Are there a set $X \subseteq \mathbb{N}$ and a constructively defined integer $n$ such that 
$(\text{card}(X) < \omega \Rightarrow X \subseteq (-\infty, n]) \land (a \text{ constructively defined algorithm decides } X \land \text{there are many elements of } X \land (\text{the infiniteness of } X \text{ is conjectured and cannot be decided by any known method}) \land (X \text{ has the simplest definition among known sets } Y \subseteq \mathbb{N} \text{ with the same set of known elements})$?

AGNIESZKA KOZDĘBA and APOLONIUSZ TYSZKA

ABSTRACT. Let $f(1) = 2$, $f(2) = 4$, and let $f(n + 1) = f(n)!$ for every integer $n \geq 2$. Edmund Landau’s conjecture states that the set $\mathcal{P}_{n^2+1}$ of primes of the form $n^2 + 1$ is infinite. Landau’s conjecture implies the following unproven statement $\Phi$: 
$\text{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq [2, f(7)]$. Let $B$ denote the system of equations:
$\{x_i! = x_k : i, k \in \{1, \ldots, 9\}\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \ldots, 9\}\}$. We write down a system $\mathcal{U} \subseteq B$ of 9 equations which has exactly two solutions in positive integers $x_1, \ldots, x_9$, namely $(1, \ldots, 1)$ and $(f(1), \ldots, f(9))$. We write down a system $\mathcal{A} \subseteq B$ of 8 equations.

Let $\Lambda$ denote the statement: if the system $\mathcal{A}$ has at most finitely many solutions in positive integers $x_1, \ldots, x_9$, then each such solution $(x_1, \ldots, x_9)$ satisfies $x_1, \ldots, x_9 \leq f(9)$. The statement $\Lambda$ is equivalent to the statement $\Phi$. It heuristically proves the statement $\Phi$. This proof does not yield that $\text{card}(\mathcal{P}_{n^2+1}) = \omega$. Let $\mathcal{F}(X)$ denote the conjunction of the first three conditions from the title. The set $X = \mathcal{P}_{n^2+1}$ satisfies the formula $\mathcal{F}(X)$.

2020 Mathematics Subject Classification: 03D20.

Key words and phrases: constructively defined integer $n$ satisfies $\text{card}(X) < \omega \Rightarrow X \subseteq (-\infty, n]$, constructively defined algorithm, current knowledge on a set $X \subseteq \mathbb{N}$, decidable and conjecturally infinite set $X \subseteq \mathbb{N}$, distinction between existing algorithms and known algorithms, known elements of a set $X \subseteq \mathbb{N}$, physical limits of computation, primes of the form $n^2 + 1$. 

1
1. Definitions and the distinction between existing algorithms and known algorithms

Algorithms always terminate. Semi-algorithms may not terminate. Examples 1–4 and the proof of Statement 1 explain the distinction between existing algorithms (i.e. algorithms whose existence is provable in ZFC) and known algorithms (i.e. algorithms whose definition is constructive and currently known to us). A definition of an integer $n$ is called constructive, if it provides a known algorithm with no input that returns $n$.

**Definition 1.** Conditions (1)–(5) concern sets $X \subseteq \mathbb{N}$.

1. There are many elements of $X$ and it is conjectured that $X$ is infinite.
2. No known algorithm with no input returns the logical value of the statement $\text{card}(X) = \omega$.
3. A known algorithm for every $k \in \mathbb{N}$ decides whether or not $k \in X$.
4. A known algorithm with no input returns an integer $n$ satisfying $\text{card}(X) < \omega \Rightarrow X \subseteq (-\infty, n]$.
5. $X$ is naturally defined i.e. $X$ has the simplest definition among known sets $Y \subseteq \mathbb{N}$ with the same set of known elements.

Condition (2) implies that no known proof shows the finiteness/infiniteness of $X$.

