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Charles L. Dodgson's Work on Trigonometry

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Abstract: There is growing interest in the mathematics of Charles L. Dodgson, alias Lewis Carroll (1832–1898). His contributions to geometry, algebra, logic, voting theory and recreational mathematics have been reviewed in recent literature. Dodgson's work in trigonometry is less known. He wrote his initial work on trigonometry, a pamphlet titled *The Formulae of Plane Trigonometry*, out of concern for his students to ease their studying burden and to improve their performance on examinations. Notable for its precision and completeness, it has all the tell-tale characteristics we associate with the serious mathematical writings he produced throughout his lifetime. He designed his later trigonometric work as the legatee of Augustus De Morgan's correspondence with would-be circle squarers to appeal to and be understood by these mathematical dilettantes. This paper surveys Dodgson's trigonometric work and what it teaches us about Dodgson's mathematical practices.

Keywords: *British mathematics, Charles L. Dodgson, circle-squaring, nineteenth century, trigonometry*

Introduction

Recent years witnessed a growing interest in the mathematics of Charles L. Dodgson, aka Lewis Carroll (1832–1898). Dodgson was the Lecturer in mathematics at Christ Church, the University of Oxford. Dodgson's fame is due to his literary works, mainly the two Alice tales: *Alice's Adventures in Wonderland* (1865) and *Through the Looking Glass* (1871).Yet, a careful look at his mathematical work shows interesting contributions to various disciplines. Recent scholarship addressed specifically Dodgson's work in geometry, algebra, logic, voting theory and recreational mathematics (Abeles, 1994; Wilson & Moktefi, 2019). It is less known that Dodgson contributed to the area of trigonometry. The aim of this paper is to survey Dodgson's trigonometric work and what it teaches about Dodgson's mathematical practices.

Dodgson's early work on trigonometry

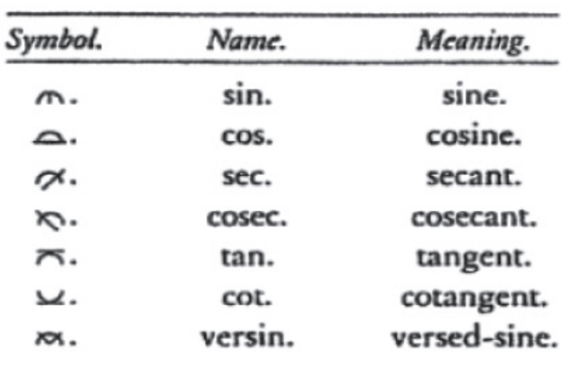
By June 1861, Dodgson had been the Mathematical Lecturer for less than six years. Understandably, his enthusiasm for the position was large, manifesting itself in several academic endeavors, especially publications and activities to assist his students. By then he already had published a book on geometry (*Syllabus of Plane Algebraical Geometry* in 1860) and two pamphlets, one on geometry (*Notes on the First Two Books of Euclid* in 1860), and one on algebra (*Notes on the First Part of Algebra* in 1861). All three reflect an educational purpose.

In his early years, just before and then soon after his appointment as the Mathematical Lecturer, Dodgson expressed some interest in trigonometry. In a diary entry on 10 May 1855, he comments that he had proved the formula '$\cos A = (b^2 + c^2 - a^2)/2bc$' using proposition 47 of Euclid's Book I (The Pythagorean theorem) and without using Book II (Wakeling, 1993, pp. 95–96). On 26 January 1856, he gave his first lecture on trigonometry (Wakeling, 1994, p. 28).

In addition to his 1861 pamphlet *The Formulae of Plane Trigonometry* (Dodgson, 1861; Abeles, 1994, pp. 121–139), Dodgson's early pieces on trigonometry include two unpublished pieces. The first piece "Formulae (Group C)" is undated and consists of three pages of trigonometric formulas (Abeles, 1994, pp. 140–143). Since it was produced with an electric pen, its earliest possible date is 1877 (the year Dodgson acquired it). At the top of the first sheet, Dodgson addressed a comment to 'the pupil', making it clear who his intended audience was. The formulas correspond to topics in sections G and L of *A Guide to the Mathematical Student in Reading, Reviewing, and Working Examples*, published in 1864 (Abeles, 1994, pp. 372–404). The second piece is an unsigned cyclostyled sheet titled "Formulae" and dated 19 March 1878 (Abeles, 1994, p. 197). It consists of 18 formulas, of which six are trigonometric. These formulas correspond to Section L of *A Guide to the Mathematical Student in Reading, Reviewing, and Working Examples*.

