we have no access to the perfect systems that physics books talk about either. I see mathematical objects as similar idealizations for the sake of how they are related, which is what the idealizations in the physics books are there for too. But it is essential that we have very clear ideas about their relations widely agreed upon so that our reasoning can be correct and be seen to be correct. We do not need to agree on their non-mathematical existence or location or when or how they were created or discovered (or which it was). These aspects are mathematically irrelevant and so mathematically neutral.

The above paragraph concerns the big leap from ordinary objects and their relations to the mathematical objects that bear the idealizations of those relations. We have access to the former and not to the latter. But we have something more powerful than mere access to the mathematical objects and relations; we decide what they shall be and stipulate that to suit ourselves (collectively, not individually). It is a smaller step to move from one set of mathematical objects and the relations among them to another set of mathematical objects and relations based on them. For example, studies in analysis from the seventeenth century to the nineteenth had produced many functions, and it was observed that functions could be added and multiplied like the values in their common ranges. This allowed the creation of altogether new structures, function spaces, based on the relations among the functions composing those spaces. Such creation is a smaller step than the move from combining sets of physical objects to the addition of their cardinal numbers. It appears to be much easier to make such intra-mathematical steps than to find altogether new physical relations to mathematize. That would appear to be why graph theory, topology, and modern algebra are so much more recent inventions than arithmetic and geometry. No one can specify in advance what might be mathematized or what cannot be. That may be because of our comparative ignorance of how it is done or because it is so creative an action that we shall never understand it.

While the above considerations go some way, I think, to clarifying the problem of access—largely by recasting it, it deals not at all with the somewhat different philosophical problem of reference. One can ask, if one takes the view that mathematical statements are made about mathematical objects, how we refer to them. Since I think that mathematical statements do not refer to specifically mathematical objects but to whatever might satisfy our axioms, the problem of reference in what might be called its platonic form does not arise. We have genuine reference to objects only in cases of applied mathematics, and there it is usually only approximate, and in informal discussion, where we are free to talk in ways that no doubt defy philosophical analysis. The same is true of the relations involved in our mathematical statements. Just as the nouns in our theorems are really pronouns standing for whatever satisfies the axioms or conventions of our theory, the relating words stand for whatever relations work in the ways specified by the theory. An example of this apparent indefiniteness that is particularly clear is plane projective geometry. One can state the axioms in terms of points lying on lines (in the plane, which itself need not be mentioned), for example, every pair of points lies on one and only one line. The obvious and intuitive content of lying on is something much like set membership because one thinks of a line as composed of the points that lie on it. But, because of the point-line duality of plane projective geometry, the word 'point' can be taken to refer to the lines in the plane, and the word 'line' can be taken to refer to the points in the plane. Then *lying on* is transferred to what intuitively one would think of as passing through, with the result that the content of the above axiom is that every pair of lines passes through one and only one point—perfectly correct. Coxeter [1955] shows that this can be done with the whole theory.[14] This example shows that there is a *whatever* aspect to the relations in mathematical theories that parallels the whatever aspect to the objects. This whatever aspect is why Russell in the quotation above[15] said 'types of relation', not just 'relations'. It was a mistake for Carnap to think that mathematics is logical syntax of language, but the mistake was not totally lacking in excuse; there is in mathematics an element of how it is possible to talk objectively and reason correctly.

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[14] This example, where it is unclear which are the sets and which are the elements, seems to me more dramatic than the mere automorphism problem of distinguishing between i and $-i$ in the complex numbers.

[15] Section 4, paragraph 5.

When we ask whether there exists a rational square root of two, we are not asking a metaphysical question. The question can be thought of as being a metaphysical one, but that is a mistake. The relational view suggests a non-metaphysical attitude that completely avoids any question about existence of rational numbers. The question is rather whether a rational number—whatever it may be—can stand in the relation that squared it equals two. Since none of the rational numbers—whatever they may be—can stand in that relation, we say that the root of two does not exist among the rationals. Only when we are happy with infinite sequences can we extend the arithmetic relations to the limits of Cauchy sequences and find that, among the Cauchy sequences, there *are* those whose limits have squares equalling two. So the root of two exists among the real numbers, but nothing was either created or found in a metaphysical sense. We simply extended our domain of discourse by making our relations apply to new materials.[16] We did not even create the Cauchy sequences; we just decided to talk about unending sequences of numbers, something we had got used to after we decided that the positive integers would be better thought of as an unending sequence. And why did we think that? Because the last positive integer would have embarrassing relations to the others.

I remarked earlier that the relational view helps with understanding the application of mathematics. This has always seemed to me one of its fundamental and obvious advantages, but the problem itself (discussed in this volume by Mark Steiner) is so little spoken of that I ought to elaborate. The simplest application of arithmetic, to make the example as simple as possible, to everyday objects requires only that they be discrete so that they can be counted rather than continuous like water. Even when what is counted *is* continuous, like time, we can agree on chunks, days for instance, that can be counted rather than measured. As soon as we recognize an order, as in the case of days, or impose an order, as in the case of pebbles, we can count them and perform arithmetic meaningfully because the relevant relations among the things are those based on order, which gave us the integers with which to do arithmetic. The relations among measurements rather than counts are more complicated, and we have created (epistemically not metaphysically) rational numbers to bear those relations to one another. This allows us to apply the arithmetic of rational numbers meaningfully to measurements. However amusing it may be to think that the hypotenuse of a $1 - 1 - \sqrt{2}$ triangle has an irrational side in mathematics, when one measures distances outside mathematics one always deals in rational numbers. Note that on this view the analogy is between relations among counts and relations among integers, and between relations among measurements and relations among rationals rather than directly between objects counted and integers or between material measured and rational numbers. One sees occasionally the questions, why should platonic mathematical objects 'apply' to ordinary physical objects, and what does it mean that they 'apply?' If one is concerned with those mathematical objects as primary, that's a very puzzling question. If one views them as being the bearers of important relations that we have decided to think about, then their application—through the relations in which they stand—is not at all mysterious.

I end with a return to disclaiming any exclusivity for the view of mathematics I have presented and discussed. In particular, I have much respect for mathematics as art. Art too is concerned with

[16] I am alluding to the standard introduction of arithmetic on equivalence classes of Cauchy sequences in which it is shown that the same arithmetic applies to ordinary numbers so that formally we replace the rationals with equivalence classes of Cauchy sequences so as to have a uniform theory.

relations but with their presentation rather than with their scientific study. You will note that I have not argued for the scientific view, lacking any premises from which to do so. I have just presented evidence and consequences. The aim is to offer a 'big picture,' as Charles Chihara puts it, a picture big enough to contain both mathematics and a context, in this case the scientific context. As a picture, its virtue is meant to be representational rather than merely aesthetic. It ought to correspond to what it represents. Like any representation, it omits much of what it represents. And non-correspondences mar it. It is not a myth, not being intended to motivate action. Let me end with a speculation. I have suggested that mathematics does not have a subject matter of things that could exercise a directing influence on our study of them by drawing themselves to our attention as matters of survival. Perhaps this is a reason why aesthetic influence is enhanced in mathematics beyond its considerable function in the natural sciences. Something has to ground our choices of topics, our choices of results to prove, our choices of proofs. What better than how attractive they are?

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