Introduction to theories without the independence property

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4th June 2008

Abstract

We present an updated exposition of the classical theory of complete first order theories without the independence property (also called NIP theories or dependent theories).

1 Introduction

The independence property was introduced by Shelah [48]. Our knowledge about theories without it was extended by Poizat [42]. Shelah answered in an appendix to his paper on simple theories, to which Poizat replied again [50, 43]. Chapter 12 of Poizat's book is an excellent account of the state of the art after this exchange [45]. Some research done much later seems to have its natural place within this classical theory of NIP. The most obvious case is Shelah's version of a theorem due to Baldwin and Benedikt, refining Poizat's result that indiscernible sequences in NIP theories do not split [4, 53].

Shelah's classical theorem which says that (it is consistent with ZFC that) theories with the independence property have strictly more types than those without, points to the independence property as a significant dividing line. Unfortunately, not much is known on the order (i. e. NIP) side of this dividing line, and those results that were not covered in Poizat's book are a bit scattered. The current renaissance of theories without the independence property is probably due to the ebbing of the boom in simplicity theory and to the great success of o-minimality and related notions. The purpose of this paper is to make explicit and easily accessible the platform from which recent research in this area starts.

Perhaps even more than the theory of stability, the theory of NIP appears pervaded by infinite combinatorics disguised as indiscernible sequences. In Sections 2 and 3 we will look at this combinatorial tool, but also at some specific classes of theories that deliver the most important examples of NIP theories. In Section 4 we will see that having the independence property or not is the last dividing line that can possibly be detected by counting types in the traditional way.

Section 5 lays the foundation for working with forking and similar notions such as almost invariance and finite satisfiability in arbitrary theories. In Section 6 we see that a theory is NIP if and only if the number of non-forking extensions of every complete type stays below a certain bound, and Section 7 explores criteria for a NIP theory to be stable.

Most sections are independent of each other, except that everything is based on Section 2 and the last two sections also depend on Section 5.

2 Definitions and basic facts

Throughout this paper we work in the monster model $\mathbb{M}$ of a complete first-order theory T. We will be ambiguous about whether the language is one- or many-sorted, using more general expressions such as 'let $\bar{a}$ and $\bar{x}$ be compatible' rather than 'let $\bar{a}$ and $\bar{x}$ have the same length'. 'Tuples' such

*The author was supported by the European Commission's Research Training Network MRTN-CT-2004-512234 (MODNET) and by Engineering and Physical Sciences Research Council grant EP/F009712/1.

as $\bar{a}$ and $\bar{x}$ can often be infinite. (The main difference between 'tuples' and sequences is that tuples consist of arbitrary elements, while sequences consist of compatible tuples.) Whenever it is important that certain tuples can, or cannot, be infinite, it should be mentioned. We write $\bar{a} \equiv_B \bar{a}'$ if $\operatorname{tp}(\bar{a}/B) = \operatorname{tp}(\bar{a}'/B)$.

An indiscernible sequence may be indexed by an arbitrary linearly ordered set, so long as it is infinite. We will call a sequence of indiscernibles dense or complete if its underlying linear order is dense or (Dedekind) complete. Model theorists traditionally use Ramsey's theorem to construct indiscernible sequences, but the following consequence of the Erdős-Rado Theorem, which was observed by Shelah, is often more convenient [50]. A detailed proof can be found, for instance, in Ben-Yaacov's thesis [6].

Theorem 1. *Let B be a set of parameters and κ a cardinal. For any sequence $(\bar{a}_i)_{i<\beth_{(2^{|T|+|B|+\kappa})^+}}$ consisting of tuples of length $|\bar{a}_i| = \kappa$ there is a B-indiscernible sequence $(\bar{a}'_j)_{j<\omega}$ with the following property: For every $k < \omega$ there are $i_0 < i_1 < \cdots < i_{k-1}$ such that $\bar{a}'_0\bar{a}'_1 \dots \bar{a}'_{k-1} \equiv_B \bar{a}_{i_0}\bar{a}_{i_1} \dots \bar{a}_{i_{k-1}}$.*

In other words, from every sufficiently long sequence a sequence of indiscernibles can be *'extracted'*. We will use this result frequently and freely.

VC dimension and alternation number

The *VC dimension* or *Vapnik-Chervonenkis dimension* of a formula $\varphi = \varphi(\bar{x}; \bar{y})$ is best defined as

$$\operatorname{vc}(\varphi(\bar{x};\bar{y})) = \max\{n < \omega \mid \exists(\bar{a}_i)_{i<n}\exists(\bar{b}_J)_{J\subseteq n}(\models \varphi(\bar{a}_i;\bar{b}_J) \Leftrightarrow i \in J)\}$$

if the maximum exists, and ∞ otherwise.[1] VC dimension satisfies a weak form of symmetry in $\bar{x}$ and $\bar{y}$. In order to express this kind of connection conveniently, we define the *opposite* $\varphi^{\mathrm{opp}} = \varphi^{\mathrm{opp}}(\bar{y};\bar{x})$ of a formula $\varphi = \varphi(\bar{x};\bar{y})$ as the same formula, but with opposite separation of variables. The *dual VC dimension* is then simply $\operatorname{vc}^*(\varphi) = \operatorname{vc}(\varphi^{\mathrm{opp}})$, i.e.

$$\operatorname{vc}^*(\varphi(\bar{x};\bar{y})) = \max\{n < \omega \mid \exists(\bar{a}_I)_{I\subseteq n}\exists(\bar{b}_j)_{j<n}(\models \varphi(\bar{a}_I;\bar{b}_j) \Leftrightarrow j \in I)\}.$$

For example, we always have $\operatorname{vc}(x = y) = \operatorname{vc}^*(x = y) = 1$, and in the theory of a dense linear order we also have $\operatorname{vc}(x < y) = \operatorname{vc}^*(x < y) = 1$.

Proposition 2. $\operatorname{vc}^*(\varphi) < 2^{\operatorname{vc}(\varphi)+1}$*; and dually* $\operatorname{vc}(\varphi) < 2^{\operatorname{vc}^*(\varphi)+1}$.

Proof. Suppose $\operatorname{vc}^*(\varphi) \geq 2^n$. We will show that $\operatorname{vc}(\varphi) \geq n$. (The claim then follows by writing $\operatorname{vc}^*(\varphi) = 2^{n+\epsilon}$, where $0 \leq \epsilon < 1$.) There are tuples $\bar{a}_I$ for $I \subseteq 2^n$ and $\bar{b}_j$ for $j < 2^n$ such that $\models \varphi(\bar{a}_I;\bar{b}_j)$ holds if and only if $j \in I$. After a mere change of notation we have tuples $\bar{b}_J$ for $J \subseteq n$ and $\bar{a}_{\mathcal{I}}$ for $\mathcal{I} \subseteq \mathcal{P}(n)$ such that $\models \varphi(\bar{a}_{\mathcal{I}};\bar{b}_J)$ holds if and only if $J \in \mathcal{I}$. For $i < n$ let $\mathcal{I}(i)$ be the principal ultrafilter on n generated by i. Then $J \in \mathcal{I}(i)$ if and only if $i \in J$. Hence $\models \varphi(\bar{a}_{\mathcal{I}(i)};\bar{b}_J)$ holds if and only if $i \in J$. Therefore $\operatorname{vc}(\varphi) \geq n$. □

This result is from Laskowski [27]. He also observed that the bipartite graph between $A = 2^n - 1$ and $B = \mathcal{P}(A)$, with edge relation defined by $\models R(a,b) \iff a \in b$, shows that the result is optimal: Clearly $\operatorname{vc}(R) = 2^n - 1$ and $\operatorname{vc}^*(R) \leq n - 1$. From this it follows that $\operatorname{vc}(R) \geq 2^{\operatorname{vc}^*(R)+1} - 1$.

