D. Küchemann (Ed.) Proceedings of the British Society for Research into Learning Mathematics, 27, 3, November 2007

FALLACIES IN MATHEMATICS

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This paper considers the application to mathematical fallacies of techniques drawn from informal logic, specifically the use of 'argument schemes'. One such scheme, for Appeal to Expert Opinion, is considered in some detail.

It might be supposed that mathematical fallacies could be defined very simply. If all mathematical reasoning is formal and deductive, then surely mathematical fallacies are merely invalid arguments? This definition has several shortcomings. Firstly, there are many invalid mathematical arguments that would not normally be described as mathematical fallacies. Secondly, much reasoning in mathematics is conducted informally. So a satisfactory account of mathematical fallacies must explain what is distinctive about formal fallacies, beyond their invalidity, and also address informal fallacies.

THE STANDARD TREATMENT

Most logic textbooks contain a chapter on informal fallacies. Characteristically this is based on the 'standard treatment' of fallacy, as an argument which 'seems to be valid but is not so' (Hamblin, 1970, p. 12). This definition may be traced back to Aristotle, for whom 'that some reasonings are genuine, while others seem to be so but are not, is evident. This happens with arguments as also elsewhere, through a certain likeness between the genuine and the sham' (Aristotle, 1995, 164a). However, the apparently subjective concept of 'seeming valid' has brought the standard treatment into disrepute.

This problem is perhaps especially acute for mathematics. As the philosopher and novelist Rebecca Goldstein has her fictitious mathematician Noam Himmel declare,

> [I]n math things are exactly the way they seem. There's no room, no logical room, for deception. I don't have to consider the possibility that maybe seven isn't really a prime, that my mind conditions seven to appear a prime. One doesn't — can't — make the distinction between mathematical appearance and reality, as one can — must — make the distinction between physical appearance and reality. (Goldstein, 1983, p. 95)

On this account, Aristotle's 'likeness between the genuine and the sham' could never arise, since there would be no sham. This would rule out mathematical fallacies, at least on the standard treatment.

However, the difference between appearance and reality that Himmel believes impossible is not that upon which the standard treatment rests. Himmel is rejecting radical scepticism about mathematical truth. His position is persuasive, although not indisputable, but it still leaves room for a weaker, error-based distinction between appearance and reality. Mathematical objects often appear to be other than they really are — through a failure of understanding. Himmel's argument might be paraphrased as the thesis that mathematical understanding guarantees mathematical truth: once one understands the concept of prime number, one realizes that seven must be prime. This is not true in natural science: understanding the concepts of ether, caloric or phlogiston does not make them real. But in both cases, if understanding is absent, truth may well be absent too. This leaves plenty of room for mathematical fallacies.

ARGUMENT SCHEMES AND FALLACIES

Many alternative theories of fallacy have been suggested. One approach employs argument schemes: 'forms of argument that model stereotypical patterns of reasoning' (Walton and Reed, 2003, p. 195). Several subtly different characterizations of the argument scheme are in use. We shall be following the influential treatment of Douglas Walton, defended in several books and articles since the early 1990s. Walton's argument schemes are presented as schematic arguments accompanied by 'critical questions'. The critical questions itemize known vulnerabilities in the argument, to which its proposer should be prepared to respond. Most of Walton's argument schemes are presumptive or defeasible, but deductive inferences can also be understood as argument schemes.

A crucial issue for the argument scheme approach is normativity: what makes an instance of an argument scheme good or bad? The answer depends on the nature of the scheme. If the scheme is deductive, then it is good precisely when it is valid, as we would expect. Defeasible schemes obviously have less normative force, which might be captured in several ways. On Walton's approach a scheme is persuasive if all of its critical questions (or at least those where the burden of proof is on the proposer of the argument) receive satisfactory answers, and otherwise not. Hence restating defeasible schemes to incorporate answers to the critical questions as additional premises will convert them into deductive schemes (Walton and Reed, 2003, p. 210). In principle, this can always be done, but at the cost of obscuring the dialectical context in which the schemes are typically employed.