**Definition 2.** Let $\beta = (((24!)!)!)!$.

**Lemma 1.** $\log_2(\log_2(\log_2(\log_2(\log_2(\log_2(\log_2(\beta))))))) \approx 1.42298$.

**Proof.** We ask Wolfram Alpha at [http://wolframalpha.com](http://wolframalpha.com) □

Edmund Landau’s conjecture states that the set $\mathcal{P}_{n^2+1}$ of primes of the form $n^2 + 1$ is infinite, see [6]–[8]. Let $[\cdot]$ denote the integer part function.

**Example 1.** The set $X = \mathcal{P}_{n^2+1}$ satisfies condition (2).

**Example 2.** The set $X = \begin{cases} \mathbb{N}, & \text{if } [\frac{\pi}{2}] \text{ is odd} \\ 0, & \text{otherwise} \end{cases}$ does not satisfy condition (2) because we know an algorithm with no input that computes $[\frac{\pi}{2}]$.

**Example 3.** ([1], [4], [5, p. 9]). The function

$$\mathbb{N} \ni n \rightarrow \begin{cases} 1, & \text{if the decimal expansion of } \pi \text{ contains } n \text{ consecutive zeros} \\ 0, & \text{otherwise} \end{cases}$$

is computable because $h = \mathbb{N} \times \{1\}$ or there exists $k \in \mathbb{N}$ such that

$$h = ((0, \ldots, k) \times \{1\}) \cup ((k + 1, k + 2, k + 3, \ldots) \times \{0\})$$

No known algorithm computes the function $h$.

**Example 4.** The set

$$X = \begin{cases} \mathbb{N}, & \text{if the continuum hypothesis is true} \\ 0, & \text{otherwise} \end{cases}$$

is decidable. No constructively existing algorithm decides $X$, which holds forever.
Definition 3. Let $\Phi$ denote the following unproven statement:
$$\text{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq [2, \beta]$$

Landau’s conjecture implies the statement $\Phi$. In Section 4, we heuristically prove the statement $\Phi$. This proof does not yield that $\text{card}(\mathcal{P}_{n^2+1}) = \omega$.

Statement 1. Condition (4) remains unproven for $X = \mathcal{P}_{n^2+1}$.

Proof. For every set $X \subseteq \mathbb{N}$, there exists an algorithm $\text{Alg}(X)$ with no input that returns
$$n = \begin{cases} 0, & \text{if } \text{card}(X) \in \{0, \omega\} \\ \max(X), & \text{otherwise} \end{cases}$$

This $n$ satisfies the implication in condition (4), but the algorithm $\text{Alg}(\mathcal{P}_{n^2+1})$ is unknown for us because its definition is ineffective.

Proving the statement $\Phi$ will disprove Statement 1. Statement 1 cannot be formalized in mathematics because it refers to the current mathematical knowledge. The same is true for Open Problem 1 and Statements 2 and 3.

Definition 4. We say that an integer $n$ is a threshold number of a set $X \subseteq \mathbb{N}$, if
$$\text{card}(X) < \omega \Rightarrow X \subseteq [2, n].$$

If a set $X \subseteq \mathbb{N}$ is empty or infinite, then any integer $n$ is a threshold number of $X$. If a set $X \subseteq \mathbb{N}$ is non-empty and finite, then the all threshold numbers of $X$ form the set $[\max(X), \infty) \cap \mathbb{N}$.

2. The physical limits of computation inspire Open Problem 1.

Open Problem 1. Is there a set $X \subseteq \mathbb{N}$ which satisfies conditions (1)–(5)?

Open Problem 1 asks: Are there a set $X \subseteq \mathbb{N}$ and a constructively defined integer $n$ such that (card($X$) < $\omega$ ⇒ $X \subseteq (-\infty, n]$) ∧ (a constructively defined algorithm decides $X$ and there are many elements of $X$) ∧ (the infiniteness of $X$ is conjectured and cannot be decided by any known method) ∧ ($X$ has the simplest definition among known sets $Y \subseteq \mathbb{N}$ with the same set of known elements)?

Statement 2. The set
$$X = \{k \in \mathbb{N} : (\beta < k) \Rightarrow (\beta, k) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$$

satisfies conditions (1)–(4). Condition (5) fails for $X$.