The VC dimension of formulas is a special case of a dimension introduced by Vapnik and Chervonenkis in a 1971 paper on statistical learning theory; it is finite if and only if a certain growth function is polynomial rather than exponential [59]. In the same year, one of Shelah's first publications contained the following definition. A formula $\varphi(\bar{x};\bar{y})$ has the *independence property* (or *IP*) if for every $n < \omega$ there are tuples $(\bar{a}_i)_{i<n}$ and $(\bar{b}_J)_{J\subseteq n}$ such that $\models \varphi(\bar{a}_i, \bar{b}_J)$ holds if and only if $i \in J$ [48, 49]. In other words, φ has the independence property if $\operatorname{vc}^*(\varphi) = \infty$ or,

[1] This definition is based on the interpretation of $\varphi(\bar{x};\bar{y})$ as representing the set system $\{\varphi(\bar{x};\bar{b}) \mid \bar{b} \in \mathbb{M}\}$ and the standard definition of VC dimension for set systems; it is also used by Grohe and Turán [12]. In a widely circulated draft of the present paper I erroneously interchanged the VC dimension with its dual.

equivalently, if $\mathrm{vc}(\varphi)=\infty$. The connection between VC dimension and the independence property was first observed twenty years later by Laskowski [27].[2]

Another related notion is the *alternation number* of a formula, defined as

$$\mathrm{alt}(\varphi(\bar{x};\bar{y})) = \max\Big\{n<\omega \;\Big|\; \exists(\bar{a}_i)_{i<\omega} \textit{ indiscernible } \exists\bar{b}\ \ \forall i<n-1 \\ \big[\models \varphi(\bar{a}_i;\bar{b}) \leftrightarrow \neg\varphi(\bar{a}_{i+1};\bar{b})\big]\Big\}$$

if the maximum exists, or ∞ otherwise. It is due to Poizat [43, 45]. Thus the alternation number of $\varphi(\bar{x};\bar{y})$ counts the maximal number of segments into which an instance $\varphi(\bar{x};\bar{b})$ of φ can cut an indiscernible sequence. This is one more than the number of alternations between truth values. For example, the alternation number of a formula with constant truth value, like $x=x\wedge y=y$, is 1, the alternation number of a linear order is 2, and $\mathrm{alt}(x=y)=3$.

Proposition 3. $\mathrm{alt}(\varphi)\le 2\,\mathrm{vc}(\varphi)+1$.

Proof. We will show that $\mathrm{alt}(\varphi)\ge 2n$ implies $\mathrm{vc}(\varphi)\ge n$. So suppose we have an indiscernible sequence $(\bar{a}_i)_{i<2n}$ and a tuple $\bar{b}$ such that $\varphi(\bar{a}_i;\bar{b})$ holds if and only if i is even, say. Given any subset $J\subseteq n$ we can find numbers $i_0<i_1<\cdots<i_{n-1}$ such that $i_0\in\{0,1\}$ and $i_{k+1}\in\{i_k+1,i_k+2\}$ (so clearly $i_{n-1}<2n$) and such that i_k is even if and only if $k\in J$. Hence $\models\varphi(\bar{a}_{i_k};\bar{b})$ if and only if $k\in J$. Now $\bar{a}_{i_0}\bar{a}_{i_1}\ldots\bar{a}_{i_{k-1}}\equiv\bar{a}_0\bar{a}_1\ldots\bar{a}_{k-1}$ by indiscernibility, and so there is a tuple $\bar{b}_J$ such that $\models\varphi(\bar{a}_k;\bar{b}_J)$ if and only if $k\in J$. □

The formula $\varphi(x;y_0\ldots y_{n-1})\equiv(x=y_0\vee\cdots\vee x=y_{n-1})$ has $\mathrm{vc}(\varphi)=n$ and $\mathrm{alt}(\varphi)=2n+1$. This shows that the inequality cannot be improved. Moreover, since $\mathrm{alt}(\varphi^{\mathrm{opp}})=3$, we see that for the alternation number there is no analogue to Proposition 2. But we still have the following result, essentially due to Poizat [42]:

Proposition 4. *The following conditions are equivalent for every formula $\varphi(\bar{x};\bar{y})$.*

1. *φ does not have the independence property.*
2. *φ^{opp} does not have the independence property.*
3. *$\mathrm{vc}(\varphi)<\infty$.*
4. *$\mathrm{alt}(\varphi)<\infty$.*
5. *For every indiscernible sequence $(\bar{a}_i)_{i<\omega}$ and every tuple $\bar{b}$ the set of indices $i<\omega$ such that $\models\varphi(\bar{a}_i;\bar{b})$ holds is finite or cofinite.*

Proof. The equivalence of 1–3 is obvious using Proposition 2. 3 implies 4 by Proposition 3. 4 implies 5: Let $(\bar{a}_i)_{i<\omega}$ be indiscernible, and suppose $\bar{b}$ is such that the set of $i<\omega$ for which $\models\varphi(\bar{a}_i;\bar{b})$ holds is neither finite nor infinite. Then clearly a suitably chosen subsequence of $(\bar{a}_i)_{i<\omega}$ witnesses that $\mathrm{alt}(\varphi)\ge n$ for all $n<\omega$. 5 implies 3: Suppose 3 does not hold, so $\mathrm{vc}(\varphi)=\infty$. By compactness, for every cardinal κ we can find a sequence $(\bar{a}_i)_{i<\kappa}$ such that for all finite subsets $K\subset\kappa$ there are tuples $(\bar{b}_J)_{J\subseteq K}$ such that for all $i\in K$ and $J\subseteq K$ we have $\models\varphi(\bar{a}_i;\bar{b}_J)$ if and only if $i\in J$. By extracting an indiscernible sequence we get a sequence $(\bar{a}'_i)_{i<\omega}$ which also has this property. By compactness there is a tuple $\bar{b}$ such that $\models\varphi(\bar{a}'_i;\bar{b})$ holds if and only if i is even, which contradicts 5. □

A complete theory is said to have the independence property if there is a formula $\varphi(\bar{x};\bar{y})$ which has it.[3] In informal contexts, it has long been customary to use the acronym *IP* for brevity.

[2] Laskowski's VC dimension is a variant: it is one more than the VC dimension in the usual sense. Following an old convention of Shelah's one could denote it by $\mathrm{VC}(\varphi)=\mathrm{vc}(\varphi)+1$. Proposition 2 can then be written as $\mathrm{VC}^*(\varphi)\le 2^{\mathrm{VC}(\varphi)}$. Laskowski also defines the *independence dimension* of φ, which in our notation is $\mathrm{VC}^*(\varphi)=\mathrm{vc}^*(\varphi)+1$.

[3] We will see later that our definition is equivalent to the original definition, which only considered formulas of the form $\varphi(x,\bar{y})$, with a single variable x [48, 49, 42].

Similarly, a complete theory that does not have the independence property is often said to be, or have, *NIP*. More recently, Shelah started calling such theories *dependent* (and theories with the independence property *independent*). Thus we can now choose between certain inelegance and massive overloading of an adjective, and the new term has encountered some resistance. It is hard to avoid taking sides; the fact that we are expanding the classical theory here seems to be a convenient excuse for sticking with the acronym in this paper, as the less anachronistic choice.

Shrinking an indiscernible sequence

We continue with a simple observation that seems to have remained unnoticed until Shelah deduced it from his version of the Baldwin-Benedikt theorem [52]. In NIP theories, if we have an indiscernible sequence and a small set over which it is not indiscernible, we can 'shrink' the sequence, i.e. find an endpiece of the sequence which is indiscernible over the set.

Let us call a sequence $(\bar{a}_i)_{i\in I}$ *uniform* for a formula $\varphi(\bar{x}_0,\dots,\bar{x}_{m-1};\bar{b})$ if for any two strictly ascending tuples $\bar{i},\bar{j}\in[I]^m$ we have $\models\varphi(\bar{a}_{i_0}\dots\bar{a}_{i_{m-1}};\bar{b})$ if and only if $\models\varphi(\bar{a}_{j_0}\dots\bar{a}_{j_{m-1}};\bar{b})$. (Thus a sequence is indiscernible over a set B if and only if it is uniform for all formulas with parameters in B.)

