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Not a difference of opinion: Wittgenstein and Turing on contradictions in mathematics

Abstract. In his 1939 Cambridge Lectures on the Foundations of Mathematics, Wittgenstein proclaims that he is not out to persuade anyone to change their opinions. I seek to further our understanding of this point by investigating an exchange between Wittgenstein and Turing on contradictions. In defending the claim that contradictory calculi are mathematically defective, Turing suggests that applying such a calculus would lead to disasters such as bridges falling down. In the ensuing discussion, it can seem as if Wittgenstein challenges Turing's claim that such disasters would occur. I argue that this is not what Wittgenstein is doing. Rather, he is scrutinizing the meaning and philosophical import of Turing's claim—showing how Turing is wavering between making an empirical prediction and a logical observation, and that it is only through this wavering that Turing can believe that he has provided a proper explanation of why contradictory calculi are mathematically defective.

1.

The aim of this paper is to further our understanding of Wittgenstein's later philosophical method through a detailed investigation of certain passages from his 1939 Cambridge Lectures on the Foundations of Mathematics.[1] The Lectures are one of the few documents where the *dialectical* nature of Wittgenstein's method is truly exhibited before us, namely in Wittgenstein's

[1] Wittgenstein (1975), henceforth 'LFM'.

discussions with Turing. That Turing occupies a privileged position in the lectures is made clear by Wittgenstein himself at the end of Lecture VI:

> Unfortunately Turing will be away from the next lecture, and therefore that lecture will have to be somewhat parenthetical. For it is no good my getting the rest to agree to something that Turing would not agree to.[2]

In effect, Wittgenstein is testing his method on Turing, who is an ideal test subject in at least two respects. First, he is a working mathematician with a philosophical bent who takes Wittgenstein seriously.[3] Second, he is subject to the sort of philosophical confusions that Wittgenstein takes to be typical of working mathematicians, and which he seeks to expose and remedy.

We are, then, witness to an actual trial run of Wittgenstein's later philosophical method. To set the stage a bit, it is interesting to take a closer look at the very first exchange between Wittgenstein and Turing during the lectures:

> [*Wittgenstein*:] We have all been taught a technique of counting in Arabic numerals. We have all of us learned to count—we have learned to construct one numeral after another. Now how many numerals have you learned to write down?
>
> *Turing*: Well, *if I were not here*, I should say $\aleph_0$.[4]

In adding the clause 'If I were not here', Turing evinces a certain wariness about Wittgenstein's philosophy. He arrives at Wittgenstein's lectures with a picture of those lectures as a space where one does not get away with saying what one would normally say without thinking twice about it. This picture is, of course, not entirely inaccurate. But it is equally important to see that

[2] (LFM: 67-68).
[3] Floyd has done stimulating work on Wittgenstein's influence on Turing. See e.g. Floyd (2013, 2016, 2017).
[4] (LFM: 31), my emphasis.

this is not what happens here. In fact, Wittgenstein replies: “I entirely agree, but that answer shows something” (LFM: 31). There is nothing amiss with saying that we have learned to write down $\aleph_0$ numerals. This is not a suspicious philosophical pronouncement, but a perfectly ordinary use of mathematical language.

This does not mean, however, that there is no potential for confusion. The danger lies in a tendency to misunderstand the grammatical role played by the symbol ‘$\aleph_0$’ in statements such as this. Here is Wittgenstein:

> In agreeing, I meant that that is the way in which the number $\aleph_0$ is used. It does not mean that Turing has learned to write down an enormous number [of numerals][5]: $\aleph_0$ is not an enormous number. […] To say that one has written down an enormous number of numerals is perfectly sensible, but to say that one has written down $\aleph_0$ numerals is nonsense.[6]

When we say ‘Turing wrote down 100 numerals on the blackboard’, the symbol ‘100’ stands proxy for the result of counting the numerals on the blackboard. When we say ‘Turing has learned to write down $\aleph_0$ numerals’, the symbol ‘$\aleph_0$’ stands proxy for a *technique*. The latter sentence is used to characterize the technique itself which Turing has mastered—a mastery that renders him capable of writing 100 numerals on the blackboard. The philosophical confusion that Wittgenstein is interested in is not internal to these statements—which are perfectly all right as they stand—but arises from wrongly assimilating them, from regarding the symbols ‘100’ and ‘$\aleph_0$’ as playing more or less the same logical role:[7]

> What kind of misunderstandings am I talking about? They arise from a tendency to assimilate to each other expressions which have very different functions in the language.

[5] Given the upcoming sentence and the context, it is clear that what is meant is not an ‘enormous number’ *simpliciter* but an ‘enormous number of numerals’.
[6] (LFM: 32).
[7] We have here a phenomenon that is already discussed in the *Tractatus*: two symbols which symbolize in different ways, but which are ostensibly used in the same way in propositions (3.323).