Proof. Condition (1) holds as $X \supseteq [0, \ldots, \beta]$ and the set $\mathcal{P}_{n^2+1}$ is conjecturally infinite. By Lemma 1, due to known physics we are not able to confirm by a direct computation that some element of $\mathcal{P}_{n^2+1}$ is greater than $\beta$, see [3]. Thus condition (2) holds. Condition (3) holds trivially. Since the set
$$\{k \in \mathbb{N} : (\beta < k) \wedge (\beta, k) \cap \mathcal{P}_{n^2+1} \neq \emptyset\}$$
is empty or infinite, the integer $\beta$ is a threshold number of $X$. Thus $X$ satisfies condition (4). Condition (5) fails for $X$ as the set of known elements of $X$ equals $\{0, \ldots, \beta\}$.

Proving Landau's conjecture will disprove Statement 2.
Theorem 1. No set $X \subseteq \mathbb{N}$ will satisfy conditions (1)-(4) forever, if for every algorithm with no input, at some future day, a computer will be able to execute this algorithm in 1 second or less.

Proof. The proof goes by contradiction. We fix an integer $n$ that satisfies condition (4). Since conditions (2)-(4) will hold forever, the semi-algorithm in Figure 1 never terminates and sequentially prints the following sentences:

(T) $n + 1 \notin X, n + 2 \notin X, n + 3 \notin X, \ldots$

Fig. 1 Semi-algorithm that terminates if and only if the set $X$ is infinite

The sentences from the sequence (T) and our assumption imply that for every integer $m > n$ computed by a known algorithm, at some future day, a computer will be able to confirm in 1 second or less that $(n, m] \cap X = \emptyset$. Thus, at some future day, numerical evidence will support the conjecture that the set $X$ is finite, contrary to the conjecture in condition (1). \hfill \Box

The physical limits of computation (3) disprove the assumption of Theorem 1.

3. Number-theoretic statements $\Psi_n$

Let $f(1) = 2$, $f(2) = 4$, and let $f(n + 1) = f(n)!$ for every integer $n \geq 2$. Let $\mathcal{U}_1$ denote the system of equations which consists of the equation $x_1! = x_1$. For an integer $n \geq 2$, let $\mathcal{U}_n$ denote the following system of equations:

$$
\begin{align*}
\forall i & \in \{2, \ldots, n-1\} \quad x_i! = x_{i+1}, \\
x_1! & = x_1, \\
x_1 \cdot x_1 & = x_2 \\
x_2 & = x_3 \quad \ldots \\
x_{n-1}! & = x_n
\end{align*}
$$

The diagram in Figure 2 illustrates the construction of the system $\mathcal{U}_n$.

Fig. 2 Construction of the system $\mathcal{U}_n$

Lemma 2. For every positive integer $n$, the system $\mathcal{U}_n$ has exactly two solutions in positive integers, namely $(1, \ldots, 1)$ and $(f(1), \ldots, f(n))$. 
Let \( B_n \) denote the following system of equations:

\[
\{ x! = x_k : i, k \in \{1, \ldots, n\} \} \cup \{ x_i \cdot x_j = x_k : i, j, k \in \{1, \ldots, n\} \}
\]

For a positive integer \( n \), let \( \Psi_n \) denote the following statement: if a system of equations \( S \subseteq B_n \) has at most finitely many solutions in positive integers \( x_1, \ldots, x_n \), then each such solution \( (x_1, \ldots, x_n) \) satisfies \( x_1, \ldots, x_n \leq f(n) \). The statement \( \Psi_n \) says that for subsystems of \( B_n \) with a finite number of solutions, the largest known solution is indeed the largest possible. The statements \( \Psi_1 \) and \( \Psi_2 \) hold trivially. There is no reason to assume the validity of the statement \( \forall n \in \mathbb{N} \setminus \{0\} \Psi_n \).