Proposition 5. *Let $(\bar{a}_i)_{i\in I}$ be an indiscernible sequence, and let $\varphi(\bar{x}_0,\dots,\bar{x}_{m-1};\bar{b})$ be a formula such that all tuples $\bar{x}_j$ are compatible with all tuples $\bar{a}_i$ and $\varphi(\bar{x}_0,\dots,\bar{x}_{m-1};\bar{y})$ does not have the independence property. Then the sequence has a non-trivial end piece which is uniform for $\varphi(\bar{x}_0,\dots,\bar{x}_{m-1};\bar{b})$.*

Proof. If not, there is an ascending tuple $\bar{i}^0\in[I]^m$ such that $\models\varphi(\bar{a}_{i^0_0},\dots,\bar{a}_{i^0_{m-1}};\bar{b})$. Since the end piece of I that is defined by $i>i^0_{m-1}$ is not uniform for $\varphi(\bar{x}_0,\dots,\bar{x}_{m-1};\bar{b})$, there is an ascending tuple $\bar{i}^1\in[I]^m$ such that $\models\neg\varphi(\bar{a}_{i^0_0},\dots,\bar{a}_{i^0_{m-1}};\bar{b})$ and $i^1_0>i^0_{m-1}$. Continuing in this way we see that $\varphi(\bar{x}_0,\dots,\bar{x}_{m-1};\bar{y})$ has infinite alternation rank, as witnessed by the indiscernible sequence $(\bar{a}_{i^k_0}\dots\bar{a}_{i^k_{m-1}})_{k<\omega}$. □

Recall that the cofinality of a linear order I is the smallest cardinal κ such that there is a subset $J\subseteq I$ of cardinality $|J|=\kappa$ which is cofinal in I, i. e., for every $i\in I$ there is a $j\in J$ such that $j\geq i$.

Corollary 6. *Let T be a NIP theory. Let $(\bar{a}_i)_{i\in I}$ be a sequence of indiscernibles. Let B be a set of parameters. If I has cofinality $\operatorname{cf} I>|T|+|\bar{a}_i|+|B|$ (any $i\in I$), then a non-trivial end piece $(\bar{a}_i)_{i\in I,i\geq j}$ of the sequence is indiscernible over B.*

Proof. For each formula $\varphi(\bar{x}_0,\dots,\bar{x}_{m-1};\bar{b})$ with parameters in $|B|$ there is a non-trivial end piece $I_{\varphi(\bar{x}_0,\dots,\bar{x}_{m-1};\bar{b})}$ of I, such that the corresponding subsequence is uniform for $\varphi(\bar{x}_0,\dots,\bar{x}_{m-1};\bar{b})$. Since there are only $|T|+|\bar{a}_i|+|B|<\operatorname{cf} I$ such formulas, the intersection of these end pieces is again a non-trivial end piece. The corresponding subsequence is indiscernible over B. □

For example we can take $\kappa=(|T|+|\bar{a}_i|+|B|)^+$. Then by regularity the non-trivial end piece actually has the same length as the original sequence. We can use this to give a simple proof of a classical theorem of Shelah. With a little bit of extra work we could make $(\bar{a}_i)_{i\in I,i\geq j}$ indiscernible over $B\cup\{\bar{a}_i\mid i<j\}$. But we will postpone this result until we get it as a corollary of Theorem 20.

Lemma 7. *Let T be a complete theory. Let $n>0$ be a natural number. Then the following conditions are equivalent.*

1. *No formula $\varphi(\bar{x};\bar{y})$ with $|\bar{y}|\leq n$ has the independence property.*
2. *Let $(\bar{a}_i)_{i<|T|^+}$ be an indiscernible sequence and B a parameter set of cardinality $|B|\leq n$. Then a non-trivial end piece $(\bar{a}_i)_{j\leq i<|T|^+}$ of the sequence is indiscernible over B.*

Proof. 1 ⇒ 2: Like Corollary 6. 2 ⇒ 1: If 1 does not hold, then there is a formula $\varphi(\bar{x};\bar{y})$ of infinite alternation rank, and such that $|\bar{y}|\leq n$. Hence there is a sequence $(\bar{a}_i)_{i<\omega}$ and a tuple $\bar{b}$ such that $\not\models\varphi(\bar{a}_i;\bar{b})\leftrightarrow\varphi(\bar{a}_{i+1};\bar{b})$ for all $i<\omega$. By compactness this contradicts 2. □

Theorem 8. *If no formula $\varphi(x;\bar{y})$, with x a single variable, has the independence property, then T is NIP.*

Proof. By symmetry of the independence property we know that also no formula $\varphi(\bar{x};y)$, with y a single variable, has the independence property. So it suffices to prove that condition 2 of the lemma for $|B| \leq n$ implies condition 2 for $|B| \leq n+1$. So suppose condition 2 holds for $|B| \leq n$, and let $B = \{b_0, b_1, b_2, \ldots, b_{n-1}, b_n\}$. Let $j < \omega$ be such that $(\bar{a}_i)_{j \leq i < \kappa}$ is indiscernible over b_n. Then $(\bar{a}_i b_n)_{j \leq i < \omega}$ is indiscernible. Let $j' < \omega$ $(j' \geq j)$ be such that $(\bar{a}_i b_n)_{j' \leq i < \omega}$ is indiscernible over $b_0 \ldots b_{n-1}$. Now clearly $(\bar{a}_i)_{j' \leq i < \omega}$ is indiscernible over B. □

Shelah originally proved this theorem using the dichotomy of Corollary 24 below and the fact that the statement of the theorem is absolute [48]. Poizat gave a much more direct proof [45], of which the above argument using Proposition 5 is a further simplification. Laskowski gave a combinatorial proof [27].

Some classes of NIP theories

The following observation can be very helpful for showing that a theory is NIP.

Remark 9. The set of formulas which do not have the independence property is closed under Boolean combinations.

Proof. Clearly $\mathrm{alt}(\varphi(\bar{x};\bar{y})) = \mathrm{alt}(\neg\varphi(\bar{x};\bar{y}))$. And it is easy to see that $\mathrm{alt}(\varphi(\bar{x};\bar{y}) \wedge \psi(\bar{x};\bar{y})) \leq \mathrm{alt}(\varphi(\bar{x};\bar{y})) + \mathrm{alt}(\psi(\bar{x};\bar{y})) - 1$. □

Together with Theorem 8 this shows:

Corollary 10. *If T is such that no atomic formula of the form $\varphi(x;\bar{y})$ has the independence property, then T is NIP.*

Remark 11. Any formula that has the independence property also has the order property. Hence stable theories are NIP.

Proof. If $\mathrm{vc}(\varphi) \geq n$, then clearly there are $\bar{a}_0, \ldots, \bar{a}_{n-1}$ and $\bar{b}_0, \ldots, \bar{b}_{n-1}$ such that $\models \varphi(\bar{a}_i, \bar{b}_j)$ if and only if $i < j$. Hence if $\mathrm{vc}(\varphi) = \infty$, then by compactness there are $(\bar{a}_i)_{i<\omega}, (\bar{b}_j)_{j<\omega}$ with this property, i. e., φ has the order property. □

As a very special case this remark shows that strongly minimal theories are NIP. Some analogues of strong minimality were defined, such as o-minimality, C-minimality and p-minimality [40, 30, 14]. These definitions can be phrased as T_0-minimality with respect to an underlying theory T_0. In the cases of o-minimality, C-minimality and p-minimality, these underlying theories are NIP, and, as we will see, NIP is inherited by T_0-minimal theories.

Let T_0 be a first-order theory, not necessarily complete. A complete theory $T \supseteq T_0$ (in a signature extending that of T_0) is called *T_0-minimal* if every T-formula $\varphi(x)$ with parameters but only one free variable is equivalent to a *quantifier-free* T_0-formula with (possibly different) parameters.[4] (In particular, T cannot have more sorts than T_0; e.g. T_0 and T are both 1-sorted.)

For example, if T_{lo} is the theory of linear orders, then a theory T containing the symbol $<$ is o-minimal if and only if it is T_{lo}-minimal. If T_C is the theory of C-structures, then a theory T containing the ternary symbol C is C-minimal if and only if it is T_C-minimal [30, 28]. The following principle was observed by Macpherson and Steinhorn. It also holds for many other stability theoretic properties [30].

