Parry Syllogisms

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Abstract Parry discusses an extension of Aristotle’s syllogistic that uses four nontraditional quantifiers. We show that his conjectured decision procedure for validity for the extended syllogistic is correct even if syllogisms have more than two premises. And we axiomatize this extension of the syllogistic.

1 Background and motivation Parry [2] discusses an extension of the syllogistic in which sentences are formed by using the quantifiers $\alpha, \iota, \eta, \omega$ in addition to the traditional quantifiers $A, E, I, O$. If $a$ and $b$ are terms and $Q$ is a quantifier then $Qab$ is a sentence. A sentence $Qab$ is an affirmative sentence if $Q$ is $A, I, \alpha, \iota$; otherwise, it is a negative sentence. Our discussion of sentences $Qab$ will be restricted to those in which $a \neq b$. As is customary $Aab$, $Eab$, $Iab$, and $Oab$ are read as ‘All $a$ are $b’$, ‘No $a$ are $b’$, ‘Some $a$ are $b’$, and ‘Some $a$ are not $b’$, respectively. $aab, iab, nab$, and $oab$ may be read as ‘There is exactly one $a$ and all $a$ are $b’$, and there is exactly one $b$ and all $b$ are $a’$, ‘There is exactly one $b$ and all $b$ are $a’$, ‘It is not true that $iab’$, and ‘It is not true that $aab’$, respectively. (So $nab$ and $oab$ may be read as disjunctive sentences.) Parry conjectures a decision procedure for validity for this extension of the traditional syllogistic, given that syllogisms have no more than two premises. We show that his decision procedure is correct even if syllogisms have more than two premises. And we axiomatize the Parry syllogistic and thus Aristotle’s syllogistic as well.

2 Preliminaries

Definition 2.1 $cd(Aab)$ (the contradictory of $Aab$) = $Oab$; $cd(Iab)$ = $Eab$, $cd(aab)$ = $oab$; $cd(iab)$ = $nab$; and $cd(cd(x)) = x$.

Definition 2.2 A pair $(W, v)$ is a model if and only if $W$ is a nonempty set and $v$ a function that maps terms into nonempty subsets of $W$ and maps sentences into $\{t, f\}$ where:

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(i) \( v(Aab) = t \) iff \( v(a) \subseteq v(b) \);
(ii) \( v(Iab) = t \) iff \( v(a) \cap v(b) \neq \emptyset \);
(iii) \( v(ab) = t \) iff \( v(b) \) has exactly one member and \( v(Aba) = t \);
(iv) \( v(\kappa ab) = t \) iff \( v(\kappa ab) = t \) and \( v(\kappa ba) = t \); and
(v) \( v(x) = t \) iff \( v(cd(x)) = f \).

A set \( X \) of sentences is consistent if and only if there is a model \( \langle W, v \rangle \) such that \( v \) assigns \( t \) to every member of \( X \); otherwise \( X \) is inconsistent. \( X \models x \) if and only if \( X \cup \{ cd(x) \} \) (or \( X, cd(x) \) for short) is inconsistent.

**Definition 2.3** Sentence \( x \) is a superordinate of sentence \( y \) if and only if \( \langle x, y \rangle \) has one of the following forms:

- \( \langle x, x \rangle \);
- \( \langle \alpha ab, \alpha ba \rangle \);
- \( \langle \eta ab, \eta ba \rangle \);
- \( \langle \omega ab, \omega ba \rangle \);
- \( \langle \kappa ab, \kappa ba \rangle \);
- \( \langle \eta ab, \eta ba \rangle \); or
- \( \langle \omega ab, \omega ba \rangle \); or
- \( \langle \eta ab, \eta ba \rangle \); or
- \( \langle \omega ab, \omega ba \rangle \); or
- \( cd(y) \) is a superordinate of \( cd(x) \) in virtue of one of the above forms. \( x \) is a subordinate of \( y \) if and only if \( y \) is a superordinate of \( x \).

The following proofs use this fact: if \( x \) is a superordinate of \( y \) (or \( y \) is a subordinate of \( x \)) then \( x \models y \) (which is short for \( \{ x \} \models y \)).