> […] Like primitive peoples, we are much more inclined to say, 'All these things, though looking different, are really the same' than we are to say, 'All these things, though looking the same, are really different.' Hence I will have to stress the differences between things, where ordinarily the similarities are stressed, though this, too, can lead to misunderstandings.[8]

When Turing hedges his statement with the clause 'If I were not here', this betrays a mismatch between Turing's expectations about Wittgenstein's lectures and Wittgenstein's actual philosophical aims.[9] Wittgenstein is not out to disagree with Turing's statement that he has learned to write down $\aleph_0$ numerals—he is rather out to investigate its logical (grammatical) status, its meaning. He wants to achieve clarity about the use of that statement that insulates us against the philosophical confusions to which such a statement is liable to give rise. Wittgenstein is aware, however, that his philosophical method remains opaque for those who are subjected to it:

> One of the greatest difficulties I find in explaining what I mean is this: You are inclined to put our difference in one way, as a difference of *opinion*. But I am not trying to persuade you to change your opinion. I am only trying to recommend a certain sort of investigation. If there is an opinion involved, my only opinion is that this sort of investigation is immensely important, and very much *against the grain* of some of you. If in these lectures I express any other opinion, I am making a fool of myself.[10]

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[8] (LFM: 15).
[9] One could say that I am making too heavy weather about Turing's statement, and that he was merely joking. To this I would reply that it may very well be true that he was joking, but that I do not believe that he was *merely* joking.
[10] (LFM: 103).

As familiar as such pronouncements—which recur throughout Wittgenstein's later work—are, they continue to vex scholars.[11] In this paper, I obviously cannot pretend to settle these matters. What I wish to do, is to look in detail at a specific discussion between Wittgenstein and Turing that revolves around the status of contradictions in mathematics, and illustrate how Wittgenstein's methodological pronouncements are key to understanding what is going on in that exchange. At issue is the idea that the presence of a contradiction in a mathematical calculus vitiates that calculus. Turing seeks to defend this idea, and advances certain statements in order to do so. I will argue, in line with what was said above, that Wittgenstein is trying to show that Turing has not properly understood the logical status of his statements—and that if he did, he would see that they cannot give him what he wants, so that he would no longer feel compelled to put them forward. Turing's responses to Wittgenstein, however, show that he did not understand what Wittgenstein was trying to do. In this vein, Turing not only served as a test subject for Wittgenstein—he can also serve as a test subject *for us*, insofar as he exemplifies the sort of wrong turns that one can take in trying to make sense of and responding to Wittgenstein's reflections.

I am certainly not the first to examine the exchange between Wittgenstein and Turing on contradictions. There are two excellent recent pieces of scholarship by Persichetti (2021) and Matthíason (2021).[12] They are mainly concerned to bring out how Wittgenstein sought to dismantle what he took to be a misplaced fear of contradictions that had swept over mathematics, by showing that contradictions are not the disaster which they were taken to be.[13] My focus, however, will be more on Wittgenstein's scrutiny of Turing's own statements—more specifically, his attempt to bring out how Turing's statements waver between empirical predictions and logical observations. In this way, I believe that my discussion adds to those of

[11] Monk, for instance, sees no way to take seriously Wittgenstein's point: "Wittgenstein, however, clearly did have very strong opinions—opinions that were, moreover, at variance with the conception of their subject held by most professional mathematicians" (1991: 420).

[12] For an earlier, much more critical account, see Chihara (1977). Schroeder (2021, Chapter 11) also discusses these issues.

[13] Of course, this quick summary cannot do justice to their accounts. I recommend both papers to anyone interested in these issues. I am merely trying to provide a quick indication of how my own discussion adds to those of Persichetti and Matthíasson. I do not have the space to compare our accounts in detail, but I will briefly indicate what I take to be some points of agreement and disagreement below.

Parischetti and Matthíason—further bringing home the philosophical richness and complexity of the Lectures, which remain understudied in the literature.

2.

Wittgenstein opens Lecture XVIII by saying: "What I ought to talk about now is the role that logic plays in mathematics, or the relation supposed to hold between logic and mathematics"[14]. In exploring this issue, Wittgenstein quickly turns his attention to contradictions. Here, I cannot do justice to all the subtle twists and turns of his discussion. My approach will be to focus on certain strands and fragments in a way that, I believe, sheds light on Wittgenstein's method and aims in the lectures—which, I assume here without argument, are representative of his later philosophy more broadly.

Wittgenstein seeks to interrogate the attitude which mathematicians take towards contradictions—an attitude of avoiding them at all costs, of regarding a contradiction as, as he puts it, "a germ which shows general illness"[15]. To find a contradiction in a system, it is thought, is like "finding a germ in an otherwise healthy body", and it "shows that the whole system or body is diseased"[16]. A contradictory calculus is regarded as *inherently* defective, as defective *qua* mathematical calculus.[17]

As my starting point, I wish to take a characteristically provocative passage, where Wittgenstein is asking the flat-footed question: why *shouldn't* we use a contradictory calculus? What would be so bad about that? Here is Wittgenstein:

> One may say, 'From a contradiction everything would follow.' The reply to that is: Well then, don't draw any conclusions from a contradiction; make that a rule. You might put it: There

[14] (LFM: 171).
[15] (LFM: 211).
[16] (LFM: 138).
[17] The metaphor of contradiction as a disease itself has a history in Wittgenstein's oeuvre: it already occurs at Wittgenstein (1979: 120). In this paper, I cannot investigate the relation between the Lectures and Wittgenstein's earlier views. See Marion & Okada (2013) for some relevant discussion.