Proposition 12. *Suppose T_0 implies that no atomic T_0-formula $\varphi(x;\bar{y})$ has the independence property. Then every T_0-minimal theory is NIP.*

[4]This definition is an obvious variant of the definition of $\mathcal{M}$-minimality [30, 28].

Proof. Towards a contradiction, suppose T is not NIP. Then by the previous theorem there is a formula of the form $\varphi(x;\bar{y})$ which has the independence property. By Proposition 4 (equivalence of 1 and 5) there is an indiscernible sequence $(a_i)_{i<\omega}$ and a tuple $\bar{b}$ such that the truth value of $\varphi(a_i;\bar{b})$ alternates infinitely. Since T is T_0-minimal, there is a quantifier-free T_0-formula $\varphi'(x;\bar{y}')$ and a tuple $\bar{b}'$ such that $\varphi(x;\bar{b})$ and $\varphi'(x;\bar{b}')$ are equivalent. Hence, again by Proposition 4, $\varphi'(x;\bar{b}')$ also has the independence property. Since the class of formulas without the independence property is closed under boolean combinations, we get a contradiction. □

Corollary 13. *All o-minimal theories are NIP.*

Proof. It suffices to check that the formula $\varphi(x;y)$ defined by $x<y$ cannot have the independence property. □

It is not much harder to show that all weakly o-minimal or quasi-o-minimal theories are NIP [29, 5]. Similarly we can also show that all C-minimal or p-minimal theories are NIP [30].

E.g. there is a lot of research on the theory of algebraically closed valued fields as a theory whose completions are controlled, in some sense, by a stable part and an o-minimal part. They are C-minimal and therefore NIP [13].

3 More careful shrinking of indiscernibles

In a relatively recent paper, Baldwin and Benedikt proved a theorem about augmenting a theory without the independence property with a new predicate for a (dense complete) sequence of indiscernibles. We already know that a formula without the independence property has finite alternation rank. Shelah gave the Baldwin-Benedikt theorem an elegant new formulation as a higher-dimensional generalisation of this fact: a sharper version of Proposition 5. Our formulation of the theorem is close to Shelah's, but the straightforward proof is more similar to the original one [4, 52].

Uniformity for a single formula

An equivalence relation $\sim$ on a linearly ordered set I is called *convex* if its classes $i/{\sim}$ are convex. It is called *finite* if it has only finitely many classes. Given a subset $J\subseteq I$ the equivalence relation $\sim_J$ on I is defined by $i\sim_J j \iff i=j$ or $[i,j]\cap J=\emptyset$ (where $[i,j]$ is the convex hull of i and j). $\sim_J$ is always a convex equivalence relation. $\sim_J$ is finite if and only if J is finite. If $\sim$ is a finite convex equivalence relation on I, there need not be a finite subset $J\subseteq I$ such that $\sim_J$ refines $\sim$. However, if I is (Dedekind) complete, as is the case when I is an ordinal, then we can take as J the set of boundary points of classes of $\sim$ [52]. (Recall that a linear order is Dedekind complete if every cut is rational. This is equivalent to the requirement that every non-empty subset bounded from above has a supremum, or that every non-empty subset bounded from below has an infimum.)

Given a general (i. e. not necessarily indiscernible) sequence $(\bar{a}_i)_{i\in I}$, a convex equivalence relation $\sim$ on I and a formula $\varphi(\bar{x}_0\dots\bar{x}_{m-1};\bar{b})$, let us say that the sequence is *uniform modulo* $\sim$ *for* $\varphi(\bar{x}_0\dots\bar{x}_{m-1};\bar{b})$ if for any two strictly ascending tuples $\bar{i},\bar{j}\in[I]^m$ satisfying $i_k\sim j_k$ for all $k<m$, we have $\models\varphi(\bar{a}_{i_0}\dots\bar{a}_{i_{m-1}};\bar{b})$ if and only if $\models\varphi(\bar{a}_{j_0}\dots\bar{a}_{j_{m-1}};\bar{b})$. Thus a sequence is indiscernible over a set B if and only if it is uniform (modulo $\sim_\emptyset$) for all relevant formulas over B, and it is k-indiscernible over B if and only if it is uniform for all relevant formulas $\varphi(\bar{x}_0\dots\bar{x}_{m-1};\bar{b})$ with $m\le k$. (Here by relevant formulas we mean the formulas of the form $\varphi(\bar{x}_0\dots\bar{x}_{m-1};\bar{b})$ such that the tuples of variables $\bar{x}_k$ are compatible with the tuples $\bar{a}_i$ in the sequence.) More generally, we will call a sequence $(\bar{a}_i)_{i\in I}$ *indiscernible modulo* $\sim$ *over* B (or B-indiscernible modulo $\sim$), if it is uniform modulo $\sim$ for all relevant formulas over B. (Of course we can abbreviate 'modulo $\sim_J$' to 'modulo J'.)[5]

[5] I have chosen not to follow Shelah's terminology precisely, since his formulation 'above J' seems to be less suggestive. Uniformity is a substitute for Shelah's Δ-indiscernibility; it is more fine-grained and therefore easier to handle. I have not used the word 'indiscernible' for this, in order to reserve it for the stronger and more natural variant in which the tuples $\bar{i},\bar{j}$ need not be ascending (but must be of the same order type).

Theorem 14. *Let $(\bar{a}_i)_{i\in I}$ be an indiscernible sequence, and let $\varphi(\bar{x}_0 \dots \bar{x}_{m-1}; \bar{b})$ be such that the formula $\varphi(\bar{x}_0 \dots \bar{x}_{m-1}; \bar{y})$ does not have the independence property.*

1. *There is a finite convex equivalence relation $\sim$ such that $(\bar{a}_i)_{i\in I}$ is uniform modulo $\sim$ for $\varphi(\bar{x}_0 \dots \bar{x}_{m-1}; \bar{b})$.*

2. *If I is complete, then there is a finite subset $J \subseteq I$ such that $(\bar{a}_i)_{i\in I}$ is uniform modulo $\sim_J$ for $\varphi(\bar{x}_0 \dots \bar{x}_{m-1}; \bar{b})$.*

Proof. Let us first note that 1 and 2 are equivalent. 1 implies 2: If $\sim$ is a finite equivalence relation on I then we can take as J the set of all suprema and infima of $\sim$-classes which are in I. J is finite, and if I is complete $\sim_J$ refines $\sim$. Conversely, any sequence of indiscernibles can be extended (in the monster model) to a complete sequence of indiscernibles, and the restriction of a finite convex equivalence relation $\sim_J$ to a sub-order is again a finite convex equivalence relation. Therefore 2 implies 1. In fact, it suffices to prove 1 for dense complete linear orders I, and therefore it suffices to prove 2 for such linear orders. So, assuming that I is dense and complete, we will prove 2.

For notational convenience we will write a_i for $\bar{a}_i$, x_k for $\bar{x}_k$, b for $\bar{b}$. For tuples $\bar{i} \in [I]^m$ we write $a_{\bar{i}} = a_{i_0}a_{i_1}\dots a_{i_{m-1}}$. Moreover, $\bar{i}[j/i_k]$ is an abbreviation for the tuple $i_0 \dots i_{k-1} j i_{k+1} \dots i_{m-1}$ in which i_k has been replaced by j. We say that $j \in I$ is a *critical point* of I if some strictly ascending tuple $\bar{i} \in [I]^m$ *witnesses* it, i. e., $j = i_k$ for some $k < m$, and every open interval containing j also contains some j' such that $\not\models \varphi(a_{\bar{i}}; b) \leftrightarrow \varphi(a_{\bar{i}[j'/i_k]}; b)$. Let $J \subseteq I$ be the set of all critical points. The two claims below finish the proof of the theorem.

Claim 1. $(\bar{a}_i)_{i\in I}$ is uniform modulo $\sim_J$ for $\varphi(\bar{x}_0 \dots \bar{x}_{m-1}; \bar{b})$.