**Theorem 2.** For every statement \( \Psi_n \), the bound \( f(n) \) cannot be decreased.

**Proof.** It follows from Lemma 2 because \( \mathcal{U}_n \subseteq B_n \). \( \square \)

**Theorem 3.** For every integer \( n \geq 2 \), the statement \( \Psi_{n+1} \) implies the statement \( \Psi_n \).

**Proof.** If a system \( S \subseteq B_n \) has at most finitely many solutions in positive integers \( x_1, \ldots, x_n \), then for every integer \( i \in \{1, \ldots, n\} \) the system \( S \cup \{ x_i! = x_{n+1} \} \) has at most finitely many solutions in positive integers \( x_1, \ldots, x_{n+1} \). The statement \( \Psi_{n+1} \) implies that \( x_i! = x_{n+1} \leq f(n + 1) = f(n)! \). Hence, \( x_i \leq f(n) \). \( \square \)

**Theorem 4.** Every statement \( \Psi_n \) is true with an unknown integer bound that depends on \( n \).

**Proof.** For every positive integer \( n \), the system \( B_n \) has a finite number of subsystems. \( \square \)

4. A conjectural solution to Open Problem [1]

**Lemma 3.** For every positive integers \( x \) and \( y \), \( x! \cdot y! = y! \) if and only if

\[
(x + 1 = y) \lor (x = y = 1)
\]

**Lemma 4.** (Wilson’s theorem, [2] p. 89). For every integer \( x \geq 2 \), \( x \) is prime if and only if \( x \) divides \( (x - 1)! + 1 \).

Let \( \mathcal{A} \) denote the following system of equations:

\[
\begin{align*}
x_2! &= x_3 \\
x_3! &= x_4 \\
x_5! &= x_6 \\
x_8! &= x_9 \\
x_1 \cdot x_1 &= x_2 \\
x_3 \cdot x_5 &= x_6 \\
x_4 \cdot x_8 &= x_9 \\
x_5 \cdot x_7 &= x_8
\end{align*}
\]

**Lemma 3** and the diagram in Figure 3 explain the construction of the system \( \mathcal{A} \).
Fig. 3  Construction of the system $\mathcal{A}$

**Lemma 5.** For every integer $x_1 \geq 2$, the system $\mathcal{A}$ is solvable in positive integers $x_2, \ldots, x_9$ if and only if $x_1^2 + 1$ is prime. In this case, the integers $x_2, \ldots, x_9$ are uniquely determined by the following equalities:

\[
\begin{align*}
    x_2 &= x_1^2 \\
    x_3 &= (x_1^2)! \\
    x_4 &= ((x_1^2)!)! \\
    x_5 &= x_1^2 + 1 \\
    x_6 &= (x_1^2 + 1)! \\
    x_7 &= (x_1^2)! + 1 \\
    x_8 &= (x_1^2)! + 1 \\
    x_9 &= (((x_1^2)! + 1)!)
\end{align*}
\]

**Proof.** By Lemma 3, for every integer $x_1 \geq 2$, the system $\mathcal{A}$ is solvable in positive integers $x_2, \ldots, x_9$ if and only if $x_1^2 + 1$ divides $(x_1^2)! + 1$. Hence, the claim of Lemma 5 follows from Lemma 4. \qed

**Lemma 6.** There are only finitely many tuples $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$, which solve the system $\mathcal{A}$ and satisfy $x_1 = 1$. This is true as every such tuple $(x_1, \ldots, x_9)$ satisfies $x_1, \ldots, x_9 \in \{1, 2\}$.

**Proof.** The equality $x_1 = 1$ implies that $x_2 = x_1 \cdot x_1 = 1$. Hence, $x_3 = x_2! = 1$. Therefore, $x_4 = x_3! = 1$. The equalities $x_5! = x_6$ and $x_5 = 1 \cdot x_5 = x_3 \cdot x_3 = x_5$ imply that $x_5, x_6 \in \{1, 2\}$. The equalities $x_8! = x_9$ and $x_8 = 1 \cdot x_8 = x_4 \cdot x_8 = x_8$ imply that $x_8, x_9 \in \{1, 2\}$. The equality $x_5 \cdot x_7 = x_8$ implies that $x_7 = \frac{x_8}{x_5} \in \{\frac{1}{2}, 1, \frac{3}{2}\} \cap \mathbb{N} = \{1, 2\}$. \qed
Conjecture 1. The statement \( \Psi_9 \) is true when is restricted to the system \( \mathcal{A} \).