> is always time to deal with a contradiction when we get to it. When we get to it, shouldn't we simply say, 'This is no use—and we won't draw any conclusions form it'?".[18]

One mathematician whom Wittgenstein undoubtedly had in the back of his mind, is Hilbert, who turned the practice of avoiding contradictions into a whole mathematical project, seeking to develop methods that allow us to make sure that a calculus contains no hidden contradictions.[19] This is evidently a development that struck Wittgenstein, and which he wished to understand philosophically.

In the lectures, Turing takes up an attitude that is broadly sympathetic to Hilbert's point of view. Turing accepts that there is something inherently defective about a contradictory calculus, so that the practice of avoiding such calculi is a *good* practice. The question remains, however, what this 'defectiveness' consists in. When Wittgenstein asks where the harm would come from if we allowed contradictory calculi, Turing replies:

> *Turing*: The real harm will not come in unless there is an application, in which case a bridge may fall down or something of that sort.
>
> *Wittgenstein*: Ah, now this idea of a bridge falling down if there is a contradiction is of immense importance. But I am too stupid to begin it now; so I will go into it next time.[20]

What we find next time—and the time after that—is a prolonged discussion of this suggestion by Turing that, as Wittgenstein paraphrases it, "the danger with a contradiction in logic or mathematics is in the application"[21].

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[18] (LFM: 209).
[19] Here is a characteristically strong statement by Hilbert: "But if we wish to restore the reputation of mathematics as the exemplar of the most rigorous science it is not enough merely to avoid the existing contradictions. The chief requirement of the theory of axioms must go farther, namely, to show that within every field of knowledge contradictions based on the underlying axiom-system are *absolutely impossible*" [original emphasis] Hilbert (1996: §33).
[20] (LFM: 211).
[21] (LFM: 211).

It is fair to say, I think, that commentators have tended to assume that Turing's point is clear—and that the main challenge is to understand how Wittgenstein seeks to respond to it. In this vein, they have rightly picked up on the fact that Wittgenstein seeks to show that contradictory calculi are not, *pace* Hilbert and Turing, inherently defective.[22] This is, however, not the whole story. It misses, I think, an important dimension of the ensuing dialectic that I will try to bring out here. Part of what Wittgenstein tries to show, I will argue, is precisely that it is not as clear as it may appear *what* exactly Turing is saying, when he says that contradictions may result in bridges falling down. As we will see, Turing wavers between making an empirical prediction and making a logical observation—and it is only *through* this wavering that he can take himself to have gotten what he wants, i.e. an explanation of why a contradictory calculus is inherently defective.

3.

Turing wants to draw a connection between the presence of a contradiction in a calculus and bad outcomes in its application. Wittgenstein, in response, seeks to scrutinize this connection.[23] At a certain point, Prince makes the following suggestion:

> Could we take this example. Suppose we have two ways of multiplying which lead to different results, only we don't notice it. Then we work out the weight of a load by one of these ways and the strength of a brass rod by the other. We come to the conclusion that the rod will not give away; and then we find that in fact it does give way.[24]

Here is Wittgenstein:

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[22] This is the main focus of Persichetti (2021) and Matthíasson (2021). See also Pérez-Escobar & Sarikaya (2022: §3.3), where the Dirac delta function is invoked to make a similar point.
[23] Even here, I have to skip over many details, selecting passages that allow me to tell what I believe is a coherent story that remains faithful to Wittgenstein's philosophical aims—faithful to the philosophical spirit that underlies the lectures, if you will. In any case, there remains much more to discuss.
[24] (LFM: 216)

> It is difficult to imagine we hadn't noticed the contradiction at all—this is important. But suppose we haven't noticed it and suppose that nothing goes wrong: the bridge doesn't fall or the brass rod doesn't break. Is our calculation wrong? I'd say: Not at all. We've done everything perfectly all right.[25]

Wittgenstein seems to be saying: even if there is a contradiction, everything may turn out fine, and then there is no real problem. How are we to interpret the status of this remark? It may appear that Wittgenstein is questioning the *truth* of Turing's claim that using a contradictory calculus would lead to bridges falling down. Perhaps it would not? If this were the case, then the difference between Turing and Wittgenstein *would* be a difference of opinion.

This is not, however, what Wittgenstein is doing. The guiding question is not so much *whether* contradictions would lead to bad results, but rather *what is meant* by saying something like that. What sort of claim is Turing making? This is not as clear as it may first appear, and Wittgenstein seeks to bring out this unclarity. Recall that what is under scrutiny is the idea that there is something *mathematically* defective about a contradictory calculus—the idea that the practice of avoiding such calculi is, we could say, vindicated by mathematics itself. It is with the aim of defending this idea, that Turing points to the application of mathematics, to the danger of bridges falling down.