Proof of Claim 1. Let $\bar{i}, \bar{j} \in [I]^m$ be two strictly ascending tuples such that $i_k \sim_J j_k$ for all $k < m$. We will show that $\models \varphi(a_{\bar{i}}; b) \leftrightarrow \varphi(a_{\bar{j}}; b)$. We will use induction on the last index k such that $i_k \neq j_k$.

Let $k < m$ be the greatest index such that $i_k \neq j_k$. (If there is no such index, the statement is trivial.) Without loss of generality $i_k < j_k$. Note that $\bar{i}[j_k/i_k]$ is also a strictly ascending tuple. Using completeness of I it is easy to see that $\models \varphi(a_{\bar{i}[j_k/i_k]}; b) \leftrightarrow \varphi(a_{\bar{i}}; b)$. (Towards a contradiction, suppose $\not\models \varphi(a_{\bar{i}[j_k/i_k]}; b) \leftrightarrow \varphi(a_{\bar{i}}; b)$. Let $j' = \inf\{j \in I \mid j > i_k \textit{ and } \not\models \varphi(a_{\bar{i}[j/i_k]}; b) \leftrightarrow \varphi(a_{\bar{i}}; b)\}$. Clearly $i_k \leq j' \leq j_k$, and $j' \in J$. Hence $[i_k, j_k] \cap J \neq \emptyset$. Therefore $i_k \not\sim_J j_k$, a contradiction.) On the other hand $\models \varphi(a_{\bar{i}[j_k/i_k]}; b) \leftrightarrow \varphi(a_{\bar{j}}; b)$ by the induction hypothesis, so we are finished.

Claim 2. J is finite.

Proof of Claim 2. Let $(M, I, <, f)$ be a structure consisting of the following data. $M \models T$ is a model containing the sequence $(a_i)_{i\in I}$. I is the index set of the set of indiscernibles, in a new sort. $<$ is the linear order on I, and $f : I \to M$ is the function given by $f(i) = a_i$. (Here we are stretching the notational convenience a bit.) For $k < m$ consider the following formula $\psi_k(\bar{u})$ (over b) in the language of $(M, I, <, f)$:

$$u_0 < u_1 < \dots < u_{m-1} \quad \wedge \quad \forall v, v' \Big(v < u_k < v' \quad \to \\ \exists w \; \big(v < w < v' \; \wedge \; \neg\big[\varphi(f(\bar{u}); b) \leftrightarrow \varphi(f(\bar{u}[w/u_k]; b)) \big] \; \big) \Big).$$

Clearly $(M, I, <, f) \models \psi_k(\bar{i})$ if and only if $\bar{i}$ witnesses that i_k is critical.

Towards a contradiction, let us assume that J is infinite. Then for some $k < m$ there is an infinite set of tuples $\bar{i}$ with pairwise different i_k such that $\models \psi_k(\bar{i})$. We fix such a k. For a very big cardinal κ let $(M', I', <, f)$ be an elementary extension of $(M, I, <, f)$ which is κ-saturated. By compactness and saturation, in I' there are arbitrarily long sequences of tuples $\bar{i}$ as above, and so we can extract a sequence $(\bar{i}^n)_{n<\omega}$ of indiscernibles over b such that $\models \psi(\bar{i}^n; b)$ and $i^n_k \neq i^{n'}_k$ for all $n < n' < \omega$.

By indiscernibility $\models \varphi(f(\bar{i}^n); b)$ holds either for all $n < \omega$ or for no $n < \omega$. To ease notation, let us assume without loss of generality that $\models \varphi(f(\bar{i}^n); b)$ holds for all $n < \omega$. Now we construct a sequence $(\bar{j}^n)_{n<\omega}$ by induction. For n even we simply define $\bar{j}^n = \bar{i}^n$. For n odd we define $j^n_{k'} = i^n_{k'}$ for all $k' \neq k$ and choose $\bar{j}^n_k$ as follows. By compactness and saturation there is an element $j_- < i^n_k$ in I' which is greater than all elements of $\mathcal{I}_n = \bigcup_{\ell<\omega} \bar{i}^\ell \cup \{j^\ell_k \mid \ell < n\}$ that are themselves less than i^n_k. There is also an element $j_+ > i^n_k$ in I' which is less than all elements of $\mathcal{I}_n$ that are themselves greater than i^n_k. Since $\models \psi(\bar{\imath}^n)$, the open interval (j_-, j_+) contains an element j^n_k such that $\models \neg\varphi(f(\bar{\imath}^n[j^n_k/i^n_k]); b)$, i. e. $\models \neg\varphi(f(\bar{\jmath}); b)$.

Now note that the sequence $(\bar{\jmath}^n)_{n<\omega}$ is indiscernible with respect to quantifier-free formulas because $(\bar{\imath}^n)_{n<\omega}$ is. As an elementary extension of $(M, I, <, f)$, the structure $(M', I', <, f)$ also has the property that the type in M' of a tuple $f(\bar{\imath})$ only depends on the quantifier-free type in I of the tuple $\bar{\imath}$. Therefore the sequence $(f(\bar{\imath}^n))_{n<\omega}$ is an indiscernible sequence in M'. But $\models \varphi(f(\bar{\imath}^n); b)$ holds if and only if n is even. Therefore the alternation rank of $\varphi(x_0 \dots x_{m-1}; y)$ is infinite, a contradiction. □

Indiscernibility

We have seen that the failure of a sequence of indiscernibles to be uniform for a single formula $\varphi(\bar{x}_0 \dots \bar{x}_{m-1}; \bar{b})$—where $\varphi(\bar{x}_0 \dots \bar{x}_{m-1}; \bar{y})$ does not have the independence property—can be described by a finite convex equivalence relation. The failure of a sequence of indiscernibles to be uniform for a 'small' set of such formulas can therefore be described by the intersection of 'few' finite convex equivalence relations, which is a convex equivalence relation with 'few' classes. However, in the presence of a dense linear order, the precise bound for the number of classes may be greater than the number of formulas: If we take I to consist of the reals, the intersection of the equivalence relations $\sim_{\{q\}}$ on I, where q is rational, has $2^{\aleph_0}$ classes—one for every real.

By compactness we can extend any sequence of indiscernibles $(\bar{a}_i)_{i\in I}$ to a complete sequence of indiscernibles $(\bar{a}_i)_{i\in \bar{I}}$, obtain a subset $J \subseteq \bar{I}$ using Theorem 14.2, and consider the restriction of $\sim_J$ to I. Remains the problem of counting the classes of $\sim_J\restriction I$.

In Theorem 20 below we would be able to ignore this minor nuisance concerning many-formulas versions of Theorem 14.1 by stating only the analogue of Theorem 14.2. Readers who are not interested in the 'moreover' part of Theorem 20.1 can jump to that theorem immediately after the following observation.

Remark 15. Let J be a subset of the completion of a linear order I. The number of classes of $\sim_J\restriction I$ is bounded by 1 plus twice the cardinality of the completion of J. In particular, it is finite if J is finite, and at most $2^{|J|}$ otherwise.

Proof. Clearly we may assume that I is complete. Let $\bar{J}$ consist of the greatest lower bounds (in I) of all non-empty subsets of J that are bounded from below in I. Then $J \subseteq \bar{J} \subseteq I$, $\bar{J}$ is Dedekind complete, and $\bar{J}$ is minimal with these properties. It does *not* follow that $\bar{J}$ is isomorphic to the completion of J (since the greatest lower bound of a subset of J may exist in J but not be the greatest lower bound as evaluated in I). But it is easy to see that the worst that can happen is that some points get a duplicate, and even that is impossible if J is finite. Therefore $\bar{J}$ has the same cardinality as the completion of J.

