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Rationale of the Mathematical Joke

Andrew Aberdein[1]

Abstract. A widely circulated list of spurious proof types may help to clarify our understanding of informal mathematical reasoning. An account in terms of argumentation schemes is proposed.

1 INTRODUCTION

A long-standing complaint of researchers into scientific practice is that the published works of scientists do not reveal, indeed willfully conceal, the cognitive processes that led to their results. R. K. Merton traces this complaint as far back as Francis Bacon [14, p. 4]. Researchers into mathematical practice must therefore have resort to less obvious sources. One neglected but potentially revealing object of inquiry is the folklore which circulates within the mathematical community: slang, proverbs, legends, and jokes [17, p. 24]. As Freud observed long ago, the last of these in particular can prove inadvertently revealing: a 'suppressed purpose can, with the assistance of the pleasure from the joke, gain sufficient strength to overcome the inhibition, which would otherwise be stronger than it' [6, p. 187]. Or, as the folklorist Gershon Legman puts it in the introduction to his *Rationale of the Dirty Joke*: 'What is suggested here is not that all these stories are necessarily true ... What is meant is that these stories and individuals do personify ... real but inexplicable peculiarities of human behavior, which they are attempting somehow to fit into a rational view of the world, whether as horror or as humor' [12, p. 22].

This paper explores one notable source of mathematical horror and/or humor: a collection of spurious proof types first compiled by Dana Angluin as [3], and reprinted in full (with two additions, but no attribution) in [17, pp. 28 f.]. The explicit and implicit arguments they embody will be analyzed in terms of 'argumentation schemes': stereotypical reasoning patterns, often accompanied by 'critical questions', which itemize possible lines of response. The most thorough presentation of this methodology to date is that of Douglas Walton and his collaborators [21]. Although argumentation schemes have been developed primarily for non-mathematical contexts, many of them lend themselves readily to mathematical application. For further discussion of this application, and some additional examples of argumentation schemes applied to mathematics, see [1, 2, 4]. The classification of argumentation schemes is an open problem, complicated, perhaps intractably, by the multiple dimensions of similarity between schemes. Walton proposes a threefold division between reasoning, source-based arguments, and the application of rules [21, pp. 348 f.]. This order is broadly followed below, but some of the categories are subdivided to highlight problematic aspects of mathematical argumentation.

[1] Humanities & Communication, Florida Institute of Technology, U.S.A., email: aberdein@fit.edu

2 RETRODUCTION

Much mathematical reasoning is sound deductive inference, secure from the anxieties that jokes such as Angluin's reflect. But not all of it. One of the most ancient techniques of mathematical problem-solving is retroduction, or working backwards. In one of its aspects, at least, this was known to ancient mathematicians as analysis, a term which has, of course, acquired other senses. Pappus defined it as follows: 'in analysis we assume that which is sought as if it were (already) done, and we inquire what it is from which this results, and again what is the antecedent cause of the latter, and so on, until by so retracing our steps we come upon something already known, or belonging to the class of first principles, and such a method we call analysis as being solution backwards (ἀνάπαλιν λύσιν)' [8, p. xlvii]. The steps which the analysis traces should be deductively sound, and once the process is complete the order of the steps may be reversed to yield a conventional deductive proof. This is familiar in the strategy of reducing an open problem to one that has already been solved. Of course, the target must be chosen carefully, as one of Angluin's spurious proofs indicates:

Proof by reduction to the wrong problem 'To see that infinite-dimensional coloured cycle stripping is decidable, we reduce it to the halting problem.' [3, p. 17]

Multi-step argumentation, often of substantial length or complexity, is characteristic of, but not unique to mathematical reasoning. It may be understood in terms of the following scheme:

2.1 Argument from Gradualism

Premise Proposition A is true (acceptable to the respondent).
Premise There is an intervening sequence of propositions, $B_1, B_2, \ldots B_{n-1}, B_n, C$ such that the following conditionals are true: If A then B_1; if B_1 then B_2; ...; if B_{n-1} then B_n; if B_n then C.
Premise The conditional 'If A then C' is not, by itself, acceptable to the respondent, nor are shorter sequences from A to C (than the one specified in the second premise) acceptable to the respondent.
Conclusion Therefore, the proposition C is true (acceptable to the respondent). [21, p. 339]

When a problem is to be reduced to a previously proved result, the task of proving C becomes that of finding a suitable theorem A, and the intervening sequence of propositions, $B_1, \ldots, B_n$ required for this scheme.

