The physical limits of computation inspire an open problem that concerns decidable sets $X \subseteq \mathbb{N}$ and cannot be formalized in $ZFC$ as it refers to the current knowledge on $X$.

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Abstract. Let $f(1) = 2$, $f(2) = 4$, and let $f(n + 1) = f(n)!$ for every integer $n \geq 2$. Edmund Landau’s conjecture states that the set $\mathcal{P}_{n^2+1}$ of primes of the form $n^2 + 1$ is infinite. Landau’s conjecture implies the following unproven statement $\Phi$: $\text{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq [2, f(7)]$. Let $B$ denote the system of equations: $\{x_1 = x_2 : i, k \in \{1, \ldots, 9\} \cup \{x_2 - x_1 : i, j, k \in \{1, \ldots, 9\}\}$. We write some system $\mathcal{U} \subseteq B$ of 9 equations which has exactly two solutions in positive integers $x_1, \ldots, x_9$, namely $(1, \ldots, 1)$ and $(f(1), \ldots, f(9))$. No known system $S \subseteq B$ with a finite number of solutions in positive integers $x_1, \ldots, x_9$ has a solution $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$ satisfying $\text{max}(x_1, \ldots, x_9) > f(9)$. We write some system $\mathcal{A} \subseteq B$ of 8 equations. Let $\Lambda$ denote the statement: if the system $\mathcal{A}$ has at most finitely many solutions in positive integers $x_1, \ldots, x_9$, then each such solution $(x_1, \ldots, x_9)$ satisfies $x_1, \ldots, x_9 < f(9)$. The statement $\Lambda$ is equivalent to the statement $\Phi$. It heuristically proves the statement $\Phi$. This proof does not yield that $\text{card}(\mathcal{P}_{n^2+1}) = \omega$. Algorithms always terminate. We explain the distinction between existing algorithms (i.e. algorithms whose existence is provable in $ZFC$) and known algorithms (i.e. algorithms whose definition is constructive and currently known to us). Assuming that the infiniteness of a set $X \subseteq \mathbb{N}$ is false or unproven, we define which elements of $X$ are classified as known. No known set $X \subseteq \mathbb{N}$ satisfies conditions (1)–(4) and is widely known in number theory or naturally defined, where this term has only informal meaning. 1. A known algorithm with no input returns an integer $n$ satisfying $\text{card}(X) < \omega \Rightarrow X \subseteq (-\infty, n]$. 2. A known algorithm for every $k \in \mathbb{N}$ decides whether or not $k \in X$. 3. No known algorithm with no input returns the logical value of the statement $\text{card}(X) = \omega$. 4. There are many elements of $X$ and it is conjectured that $X$ is infinite. 5. $X$ has the simplest definition among known sets $Y \subseteq \mathbb{N}$ with the same set of known elements. Conditions (2)–(5) hold for $X = \mathcal{P}_{n^2+1}$, condition (1) holds assuming the statement $\Lambda$. Conditions (1)–(4) hold for $X = \{k \in \mathbb{N} : (10^6 < k \Rightarrow (f(10^6), f(k)) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$, condition (5) fails as the set of known elements of $X$ equals $[0, \ldots, 10^6]$. No set $X \subseteq \mathbb{N}$ will satisfy conditions (1)–(4) forever, if for every algorithm with no input, at some future day, a computer will be able to execute this algorithm in 1 second or less. The physical limits of computation disprove this assumption.

2020 Mathematics Subject Classification: 03D20.

Key words and phrases: conjecturally infinite set $X \subseteq \mathbb{N}$, constructively defined integer $n$ satisfies $\text{card}(X) < \omega \Rightarrow X \subseteq (-\infty, n]$, current knowledge on a set $X \subseteq \mathbb{N}$, distinction between existing algorithms and known algorithms, known elements of a set $X \subseteq \mathbb{N}$ whose infiniteness is false or unproven, physical limits of computation, primes of the form $n^2 + 1$, $X$ is decidable by a constructively defined algorithm.