In a letter to his sister, Mary, dated 20 February 1861, Dodgson lists seven works, some finished and some in progress. In a note to this letter, Morton Cohen suggest that number 5, "Collection of formulae (1/2 done)", is probably *The Formulae of Plane Trigonometry* (Cohen, 1979, pp. 47–48). In *The Lewis Carroll Handbook*, 11 June 1861 is the date given for its Preface (Williams *et al.*, 1979, p. 19). However, it is unlikely that the "Collection of formulae" represents the entire *The Formulae of Plane Trigonometry* pamphlet. More likely is that this "Collection" was a part of the arrangement of "Formulae of Pure Mathematics", a work that Dodgson announced in the Preface of *The Formulae of Plane Trigonometry* that he was preparing for publication, but it was never published.

The Formulae of Plane Trigonometry has all the tell-tale characteristics we associate with the serious mathematical writings he produced throughout his lifetime. Fundamental to his approach is the need for precision. For him the main goal of trigonometry is to determine an angle, and we can do that once we have a clear idea of all the different meanings the term 'angle' has. More importantly, he has constructed a notational system for the familiar trigonometric ratios that he would like to substitute for the standard ones. To accomplish this, in the Preface of this pamphlet, he reached out to other mathematicians for their opinion about his new notation (This topic will be developed further later in this paper). Consistent with Dodgson's artistic sensibilities, these symbols have an artistic flavor. And he makes the following claims for them: they suggest their meaning; they can be written easily; they are connected with each other, and, he believes, are different from any existing symbols. Dodgson's proposed new symbols for the trigonometric ratios are given in Figure 1[1].



Symbol.	Name.	Meaning.
	sin.	sine.
	cos.	cosine.
	sec.	secant.
	cosec.	cosecant.
	tan.	tangent.
	cot.	cotangent.
	versin.	versed-sine.

Figure 1. Table of trigonometric symbols (Abeles, 1994, p. 131)

[1] *Versine x = 1 – cos x* always has a positive value. It was used by navigators who employed logarithms to calculate distance from the coordinates of their current position to those of their destination.

Dodgson planned to publish a collection of mathematical formulas and so he wants to know if there are additional or better ones that should be included. Formulas, plenty of them, were essential to a student's performance on the examinations required for the B.A. degree at Oxford.

The Formulae of Plane Trigonometry: the subjects of plane trigonometry

In the section 'Preliminary remarks' of *The Formulae of Plane Trigonometry*, Dodgson gives the three subjects that comprise plane trigonometry: Part I. Goniometry, the measurement of angles by angular units, either a right angle or a 'radial angle', about 57.3 degrees; Part II. Goniometry by ratios, the indication, but not the measurement of angles; Part III. Trigonometry, the properties of rectilinear figures.

In Part II, Dodgson distinguishes three types of angles: the 'geometrical angle' that Euclid treats, i.e. an absolute magnitude without regard to direction (Fig. 2); the 'angle of position' which can be positive or negative depending on which way it's measured the shortest way around (Fig. 3); the 'angle of revolution', an extension of the 'angle of position' (Fig. 4) (Figures 2–4 are reproduced in Abeles, 1994, pp. 128–129).

Summarizing, Dodgson states that to measure a 'geometrical angle *A*', the magnitude of one of cos *A*, tan *A*, sec *A*, cot *A* is needed. To determine an 'angle of position *A*', one of cos *A*, tan *A*, sec *A*, cot *A*, sin *A*, cosec *A*, and the signs of two that are not reciprocals is needed. To determine an 'angle of revolution', an angular unit must be given.