Maybe bridges will fall down, maybe they wouldn't. Whatever is the case, however, this cannot furnish us with a *mathematical* reason to avoid contradictions. That is: if Turing is making an *empirical prediction* about what would happen when we applied a contradictory calculus, then he has converted what began as a mathematical issue into an empirical issue.[26] Wittgenstein wants to know: why do we avoid contradictions *in mathematics*? This question

[25] (LFM: 217)

[26] So far, I am relying on the fact that—*qua* empirical prediction—the statement that bridges will fall down is contingent. As we will see, however, Turing will transition into making claims about what *could* happen or what *must* happen. Such statements require a different treatment. In this footnote and in the subsequent discussion, I am indebted to an anonymous referee for showing me the need to better clarify my approach to these distinctions.

cannot be properly answered by making empirical predictions about the results of applying contradictory calculi. Even if it were the case that using a contradictory calculus is an unreliable method for building bridges, this would still not amount to a mathematical defect.

Here is Wittgenstein again:

> The question is: Why are people afraid of contradictions? It is easy to understand why they should be afraid of contradictions in orders, descriptions, etc., *outside* mathematics. The question is: Why should they be afraid of contradictions inside mathematics? Turing says, 'Because something may go wrong with the application.' But nothing need go wrong. And if something does go wrong—if the bridge breaks down—then your mistake was of the kind of using a wrong natural law.[27]

Applying a mathematical calculus in such a way that bridges fall down, this is not a *mathematical* mistake. It does not reveal that there was something mistaken about the calculus itself. One has used a calculus which, it turns out, is not a suitable calculus for building bridges—and this is like using a wrong natural law. By itself, a mathematical calculus does not proclaim to be useful for building bridges. Rather, it is we who decide to use certain calculi for this purpose. If the calculus turns out not to be suitable for that purpose, this does not reveal that the calculus is somehow mathematically defective—it only reveals that it does not have the kind of application we thought it did.[28] Perhaps it is true that no contradictory calculus is useful for building bridges—this would still only establish just that: contradictory calculi are not useful for building bridges.[29]

[27] (LFM: 217)
[28] As Schroeder puts the point: "A tool may be unsuitable for a given purpose without being intrinsically a bad tool" (2021: 199).
[29] Thus, I would say that the reply of Marion & Okada (2013: 76) that *both* consistent and inconsistent calculi may lead to bridges falling down does not latch onto the main point. Even if, as a matter of fact, only inconsistent calculi led to bridges falling down, this would still not show the latter to be mathematically defective.

Turing, however, does not wish to give up on the idea that there is something inherently defective about a contradictory calculus. The discussion continues:

> [*Wittgenstein:*] Is Prince's case a case of a 'hidden contradiction'? And if something is a 'hidden contradiction', does it do any harm while it is—as you might say—hidden?
> You might say that with an open contradiction we would not know what to do; we would not know what use to make of it. And what about a 'hidden contradiction'? Is it there as long as it is hidden?
> *Turing*: You cannot be confident about applying your calculus until you know that there is no hidden contradiction in it.[30]

What exactly is Turing trying to say here? Of course, Turing is not making a psychological observation about the conditions under which we feel confident in applying a calculus. He is making a normative claim: we *shouldn't* be confident about applying a calculus unless we know that there is no hidden contradiction in it. That is: we do not have the *right* to feel at ease, until we have assured ourselves of the absence of any hidden contradiction. If we just blindly apply a calculus without such assurance, we are being reckless and irresponsible. There is a clear moralistic dimension to Turing's statement—a dimension that Wittgenstein, throughout his life, took to be symptomatic of philosophical confusion. Where philosophy becomes moralistic, we can be sure that a wrong turn has been taken.

Wittgenstein replies:

> *Wittgenstein*: There seems to me to be an enormous mistake there. For your calculus gives certain results, and you want the bridge not to break down. I'd say things can go wrong in only two ways: either the bridge breaks down or you have made a mistake in your

[30] (LFM: 217).

calculation—for example, you multiplied wrongly. But you seem to think that there may be a third thing wrong: the calculus is wrong.

Turing: No. What I object to is the bridge falling down.

Wittgenstein: But how do you know that it will fall down? Isn't that a question of physics? It may be that if one throws dice in order to calculate the construction of the bridge it will never fall down.[31]

Turing says: 'If we use a contradictory calculus in the application of mathematics, something will go wrong somewhere'. Does Wittgenstein disagree with this statement, accusing Turing of having made a false claim? No. What Wittgenstein interrogates, is the philosophical use to which Turing seeks to put his statement. There is something that Turing wants this statement to do for him: explain why there is something inherently defective about a contradictory calculus, thereby vindicating the mathematical practice of avoiding such calculi, as well as the mathematical practice of developing methods to seek out hidden contradictions. What Wittgenstein is trying to bring out in the above remarks, is that *if* we understand Turing's statement as making an empirical prediction about what *would* happen, then it cannot do for him what he wants it to do—*irrespective* of whether that prediction turns out to be true or false, so that there is no question of agreeing or disagreeing with Turing's prediction.[32] Wittgenstein isn't contesting Turing's claim that bridges will fall down—he is contesting Turing's *use* of that claim in support of a certain philosophical picture of our mathematical practice.