### 3 Decision procedure for validity

**Distribution** is defined by the following table:

<table>
<thead>
<tr>
<th>( ab )</th>
<th>( a )</th>
<th>( b )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Eab, aab )</td>
<td>( d )</td>
<td>( d )</td>
</tr>
<tr>
<td>( Aab, \eta ab )</td>
<td>( d )</td>
<td></td>
</tr>
<tr>
<td>( Qab, \iota ab )</td>
<td>( d )</td>
<td></td>
</tr>
<tr>
<td>( Iab, \omega ab )</td>
<td>( d )</td>
<td></td>
</tr>
</tbody>
</table>

So, for example, \( a \) is distributed in \( Eab \) and \( b \) is undistributed in \( \omega ab \). The following proofs use this fact: if \( x \) is a superordinate of \( y \) and term \( a \) is distributed in \( y \) then \( a \) is distributed in \( x \).

**Definition 3.1** A set \( C \) of sentences is a chain if and only if it has form \( Q_1[a_1a_2], \ldots, Q_n[a_{n-1}a_n], Q_{n+1}[a_na_1], \ldots, \) where each term \( a_i \) occurs exactly twice and no term occurs twice in a sentence.

**Theorem 3.2** A chain \( C \) is inconsistent if and only if:

(i) exactly one negative sentence occurs in \( C \);
(ii) every term is distributed at least once in \( C \); and
(iii) if \( \eta \) occurs in \( C \) so does \( \alpha \) or \( \iota \).

**Proof:** (Only if)

**Case 1:** (i) is not satisfied.

**Subcase 1:** No negative sentence occurs in \( C \). Construct chain \( C' \) by replacing every affirmative quantifier in \( C \) with \( \alpha \). \( C' \) is consistent given model \( \langle \{ 1 \}, v \rangle \), where \( v \) assigns \( \{ 1 \} \) to every term. So \( C \) is consistent since \( aab \models Qab \) if \( Q \) is affirmative.

**Subcase 2:** More than one negative sentence occurs in \( C \).
Definition 3.3 A set $X$ of sentences has form $\alpha[a_1 - a_n]$ if and only if either $X = \emptyset$ and $a_1 = a_n$ or $X$ has form $\alpha[a_1a_2], \ldots, \alpha[a_{n-1}a_n]$. A set $X$ of sentences has form $\alpha a_1 - a_n$ if and only if either $X = \emptyset$ and $a_1 = a_n$ or $X$ has form $\alpha a_1a_2, \ldots, \alpha a_{n-1}a_n$.

Construct chain $C'$ with form $\alpha[a_1 - a_2], Ea_2a_3, \ldots, \alpha[a_{n-1}a_n]$, $Ea_na_1$, where each sentence in $C$ is a subordinate of a sentence in $C'$ and each sentence in $C'$ is a superordinate of a sentence in $C$. $C'$ is consistent given model $\{\{1, 2, 3\}, v\}$, where $v(x) = \{1\}$ if $x$ is $a_2$ or $x$ is a term in a member of $\alpha[a_1 - a_2]$; $v(x) = \{2\}$ if $x$ is $a_{2n}$ or $x$ is a term in a member of $\alpha[a_{2n} - a_{2n-1}]$, where $n$ is even and $n > 2$; and $v(x) = \{3\}$ if $x$ is any other term. So $C$ is consistent.

So, for example, $Eab, Acb, \eta cd, Ide, Oef, \alpha fg, \omega gh, \theta ha$ is consistent since $Eab, \alpha, Ecd, ade, Eef, \alpha fg, Egh, \alpha ha$ is consistent given model $\{\{1, 2, 3\}, v\}$ such that $v(b) = \{1\}$, $v(c) = \{1\}$, $v(d) = \{2\}$, $v(e) = \{2\}$, $v(f) = \{3\}$, $v(g) = \{3\}$, $v(h) = \{2\}$, and $v(a) = \{2\}$.

Case 2: (i) is satisfied, but (ii) is not.

Subcase 1: A term $a$ is undistributed in two affirmative sentences in chain $C$. Construct chain $C'$ with form $Ede, \alpha[e - b], iab, iac, \alpha[c - d]$, where each sentence in $C$ is a subordinate of a sentence in $C'$ and each sentence in $C'$ is a superordinate of a sentence in $C$. $C'$ is consistent given $\{\{1, 2\}, v\}$, where $v(x) = \{1\}$ if $x = e$ or $x = b$ or $x$ is a term in $\alpha[e - b]$, $v(x) = \{2\}$ if $x = c$ or $x = d$ or $x$ is a term in $\alpha[c - d]$, and $v(x) = \{1, 2\}$ if $x$ is any other term. So $C$ is consistent.

Subcase 2: A term $a$ is undistributed in an affirmative sentence and a negative sentence. Construct chain $C'$ with form $\alpha[c - b], iab, Oac$, where each sentence in $C$ is a subordinate of a sentence in $C'$ and each sentence in $C'$ is a superordinate of a sentence in $C$. $C'$ is consistent given $\{\{1, 2\}, v\}$, where $v(a) = \{1, 2\}$ and for every term $x$ other than $a$, $v(x) = \{1\}$. So $C$ is consistent.