In Part III, Dodgson gives formulas for each of Parts I, II, and III of his pamphlet. Part I has one formula—connecting English degrees with French grades and with radial angles. Part II has twenty-four formulas, divided into five groups. Formulae in the first group consist of six formulas concerning one angle; in the second group the eight formulas concern two or more angles; in the third group, on the powers of co-indicants, there are four formulas. The fourth group has one formula, the summation of series of co-indicants. The fifth group connects the co-indicants of an angle with its radial measure. Here there are five formulas, including number 23, John Machin's (1838–1916) arctangent series to find ∏, and number 21, Gregory's (James Gregory 1638–1675) arctangent series

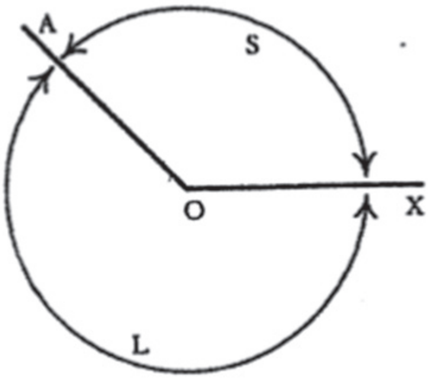


Figure 2. The geometrical angle

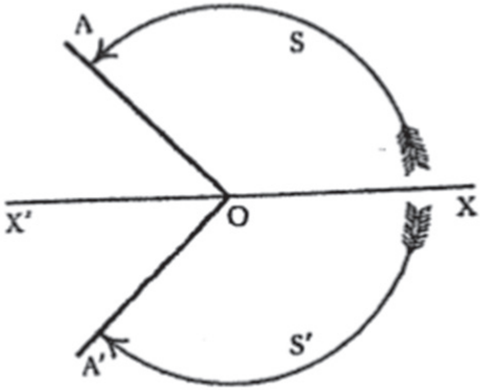


Figure 3. The angle of position

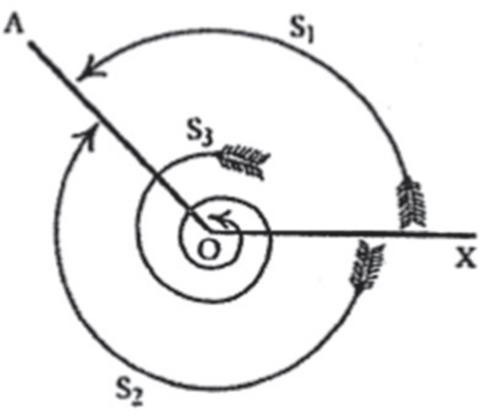


Figure 4. The angle of revolution

whose appearance in this 1861 publication contradicts Morris Kline's statement that Gregory's work was generally unknown before 1939 (Kline, 1972, p. 355)[2]. In the nineteen formulas of Part III, ten deal with triangles, four with quadrilaterals, and five with regular polygons.

Oddly, the mathematician Augustus De Morgan (1806–1871), in his review of *The Formulae of Plane Trigonometry*, did not comment on any of the book's text, perhaps because he considered the topics to be standard ones and thought that Dodgson did not provide any new insights into them (De Morgan, 1861).

[2] This observation is due to Victor Katz, for which the author is grateful.

Later work on trigonometry

Subsequent published work on trigonometry occurred more than thirty years later, in Dodgson's twenty-four 'Pillow Problems', where he used his new notational symbols for the trigonometric relations; and in Question 11530 submitted to *The Educational Times* in 1892.

The first of four editions of *Curiosa Mathematica, Part II. Pillow Problems* appeared in 1893. Of its 72 problems, 20 are on plane trigonometry, another four on solid trigonometry. The earliest, no. 55, is dated 20 December 1874, the latest, no. 37, bears the date December 1891. However, nos. 6 and 70 are undated, so it's possible either one could have an earlier or later date (Dodgson, 1895).

Dodgson contributed eleven items to *The Educational Times*, a monthly London periodical that began in November 1848 to publish several columns on mathematical problems and their solutions (Grattan-Guinness, 1992). Question 11530 appeared on 1 May 1892 (Dodgson, 1893; Abeles, 1994, pp. 150–154). It requires an 'investigation' of the trigonometric formula,

$$tan^{-1}\, 1/a \;=\; tan^{-1}\, 1/(a+x) + \; tan^{-1}\, 1/(a+y) = tan^{-1}\; (2a + x + y) \,/\, \{a^2 \;+\; a(x + y) \;+ xy -1\}$$

Dodgson remarked that using this formula he obtained the limits 3.141597 and 3.141583 for ∏. In the published solution by the British mathematician, H. J. Woodall, he remarks that the formula appears in Prof. William Wallace's article, "Algebra" in the 7th (1842) and 8th (1860) editions of the *Encyclopedia Britannica* (Abeles, 1994, p. 151).