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[31] (LFM: 218).

[32] Thus, the point of the remark about the dice is to bring out *that* we are concerned with an empirical question which is irrelevant to the issue at hand. Turing is understanding the use of a contradictory calculus as something akin to throwing dice, and is saying something like: 'If you do it like that, it is very likely that the bridge will fall down'. Persichetti (2021: 3795) and Matthíasson (2021: 555) have argued that there need not be anything unreliable about using a contradictory calculus in the application of mathematics—that it need not be like using dice at all. See also, again, Pérez-Escobar & Sarikaya (2022: §3.3). I do not disagree with any of this. But I do not think this is Wittgenstein's main point here. He is not investigating *whether* using a contradictory calculus is like using dice. Rather, he is saying: *if* we are trying to vindicate the idea that a contradictory calculus is inherently defective *qua* mathematical calculus, then the question whether using a contradictory calculus is like using dice is a red herring anyway.

Under this pressure from Wittgenstein, Turing moves away—in his next reply—from the realm of empirical predictions:

> *Turing*: If one takes Frege's symbolism and gives someone the technique of multiplying in it, then by using a Russell paradox he could get a wrong multiplication.[33]

Turing shifts from making empirical predictions about what *would* happen in applying a contradictory calculus, to making an observation about what one *could* do within such a calculus.[34] One *could* use Russell's paradox to generate any multiplication one likes. These multiplications include multiplications that—if they were used as a benchmark for building bridges—would lead to bridges falling down. But notice the shift here. There are two very different statements that must be distinguished. The first is the claim that applying a contradictory calculus will lead to bad results. This is an empirical prediction. Now, there is the claim that one can use a contradictory calculus to yield multiplications that are such that they will cause bridges to fall down. This is no longer an empirical claim—it is a *logical* observation about what one can infer within a contradictory calculus.

The problem with the empirical claim was that it does not furnish us with a mathematical reason against contradictions. What about this logical observation? The problem here is that it merely points our attention to what was known all along: that one can use a contradictory calculus to infer anything whatsoever. Turing's point now seems to be: you *could* use the calculus in this way, and if you do, bridges will fall down. But here, we can simply repeat Wittgenstein's earlier flat-footed answer: then just don't use it like that! Again, we have not been shown a mathematical defect in such a calculus. Rather, we have been given a reason not to apply it in certain ways.

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[33] (LFM: 218).

[34] This is similar to Chihara, who—after presenting an inconsistent calculus—writes: "Indeed, it is not hard to see how, by relying on such a system in reasoning about, say, the number of steel beams of such and such tested strength needed in a bridge to support a load of N tons, a disaster *could* result" [my emphasis] (1977: 377). Matthíasson (2021: 553ff.) has a good critical discussion of Chihara's account.

Wittgenstein seeks to bring this out by replying: “This would come down to doing something which we would not call multiplying”[35]. Turing talks about giving someone the technique of multiplying in Frege’s calculus, and of then using Russell’s paradox to get a wrong multiplication. But no one in their right mind would understand the latter as an exercise of the technique of multiplication. If *this* is what is worrisome about contradictory calculi, we need not worry about them at all, since this is not the sort of thing anyone does or even feel tempted to do.[36] If an engineer were to build bridges in this way, this would not be an honest mistake—we would rather regard them as out of their mind. The problem would not lie with Frege’s calculus, but with the engineer[37]—after witnessing this, we would certainly not trust them with a consistent calculus either. It would be ridiculous to respond: ‘all we have to do, is make sure that we give them a consistent calculus next time, and then everything will be all right’.

In fact, Turing’s reply reinforces precisely the provocative suggestion that Wittgenstein had made earlier: that a contradiction only leads to bad results if it lies out in the open—and that all we need to do in such a case, is simply not use it in drawing inferences. It is true that one *can* use Frege’s calculus to infer anything whatsoever from Russell’s paradox. But why *would* anyone us it in that way, if their aim is to calculate the right parameters for building bridges—if their aim is to *multiply*?

We find a similar dialectic towards the end of Lecture XXII:

> [*Wittgenstein to Turing*] Before we stop, could you say whether you really think that it is the contradiction which gets you into trouble—the contradiction in logic? Or do you see that it is something quite different?—I don’t say that a contradiction may not get you into trouble. Of course it may.

[35] (LFM: 218).
[36] Compare: “Or do you mean the contradiction may tempt one into trouble? As a matter of fact it doesn’t. No one has ever yet got into trouble from a contradiction in logic” (LFM: 219).
[37] Schroeder (2021: 194-195) and Persichetti (2021: 3789ff.) make similar points.