Uncontentious deductive examples are not the only varieties of backward reasoning to arise in mathematics:

Proof by importance A large body of useful consequences all follow from the proposition in question. [3, p. 17]

This piece of reasoning is interestingly ambiguous. On the one hand, it might be perceived as consequentialist practical reasoning:

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2.2 Argument from Positive Consequences

Premise If A is brought about, then good consequences will plausibly occur.
Conclusion Therefore, A should be brought about.

Critical Questions:

1. How strong is the likelihood that the cited consequences will (may, must) occur?
2. What evidence supports the claim that the cited consequences will (may, must) occur, and is it sufficient to support the strength of the claim adequately?
3. Are there other opposite consequences (bad as opposed to good, for example) that should be taken into account? [21, pp. 332 f.]

Such reasoning is employed in defence of contested axioms or principles, such as the axiom of choice or the law of excluded middle, but precisely because as axioms they defy proof. As a spurious proof, 'proof by importance' seems rather to work by suggesting that a widely employed but unproven conjecture must be true, because of its extensive coherence with other results. This sort of hunch was employed in defence of, for example, the Taniyama-Shimura hypothesis prior to its confirmation by Andrew Wiles.

But reasoning backwards from the consequences of a conjecture may also be understood as abductive reasoning. George Pólya discusses the importance of abduction for Euler in convincing himself of the correctness of his novel proof that $\sum 1/n^2 = \pi^2/6$ through representation of sin x as an infinite product: 'Euler does not reexamine the grounds for his conjecture ... he examines only its consequences. He regards the verification of any such consequence as an argument in favor of his conjecture ... In scientific research as in ordinary life, we believe, or ought to believe, a conjecture more or less according as its observable consequences agree with the facts. In short, Euler seems to think the same way as reasonable people, scientists or non-scientists, usually think' [16, vol. 1, p. 22]. Of course, despite the importance of this informal procedure for Euler, he never mistook it for proof. Indeed he kept worrying away at the problem until he found an independent proof in more conventional terms.

Euler's inference, and some cases of Angluin's 'proof by importance', might thus be characterized by the positive form of the following abductive argumentation scheme:

2.3 Argument from Evidence to a Hypothesis

Premise If A (a hypothesis) is true, then B (a proposition reporting an event) will be observed to be true.
Premise B has been observed to be true [false], in a given instance.
Conclusion Therefore, A is true [false].

Critical Questions:

1. Is it the case that if A is true, then B is true?
2. Has B been observed to be true (false)?
3. Could there be some reason why B is true, other than its being because of A being true? [21, pp. 331 f.]

The falsification variant is *modus tollens*, and thus uncontroversial. From a strict falsificationist perspective, the best that can be said of any hypothesis is that it has not (yet) been falsified. The defeasible acceptance of a hypothesis on the grounds of non-falsification is another application of the positive variant of this scheme. For empirical science, this may come close enough to confirmation for practical cases. In mathematics the effect is less convincing:

Proof by accumulated evidence Long and diligent search has not revealed a counterexample. [3, p. 17]

Of course this is no sort of 'proof'. If the scope of the search is explicitly stated, a more limited result could be confirmed. But counterexamples revealed only by *extraordinarily* long and diligent searches are common enough to undercut any hope that such reasoning may hold in general. For example, Euler suggested in 1769 that $a^4 + b^4 + c^4 = d^4$ has no integer solutions, but it was not until 1986 that a counterexample in the millions was uncovered [20, p. 293]. It is because of such counterexamples, which have few empirical analogues, that the third critical question can only be fully answered in mathematical cases by proving the hypothesis.