Dodgson became interested in the calculation of ∏ when, after De Morgan's death, he took over the unenviable position of correspondent to the many would-be mathematicians who claimed they had 'squared' the circle. Both De Morgan and Dodgson knew that ∏ is not a rational number and so the area of a unit circle cannot be made equal to the area of a unit square. Carl L. F. Lindemann (1852–1939) first proved the irrationality of ∏ in 1882. In a diary entry of 26 August 1879, Dodgson wrote that he tried to find lower and upper limits for ∏ [trigonometrically] for the benefit of circle-squarers (Wakeling, 2003, p. 203). His method required dividing a 45 degree angle (∏/4) into angles with rational tangents. Using the series,

$$\{1/k\} = \{1/(k + x)\} + \{1/(k + y)\},\; xy \;=\; k^2 + 1$$

he tried to find the number of these angles that would produce the best value for ∏. By August 1881, his best approximation, ∏ < 3.1450, used eight angles. He continued to work with his method, achieving the limits 3.141583, 3.141597 on July 22, 1884 (Wakeling, 2004, p. 129). His method produced a relatively accurate, intuitively sensible 'do-it-yourself-kit' for the late nineteenth century would-be circle-squarers whose backgrounds included Euclidean geometry.

Dodgson continued to work on this topic despite Lindemann's proof, probably because dilettante circle-squarers would not be aware of the proof and even if they were, they could not easily apprehend it and its application to the circle-squaring problem. To this end, Dodgson took further, more important steps. He wrote a treatise titled *The Limits of Circle-Squaring. Simple Facts For Circle-Squarers*, but it was never published (Abeles, 1994, pp. 144–147). In the introductory chapter, dated 20 April 1882, Dodgson explained his reasons for writing this treatise (Madan, 1932, pp. 121–125). He was not satisfied in merely disproving each circle-squarer's 'theorem' because such a person would not even listen to nor believe anyone refuting his 'proof', and, more importantly, the treatise would free Dodgson from having to respond to every new circle-squarer's fallacious arguments (Abeles, 1993, p. 153; 1994, pp. 144–147). And in 1888, in the preface to his book, *Curiosa Mathematica. Part I. A New Theory of Parallels*, he describes his task of refuting the circle-squaring crowd (Dodgson, 1888).

Dodgson's outreach to the mathematical community

Throughout his career, Dodgson sought opinions from other mathematicians whenever he had an innovation that he wanted the mathematical community to adopt. The earliest is in the Preface to *The Formulae of Plane Trigonometry* discussed earlier in this paper. Here he asks three questions of his readers. First, are there any objections to his symbols? If not, his second question asks if the reader can suggest better symbols. Question 3 asks *if* the symbols did appear in a published work, would the reader not object to use or recommend that work?

In his review of *The Formulae of Plane Trigonometry*, Augustus De Morgan answers Dodgson's questions. To the first, he answers that he has no objections to them, but he objects to their introduction in an elementary work particularly by a teacher who can compel their use. He answers the second question in the negative because the new symbols require two, rather than one, pen stroke. To the third question he claims the proposed changes require more discussion, and he makes a case for the British Association taking up the topic of mathematical notation in general for discussing improvements in notation. In spite of these comments, De Morgan considers Dodgson's notations to be "one of the best which could be suggested" (De Morgan, 1861, p. 113).

Dodgson's *Circular to Mathematical Friends*, issued in June 1862, is another instance of him asking his colleagues for their opinions on mathematical subjects. Dodgson asked them to comment on the accompanying sheets (e.g., Are any subjects omitted? Are they in proper order?), which displayed all the topics in pure mathematics together with a list of 2,000 examples referenced by topic (Abeles, 1994, pp. 350–351). In a similar manner, Dodgson issued in 1877 a *Circular accompanying A Method of Taking Votes on More Than Two Issues*, the third pamphlet in his work on voting theory, in which he asks his Oxford colleagues to offer criticisms of this 1876 pamphlet (Abeles, 2001, p. 59; Black, 1958, p. 234). Dodgson had distributed all three of his pamphlets not only to colleagues in Christ Church and in the University, but also to members of Parliament. And he wrote many letters to the newspapers, the *Pall Mall Gazette*, and the *St. James's Gazette*, explaining his views. Several well-known political figures replied to these letters (see Abeles, 2001, pp. 105–175).