> *Turing:* I think that with the ordinary kind of rules which one uses in logic, if one can get into contradictions, then one can get into trouble.[38]

Again, Turing is tying the presence of a contradiction to a certain logical possibility of getting into trouble—presumably the already mentioned possibility of using the contradiction to yield random multiplications. As before, Wittgenstein tries to make clear that this logical possibility does not get Turing what he needs:

> *Wittgenstein:* But does this mean that with contradictions one *must* get into trouble? […] If a contradiction may lead you into trouble, so may anything. It is no more likely to do so than anything else.[39]

To say that we *can* get into trouble, is vacuous—there is nothing in such a statement for Wittgenstein to disagree with. But neither does such a statement give Turing what he wants. For Turing to get what he wants, he would need to be able to say that we *must* get into trouble. And we do, in fact, find Turing come close to saying something like that:[40]

> *Turing:* Although you do not know that the bridge will fall down if there are no contradictions, yet it is almost certain that if there are contradictions it will go wrong somewhere.[41]

Given what has been said before, this statement should strike us. Turing started out by making predictions about what would happen if we applied a contradictory calculus. Wittgenstein pointed out to him that that is a matter of physics. Turing then made a logical observation about

[38] (LFM: 219).
[39] (LFM: 219).
[40] This passage comes before the one just discussed. This is indicative of how Turing—under pressure from Wittgenstein—is struggling to find the right way to express what he is trying to say. Of course, from Wittgenstein's point of view, the reason for this is that there is no such right way, because Turing's ideas are confused.
[41] (LFM: 218)

how a contradictory calculus *could* be applied. Wittgenstein pointed out to him that this need not worry us, since it wouldn't even pass as a proper application. One way to read what Turing is saying here, is as simply moving back towards a merely contingent prediction about what would happen. In this case, we would fall back into the previous dialectic.

I believe, however, that something more complex is going on, as shown by Turing's use of the phrase 'almost certain'.[42] Turing seems to want to say that a contradiction *must* get us into trouble, which would constitute a reason to regard a contradictory calculus as inherently defective. At this point, it may seem that Wittgenstein cannot but disagree with such a claim. Again, however, I think that talking in terms of 'disagreement' or 'difference of opinion' obscures the true nature of the dialectic.

We can see this as follows. The claim that we *must* get into trouble if we use a contradictory calculus does not have the status of an empirical prediction, but of an *a priori* principle. Turing is trying to make what could be called a *logical prediction*: he wants to say that a contradictory calculus will *inevitably* get us into trouble, in a way that is not a mere matter of physics. Only so can it seem that we do have a reason to regard a contradictory calculus as inherently defective. But this means that we lose our hold on a self-standing sense of the 'trouble' that a contradictory calculus supposedly gets us into. As soon as independent criteria are given for 'getting into trouble'—such as bridges falling down—the claim reverts back to being an empirical prediction. The only way in which it can be the case that we *must* get into trouble, is if the contradictoriness of the calculus *itself* functions as the criterion for 'getting into trouble'. In other words: To say 'we will get into trouble', just becomes a way of saying that the calculus *is* contradictory—no different criterion is in play anymore. Used in this way, then, the expression 'we will get into trouble' does not give us a self-standing reason to reject a contradictory calculus, but simply becomes a way of saying that we don't want anything to do with it.

Crucially, and revealingly, this was in fact Wittgenstein's point from the start:

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[42] In this paragraph and the next, I am especially indebted to the aforementioned comments of an anonymous referee.

> We most naturally compare a contradiction to something which jams. I would say that anything which we give and conceive to be an explanation of why a contradiction does not work is always just another way of saying that we do not want it to work.[43]

Although this passage comes long before the discussion of bridges falling down, it fits that discussion perfectly. Turing is trying to give an explanation of why a contradictory calculus does not work: because it would lead to disaster in the application of mathematics. Properly scrutinized, however, his purported explanation turns out to be a convoluted way of saying that Turing does not *want* it to work—or better: that he does not want to work *with it*. Once the voluntaristic nature of Turing's claim is brought to light, however, it again becomes apparent that there is nothing for Wittgenstein to disagree with: Turing is free to not want to work with contradictory calculi. His mistake is that he thinks that he has provided reasons that should compel others to do the same.

Wittgenstein does not disagree with Turing's empirical prediction that applying a contradictory calculus would lead to bridges falling down—this would be a matter for empirical investigation. He does not disagree with Turing's logical observation that one *could* use a contradictory calculus to arrive at multiplications that would, if used in an application, lead to bridges falling down. Nor, finally, does Wittgenstein disagree with Turing's attitude of not wanting to work with contradictory calculi. The problem, rather, lies in how Turing thinks he can bring these together. By conflating the empirical prediction and the logical observation—by obscurely wavering between the two—Turing mistakenly believes that is able to show that there is something inherently defective about a contradictory calculus *qua* mathematical calculus, which in turn is taken to support the Hilbertian project of developing methods for finding hidden contradictions. It is this that Wittgenstein wants to show to be confused.

[43] (LFM: 187).