3 CITATION

Source-based arguments are a rich source of mathematical anxiety and humour. The most numerous cases on Angluin's list are those which turn on deviant citation practices. As Paul Renteln and Alan Dundes note, this prevalence 'hints at the anxieties felt by many mathematicians regarding the degree to which mathematical truth is dependent upon the trustworthiness of previous results. This anxiety is exacerbated by the fact that some mathematicians have a less rigorous proof style than other mathematicians' [17, p. 28]. Several subvarieties may be discerned, many of them painfully familiar, in which citation fails for distinct reasons. Firstly there can be references to things which do not exist at all, or which do not say what they are supposed to:

Proof by wishful citation The author cites the negation, converse, or generalization of a theorem from the literature to support his claim.
Proof by ghost reference Nothing even remotely resembling the cited theorem appears in the reference given. [3, pp. 16 f.]

A more subtle issue is that the level of confidence which the source ascribes to the claim may not be adequately reflected in the citation of that claim. Angluin's juxtaposition of the next pair of examples nicely illustrates an anxiety about the credibility arbitrage which may lie behind some references:

Proof by eminent authority 'I saw Karp in the elevator and he said it was probably NP-complete.'
Proof by personal communication 'Eight-dimensional coloured cycle stripping is NP-complete [Karp, personal communication].' [3, p. 16]

In other cases, citations may send the diligent on a wild goose chase which is practically (or even logically) impossible to complete:

Proof by reference to inaccessible literature The author cites a simple corollary of a theorem to be found in a privately circulated memoir of the Slovenian Philological Society, 1883.
Proof by mutual reference In reference A, Theorem 5 is said to follow from Theorem 3 in reference B, which is shown to follow from Corollary 6.2 in reference C, which is an easy consequence of Theorem 5 in reference A. [3, p. 17]

Most transparently, the citation may refer to future activity:

Proof by forward reference Reference is usually to a forthcoming paper by the author, which is often not as forthcoming as at first. [3, p. 17]
Proof by deferral 'We'll prove this later in the course.' [17, p. 28]

All of these different cases may be understood as ineffective applications of the same argumentation scheme, that for Appeal to Expert Opinion.

3.1 Appeal to Expert Opinion

Major Premise Source E is an expert in subject domain S containing proposition A.
Minor Premise E asserts that proposition A is true (false).
Conclusion A is true (false).

Critical Questions:

1. Expertise Question: How credible is E as an expert source?
2. Field Question: Is E an expert in the field that A is in?
3. Opinion Question: What did E assert that implies A?
4. Trustworthiness Question: Is E personally reliable as a source?
5. Consistency Question: Is A consistent with what other experts assert?
6. Backup Evidence Question: Is E's assertion based on evidence? [21, p. 310]

Citation is indispensable in serious mathematical practice, but must be conducted with care. Walton's list of critical questions, although designed primarily for a forensic context, provides a good starting point for an analysis of what comprises such care. Good citation requires a positive answer to all six questions. All of the above examples block a satisfactory answer to one or more of these questions. For most of them, at least the Opinion Question could not be fully answered. (Mutual reference may be an exception, but it certainly fails the Backup Evidence Question, as do several of the others.)

All of these cases exhibit defective argumentation by the prover, whose use of the scheme is exposed as unwarranted by his inability to answer the critical questions. But the prover must also have a respondent, the audience for his proof. Pseudo-proofs can arise from the argumentational shortcomings of the respondent too:

Proof by vehement assertion It is useful to have some kind of authority relation to the audience. [3, p. 17]

Here the prover leads his audience into employing the scheme for Appeal to Expert Opinion, with him as the expert. In effect, he is appealing to his own authority. Something similar may be happening in the next case:

Proof by vigorous handwaving Works well in a classroom or seminar setting. [3, p. 16]

In each of these cases the audience is browbeaten into acquiescence, something they share with an even more common scenario:

Proof by intimidation 'Trivial' [3, p. 16]

In cases where the result is not trivial, and perhaps not correct, this commonplace usage may be understood as a profoundly compressed enthymematic instance of a different scheme, for Argument from Danger.