Case 3: (i) and (ii) are satisfied, but (iii) is not. Then $C$ has form $\eta ab, Ab - a$. So $C$ is consistent given $\{\{1, 2\}, v\}$, where for every term $x$, $v(x) = \{1, 2\}$.

(If) We show that every chain that satisfies conditions (i) to (iii) is “reducible” to a chain with two members that satisfies these conditions. Then we rely on the inconsistency of these two-membered chains. □

Definition 3.4 A chain $x, y, X$ is $1$-reducible to a set $z, X$ if and only if $x, y \models z$.

So, for example, chain $Eab, ibc, iac$ is $1$-reducible to $Eac, iac$.

Lemma 3.5 If a chain $C$ with $n$ $(n \geq 3)$ sentences satisfies conditions (i) to (iii) then it is $1$-reducible to a chain with $n - 1$ sentences that satisfies conditions (i) to (iii).

Proof: Assume the antecedent where $C = Q_1ab, Q_2[bc], X$. There are exactly four cases to consider since exactly one negative sentence occurs in $C$.

Case 1: $Q_1 = E$.

Subcase 1: $c$ is distributed. Then $Q[bc]$ is a superordinate of $Acb$. Since $Eab, Acb \models Eac, C$ is $1$-reducible to $Eac, X(C')$. $C'$ obviously satisfies conditions (i) and
(iii). Condition (ii) is satisfied since any every term that occurs in $C'$ that is distributed in $C$ is also distributed in $C'$.

**Subcase 2:** $c$ is undistributed. Then $Q[bc]$ is a superordinate of $Ibc$. $Eab, Ibc \models Oca$.

**Case 2:** $Q_1 = O$.

**Subcase 1:** $c$ is distributed. Then $Q[bc]$ is a superordinate of $Acb$. $Oab, Acb \models Oac$.

**Subcase 2:** $c$ is undistributed. Then $Q[bc]$ is $I[bc]$. $Abc$, or $icb$. Then there must be some affirmative sentence $Q_2[dc]$ in which $c$ is distributed. Then $Q_2[dc]$ is a superordinate of $Acd$. Suppose $d$ is distributed. $I[bc], \alpha[cd] \models ibd$. Suppose $d$ is undistributed. $I[bc], Acd \models Ibd$.

**Case 3:** $Q_1 = \eta$.

**Subcase 1:** $c$ is distributed. Then $Q[bc]$ is $\alpha[bc]$. $\etaab, \alpha[bc] \models Eac$.

**Subcase 2:** $c$ is undistributed. Then $Q[bc]$ is $Abc$ or $icb$. Suppose the former. $\etaab, Abc \models \etaac$. Suppose the latter. $\etaab, icb \models Oca$.

**Case 4:** $Q_1 = \omega$.

**Subcase 1:** $c$ is distributed. Then $Q[bc]$ is $\omega[bc]$. $\omegaab, \omegabc \models Oac$.

**Subcase 2:** $c$ is undistributed. Then $Q[bc]$ is a superordinate of $Abc$. $\omegaab, Abc \models \omegaac$.

**Lemma 3.6** Suppose $C_1, \ldots, C_n$ is a sequence of chains such that $C_1$ satisfies conditions (i) to (iii) and $C_j$ is $I$-reducible to $C_{j+1}$. Then $C_n$ satisfies conditions (i) to (iii) and $C_1$ is inconsistent if $C_n$ is inconsistent.

**Proof:** Use induction, relying on the preceding lemma and this fact: if $X, y$ is inconsistent and $Y \models y$ then $X, Y$ is inconsistent.

**Lemma 3.7** Every two-membered chain that satisfies conditions (i) to (iii) is inconsistent.

**Proof:** The only two-membered chains that satisfy conditions (i) to (iii) are: $Eab, Iab; Oab, Aab; \etaab, \lab; \omegaab, \aab$; and their superordinates, where ‘chains that are superordinates of chains’ is defined in the natural way. It is easily shown that these chains are inconsistent.

**Corollary 3.8** (Smiley [3]) A chain in which no quantifiers other than $A$, $E$, $I$, or $O$ occur is inconsistent if and only if it has one of the following forms: (i) $Oab$, $Aa - b$, (ii) $Eab$, $Ac - a$, $Ac - b$, or (iii) $Eab, I[cd]$, $Ac - a$, $Ad - b$. 
Definition 3.9 Suppose $X$ is a set of sentences and $x$ is a sentence. $(X, x)$ is a syllogism if and only if $X, cd(x)$ is a chain. $X \models y$ (‘$X$, so $x$’ is valid) if and only if $X, cd(x)$ is inconsistent.