Theorem 5. Conjecture [1] proves the following implication: if there exists an integer \( x_1 \geq 2 \) such that \( x_1^2 + 1 \) is prime and greater than \( f(7) \), then the set \( \mathcal{P}_{n^2+1} \) is infinite.

Proof. Suppose that the antecedent holds. By Lemma [5], there exists a unique tuple \((x_2, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^8\) such that the tuple \((x_1, x_2, \ldots, x_9)\) solves the system \( \mathcal{A} \). Since \( x_1^2 + 1 > f(7) \), we obtain that \( x_1^2 \geq f(7) \). Hence, \( (x_1^2)! \geq f(7)! = f(8) \). Consequently, \( x_9 = ((x_1^2)! + 1) \geq (f(8) + 1)! > f(8)! = f(9) \), Conjecture [1] and the inequality \( x_9 > f(9) \) imply that the system \( \mathcal{A} \) has infinitely many solutions \((x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9\). According to Lemmas [5] and [6], the set \( \mathcal{P}_{n^2+1} \) is infinite. □

Theorem 6. Conjecture [7] implies the statement \( \Phi \).

Proof. It follows from Theorem 5 and the equality \( f(7) = (((24!)!)!)) \). □

Theorem 7. The statement \( \Phi \) implies Conjecture [7]

Proof. By Lemmas [5] and [6] if positive integers \( x_1, \ldots, x_9 \) solve the system \( \mathcal{A} \), then

\[
(x_1 \geq 2) \land (x_5 = x_1^2 + 1) \land (x_5 \text{ is prime})
\]

or \( x_1, \ldots, x_9 \in \{1, 2\} \). In the first case, Lemma [5] and the statement \( \Phi \) imply that the inequality \( x_5 \leq (((24!)!)!)! = f(7) \) holds when the system \( \mathcal{A} \) has at most finitely many solutions in positive integers \( x_1, \ldots, x_9 \). Hence, \( x_2 = x_5 - 1 < f(7) \) and \( x_3 = x_2! < f(7)! = f(8) \). Continuing this reasoning in the same manner, we can show that every \( x_i \) does not exceed \( f(9) \). □

Statement 3. Conditions (1)–(3) and (5) hold for \( X = \mathcal{P}_{n^2+1} \). The statement \( \Phi \) implies that condition (4) holds for \( X = \mathcal{P}_{n^2+1} \).

Proof. The set \( \mathcal{P}_{n^2+1} \) is conjecturally infinite. There are 2199894223892 primes of the form \( n^2 + 1 \) in the interval \([2, 10^{28}]\), see [7]. These two facts imply condition (1). By Lemma [1], due to known physics we are not able to confirm by a direct computation that some element of \( \mathcal{P}_{n^2+1} \) is greater than \( f(7) = (((24!)!)!)! = \beta \), see [3]. Thus condition (2) holds. Conditions (3) and (5) hold trivially. The statement \( \Phi \) implies that \( \beta \) is a threshold number of \( \mathcal{P}_{n^2+1} \). Hence, the statement \( \Phi \) implies that condition (4) holds for \( X = \mathcal{P}_{n^2+1} \). □

Acknowledgement. Agnieszka Kozdęba prepared three diagrams. Apoloniusz Tyszka wrote the article.
References


Agnieszka Kozdeba
Faculty of Environmental Engineering and Land Surveying
Hugo Kołłątaj University
Balicka 253C, 30-198 Kraków, Poland
Institute of Mathematics
Jagiellonian University
Łojasiewicza 6, 30-348 Kraków, Poland
E-mail address: Agnieszka.Kozdeba@im.uj.edu.pl

Apoloniusz Tyszka
Technical Faculty
Hugo Kołłątaj University
Balicka 116B, 30-149 Kraków, Poland
E-mail address: rttyszka@cyf-kr.edu.pl