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A NOTE ON OMITTING THE REPLACEMENT SCHEMA

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In [1] Heath considers a formalisation of primitive recursive arithmetic similar to that given in Goodstein [2], in which the replacement schema (Goodstein's \mathbf{Sb}_2) is deduced from special cases of itself, using a double recursive uniqueness rule. The deduction of \mathbf{Sb}_2 given in [1] is, however, incomplete. This is rectified in the present note. The special cases of \mathbf{Sb}_2 taken by Heath are:

- (i) $A = B \vdash SA = SB$
- (ii) $A = B \vdash x + A = x + B$
- (iii) $A = B \vdash A + x = B + x$
- (iv) $A = B \vdash x \dot{-} A = x \dot{-} B$
- (v) $A = B \vdash A \dot{-} x = B \dot{-} x$

Remark In fact either (ii) or (iii) can be omitted since $x + y = y + x$ can be proved without using (ii) or (iii) and then one can be derived from the other.

In order to derive the full \mathbf{Sb}_2 , i.e., $A = B \vdash f(A) = f(B)$, for any primitive recursive function f , it is necessary to show that the substitution theorem, $x = y \rightarrow f(x) = f(y)$, persists under definition by a primitive recursive schema. Heath shows that it persists under the recursion without parameter, which I shall call \mathbf{R} ,

$$\begin{aligned} f(0) &= (0), \\ f(Sx) &= g(x, f(x)), \end{aligned}$$

i.e., that from $x = y \ \& \ w = z \rightarrow g(x, w) = g(y, z)$ we can deduce $x = y \rightarrow f(x) = f(y)$. He then quotes a theorem of R. M. Robinson that all primitive recursive functions are generated from 0, x , Sx , $x + y$ and $x \dot{-} y$ by substitution and the recursion \mathbf{R} . To complete the proof it would be sufficient to show that Robinson's reduction of primitive recursion can be carried out in the restricted primitive recursive arithmetic (i.e., without full \mathbf{Sb}_2). This would involve defining the pairing functions $J(x, y)$, $K(x)$ and $L(x)$ given by Robinson, deriving their main properties, e.g. $L(Sx) \neq 0 \rightarrow K(Sx) = K(x)$ & $L(Sx) = S(Lx)$, and checking that the substitution theorem is satisfied by them. This part was omitted by Heath, and it is not clear that this programme could be carried out.

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However it is fairly easy to check that the substitution theorem persists under full recursion, by a simple adaptation of Heath's proof for the recursion scheme **R**, as the following theorem shows.

Theorem Suppose f is defined by primitive recursion from h and g , i.e.,

$$f(u_0, \dots, u_n, 0) = h(u_0, \dots, u_n) \tag{a}$$

$$f(u_0, \dots, u_n, Sx) = g(u_0, \dots, u_n, x, f(u_0, \dots, u_n, x)) \tag{b}$$

and the substitution theorem has already been proved for h and g , i.e.,

$$u_0 = v_0 \ \& \ \dots \ \& \ u_n = v_n \rightarrow h(u_0, \dots, u_n) = h(v_0, \dots, v_n) \tag{c}$$

and

$$u_0 = v_0 \ \& \ \dots \ \& \ u_{n+2} = v_{n+2} \rightarrow g(u_0, \dots, u_{n+2}) = g(v_0, \dots, v_{n+2}) \tag{d}$$

Then the substitution theorem holds for f , i.e.,

$$u_0 = v_0 \ \& \ \dots \ \& \ u_{n+1} = v_{n+1} \rightarrow f(u_0, \dots, u_{n+1}) = f(v_0, \dots, v_{n+1})$$

Proof

Lemma I $u_0 = v_0 \ \& \ \dots \ \& \ u_n = v_n \rightarrow f(u_0, \dots, u_n, x) = f(v_0, \dots, v_n, x)$

By induction on x , prove the basis

$$u_0 = v_0 \ \& \ \dots \ \& \ u_n = v_n \rightarrow f(u_0, \dots, u_n, 0) = f(v_0, \dots, v_n, 0) \tag{by hypotheses (a) and (c)}$$

and the step

$$u_0 = v_0 \ \& \ \dots \ \& \ u_n = v_n \ \& \ (u_0 = v_0 \ \& \ \dots \ \& \ u_n = v_n \rightarrow f(u_0, \dots, u_n, x) = f(v_0, \dots, v_n, x)) \rightarrow f(u_0, \dots, u_n, Sx) = f(v_0, \dots, v_n, Sx) \tag{by hypotheses (b) and (d)}$$

Lemma II $x = y \rightarrow f(u_0, \dots, u_n, x) = f(u_0, \dots, u_n, y)$

By double induction on x and y , prove

$$x = 0 \rightarrow f(u_0, \dots, u_n, x) = f(u_0, \dots, u_n, 0)$$

and

$$0 = y \rightarrow f(u_0, \dots, u_n, 0) = f(u_0, \dots, u_n, y)$$

by schema **F** on x and y respectively. Then use the deduction theorem to prove

$$(x = y \rightarrow f(u_0, \dots, u_n, x) = f(u_0, \dots, u_n, y)) \rightarrow (Sx = Sy \rightarrow f(u_0, \dots, u_n, Sx) = f(u_0, \dots, u_n, Sy))$$

Assume $x = y \rightarrow f(u_0, \dots, u_n, x) = f(u_0, \dots, u_n, y)$ and $Sx = Sy$ and without using **Sb**₁ on any of the variables u_0, \dots, u_n, x, y , deduce, in turn,

$$\begin{aligned} x = y & \\ f(u_0, \dots, u_n, x) = f(u_0, \dots, u_n, y) & \tag{by modus ponens} \\ g(u_0, \dots, u_n, x, f(u_0, \dots, u_n, x)) = g(u_0, \dots, u_n, y, f(u_0, \dots, u_n, y)) & \tag{by hypothesis (d)} \end{aligned}$$

Therefore

$$f(u_0, \dots, u_n, Sx) = f(u_0, \dots, u_n, Sy) \quad \text{by hypothesis (b).}$$

The theorem follows from Lemmas I and II.

REFERENCES

- [1] Heath, I. J., "Omitting the replacement schema in recursive arithmetic," *Notre Dame Journal of Formal Logic*, vol. VIII (1967), pp. 234-238.
- [2] Goodstein, R. L., "Logic-free formalization of recursive arithmetic," *Mathematica Scandinavia*, vol. 2 (1954), pp. 247-261.

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