1. Definitions and the distinction between existing algorithms and known algorithms

Let $f(1) = 2$, $f(2) = 4$, and let $f(n + 1) = f(n)!$ for every integer $n \geq 2$. Algorithms always terminate. Semi-algorithms may not terminate. Examples [1–4] and the proof of Statement [1] explain the distinction between existing algorithms (i.e. algorithms whose
existence is provable in \( ZFC \) and known algorithms (i.e. algorithms whose definition is constructive and currently known to us). A definition of an integer \( n \) is called constructive, if it provides a known algorithm with no input that returns \( n \). Definition 1 applies to sets \( X \subseteq \mathbb{N} \) whose infiniteness is false or unproven.

**Definition 1.** We say that a non-negative integer \( k \) is a known element of \( X \), if \( k \in X \) and we know an algebraic expression that defines \( k \) and consists of the following signs: 1 (one), + (addition), − (subtraction), \( \cdot \) (multiplication), \( / \) (division), ^ (exponentiation), ! (factorial), ( (left paranthesis), ) (right paranthesis).

Let \( T \) denote the set of twin primes. The known elements of the set \( \left\{ 0, 13, \frac{2^{100} - 1}{3} \right\} \bigcup \left( [[[((9!)!)!]!] + 1, \infty) \cap T \right) \) form the set \( \left\{ 0, 13, \frac{2^{100} - 1}{3} \right\} \). Let \( t \) denote the largest twin prime that is smaller than \( (((((9!)!)!)!)!)! \). The number \( t \) is an unknown element of \( T \).

**Definition 2.** Conditions (1)–(5) concern sets \( X \subseteq \mathbb{N} \).

1. A known algorithm with no input returns an integer \( n \) satisfying \( \text{card}(X) < \omega \) \( \Rightarrow \) \( X \subseteq (-\infty, n] \).
2. A known algorithm for every \( k \in \mathbb{N} \) decides whether or not \( k \in X \).
3. No known algorithm with no input returns the logical value of the statement \( \text{card}(X) = \omega \).
4. There are many elements of \( X \) and it is conjectured that \( X \) is infinite.
5. \( X \) has the simplest definition among known sets \( Y \subseteq \mathbb{N} \) with the same set of known elements.

Condition (3) implies that no known proof shows the finiteness/infiniteness of \( X \). No known set \( X \subseteq \mathbb{N} \) satisfies conditions (1)–(4) and is widely known in number theory or naturally defined, where this term has only informal meaning.

**Definition 3.** Let \( \beta = (((24!)!)!)! \).

**Lemma 1.** \( \log_2(\log_2(\log_2(\log_2(\log_2(\log_2(\log_2(\log_2(\beta)))))))) \approx 1.42298 \).

**Proof.** We ask Wolfram Alpha at [https://wolframalpha.com](https://wolframalpha.com). \( \square \)

Edmund Landau’s conjecture states that the set \( \mathcal{P}_{n^2 + 1} \) of primes of the form \( n^2 + 1 \) is infinite, see [8]–[10]. Let \( [\cdot] \) denote the integer part function.

**Example 1.** The set \( X = \mathcal{P}_{n^2 + 1} \) satisfies condition (3).

**Example 2.** The set \( X = \left\{ \begin{array}{ll} \mathbb{N}, & \text{if } \left[ \frac{\beta}{\pi} \right] \text{ is odd} \\ 0, & \text{otherwise} \end{array} \right. \) does not satisfy condition (3) because we know an algorithm with no input that computes \( \left[ \frac{\beta}{\pi} \right] \). The set of known elements of \( X \) is empty. Hence, condition (5) fails for \( X \).

**Example 3.** ([11], [5], [7] p. 9). The function
\[
\mathbb{N} \ni n \mapsto \begin{cases} 1, & \text{if the decimal expansion of } \pi \text{ contains } n \text{ consecutive zeros} \\ 0, & \text{otherwise} \end{cases}
\]
is computable because \( h = \mathbb{N} \times \{1\} \) or there exists \( k \in \mathbb{N} \) such that
\[
h = (\{0, \ldots, k\} \times \{1\}) \cup (\{k + 1, k + 2, k + 3, \ldots\} \times \{0\})
\]
No known algorithm computes the function \( h \).
The physical limits of computation inspire an open problem

Example 4. The set

\[ X = \begin{cases} \mathbb{N}, & \text{if the continuum hypothesis holds} \\ \emptyset, & \text{otherwise} \end{cases} \]

is decidable. This \( X \) satisfies conditions (1) and (3) and does not satisfy conditions (2), (4), and (5). These facts will hold forever.