It suffices to show that the number of classes of $\sim_{\bar{J}}$ is bounded by 1 plus twice the size of $\bar{J}$. First, we have $|\bar{J}|$ 1-element classes consisting of an element of $\bar{J}$ each. Next, there may be one class which is greater than all elements of $\bar{J}$. Each of the remaining classes is disjoint from $\bar{J}$ and bounded from above by an element of $\bar{J}$, hence by a least element $b \in \bar{J}$. Since b determines the class, the number of remaining classes is at most $|J|$. □

A linear order is called *scattered* if it does not contain a copy of the rationals as a suborder. For example all ordinals are scattered. Note that every suborder of a scattered linear order is scattered. An *ordered sum* of linear orders I_k, indexed by a linear order K, is the obvious linear order defined on the disjoint union $\bigcup_{k\in K} I_k$. A classical theorem of Hausdorff implies that being scattered or not is a surprisingly strong dichotomy [15, 41, 47, 16].

Theorem 16 (Hausdorff's Theorem on scattered linear orders)**.**

1. *Every linear order has a unique decomposition as an ordered sum $\bigcup_{k\in K} I_k$, where each I_k is non-empty and scattered, and K is dense unless $|K| \leq 1$.*

2. *The class of scattered linear orders is the smallest class of linear orders which contains the 1-element order and is closed under isomorphism, order-reversing, and ordinal-indexed ordered sums.*

Proof. (Sketch only.) 1. Let $\mathcal{A}$ be the smallest class of linear orders which contains the 1-element order and is closed under isomorphism, order-reversing and ordinal-indexed ordered sums. Given a linear order I, the relation defined by $i \sim j \iff [i,j] \in \mathcal{A}$ is a convex equivalence relation, with classes $(i/{\sim}) \in \mathcal{A}$. We can take $K = I/{\sim}$ and $I_k = k$. 2. Let $\mathcal{S}$ be the class of scattered linear orders. We can easily verify $\mathcal{A} \subseteq \mathcal{S}$ using the inductive definition of $\mathcal{A}$. On the other hand, if we decompose $I \in \mathcal{S}$ as in 1, then clearly $|K| \leq 1$. Hence by uniqueness of the decomposition, I consists of a single $\sim$-class $I \in \mathcal{A}$. □

We prove some straightforward consequences that are probably well known, although I could not find a reference for them.

Remark 17. If $(I_k)_{k\in K}$ is a sequence of complete linear orders I_k with endpoints, indexed by a complete linear order K, then the ordered sum of the sequence is complete.

Proposition 18. *The completion of a scattered linear order is scattered and has the same cardinality.*

Proof. The class of linear orders whose completion is scattered contains the 1-element order, and it is clearly closed under isomorphism and order-reversing. It follows from the previous remark that it is also closed under ordinal-indexed ordered sums. Using Hausdorff's Theorem it follows that the completion of a scattered linear order is scattered.

Next we show that for scattered linear orders completion is cardinality-preserving. The class of linear orders with cardinality-preserving completions contains all finite linear orders, and it is clearly closed under isomorphism and order-reversing. It remains to check that it is closed under ordinal-indexed ordered sums of sequences $(I_k)_{k<\alpha}$. We may assume that each term I_k is non-empty. Now for each I_k consider $\bar{I}_k$, the smallest complete linear order with endpoints which contains I_k. Note that $\left|\bar{I}_k\right| = |I_k|$. Thus the ordered sum of $(I_k)_{k<\alpha}$ is embedded in the ordered sum of $(\bar{I}_k)_{k<\alpha}$, which is complete and has the same cardinality. □

It easily follows from this that a linear order is scattered if and only if every suborder has a cardinality-preserving completion. But we will not use this nice characterisation.

Proposition 19. *Let J be a subset of the completion of a scattered linear order I. The number of classes of $\sim_J\restriction I$ is at most $2|J|+1$.*

Proof. Since I is scattered its completion is also scattered, and so is J. Therefore the completion of J has cardinality $|J|$. Apply Remark 15. □

Theorem 20. *Suppose T has NIP. Let $(\bar{a}_i)_{i\in I}$ be a sequence of indiscernibles. Let B be a set of parameters, and let $\kappa = |T| + |\bar{a}_i| + |B|$ (any $i \in I$).*

1. *There is a convex equivalence relation $\sim$ on I with at most 2^κ classes such that the sequence $(\bar{a}_i)_{i\in I}$ is indiscernible modulo $\sim$ over B.*
   Moreover, if I is scattered, then we can find such $\sim$ with at most κ classes.

2. *If I is complete, then there is a subset $J \subseteq I$ of cardinality $|J| \leq \kappa$ such that the sequence $(\bar{a}_i)_{i\in I}$ is indiscernible modulo J over B.*

Proof. 1 follows from 2 (using Proposition 19 for the 'moreover' part), so we only prove 2. There are $|T| + |\bar{a}_i| + |B|$ formulas $\varphi(\bar{x}_0, \ldots, \bar{x}_{m-1})$ with parameters in B for which $\bar{x}_k$ and $\bar{a}_i$ are compatible. For each of them we get a set J_φ from Theorem 14. If we take $J = \bigcup_\varphi J_\varphi$, then the sequence is uniform modulo J for every such φ, hence indiscernible modulo J over B. □

Corollary 21. *Let T be a NIP theory. Let $(\bar{a}_i)_{i\in I}$ be a sequence of indiscernibles. Let B be a set of parameters. If I has cofinality $\operatorname{cf} I > |T| + |\bar{a}_i| + |B|$ (any $i \in I$), then a non-trivial end piece $(\bar{a}_i)_{i\in I, i\geq j}$ of the sequence is indiscernible over $B \cup \{\bar{a}_i \mid i < j\}$.*

Proof. We may assume that I is complete. Let J be as in part 2 of the theorem. Since $|J| < \operatorname{cf} I$, we have $j = \sup J \in I$. □

By Proposition 4 and compactness, it is not hard to rephrase Theorem 20.1, Theorem 20.2 or Corollary 6 as characterisations of NIP. These characterisations even work with larger bounds (instead of 2^κ, κ or $|T| + |\bar{a}_i| + |B|$, respectively). But if we make the bounds smaller, we get new, stronger variants of NIP such as the notion of a *strongly dependent* theory. (See Section 8 for more on this.)

4 Counting types

For notational convenience we will discuss only 1-sorted theories in this section. The *stability function* of a complete theory T is the function

$$g_T(\kappa) \quad = \sup_{M \models T,\ |M|=\kappa} |\mathrm{S}(M)| \quad = \sup_{M \models T,\ |M|=\kappa} \left|\mathrm{S}^1(M)\right|.$$

As usual, $\mathrm{S}^n(M)$ denotes the set of all complete n-types over M, i.e. the set of all complete types over n in some fixed set of n free variables, and $\mathrm{S}(M) = \bigcup_{n<\omega} S^n(M)$. The *stability spectrum*, which we will not use, is the set of infinite cardinals κ such that $g_T(\kappa) = \kappa$. Clearly $\kappa \le g_T(\kappa) \le 2^\kappa$ for all $\kappa \ge |T|$.

The classification of the possible stability functions (in the countable case) was finished by Keisler, but much of the work was done earlier by Shelah [22, 48]. For countable T, only the following six functions are possible.

$$\kappa \quad \kappa + 2^{\aleph_0} \quad \kappa^{\aleph_0} \quad \operatorname{ded}\kappa \quad (\operatorname{ded}\kappa)^{\aleph_0} \quad 2^\kappa,$$

where $\operatorname{ded}(\kappa)$ is defined as the supremum of the cardinalities of all linear orders which contain a dense subset of cardinality κ.

The five dividing lines between these functions are total transcendence, superstability, stability, multi-order and the independence property [21]. But there is some fine print. Since $\kappa < \operatorname{ded}\kappa \le (\operatorname{ded}\kappa)^{\aleph_0} \le 2^\kappa$ for all infinite cardinals κ, the generalised continuum hypothesis implies a collapse of the last three functions; I believe it is still unknown whether it is consistent that $\operatorname{ded}\kappa < (\operatorname{ded}\kappa)^{\aleph_0}$ for some cardinal κ.

We will now look more closely at the last dividing line. We will not need any previous results. For a formula $\varphi(\bar{x};\bar{y})$ and a set B, $\mathrm{S}_\varphi(B)$ denotes the set of all complete φ-types (i.e. maximal consistent sets of instances $\varphi(\bar{x};\bar{b})$ of φ and instances $\neg\varphi(\bar{x};\bar{b})$ of $\neg\varphi$) over B.