Turing, however, does not realize what Wittgenstein is trying to do. This comes out nicely in the following exchange, towards the end of Lecture XXII:

> *Turing:* You seem to be saying that if one uses a little common sense, one will not get into trouble.
>
> *Wittgenstein:* No, that is *NOT* what I mean at all.—The trouble described is something you get into if you apply the calculation in a way that leads to something breaking. This you can do with *any* calculation, contradiction or no contradiction.[44]

Turing understands Wittgenstein's remarks as concerned with the question: how do we prevent contradictions from getting us into trouble? He takes Wittgenstein to be saying something like: we could get into trouble, all right, but if we do so and so, we can keep ourselves out of trouble. But this is to miss Wittgenstein's point. He is not interested in devising methods for keeping ourselves out of trouble. He is, rather, interested in understanding what *sort* of 'trouble' contradictions are supposed to get us into.[45] Turing is assuming that the picture of contradictions getting us into trouble is a clear picture, and that the only issue is how to avoid landing ourselves in such a situation. Wittgenstein, on the other hand, is trying to bring out precisely that Turing has not yet gotten hold of a clear picture at all, and that this is betrayed by his own wavering between making empirical predictions and advancing logical observations—between saying something like 'using a contradictory calculus to build bridges would result in them falling down' and saying something like 'a contradictory calculus can be used to build bridges in such a way that they would fall down'. To bring this to Turing's attention, is not a matter of changing his opinions, but of relieving him of unclarity *in* his opinions.

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[44] (LFM: 219).

[45] Matthíasson has a good discussion of why, according to Wittgenstein, deriving a contradiction "is not a *special* kind of mistake that we should distinguish from any other kind of wrong modelling of reality by a formal system" (2021: 554). Making this clear to Turing is one aspect of the exercise of getting him to realize that he has no clear conception of how contradictions get us into 'trouble'.

4.

I wish to end by zooming out a bit and connecting my previous discussion to Wittgenstein's overarching discussion of contradictions. Of course, I should again emphasize that I will only be touching the surface of what are deeply complex philosophical issues.[46]

Let us begin with an exchange between Wittgenstein and Turing that arises from considering contradictory orders such as 'sit down and do not sit down'. Our technique of giving orders does not furnish such a contradictory order with a clear meaning: we do not know what to do with it. Here are Turing and Wittgenstein:

> *Turing:* I should say that if one teaches people to carry out orders of the form 'p and not-q' then the most natural thing to do when ordered 'p and not-p' is to be dissatisfied with anything which is done.
>
> *Wittgenstein:* I entirely agree. But there is just one point: does 'natural' mean 'mathematically natural'?
>
> *Turing:* No.
>
> *Wittgenstein:* Exactly. 'Natural' there is not a mathematical term. It is not mathematically determined what is the natural thing to do.[47]

The point is this: mathematics itself does not tell us what to do with contradictions—whether to discard them, reinterpret them, etc.[48] We have mathematical techniques which, we could

[46] Although she does not focus on mathematics, McGinn (2021) has many discussions that revolve around the same nexus of issues. One thing she does, is to investigate what it could mean, exactly, to call a certain continuation of a technique 'natural'—a point that will come up below.
[47] (LFM: 186-187).
[48] That Turing agrees with Wittgenstein here, I take to be in tension with their discussion of bridges falling down—where Turing does seem to want to say that there is something 'mathematically natural' about avoiding contradictions. This tension further testifies of the unclarity in Turing's views. An anonymous referee suggested that perhaps Turing is merely denying that 'natural' means 'mathematically natural' because the contradiction in question is purely logical, and does not involve any mathematics—leaving room for the idea that the dissatisfaction is nevertheless 'logically natural'. If this is Turing's thought, this would indeed remove the tension—but only at the cost of misunderstanding Wittgenstein's point, who is here using 'mathematically natural' in a sense that does not depend on the specific content of the contradiction in question.

say, reach up to but nevertheless stop short of contradictions. These techniques give rise to contradictions—furnish us with techniques for generating contradictions—but do not determine what to do with them.[49] We know what the order 'do p and do q' means—this is taken care of by our technique. But we do not know what the order 'do p and do not-p' means—our technique does not tell us what to do with such an order.[50] The confusion that Wittgenstein seeks to lay bare arises when we think that we do know what it means—and that it is *because* of its meaning, that it must be discarded. Here is one passage in which Wittgenstein addresses this confusion:

> One is tempted to say, 'A contradiction not only doesn't work—it *can't* work.' One wants to say, 'Can't you see? I can't sit and not sit at the same time.' One even uses the phrase 'at the same time'—as when one says, 'I can't talk and eat at the same time.' The temptation is to think that if a man is told to sit and not to sit, he is asked to do something which he quite obviously can't do.
>
> Hence we get the idea of the proposition as well as the sentence. The idea is that when I give you an order, there are the words—then something else, the sense of the words—then your action. And so with 'Sit and don't sit', it is supposed that besides the words and what he does, there is also the *sense* of the contradiction—that something which he can't obey.
>
> One is inclined to say that the contradiction leaves you no room for action, thinking that one has now *explained* why the contradiction doesn't work.[51]

The confusion can be described as follows: one turns the correct observation that our techniques do not determine what to do with a contradiction into the idea that our techniques determine that nothing can be done with a contradiction. Wittgenstein's point is: we may indeed