3.2 Argument from Danger

Premise If you (the respondent) bring about A, then B will occur.
Premise B is a danger to you.
Conclusion Therefore (on balance) you should not bring about A. [21, p. 334]

The argument induced by the dogmatic declaration 'Trivial!' may be unpacked as an instance of this scheme: 'If you ask for clarification, you will appear to be a fool. Appearing a fool is a danger to you. Therefore (on balance) you should not ask.' (Those in no danger of appearing a fool can ask devastatingly simple questions which might, from a humbler source, provoke mere condescension.) However, where the result seems trivial to the audience too, there can be no intimidation in drawing attention to its triviality; rather, this may be understood as a based on a different source: (shared) intuition.

4 INTUITION

Like citation, intuition is ubiquitous, but problematic. As Reuben Hersh observes 'If we look at mathematical practice, the intuitive is everywhere. . . . The word intuition, as mathematicians use it, carries a heavy load of mystery and ambiguity. Sometimes it's a dangerous illegitimate substitute for rigorous proof. Sometimes it's a flash of insight that tells the happy few what others learn with great effort' [9, p. 61]. Unsurprisingly, this uneasy ambiguity manifests in jokes:

Proof by appeal to intuition Cloud-shaped drawings frequently help here. [3, p. 17]
Proof by seduction 'Convince yourself that this is true!' [17, p. 28]

The second example may be found, almost verbatim, even in so august an author as G. H. Hardy: 'The reader will convince himself of the truth of the following assertion' [7, p. 4].

Intuition can be private, but if it is intended to convince others it must be shared; that is, it must be obvious. Hardy (crediting J. E. Littlewood) offers a partial resolution of the underlying ambiguity of intuitive obviousness:[2]

> When one says 'such and such a theorem is almost obvious' one may mean one or other of two things. One may mean 'it is difficult to doubt the truth of the theorem,' 'the theorem is such as common sense instinctively accepts,' as it accepts, for example, the truth of the propositions '$2 + 2 = 4$' or 'the base angles of an isosceles triangle are equal.' That a theorem is 'obvious' in this sense does not prove that it is true, since the most confident of the intuitive judgments of common sense are often found to be mistaken; and even if the theorem is true, the fact that it is also 'obvious' is no reason for not also proving it, if a proof can be found. The object of mathematics is to prove that certain premises imply certain conclusions; and the fact that the conclusions may be as 'obvious' as the premises never detracts from the necessity, and often not even from the interest of the proof.
>
> But sometimes (as for example here [If $\phi(n)$ and $\psi(n)$ tend to limits a, b, then $\phi(n) + \psi(n)$ tends to the limit $a + b$.]) we mean by 'this is almost obvious' something quite different from this. We mean 'a moment's reflection should not only convince the reader of the truth of what is stated, but should also suggest to him the general lines of a rigorous proof.' And often, when a statement is 'obvious' in this sense, one may well omit the proof, not because the proof is unnecessary, but because it is a waste of time to state in detail what the reader can easily supply for himself [7, p. 130].

Thus a claim of intuitive obviousness is a knowledge claim resting on an equivocal basis. Both of the senses Hardy distinguishes may be characterized by the same scheme:

[2] Hersh also analyses the ambiguity of intuition, identifying six different senses [9, pp. 61 f.]. Although Hersh ranges wider, Hardy is more incisive: he subdivides the third of Hersh's senses.

4.1 Argument from Position to Know

Major Premise Source a is in position to know about things in a certain subject domain S containing proposition A.
Minor Premise a asserts that A is true (false).
Conclusion A is true (false).

Critical Questions:

1. Is a in position to know whether A is true (false)?
2. Is a an honest (trustworthy, reliable) source?
3. Did a assert that A is true (false)? [21, p. 309]

In the second of Hardy's senses, the prover and (so he assumes) the respondent are in a position not only to know but to prove the 'obvious' result, and the critical questions may be easily answered in the affirmative. In the first sense, an adequate characterization of the basis for the prover's knowledge claim is more fugitive. It may be, as Hersh suggests, 'reliable mathematical belief without the slightest dream of being formalized' [9, p. 61]. Since he views this faculty as reliable, Hersh would still answer the critical questions affirmatively. Sceptics about such reliability would demur from one or both of the first two questions.

