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## A NOTE ON OMITTING THE REPLACEMENT SCHEMA

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In [1] Heath considers a formalisation of primitive recursive arithmetic similar to that given in Goodstein [2], in which the replacement schema (Goodstein's  $\mathbf{Sb}_2$ ) is deduced from special cases of itself, using a double recursive uniqueness rule. The deduction of  $\mathbf{Sb}_2$  given in [1] is, however, incomplete. This is rectified in the present note. The special cases of  $\mathbf{Sb}_2$  taken by Heath are:

- (i)  $A = B \vdash SA = SB$
- (ii)  $A = B \vdash x + A = x + B$
- (iii)  $A = B \vdash A + x = B + x$
- (iv)  $A = B \vdash x \dot{-} A = x \dot{-} B$
- (v)  $A = B \vdash A \dot{-} x = B \dot{-} x$

*Remark* In fact either (ii) or (iii) can be omitted since  $x + y = y + x$  can be proved without using (ii) or (iii) and then one can be derived from the other.

In order to derive the full  $\mathbf{Sb}_2$ , i.e.,  $A = B \vdash f(A) = f(B)$ , for any primitive recursive function  $f$ , it is necessary to show that the substitution theorem,  $x = y \rightarrow f(x) = f(y)$ , persists under definition by a primitive recursive schema. Heath shows that it persists under the recursion without parameter, which I shall call  $\mathbf{R}$ ,

$$\begin{aligned} f(0) &= (0), \\ f(Sx) &= g(x, f(x)), \end{aligned}$$

i.e., that from  $x = y \ \& \ w = z \rightarrow g(x, w) = g(y, z)$  we can deduce  $x = y \rightarrow f(x) = f(y)$ . He then quotes a theorem of R. M. Robinson that all primitive recursive functions are generated from 0,  $x$ ,  $Sx$ ,  $x + y$  and  $x \dot{-} y$  by substitution and the recursion  $\mathbf{R}$ . To complete the proof it would be sufficient to show that Robinson's reduction of primitive recursion can be carried out in the restricted primitive recursive arithmetic (i.e., without full  $\mathbf{Sb}_2$ ). This would involve defining the pairing functions  $J(x, y)$ ,  $K(x)$  and  $L(x)$  given by Robinson, deriving their main properties, e.g.  $L(Sx) \neq 0 \rightarrow K(Sx) = K(x)$  &  $L(Sx) = S(Lx)$ , and checking that the substitution theorem is satisfied by them. This part was omitted by Heath, and it is not clear that this programme could be carried out.

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However it is fairly easy to check that the substitution theorem persists under full recursion, by a simple adaptation of Heath's proof for the recursion scheme **R**, as the following theorem shows.

**Theorem** Suppose  $f$  is defined by primitive recursion from  $h$  and  $g$ , i.e.,

$$f(u_0, \dots, u_n, 0) = h(u_0, \dots, u_n) \tag{a}$$

$$f(u_0, \dots, u_n, Sx) = g(u_0, \dots, u_n, x, f(u_0, \dots, u_n, x)) \tag{b}$$

and the substitution theorem has already been proved for  $h$  and  $g$ , i.e.,

$$u_0 = v_0 \ \& \ \dots \ \& \ u_n = v_n \rightarrow h(u_0, \dots, u_n) = h(v_0, \dots, v_n) \tag{c}$$

and

$$u_0 = v_0 \ \& \ \dots \ \& \ u_{n+2} = v_{n+2} \rightarrow g(u_0, \dots, u_{n+2}) = g(v_0, \dots, v_{n+2}) \tag{d}$$

Then the substitution theorem holds for  $f$ , i.e.,

$$u_0 = v_0 \ \& \ \dots \ \& \ u_{n+1} = v_{n+1} \rightarrow f(u_0, \dots, u_{n+1}) = f(v_0, \dots, v_{n+1})$$

*Proof*

**Lemma I**  $u_0 = v_0 \ \& \ \dots \ \& \ u_n = v_n \rightarrow f(u_0, \dots, u_n, x) = f(v_0, \dots, v_n, x)$

By induction on  $x$ , prove the basis

$$u_0 = v_0 \ \& \ \dots \ \& \ u_n = v_n \rightarrow f(u_0, \dots, u_n, 0) = f(v_0, \dots, v_n, 0) \tag{by hypotheses (a) and (c)}$$

and the step

$$u_0 = v_0 \ \& \ \dots \ \& \ u_n = v_n \ \& \ (u_0 = v_0 \ \& \ \dots \ \& \ u_n = v_n \rightarrow f(u_0, \dots, u_n, x) = f(v_0, \dots, v_n, x)) \rightarrow f(u_0, \dots, u_n, Sx) = f(v_0, \dots, v_n, Sx) \tag{by hypotheses (b) and (d)}$$

**Lemma II**  $x = y \rightarrow f(u_0, \dots, u_n, x) = f(u_0, \dots, u_n, y)$

By double induction on  $x$  and  $y$ , prove

$$x = 0 \rightarrow f(u_0, \dots, u_n, x) = f(u_0, \dots, u_n, 0)$$

and

$$0 = y \rightarrow f(u_0, \dots, u_n, 0) = f(u_0, \dots, u_n, y)$$

by schema **F** on  $x$  and  $y$  respectively. Then use the deduction theorem to prove

$$(x = y \rightarrow f(u_0, \dots, u_n, x) = f(u_0, \dots, u_n, y)) \rightarrow (Sx = Sy \rightarrow f(u_0, \dots, u_n, Sx) = f(u_0, \dots, u_n, Sy))$$

Assume  $x = y \rightarrow f(u_0, \dots, u_n, x) = f(u_0, \dots, u_n, y)$  and  $Sx = Sy$  and without using **Sb**<sub>1</sub> on any of the variables  $u_0, \dots, u_n, x, y$ , deduce, in turn,

$$\begin{aligned} x = y & \\ f(u_0, \dots, u_n, x) = f(u_0, \dots, u_n, y) & \tag{by modus ponens} \\ g(u_0, \dots, u_n, x, f(u_0, \dots, u_n, x)) = g(u_0, \dots, u_n, y, f(u_0, \dots, u_n, y)) & \tag{by hypothesis (d)} \end{aligned}$$

Therefore

$$f(u_0, \dots, u_n, Sx) = f(u_0, \dots, u_n, Sy) \quad \text{by hypothesis (b).}$$

The theorem follows from Lemmas I and II.

#### REFERENCES

- [1] Heath, I. J., "Omitting the replacement schema in recursive arithmetic," *Notre Dame Journal of Formal Logic*, vol. VIII (1967), pp. 234-238.
- [2] Goodstein, R. L., "Logic-free formalization of recursive arithmetic," *Mathematica Scandinavia*, vol. 2 (1954), pp. 247-261.

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