[49] There is an echo here of the *Tractatus* point that tautology and contradiction are the limiting cases of the symbolism (4.466). In the *Tractatus*, however, Wittgenstein still believed that the existing practice also determines what to do with its own limiting cases.
[50] Perischetti expresses the point by saying that contradictions are "parts of a grammar that miss a clear use defined by a rule" (2021: 3797)
[51] (LFM: 185).

be determined *to* do nothing with contradictions and throw them out—and this may be the most natural and most intelligent thing to do, for a whole battery of reasons—but this is nevertheless not something that is dictated by our techniques themselves. Wittgenstein writes:

> You might ask: What are we convinced of when we are convinced of the truth of a logical proposition? How do we become convinced of, say, the law of contradiction? We first learn a certain technique of using words. Then the most natural continuation for us is to eliminate certain sentences which we don't use—like contradictions. This hangs together with certain other techniques.[52]

We do not want contradictions to work because that is the most natural way of continuing our techniques. No use suggests itself to us—nothing speaks for it, and much against it. Still, it is not mathematics itself that dictates this continuation of our techniques. There is a point of responsibility and self-determination here: *we* are responsible for our practice of avoiding contradictions.[53] And, crucially, this is something we have every right to do. It can be all but irresistible to read Wittgenstein as trying to argue that contradictions are perfectly all right, that we should stop avoiding contradictions.[54] As I understand him, this is not his point.[55] He is not attacking the practice of avoiding contradictions, but is rather trying to understand its nature—its true grounds, if you will. And this is equally to reveal the groundlessness of that practice, as long as one is stuck—as Turing is—with a certain picture of what a suitable ground could be.

[52] (LFM: 201).
[53] There is a parallel with Kant here, to be explored on other occasions. Both Kant and Wittgenstein are concerned to show that the human faculties are not merely passive in ways in which they were previously taken to be.
[54] In this vein, Monk (1991: 420) reads Wittgenstein as challenging the law of contradiction. What Wittgenstein is really doing, however, is investigating its status. It is, however, characteristic of this kind of investigation that it will appear as challenging the law of contradiction, precisely because it removes the false picture of necessity—the false picture of what its necessity *consists in*—that accompanies it.
[55] It is certainly true that Wittgenstein acknowledges that it is *possible* to develop our techniques in alternative ways—by developing, for instance, paraconsistent calculi that operate with contradictions. We *could* do that, but Wittgenstein is not telling us *to* do that. Thus, I am more wary than Persichetti (2021: 3791-3792) and Matthíasson (2021: 538) of portraying Wittgenstein as an advocate *avant la lettre* of paraconsistent logic.

Wittgenstein, we could say, is trying to counteract misleading pictures of what gives us the *right* to avoid contradictions.

That the rejection of contradictory calculi is not dictated by mathematics itself, this is what Turing cannot accept—and what proves difficult for anyone to accept. Turing wants to furnish grounds—to provide explanations—where no grounds or explanations of the sought after kind are available. It is as if we can perfectly accept notions of responsibility and self-determination with regards to other practices—painting and classical music, say—but not with regards to logic and mathematics. These, at least, we seem to want to say, leave us no choice in how to proceed.

That we are responsible for these practices as well—for their continuation beyond what they themselves dictate to us—does not mean, however, that these continuations are arbitrary. Quite the opposite: what we naturally agree in doing is not at all arbitrary, but belongs to the very fabric of our being, to our humanity:

> Mathematical truth isn't established by their all agreeing that it's true—as if they were witnesses of it. Because they all agree in what they do, we lay it down as a rule, and put it in the archives. Not until we do that have we got to mathematics. One of the main reasons for adopting this as a standard, is that it's the natural way to do it, the natural way to go—for all these people.[56]

The special status of mathematics is not that it's the only practice that dictates its own continuation—that leaves us no choice. Rather, one could say that mathematics is precisely the domain where we naturally agree *in* continuing the techniques that we have been taught[57]—that a mathematical technique *is* a technique that exhibits such agreement.

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[56] (LFM: 107).

[57] That this is characteristic of mathematics is a point that Wittgenstein emphasizes many times. For instance: "The fact is, all grown-ups count alike and do not, when asked to count objects, constantly hesitate and say, 'Now did I leave out a number in counting?'" (LFM: 101). See also, of course, the well-known §240 from the *Philosophical Investigations* about mathematicians not coming to blows with each other.

Such agreement, it should be emphasized, is not a matter of opinion:

> 'So you are saying that human agreement decides what is true and what is false?'—What is true or false is what human beings say; and it is in their *language* that human beings agree. This is agreement not in opinions, but rather in form of life.[58]

We do not have an *opinion* about how to continue our technique of counting. We simply continue it. And that we all continue in the same way—that we all naturally agree in (not: about!) how to proceed—this belongs to our form of life. That there are these techniques—these *mathematical* techniques—in which we all, given a certain teaching and upbringing, proceed in the same way, this is part of our form of life, of the human form of life. This is not to undermine the inexorability of mathematics, but rather to confirm it—to bring it plainly into view for the first time.

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