Later on, Dodgson made available to interested logicians at his printers', Messrs. Parker on Broad Street, *A Challenge to Logicians*, a single sheet dated October 1892 containing a single sequence problem with eleven premises involving fourteen classes (Abeles, 2010, pp. 109–110). The problem is similar in structure to a hypothetical, a topic that highly intrigued him in the early 1890s (Moktefi & Abeles, 2016). It subsequently led to the 'Barbershop problem', the first of two papers he published under his pseudonym, Lewis Carroll, in *Mind*. Many logicians responded with comments, including William E. Johnson (1858–1931) and Hugh MacColl (1837–1909) (Moktefi, 2007; Abeles, 2010, pp. 111–183). In 1895, Dodgson also sent his *Logical Nomenclature. Desiderata* to many logicians inviting them to respond to the technical questions he proposed. There are two versions of this piece, one shorter than the other—four questions in the shorter version, eight questions in the longer version. John Venn (1834–1923) answered five of the eight questions (Abeles, 2010, pp. 49–52).

Conclusion

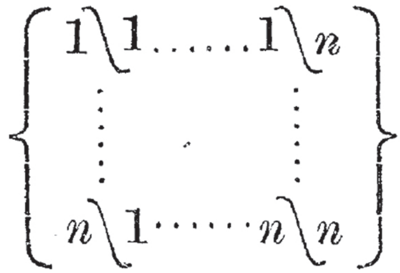


Figure 5. An algebraical matrix (Dodgson, 1867, p. 9)

It was in *The Formulae of Plane Trigonometry* that Dodgson first demonstrated his concern with improving mathematical symbolization. He continued to be intrigued by this topic throughout his lifetime. Several years after writing *The Formulae of Plane Trigonometry*, he approached in his 1867 book, *Elementary Treatise on Determinants*, the symbolization problem in linear algebra in a similar manner, defining a new symbol for a matrix and its entries (Abeles, 2008; Rice, 2019). He called a matrix a 'Block' because he thought 'matrix' means the *form* into which algebraical quantities are placed, rather than the actual 'assemblage' of them (Fig. 5). He found his symbol for a matrix "simple, distinct, and easy to be written" (Dodgson, 1867, p. iv).

Much later, he returned to the same theme in his two logic books, published in 1886 and 1896, *The Game of Logic* and *Symbolic Logic, Part I, Elementary*, respectively (Carroll, 1958). He introduced a new notation to make logical calculations easier and more accessible to a wide audience that customarily considered the subject to be difficult (Moktefi, 2015). His subscript notation differs from that of any of his contemporaries and reflects his criteria for good symbols: simple, distinct, and easily written (Abeles, 2005; Moktefi, 2019; For connections between geometry and logic in Dodgson's work, see Abeles & Moktefi, 2011). In Figure 6, Dodgson argues that when a sequence of premises is joined to the denial of the derived conclusion in a triliteral argument, the result is absurdity.

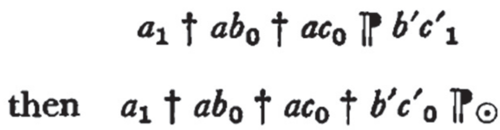


Figure 6. Logic symbols for a triliteral argument (Bartley, 1986, p. 278)

Alas, none of his new symbol systems were adopted by the mathematical community. And his trigonometric symbols appeared only in his *Curiosa Mathematica, Part II. Pillow Problems*, which was published five years before his death.

Dodgson's later trigonometric work revolving around the circle-squaring problem, demonstrates his precise and thorough approach, designed to appeal and to be understood by would-be circle squarers. Alas, even had he published his projected book *The Limits of Circle-Squaring. Simple Facts For Circle-Squarers*, it might not have deterred the prestigious journal, *Nature*, from publishing, in 1914, T. M. P. Hughes' article: "The diameter of a circle equal in area to any given square" (Hughes, 1914).

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