5 META-ARGUMENT

A prominent feature of mathematics is the reification of arguments as themselves the objects of mathematical inquiry. However, arguments which bear upon the status of other arguments comprise a variety of source-based argumentation which does occur in non-mathematical contexts. Angluin's list contains two cases which exploit this practice in different ways.

Proof by metaproof A method is given to construct the desired proof. The correctness of the method is proved by any of these techniques. [3, p. 17]

Here the source from which the argument at issue arises is itself an argument, the presumed virtues of which would indeed transfer to the object argument, but of course, only if the method is secure.

Proof by funding How could three different government agencies be wrong? [3, p. 16]

The existence of funding, like the eminence of the author's institution, and other 'esteem indicators', might be taken as indirect evidence for the soundness of his results. While this evidence is pretty weak, similar evidence can be persuasive. One empirical study presented samples of undergraduates and research mathematicians with a heuristic argument for the presence of one million consecutive sevens in the decimal expansion of π. Correctly attributing its authorship to Field's Medallist Tim Gowers significantly increased the positive appraisal of the argument in both groups [10]. As one research subject commented, 'We are told the argument is made by a reputable mathematician, so we implicitly assume that he would tell us if he knew of any evidence or convincing arguments to the contrary' [10, p. 42]. This demonstrates the perceived relevance of the author's ethos to the acceptance of his arguments, an application of the following scheme:

5.1 Ethotic Argument

Premise If x is a person of good (bad) moral character, then what x says should be accepted as more plausible (rejected as less plausible).
Premise a is a person of good (bad) moral character.
Conclusion Therefore, what a says should be accepted as more plausible (rejected as less plausible).

Critical Questions:

1. Is a a person of good (bad) moral character?
2. Is character relevant in the dialogue?
3. Is the weight of presumption claimed strongly enough warranted by the evidence given? [21, p. 336]

Walton's version is not quite suitable for our purposes: the ethos of a must encompass epistemic instead of (or as well as) moral virtue (as Walton explores elsewhere: cf. [21, p. 359]). It is not Tim Gowers's blameless personal life which leads us to trust his arguments, but rather his demonstrably high standards as a working mathematician. Notoriously, academics can have deplorable ethical standards, while still producing work of genuine value. This may be because, unlike politicians, say, whose weak morals can prove more problematic, they back up their claims with independently verifiable argumentation. Does this obviate the need for consideration of their ethos, even restricted to its epistemic aspects? Only if the argument is fully worked out: if we are invited to accept aspects of the reasoning on trust, it is not unreasonable to ask whether the reasoner is trustworthy.

Ethotic Argument is closely related to Appeal to Expert Opinion. Indeed, the research subject's comments quoted above could also be construed as positive answers to some of the critical questions for the latter scheme. As the commenter is inferring the absence of counter-arguments from the absence of evidence for them, he might also be taken as employing a different form of source-based argument, an Argument from Ignorance.

6 EPISTEMIC CLOSURE

Arguments from Ignorance are typically classified as fallacious, but they can be convincing in epistemically closed domains. Mathematics, however, is not such a domain, so legitimate mathematical applications of the following scheme are somewhat scarce.

6.1 Argument from Ignorance

Major Premise If A were true, then A would be known to be true.
Minor Premise It is not the case that A is known to be true.
Conclusion Therefore A is not true.

Critical Questions:

1. How far along has the search for evidence progressed?
2. Which side has the burden of proof in the dialogue as a whole? In other words, what is the ultimate *probandum* and who is supposed to prove it?
3. How strong does the proof need to be in order for this party to be successful in fulfilling the burden? [21, p. 327]

This scheme is an instance of *modus tollens*, so its plausibility just turns on the acceptability of its premises. The major premise states the epistemic closure principle, which can be analyzed as a conjunction of two conditionals: (i) If A were true, then A would be known to be true by someone; (ii) If A were known to be true by someone, then A would be known to be true (by me). In much everyday, social knowledge this is unproblematic: (i) is trivially true and (ii) is a straightforward claim about the speaker's position within an epistemic community.