Definition 4. Let \( \Phi \) denote the following unproven statement:

\[ \text{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq [2,\beta] \]

Landau’s conjecture implies the statement \( \Phi \). Theorem 6 heuristically justifies the statement \( \Phi \). This proof does not yield that \( \text{card}(\mathcal{P}_{n^2+1}) = \omega \).

Statement 1. Condition (1) remains unproven for \( X = \mathcal{P}_{n^2+1} \).

Proof. For every set \( X \subseteq \mathbb{N} \), there exists an algorithm \( \text{Alg}(X) \) with no input that returns

\[ n = \begin{cases} 0, & \text{if card}(X) \in \{0, \omega\} \\ \max(X), & \text{otherwise} \end{cases} \]

This \( n \) satisfies the implication in condition (1), but the algorithm \( \text{Alg}(\mathcal{P}_{n^2+1}) \) is unknown for us because its definition is ineffective. □

Proving the statement \( \Phi \) will disprove Statement 1. Statement 1 cannot be formalized in ZFC because it refers to the current mathematical knowledge. The same is true for Open Problem 1 and Statements 2 and 3.

Definition 5. We say that an integer \( n \) is a threshold number of a set \( X \subseteq \mathbb{N} \), if

\[ \text{card}(X) < \omega \Rightarrow X \subseteq (-\infty, n] \]

If a set \( X \subseteq \mathbb{N} \) is empty or infinite, then any integer \( n \) is a threshold number of \( X \). If a set \( X \subseteq \mathbb{N} \) is non-empty and finite, then the all threshold numbers of \( X \) form the set \( [\max(X), \infty) \cap \mathbb{N} \).

2. The physical limits of computation inspire Open Problem

Open Problem 1. Is there a set \( X \subseteq \mathbb{N} \) which satisfies conditions (1)–(5)?

Open Problem 1 asks: Are there a set \( X \subseteq \mathbb{N} \) and a constructively defined integer \( n \) such that \( \text{card}(X) < \omega \Rightarrow X \subseteq (-\infty, n) \) \( \land \) \( X \) is decidable by a constructively defined algorithm \( \land \) (there are many elements of \( X \)) \( \land \) (the infiniteness of \( X \) is conjectured and cannot be decided by any known method) \( \land \) (\( X \) has the simplest definition among known sets \( Y \subseteq \mathbb{N} \) with the same set of known elements)?

Statement 2. The set

\[ X = \{ k \in \mathbb{N} : (10^6 < k) \Rightarrow (f(10^6), f(k)) \cap \mathcal{P}_{n^2+1} \neq \emptyset \} \]

satisfies conditions (1)–(4). Condition (5) fails for \( X \).

Proof. Condition (4) holds as \( X \supseteq \{0, \ldots, 10^6\} \) and the set \( \mathcal{P}_{n^2+1} \) is conjecturally infinite. By Lemma 1 due to known physics we are not able to confirm by a direct computation that some element of \( \mathcal{P}_{n^2+1} \) is greater than \( f(10^6) > f(7) = \beta \), see [4]. Thus condition (3) holds. Condition (2) holds trivially. Since the set

\[ \{ k \in \mathbb{N} : (10^6 < k) \land (f(10^6), f(k)) \cap \mathcal{P}_{n^2+1} \neq \emptyset \} \]

is empty or infinite, the integer \( 10^6 \) is a threshold number of \( X \). Thus \( X \) satisfies condition (1). Condition (5) fails for \( X \) as the set of known elements of \( X \) equals \( \{0, \ldots, 10^6\} \). □
Proving Landau’s conjecture will disprove Statement 2.

**Theorem 1.** No set $X \subseteq \mathbb{N}$ will satisfy conditions (1)-(4) forever, if for every algorithm with no input, at some future day, a computer will be able to execute this algorithm in 1 second or less.

*Proof.* The proof goes by contradiction. We fix an integer $n$ that satisfies condition (1).

Since conditions (1)-(3) will hold forever, the semi-algorithm in Figure 1 never terminates and sequentially prints the following sentences:

$$(T) \quad n+1 \notin X, \ n+2 \notin X, \ n+3 \notin X, \ldots$$

![Fig. 1] Semi-algorithm that terminates if and only if $X$ is infinite

The sentences from the sequence (T) and our assumption imply that for every integer $m > n$ computed by a known algorithm, at some future day, a computer will be able to confirm in 1 second or less that $(n,m] \cap X = \emptyset$. Thus, at some future day, numerical evidence will support the conjecture that the set $X$ is finite, contrary to the conjecture in condition (4).