Remark 22. If $\varphi(\bar{x};\bar{y})$ has the independence property, then for every $\kappa \ge |T|$ there is a model M of cardinality $|M| = \kappa$ such that $|\mathrm{S}_\varphi(M)| = 2^\kappa$.

Shelah proved the following theorem, which provides a converse to the remark under the set-theoretical hypothesis $\operatorname{ded}\kappa < 2^\kappa$ [48].

Theorem 23. *Let $\varphi(\bar{x};\bar{y})$ be a formula and A an infinite set such that $|\mathrm{S}_\varphi(A)| > \operatorname{ded}|A|$. Then $\varphi(\bar{x};\bar{y})$ has the independence property.*

Proof. To avoid some boring complications, we assume that $\bar{y} = y$ is a single variable. Let $B \subseteq A$ be a subset of minimal cardinality $\mu = |B|$ such that still $|\mathrm{S}_\varphi(B)| > \operatorname{ded}|A|$. Let $(b_i)_{i<\mu}$ be an enumeration of B, and let $B_\alpha = \{b_i \mid i < \alpha\}$ for all $\alpha < \mu$. We will need the following sets.

$$S_\alpha = \left\{p \in \mathrm{S}_\varphi(B_\alpha) \;\middle|\; |\mathrm{S}_\varphi(B){\restriction}p| > \operatorname{ded}|A| \right\} \quad \text{for all } \alpha < \mu$$
$$S_\mu = \left\{p \in \mathrm{S}_\varphi(B) \;\middle|\; \forall\alpha < \mu :\ p{\restriction}B_\alpha \in S_\mu\right\}$$
$$S_{<\mu} = \bigcup_{\alpha<\mu} S_\alpha \qquad S_{\le\mu} = \bigcup_{\alpha\le\mu} S_\alpha.$$

Here and in the following we are slightly abusing notation by writing $W\restriction q = \{p \in W \mid p \supseteq q\}$ if q is a type and W is a set of types.

First we observe that for any $q \in S_\alpha$ (where $\alpha < \mu$) we have $|S_\mu\restriction q| > \operatorname{ded}|A|$. Towards proving this, using the identity

$$\mathrm{S}_\varphi(B) \setminus S_\mu = \bigcup_{\alpha<\mu} \bigcup_{q\in \mathrm{S}_\varphi(B_\alpha)\setminus S_\alpha} \mathrm{S}_\varphi(B)\restriction q$$

we can check that $|\mathrm{S}_\varphi(B) \setminus S_\mu| \leq \operatorname{ded}|A|$. (The index sets in the second union have cardinality $\leq \operatorname{ded}|A|$ because $|\mathrm{S}_\varphi(B_\alpha)| \leq \operatorname{ded}|A|$ by minimality of μ.) Now the claim easily follows from $\mathrm{S}_\varphi(B)\restriction q \subseteq S_\mu\restriction q \cup (\mathrm{S}_\varphi(B) \setminus S_\mu)$. We note for later use that

$$|S_\mu\restriction q| > \operatorname{ded}\mu.$$

With this out of the way, we define a linear order on $S_{\leq\mu}$, as follows. For $p, q \in S_{\leq\mu}$ such that $p \neq q$ there is a minimal $\alpha < \mu$ such that $p \in S_\alpha$, or $q \in S_\alpha$, or $p\restriction B_{\alpha+1} \neq q\restriction B_{\alpha+1}$. We set $p < q$ if one of the following holds.

- $\varphi(\bar{x}; b_\alpha) \in p$ and $\neg\varphi(\bar{x}; b_\alpha) \in q$,
- $p \in S_\alpha$ and $\neg\varphi(\bar{x}; \bar{b}_\alpha) \in q$, or
- $\varphi(\bar{x}; \bar{b}_\alpha) \in p$ and $q \in S_\alpha$.

Otherwise clearly $q < p$, and it is easy to check that this is in fact a linear order. We will use it to prove the following claim, which concludes the theorem.

Claim. Let $n < \omega$. For all $q \in S_{<\mu}$ there are elements $c^q_0, \ldots, c^q_{n-1} \in B$ such that for all $w \subseteq n$ the set $q(\bar{x}) \cup \{\varphi(\bar{x}; c^q_i) \mid i \in w\} \cup \{\neg\varphi(\bar{x}; c^q_i) \mid i \in n \setminus w\}$ is consistent.

The proof is by induction, the case $n = 0$ being trivially true. So suppose the claim is true for all numbers smaller than n. Note that $S_{<\mu}\restriction q \subset S_{\leq\mu}\restriction q$ is a dense subset. Since $|S_{\leq\mu}\restriction q| > \operatorname{ded}\mu$, it follows that $|S_{<\mu}\restriction q| > \mu$, and so $|S_\alpha\restriction q| > \mu$ for an ordinal $\alpha < \mu$.

For every type $p \in S_\alpha\restriction q$ the induction hypothesis yields a tuple $c^p_0, \ldots, c^p_{n-2} \in B$ such that for all $w \subseteq n-1$ the set $p(\bar{x}) \cup \{\varphi(\bar{x}; c^p_i) \mid i \in w\} \cup \{\neg\varphi(\bar{x}; c^p_i) \mid i \in (n-1) \setminus w\}$ is consistent. Since $\left|B^{n-1}\right| = \mu < |S_\alpha\restriction q|$, we can find two distinct types $p_1, p_2 \in S_\alpha\restriction q$ giving rise to the same tuple. For some $\beta < \mu$ we have $\varphi(\bar{x}; b^\beta) \in p_1$ and $\neg\varphi(\bar{x}; b^\beta) \in p_2$. (If not, exchange p_1 and p_2.) Now it is easy to see that $c^q_0 = c^{p_1}_0 = c^{p_2}_0, \ldots, c^q_{n-2} = c^{p_1}_{n-2} = c^{p_2}_{n-2}, c^q_{n-1} = b^\beta$ are as in the claim. □

Corollary 24. *If T does not have the independence property, then*

$$g_T(\kappa) \leq (\operatorname{ded}\kappa)^{|T|}.$$

If T has the independence property, then

$$g_T(\kappa) = 2^\kappa.$$

A result of Mitchell says that for any cardinal κ such that $\operatorname{cf}\kappa > \aleph_0$, there is a cardinal preserving Cohen extension of the set-theoretical universe such that $\operatorname{ded}\kappa < 2^\kappa$ holds in the extension [31]. If we apply this to a suitable $\kappa > |T|$, we see that having the independence property or not can always be detected by counting types, just not (in general) in the original model of ZFC.

5 Forking and related notions

Shelah's notion of forking, although somewhat complicated, is a fundamental tool for stable theories. A big breakthrough for simple theories was Byunghan Kim's discovery that forking is symmetric in that context as well [25]. Forking in unstable NIP theories is not symmetric, but recently it is becoming more and more clear that it plays a fundamental, albeit still mysterious, role even when it is not symmetric. Therefore it makes sense to develop the theory of forking and some related notions for arbitrary theories. Most of this can be found in Shelah's book [49], but we will be a bit more methodical in our presentation.

Three preindependence relations

A (partial) type $p(\bar{x})$ is *finitely satisfied* in a set C if for every finite subset $p_0(\bar{x}) \subseteq p(\bar{x})$ there is a tuple $\bar{a} \in C$ which realises $p_0(\bar{x})$. An important case is when p is a complete type over a set B and $M \subseteq B$ is a model in which p is finitely satisfied. In this case the type p is called a *coheir* of its restriction to M. We write $A \mathop{\smile\kern-0.9em|}^{\mathrm{u}}_{C} B$ if $\operatorname{tp}(\bar{a}/BC)$ is finitely satisfied in C for every tuple $\bar{a} \in A$.[6]

We will say that a complete type $p(\bar{x})$ over B *quasidivides*[7] over a set C if there are tuples $\bar{b}_0, \bar{b}_1 \in B$ that start an indiscernible sequence over C, but no sequence starting with $\bar{b}_0, \bar{b}_1$ is indiscernible over $\bar{a}C$ for a realisation $\bar{a}$ of $p(\bar{x})$. We write $A \mathop{\smile\kern-0.9em|}^{\mathrm{s}}_{C} B$ if $\operatorname{tp}(\bar{a}/BC)$ does not quasidivide over C for any $\bar{a} \in A$.