In mathematical cases, (i) is characteristically false: there are few unknown A of which it can be said that if A were true, then A would be known to be true by someone. A notable exception is the use of computer assistance. If a computer is known to have calculated the logical closure of a set of axioms, or determined which of a wide range of cases have a particular property, then an unsuccessful search through the resultant data set is conclusive: the truth of the closure principle permits the deduction of the conclusion. Even when the closure principle is not known with certainty, it may often be practically assumed. As the science writer George Szpiro remarks of *Mathematica*, 'You can enter a function that you want to integrate, and out comes the correct expression. And if it does not, you may be reasonably sure that a solution to your question simply does not exist' [18, p. 198].

However, (ii) is often plausible in mathematics. Experts may confidently assert of many propositions in their field, that if a proof were known, it would be known by them. Even non-experts can be confident about some propositions: if a counter-example to Goldbach's conjecture is found, we'll all know about it. So I may safely infer from my ignorance that no such number has been found, but not of course that it does not exist. Hence the *modus tollens* stops at (ii), and the safe conclusion is not that A is not true, but that nobody knows whether A is true.

Angluin offers an interesting variant of this argument:

Proof by cosmology The negation of the proposition is unimaginable or meaningless. Popular for proofs of the existence of God. [3, p. 17]

This 'proof' employs a weaker closure principle: 'If $\neg A$ were true, then $\neg A$ would be imaginable.' The absence of an objective measure of imaginability makes this principle implausible.

Arguments from Ignorance can also be induced in the respondent:

Proof by cumbersome notation Best done with access to at least four alphabets and special symbols.
Proof by exhaustion An issue or two of a journal devoted to your proof is useful.
Proof by obfuscation A long plotless sequence of true and/or meaningless syntactically related statements. [3, p. 16]

These three cases encourage the respondent to pass the buck, rather than tackle a purported proof of conspicuous rebarbativeness. The Argument from Ignorance would run 'If there was something wrong with this proof, someone else would have noticed. Nobody seems to have noticed anything. So, it must be OK.'

7 GENERALIZATION

Much mathematical practice concerns the correct application of established rules. A notable example is the move from particular to general results. However, as with other mathematical activities, generalization lends itself to abuse when applied carelessly:

Proof by example The author gives only the case $n = 2$ and suggests that it contains most of the ideas of the general proof.
Proof by omission 'The reader may easily supply the details.'
'The other 253 cases are analogous.'
'. . . ' [3, p. 16]

These patterns of reasoning may be found in both historical and contemporary mathematics. John Wallis, for instance, routinely argued from specific examples to the conclusion that a procedure may be applied indefinitely. He 'relied boldly and confidently on his really astounding intuition as to the correlation between the sums of different series. He called [this] "modus inductionis": later it was termed "incomplete induction."' [15, p. 38].

Standards of rigour have changed profoundly since Wallis's day. But instances of proofs by example or omission still arise in ostensibly rigorous contexts, although not always without reproach. Take the dispute over Wu-Yi Hsiang's alleged proof of Kepler's conjecture, that the maximum density of a packing of congruent spheres in three dimensions is $\pi/\sqrt{18}$. In his review, Gábor Fejes Tóth complains that 'we are given arguments such as "the most critical case is. . . " followed by a statement that "the same method will imply the general case." The problem with arguments of this kind is not only that they require the reader to redo some pages of calculations, but, notoriously, that they occur at places where we expect difficulties and most frequently it is impossible to see how the same method works in the general case' [5]. Both cases exemplify the following scheme.

7.1 Argument from Example

Premise In this particular case, the individual a has property F and also property G.
Conclusion Therefore, generally, if x has property F, then it also has property G.