Fallacies may be understood as argument schemes used inappropriately. This cashes out the troublesome concept of 'seeming valid' in the stereotypical character of the schemes. Essentially, they have been chosen as representative of the sort of arguments generally found convincing. The use of an argument scheme can be fallacious in two distinct ways. Firstly, some schemes are invariably bad. They are distinguished from other invalid arguments by their tempting character, presumably a consequence of their similarity to a valid scheme. This is typical of formal fallacies, such as the quantifier shift fallacy. Secondly, argument schemes with legitimate instances can be misused, when deployed in circumstances that preclude a satisfactory answer to the critical questions. For example, the argument scheme for Appeal to Expert Opinion (discussed below) is associated with questions addressing the source of the opinion. The traditional *ad verecundiam* fallacy arises when these questions are not properly answered. One attraction of the argument scheme approach to fallacy theory is that it rehabilitates the conventional fallacies as defective examples of defeasible but sometimes persuasive argument schemes.

APPLICABILITY TO MATHEMATICS

There are two ways to apply argument schemes to mathematics. Firstly, specifically mathematical argument schemes could capture patterns of reasoning unique to that context. Secondly, context-independent argument schemes could explicate the connexions between mathematical practice and ordinary reasoning. We shall concentrate on the second approach.

Argument schemes may be loosely grouped in terms of the conclusions they establish. These include particular and general propositions to be accepted or rejected, actions to be performed, assessments of other arguments, causal claims, rules to be followed or ignored, and commitments to be ascribed to agents. Arguments of most or all of these kinds may be found in mathematical reasoning: mathematics is not just the derivation of conclusions from axioms.

There is not enough space to consider all, or indeed more than one, of these schemes. We shall concentrate on exploring how argument schemes may explicate the role of authority in mathematics.

Argument Scheme for Appeal to Expert Opinion

Major Premise: Source *E* is an expert in subject domain *D* containing proposition *A*.

Minor Premise: *E* asserts that proposition *A* (in domain *D*) is true (false).

Conclusion: *A* may plausibly be taken to be true (false).

Critical Questions:

(1) *Expertise Question*: How credible is *E* as an expert source?

(2) *Field Question*: Is *E* an expert in the field that *A* is in?

(3) *Opinion Question*: What did *E* assert that implies *A*?

(4) *Trustworthiness Question*: Is *E* personally reliable as a source?

(5) *Consistency Question*: Is *A* consistent with what other experts assert?

(6) *Backup Evidence Question*: Is *E*'s assertion based on evidence? (Walton, 2006, pp. 87 f.)

Mathematicians have a schizophrenic attitude to authority. On the one hand, one should puzzle everything out for oneself. Reluctance to do so may betray an ‘authoritarian proof scheme’: ‘the view that mathematics is a collection of truths, with little or no concern and appreciation for the origin of the truths’; at worst ‘either the student is helpless without an authority at hand, or the student regards a justification of a result as valueless and unnecessary’ (Harel and Sowder, 1998, pp. 247; 275). On the other hand, division of labour is unavoidable, and much to be desiderated. Some results, without being seriously in doubt, are routinely supported by rote appeal to authority, which can sometimes prove fugitive.

This antithesis may be partially resolved by appeal to context. The authoritarian proof scheme is bad because it substitutes authority where reasoning is required — chiefly student exercises, that is a pedagogic context. Conversely, appeal to settled results is a sensible allocation of resources in original work. This distinction may be expressed in terms of the characteristic response to the critical questions. Professional mathematicians may be trusted to demand satisfactory answers for these questions, which students in the grip of the authoritarian proof scheme do not even think to ask. Hence, the student usage is fallacious, but the professional usage is not.

A recent discussion of mathematical folklore remarks on 'the anxieties felt by many mathematicians regarding the degree to which mathematical truth is dependent upon the trustworthiness of previous results. This anxiety is exacerbated by the fact that some mathematicians have a less rigorous proof style than other mathematicians' (Renteln and Dundes, 2005, p. 28). The authors reproduce (without attribution) a widely circulated list of spurious proof methods collated by Dana Angluin, which includes several abuses of Appeal to Expert Opinion. For example,

> Proof by eminent authority: 'I saw Karp in the elevator and he said it was probably NP-complete.'
>
> Proof by personal communication: 'Eight-dimensional colored cycle stripping is NP-complete [Karp, personal communication].' (Angluin, 1983, p. 16)

These are failures of the opinion question. They comprise appeals to an authority which may not say what he is reputed to say. Their juxtaposition neatly illustrates how such legerdemain can come about. Another potential abuse of this scheme is the appeal to one's own authority:

> Proof by vigorous handwaving: Works well in a classroom or seminar setting.
>
> Proof by vehement assertion: It is useful to have some kind of authority relation to the audience. (Angluin, 1983, pp. 16 f.)