□

The physical limits of computation ([4]) disprove the assumption of Theorem 1.

3. Number-theoretic statements $\Psi_n$

Let $\mathcal{U}_1$ denote the system of equations which consists of the equation $x_1! = x_1$. For an integer $n \geq 2$, let $\mathcal{U}_n$ denote the following system of equations:

$$\begin{cases} 
    x_1! = x_1 \\
    x_1 \cdot x_1 = x_2 \\
    \forall i \in \{2, \ldots, n-1\} \ x_i! = x_{i+1} 
\end{cases}$$

The diagram in Figure 2 illustrates the construction of the system $\mathcal{U}_n$.

![Fig. 2] Construction of the system $\mathcal{U}_n
Lemma 2. For every positive integer \( n \), the system \( U_n \) has exactly two solutions in positive integers, namely \((1, \ldots, 1)\) and \((f(1), \ldots, f(n))\).

Let \( B_n \) denote the following system of equations:
\[
\left\{ x_i! = x_k : i, k \in \{1, \ldots, n\} \right\} \cup \left\{ x_i \cdot x_j = x_k : i, j, k \in \{1, \ldots, n\} \right\}
\]
For every positive integer \( n \), no known system \( S \subseteq B_n \) with a finite number of solutions in positive integers \( x_1, \ldots, x_n \) has a solution \((x_1, \ldots, x_n)\) satisfying \( \max(x_1, \ldots, x_n) > f(n) \). For a positive integer \( n \), let \( \Psi_n \) denote the following statement: if a system \( S \subseteq B_n \) has at most finitely many solutions in positive integers \( x_1, \ldots, x_n \), then each such solution \((x_1, \ldots, x_n)\) satisfies \( x_1, \ldots, x_n \leq f(n) \). The statement \( \Psi_n \) says that for subsystems of \( B_n \) with a finite number of solutions, the largest known solution is indeed the largest possible. The statements \( \Psi_1 \) and \( \Psi_2 \) hold trivially. There is no reason to assume the validity of the statement \( \forall n \in \mathbb{N} \setminus \{0\} \Psi_n \).

Theorem 2. For every statement \( \Psi_n \), the bound \( f(n) \) cannot be decreased.

Proof. It follows from Lemma 2 because \( U_n \subseteq B_n \). \( \square \)

Theorem 3. For every integer \( n \geq 2 \), the statement \( \Psi_{n+1} \) implies the statement \( \Psi_n \).

Proof. If a system \( S \subseteq B_n \) has at most finitely many solutions in positive integers \( x_1, \ldots, x_n \), then for every integer \( i \in \{1, \ldots, n\} \) the system \( S \cup \{x_i! = x_{n+1}\} \) has at most finitely many solutions in positive integers \( x_1, \ldots, x_{n+1} \). The statement \( \Psi_{n+1} \) implies that \( x_i! = x_{n+1} \leq f(n+1) = f(n)! \). Hence, \( x_i \leq f(n) \). \( \square \)

Theorem 4. Every statement \( \Psi_n \) is true with an unknown integer bound that depends on \( n \).

Proof. For every positive integer \( n \), the system \( B_n \) has a finite number of subsystems. \( \square \)

4. A conjectural solution to Open Problem [1]

Lemma 3. For every positive integers \( x \) and \( y \), \( x! \cdot y = y! \) if and only if \( (x + 1 = y) \lor (x = y = 1) \)

Lemma 4. (Wilson’s theorem, [2, p. 89]). For every integer \( x \geq 2 \), \( x \) is prime if and only if \( x \) divides \( (x - 1)! + 1 \).