Critical Questions:

1. Is the proposition claimed in the premise in fact true?
2. Does the example cited support the generalization it is supposed to be an instance of?
3. Is the example typical of the kinds of cases the generalization covers?
4. How strong is the generalization?
5. Do special circumstances of the example impair its generalizability? [21, p. 314]

This scheme underpins incomplete or enumerative induction. This is, of course, formally invalid, but still has a place in informal mathematical reasoning, for example in hypothesis formation: a is F and also G; I can't think of anything which is F and not G; Are all Fs G? The concern with enumerative induction is that it may be mistaken for proof. Some empirical research is worrisome: one study found that 80% of trainee elementary mathematics teachers considered arguments based on specific instances to be mathematical proofs [13].

But enumerative induction is not the only application of this scheme. While Wallis might be described as employing induction, Hsiang is attempting something else, but is still employing Argument from Example. His generalizing moves are contentious because his examples are said to provide insufficient support, and to be poorly chosen, that is to fail the second and third critical questions, not because generalization is in principle non-rigorous. Where the examples are genuinely typical, and the method transparent, application of this scheme is consistent with mathematical rigour. As Jamie Tappenden observes, 'generalizing is much more than just picking constants and replacing them with variables. Articulating the right structures, which then can be generalized, is an incredibly involved process and it is hard to get it right' [19, pp. 264 f.].

A special case of generalization arises with diagrammatic proof. Angluin's list contains one example that articulates the conservative attitude:

Proof by picture A more convincing form of proof by example. Combines well with proof by omission. [3, p. 17]

Defenders of the legitimacy of diagrammatic proof acknowledge the centrality of the 'Generalization Problem': 'After we have proved a theorem using a diagram, how can we legitimately generalize the configuration of this diagram to a wide (usually infinite) class of configurations, and to what class exactly?' [11, p. 81]. However, they point out that similar issues arise in non-diagrammatic inference, and may be resolved in a similar fashion in the diagrammatic case, by the painstaking assurance that the particular case really does generalize, that is by answering the critical questions of the scheme for Argument from Example.

8 DEFINITION

A less contentious application of a rule is the employment of a definition. Well-chosen definitions can be pivotal to mathematical progress. At a superficial level, this choice can appear arbitrary: mathematicians have more flexibility in the concepts they take as primitive than scientists in any other domain. On the other hand, appeals to the naturalness or otherwise of definitions are frequent, and a change of definitions can revolutionize the understanding of a hitherto intractable problem. Of course, this strategy can be abused.

Proof by semantic shift Some of the standard but inconvenient definitions are changed for the statement of the result. [3, p. 17]

Instead of defining the same problem with different terms, a different (typically much easier) problem is described by the same terms. Such sharp practice is a straightforward misapplication of the scheme for Argument from Definition.

8.1 Argument from Definition to Verbal Classification

Individual Premise a fits definition D.
Classification Premise For all x, if x has property D, then x can be classified as having property G.
Conclusion a has property G.

Critical Questions:

1. What evidence is there that D is an adequate definition, in light of other possible alternative definitions that might exclude a's having G?
2. Is the verbal classification in the classification premise based merely on a stipulative or biased definition that is subject to doubt? [21, p. 319]

If the non-standard definition D is poorly motivated, the example will fail to answer the first question; if the non-standard definition is smuggled past a classification premise only known to hold for the standard definition, the example would (also) fail to answer the second question.

9 CONCLUSION & FUTURE WORK

We have seen how the anxieties of practicing mathematicians, implicit within their folk humour, draw attention to those aspects of their informal practice where their argumentation carries the greatest risk. We have also seen the utility of a diverse selection of argumentation schemes, developed for non-mathematical applications, in the analysis of a wide-ranging collection of spurious varieties of mathematical proof. This may be read as further corroboration that 'mathematical reasoning is already in accord with principles and techniques from informal logic—even if this is unnoticed by the practitioners' [4, p. 150].

The argumentation scheme methodology has ancient roots, but is still a work in progress. In recent years, it has attracted much attention from researchers in artificial intelligence. In particular, schemes have been implemented within systems of defeasible argumentation, and as strategies in dialogue games [21, pp. 370; 383]. The application of schemes to mathematical argumentation is still in its infancy, but the comparative maturity of the methodology in other domains offers an inviting prospect for future research.

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