Since the source of authority is identical with the arguer, the opinion question receives a clear answer, but any or all of the other critical questions may not.

Empirical research appears to confirm the persuasiveness of Appeal to Expert Opinion in mathematics. Matthew Inglis and Juan Pablo Mejia-Ramos (2006) studied the reception of a version of the following argument:

> All the evidence is that there is nothing very systematic about the sequence of digits of pi. Indeed, they seem to behave much as they would if you just chose a sequence of random numbers between 0 to 9. This hunch sounds vague, but it can be made precise as follows: there are various tests that statisticians perform on sequences to see whether they are likely to have been generated randomly, and it looks very much as though the sequences of digits of pi would pass these tests. Certainly the first few million do. One obvious test is to see whether any given short sequence of digits, such as 137, occurs with about the

> right frequency in the long term. In the case of the string 137 one would expect it to crop up about one thousandth of the time in the decimal expansion of pi. [...]
>
> Experience strongly suggests that short sequences in the decimal expansion of the irrational numbers that crop up in nature, such as pi, e or the square root of 2, *do* occur with the correct frequencies. And if that is so, then we would expect a million sevens to occur in the decimal expansion of pi about $10^{-1000000}$ of the time – and it is of course no surprise that we will not actually be able to check that directly. And yet, the argument that it does eventually occur, while not a proof, is pretty convincing. (Gowers, 2006, p. 194).

They discovered that adding the phrase ‘taken from a talk by Prof. Timothy Gowers, University of Cambridge’ significantly increased the degree to which either undergraduates or research mathematicians considered themselves persuaded by this argument. This suggests that an Appeal to Expert Opinion can be successful, but does not settle whether it is fallacious. Consideration of the argument scheme may help to answer this question.

The Gowers argument contains at least three (implicit or explicit) appeals to authority. Firstly, Gowers’s authority may help support the conclusion of his argument, which does not pretend to be more than heuristic. Secondly, Gowers relies on premises for which his authority may also lend support. Thirdly, Gowers himself explicitly appeals to the authority of ‘statisticians’ to support the claim that the first few million digits of π are normal. Inglis and Mejia-Ramos are primarily concerned with the first of these appeals, although they concede that the others may have influenced their respondents.

For each appeal we may ask how well the critical questions for this scheme may be answered, and whether identifying Gowers as the author would help. For the first two appeals the authority is Gowers. His expertise is beyond question, and is clearly in the right field, so respondents who know his identity can answer the first two questions much more satisfactorily. The opinion question does not arise in either case, since no third party is invoked. The trustworthiness question speaks directly to the persuasiveness of the argument: as one of the research subjects comments, ‘We are told the argument is made by a reputable mathematician, so we implicitly assume that he would tell us if he knew of any evidence or convincing arguments to the contrary’ (Inglis and Mejia-Ramos, 2006, p. 47). Knowing that Gowers wrote the argument does not directly address the last two critical questions, but the respondents are still better placed to check than they would be with an anonymous argument.

In the case of Gowers’s own appeal to authority, identifying him does not identify the authority. However, his eminent respectability suggests that the sources he cites are credible, and accurately cited, so the respondents for whom Gowers was named would be better placed to answer the critical questions for this appeal too.

These reflections suggest that the appeals to authority in the passage investigated by Inglis and Mejia-Ramos are non-fallacious instances of the argument scheme for Appeal to Expert Opinion, and thus that their respondents were rationally persuaded.

CONCLUSION AND FURTHER WORK

We have seen that argument schemes may be used to distinguish fallacious from non-fallacious instances of informal reasoning in mathematics, thereby resisting over-broad ascription of fallacy. They also suggest new avenues for empirical research. For example, it would be possible to construct experiments to assess the comparative significance of satisfactory answers to each of the critical questions in an Appeal to Expert Opinion. Moreover, at least two dozen more argument schemes may be found in mathematical reasoning, and they await empirical investigation.

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