Theorem 3.10 Theorem 3.2 provides a decision procedure for determining whether any syllogism is valid.

Proof: Note that $X \models x$ if and only if $X, cd(x)$ is inconsistent. □

Theorem 3.11 A syllogism is valid if and only if there is no countermodel with a three-membered domain.

Proof: Given the above proof of Theorem 3.2 every consistent chain can be shown to be consistent by using a three-membered model. □

Definition 3.12

(i) Dilution If $X \vdash x$ then $X, Y \vdash x$.

(ii) Cut If $X \vdash x$ and $Y, x \vdash y$ then $X, Y \vdash y$.

(iii) Antilogism $X, x \vdash y$ then $X, cd(y) \vdash cd(x)$.

(iv) Reductio If $X, x \vdash y$ and $X, x \vdash cd(y)$ then $X \vdash cd(x)$.

(v) Superordination $x \vdash y$ if $x$ is a superordinate of $y$.

(vi) Basic syllogisms $Aab, Abc \vdash Aac$; $Iab, Abc \vdash Iac$; $Iab, abc \vdash iac$; $Aab, abc \vdash aac$; and $Aab, icb \vdash ca$.

(vii) $X \vdash y$ if $X \vdash y$ in virtue of (i) to (vi).

Theorem 3.13 If $X, x$ is a syllogism then $X \models x$ if and only if $X \vdash x$.

Proof: (If) Straightforward. (Only if) Assume the antecedent. Given the proof of Theorem 3.2 there is a sequence of chains $C_1(X, cd(x)), \ldots, C_n(y, z)$ such that $(y, z)$ satisfies conditions (i) to (iii) of Theorem 3.2 and $C_j$ is 1-reducible to $C_{j+1}$. $C_n \vdash y$ and $C_n \vdash cd(y)$ (by Superordination and Dilution); $C_j \vdash C_{j+1}$ (by Basic syllogisms, Superordination, Antilogism, and Cut); and $C_1 \vdash y$ and $C_1 \vdash cd(y)$ (by repeated uses of Cut). So $X \vdash x$ (by Reductio). □

We illustrate the algorithm for showing that $X \vdash x$ given $(X, x)$ is a valid syllogism by considering the Pseudo-Scotus valid syllogism mentioned in note 1: $\vdash II \vdash I$. $Eab \vdash Eab$ (by Reflexivity); $tabl \vdash cd(Eab)$ (by Superordination). So $Eab, tab \vdash Eab$ and $Eab, tab \vdash cd(Eab)$ (by Dilution). $Eac, icb \vdash Eab$ (by Antilogism) since $Iab, icb \vdash Iac$ (by Cut) since $icb \vdash Abc$ (by Superordination) and $Iab, Abc \vdash Iac$ (by Basic syllogisms). So $Eac, icb, tab \vdash Eab$ and $Eac, icb, tab \vdash cd(Eab)$ (by Cut). So $icb, tab \vdash Iac$ (by Reductio).

Departing from the algorithm, $icb, tab \vdash Iac$ (by Cut) since $icb \vdash Abc$ and $tab \vdash Iab$ (by Superordination) and $Abc, Iab \vdash Iac$ (by Basic syllogisms).

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NOTES

1. See Parry [2] for alternative ways of reading the nontraditional sentences and for a useful discussion of the history of them. He reads $\text{tab}$ as ‘Some b is every a.’ And he claims that the earliest known example of a syllogism with such nontraditional sentences is due to Pseudo-Scotus: ‘Something that moves in a circle is every moon. Something shining is every moon. So something shining moves in a circle.’ An alternative formulation of the syllogism is: ‘There is exactly one moon and all moons are things that move in a circle. There is exactly one moon and all moons are things that shine. So some things that shine are things that move in a circle.’ So this nontraditional syllogism has mood and figure $\text{II} - 2$.

2. Our axiomatization of Aristotle’s syllogistic is similar to Smiley [3].

3. Parry [2] conjectures that this theorem holds for chains with three members given the following fourth condition is added: if $\omega$ occurs then $\alpha$ or $\iota$ occurs. Given our theorem the fourth condition is superfluous.

4. Johnson [1] proves the special case of this theorem that involves only the Aristotelian quantifiers.

REFERENCES


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