Let \( \mathcal{A} \) denote the following system of equations:
\[
\begin{align*}
x_2! &= x_3 \\
x_3! &= x_4 \\
x_5! &= x_6 \\
x_8! &= x_9 \\
x_1 \cdot x_1 &= x_2 \\
x_3 \cdot x_5 &= x_6 \\
x_4 \cdot x_8 &= x_9 \\
x_5 \cdot x_7 &= x_8
\end{align*}
\]

Lemma 4 and the diagram in Figure 3 explain the construction of the system \( \mathcal{A} \).
Lemma 5. For every integer $x_1 \geq 2$, the system $\mathcal{A}$ is solvable in positive integers $x_2, \ldots, x_9$ if and only if $x_1^2 + 1$ is prime. In this case, the integers $x_2, \ldots, x_9$ are uniquely determined by the following equalities:

\[
\begin{align*}
x_2 &= x_1^2 \\
x_3 &= (x_1^2)! \\
x_4 &= ((x_1^2))! \\
x_5 &= x_2 + 1 \\
x_6 &= (x_2^2 + 1)! \\
x_7 &= (x_5)! + 1 \\
x_8 &= (x_7)! + 1 \\
x_9 &= (((x_1^2))! + 1)!
\end{align*}
\]

Proof. By Lemma 3 for every integer $x_1 \geq 2$, the system $\mathcal{A}$ is solvable in positive integers $x_2, \ldots, x_9$ if and only if $x_1^2 + 1$ divides $(x_1^2)! + 1$. Hence, the claim of Lemma 5 follows from Lemma 4. \hfill \Box

Lemma 6. There are only finitely many tuples $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$, which solve the system $\mathcal{A}$ and satisfy $x_1 = 1$. This is true as every such tuple $(x_1, \ldots, x_9)$ satisfies $x_1, \ldots, x_9 \in \{1, 2\}$.

Proof. The equality $x_1 = 1$ implies that $x_2 = x_1 \cdot x_1 = 1$. Hence, $x_3 = x_2! = 1$. Therefore, $x_4 = x_3! = 1$. The equalities $x_5! = x_6$ and $x_6 = 1 \cdot x_5 = x_3 \cdot x_5 = x_6$ imply that $x_5, x_6 \in \{1, 2\}$. The equalities $x_8! = x_9$ and $x_8 = 1 \cdot x_8 = x_4 \cdot x_8 = x_9$ imply that $x_8, x_9 \in \{1, 2\}$. The equality $x_5 \cdot x_7 = x_8$ implies that $x_7 = \frac{x_8}{x_5} \in \left\{ \frac{1}{1}, \frac{1}{2}, \frac{1}{1}, \frac{1}{2} \right\} \cap \mathbb{N} = \{1, 2\}$. \hfill \Box
Conjecture 1. The statement \( \Psi_9 \) is true when is restricted to the system \( \mathcal{A} \).

Theorem 5. Conjecture 1 proves the following implication: if there exists an integer \( x_1 \geq 2 \) such that \( x_1^2 + 1 \) is prime and greater than \( f(7) \), then the set \( \mathcal{P}_{n^2+1} \) is infinite.

Proof. Suppose that the antecedent holds. By Lemma 5 there exists a unique tuple \((x_2, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^8\) such that the tuple \((x_1, x_2, \ldots, x_9)\) solves the system \( \mathcal{A} \). Since \( x_1^2 + 1 > f(7) \), we obtain that \( x_1^2 > f(7) \). Hence, \( (x_1^2)! > f(7)! = f(9) \). Consequently, \( x_9 = ((x_1^2)! + 1)! > (f(8) + 1)! > f(8)! = f(9) \). Conjecture 1 and the inequality \( x_9 > f(9) \) imply that the system \( \mathcal{A} \) has infinitely many solutions \((x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9\). According to Lemmas 5 and 6, the set \( \mathcal{P}_{n^2+1} \) is infinite.

Theorem 6. Conjecture 1 implies the statement \( \Phi \).

Proof. It follows from Theorem 5 and the equality \( f(7) = (((24!)!)!)! \).

Theorem 7. The statement \( \Phi \) implies Conjecture 1.

Proof. By Lemmas 5 and 6 if positive integers \( x_1, \ldots, x_9 \) solve the system \( \mathcal{A} \), then 
\[
(x_1 \geq 2) \land (x_5 = x_1^2 + 1) \land (x_5 \text{ is prime})
\]
or \( x_1, \ldots, x_9 \in \{1, 2\} \). In the first case, Lemma 5 and the statement \( \Phi \) imply that the inequality \( x_5 \leq (((24!)!)!)! = f(7) \) holds when the system \( \mathcal{A} \) has at most finitely many solutions in positive integers \( x_1, \ldots, x_9 \). Hence, \( x_2 = x_5 - 1 < f(7) \) and \( x_3 = x_5! < f(7)! = f(8) \). Continuing this reasoning in the same manner, we can show that every \( x_i \) does not exceed \( f(9) \).

Statement 3. Conditions (2) – (5) hold for \( X = \mathcal{P}_{n^2+1} \), condition (1) holds assuming the statement \( \Phi \).

Proof. The set \( \mathcal{P}_{n^2+1} \) is conjecturally infinite. There are 2199894223892 primes of the form \( n^2 + 1 \) in the interval \([2, 10^{28}]\), see [9]. These two facts imply condition (4). By Lemma 1 due to known physics we are not able to confirm by a direct computation that some element of \( \mathcal{P}_{n^2+1} \) is greater than \( f(7) = (((24!)!)!)! = \beta \), see [4]. Thus condition (3) holds. Conditions (2) and (5) hold trivially. The statement \( \Phi \) implies that \( \beta \) is a threshold number of \( \mathcal{P}_{n^2+1} \). Hence, the statement \( \Phi \) implies condition (1) for \( X = \mathcal{P}_{n^2+1} \).

Proving Landau’s conjecture will disprove Statement 3.

5. Open Problems

The set \( X = \mathcal{P}_{n^2+1} \) satisfies the conjunction
\[
\neg(\text{Condition 1}) \land (\text{Condition 2}) \land (\text{Condition 3}) \land (\text{Condition 4}) \land (\text{Condition 5})
\]
The set \( X = \{0, \ldots, f(7)\} \cup \mathcal{P}_{n^2+1} \) satisfies the conjunction
\[
\neg(\text{Condition 1}) \land (\text{Condition 2}) \land (\text{Condition 3}) \land (\text{Condition 4}) \land \neg(\text{Condition 5})
\]
The set \( X = \begin{cases} \mathbb{N}, & \text{if } (f(9^8), f(9^9)) \cap \mathcal{P}_{n^2+1} \neq \emptyset \\ \{0, \ldots, 10^9\}, & \text{otherwise} \end{cases} \) satisfies the conjunction
\[
(\text{Condition 1}) \land (\text{Condition 2}) \land \neg(\text{Condition 3}) \land (\text{Condition 4}) \land \neg(\text{Condition 5})
\]

Open Problem 2. Is there a set \( X \subseteq \mathbb{N} \) that satisfies the conjunction
\[
(\text{Condition 1}) \land (\text{Condition 2}) \land \neg(\text{Condition 3}) \land (\text{Condition 4}) \land (\text{Condition 5})?
\]
The numbers $2^{2^k} + 1$ are prime for $k \in \{0, 1, 2, 3, 4\}$. It is open whether or not there are infinitely many primes of the form $2^{2^k} + 1$, see [6, p. 74]. It is open whether or not there are infinitely many composite numbers of the form $2^{2^k} + 1$, see [6, p. 74]. Most mathematicians believe that $2^{2^k} + 1$ is composite for every integer $k \geq 5$, see [3, p. 23].

The set

$$X = \begin{cases} \mathbb{N}, & \text{if } (f(9^8), f(9^9)) \cap P_{n^2+1} \neq \emptyset \\ \{0, \ldots, 10^6\} \cup \{n \in \mathbb{N} : n \text{ is the sixth prime number of the form } 2^{2^k} + 1\}, & \text{otherwise} \end{cases}$$

satisfies the conjunction

$$\neg(\text{Condition 1}) \land (\text{Condition 2}) \land \neg(\text{Condition 3}) \land (\text{Condition 4}) \land \neg(\text{Condition 5})$$

**Open Problem 3.** Is there a set $X \subseteq \mathbb{N}$ that satisfies the conjunction

$$\neg(\text{Condition 1}) \land (\text{Condition 2}) \land \neg(\text{Condition 3}) \land (\text{Condition 4}) \land (\text{Condition 5})?$$

It is possible, although very doubtful, that at some future day, the set $X = P_{n^2+1}$ will solve Open Problem 2. The same is true for Open Problem 3. It is possible, although very doubtful, that at some future day, the set $X = \{k \in \mathbb{N} : 2^{2^k} + 1 \text{ is composite}\}$ will solve Open Problem 1. The same is true for Open Problems 2 and 3.

**Acknowledgement.** Agnieszka Kozdęba prepared three diagrams. Apoloniusz Tyszka wrote the article.

**References**