A formula $\varphi(\bar{x}; \bar{b})$ *divides* over a set C if there is a sequence $(\bar{b}_i)_{i<\omega}$ of realisations of $\operatorname{tp}(\bar{b}/C)$ such that for some $n < \omega$ every n-element subset of the set $\{\varphi(\bar{x}; \bar{b}_i) \mid i < \omega\}$ is inconsistent. A complete type $p(\bar{x})$ divides over C if it contains a formula that divides over C [49]. We write $A \mathop{\smile\kern-0.9em|}^{\mathrm{d}}_{C} B$ if $\operatorname{tp}(\bar{a}/BC)$ does not divide over C for any tuple $\bar{a} \in A$.

In all three definitions it was deliberately left open whether the tuples must be finite or can be infinite—in each case it is easily seen that it does not matter. The following characterisations are also easy, and left to the reader.

Remark 25.

1. $A \mathop{\smile\kern-0.9em|}^{\mathrm{u}}_{C} B$ if and only if
   every formula $\varphi(\bar{x})$ over BC satisfied by some $\bar{a} \in A$
   is also satisfied by some $\bar{a}' \in C$.

2. $A \mathop{\smile\kern-0.9em|}^{\mathrm{s}}_{C} B$ if and only if
   for every indiscernible sequence $(\bar{b}_i)_{i<\omega}$ over C such that $\bar{b}_0, \bar{b}_1 \in BC$
   there is an indiscernible sequence $(\bar{b}'_i)_{i<\omega}$ over AC such that $\bar{b}'_0\bar{b}'_1 = \bar{b}_0\bar{b}_1$.

3. $A \mathop{\smile\kern-0.9em|}^{\mathrm{d}}_{C} B$ if and only if
   for every indiscernible sequence $(\bar{b}_i)_{i<\omega}$ over C such that $\bar{b}_0 \in BC$
   there is $(\bar{b}'_i)_{i<\omega} \equiv_{\bar{b}_0 C} (\bar{b}_i)_{i<\omega}$ which is indiscernible over AC.

Preindependence relations were first defined in my thesis. I chose the axioms for the technical reason that they are, in any complete theory, the greatest common denominator of the relations $\mathop{\smile\kern-0.9em|}^{\mathrm{u}}$, $\mathop{\smile\kern-0.9em|}^{\mathrm{s}}$, $\mathop{\smile\kern-0.9em|}^{\mathrm{d}}$, and also of many other similarly defined notions. This includes notions derived in an equally obvious way from splitting, forking, semiforking, thorn-forking, thorn-dividing, and modular pairs in the lattice of algebraically closed sets. Taking this ubiquity into account, the axioms are remarkably strong in their combination [1, 2].

A *preindependence relation* is a ternary relation $\mathop{\smile\kern-0.9em|}$ between the small subsets of the monster model, which satisfies the following subset of the traditional forking calculus for stable (and simple) theories.

(invariance) If $A \mathop{\smile\kern-0.9em|}_{C} B$ and $(A', B', C') \equiv (A, B, C)$, then $A' \mathop{\smile\kern-0.9em|}_{C'} B'$.

(monotonicity) If $A \mathop{\smile\kern-0.9em|}_{C} B$, $A' \subseteq A$ and $B' \subseteq B$, then $A' \mathop{\smile\kern-0.9em|}_{C} B'$.

(base monotonicity) Suppose $D \subseteq C \subseteq B$. If $A \mathop{\smile\kern-0.9em|}_{D} B$, then $A \mathop{\smile\kern-0.9em|}_{C} B$.

(transitivity) Suppose $D \subseteq C \subseteq B$. If $B \mathop{\smile\kern-0.9em|}_{C} A$ and $C \mathop{\smile\kern-0.9em|}_{D} A$, then $B \mathop{\smile\kern-0.9em|}_{D} A$.

(normality) $A \mathop{\smile\kern-0.9em|}_{C} B$ implies $AC \mathop{\smile\kern-0.9em|}_{C} B$.

(strong finite character) If $A \not\mathop{\smile\kern-0.9em|}_{C} B$, then there are finite tuples $\bar{a} \in A$, $\bar{b} \in B$ and a formula $\varphi(\bar{x}) \in \operatorname{tp}(\bar{a}/BC)$ such that $\bar{a}' \not\mathop{\smile\kern-0.9em|}_{C} \bar{b}$ for all $\bar{a}'$ that satisfy $\varphi(\bar{x})$.

[6] The letter 'u' is for 'ultrafilter', see Proposition 29.

[7] In an earlier version I called this notion 'strong splitting', although it is a bit weaker than Shelah's notion of this name [49]. It follows from Proposition 34 below that for global types the two definitions are equivalent. Using quasidividing rather than strong splitting was necesary to ensure that $\mathop{\smile\kern-0.9em|}^{\mathrm{s}}$ satisfies the transitivity axiom. Unfortunately there are types that quasidivide but do not split. The letter 's' is still reminiscent of the other term.

Proposition 26. $\mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{u}}$ *is a preindependence relation.*

Proof. Invariance, monotonicity, base monotonicity and strong finite character are obvious. Transitivity: Suppose $D \subseteq C \subseteq B$, $B \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{u}}_C A$ and $C \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{u}}_D A$. Then every formula over AD which is satisfied in B is (over AC and therefore) satisfied in C, hence satisfied in D. Therefore $B \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{u}}_D A$. Normality: Suppose $A \mathop{\smile\!\!\!\!\!\!\;|}_C B$, and φ is a formula over BC. If φ is satisfied by a tuple $\bar{a}\bar{c} \in AC$, where $\bar{a} \in A$ and $\bar{c} \in C$, then the formula $\varphi(\bar{x}, \bar{c})$ is over BC, hence satisfied in C. Therefore φ is satisfied in AC. □

Proposition 27. $\mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{s}}$ *is a preindependence relation.*

Proof. Invariance, monotonicity, transitivity and normality are clear. Base monotonicity: Suppose $D \subseteq C \subseteq B$ and $A \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{s}}_D B$. Let $(\bar{b}_i)_{i<\omega}$ be indiscernible over C, with $\bar{b}_0, \bar{b}_1 \in B$. Let $\bar{c}$ be an enumeration of C. Then $(\bar{b}_i\bar{c})_{i<\omega}$ is an indiscernible sequence over D. Hence there is an AD-indiscernible sequence $(\bar{b}'_i\bar{c}'_i)_{i<\omega}$ such that $\bar{b}'_0\bar{c}'_0\bar{b}'_1\bar{c}'_1 = \bar{b}_0\bar{c}\bar{b}_1\bar{c}$. It follows that $(\bar{b}'_i)_{i<\omega}$ is AC-indiscernible and $\bar{b}'_0\bar{b}'_1 = \bar{b}_0\bar{b}_1$. Therefore $A \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{s}}_C B$. Strong finite character: If $A \not\mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{s}}_C B$, then there are $\bar{b}_0, \bar{b}_1 \in B$ which witness this: There is a C-indiscernible sequence $(\bar{b}_i)_{i<\omega}$ but no such AC-indiscernible sequence. Hence the type expressing that $\bar{b}_0, \bar{b}_1, \bar{y}_2, \bar{y}_3, \bar{y}_4, \ldots$ is AC-indiscernible is inconsistent. For this inconsistency a single formula $\varphi(\bar{a}, \bar{y}_0\bar{y}_1) \in \mathrm{tp}(\bar{b}_0\bar{b}_1/AC)$ (φ over C) is sufficient. Now clearly $\varphi(\bar{x}, \bar{b}_0\bar{b}_1) \in \mathrm{tp}(\bar{a}/BC)$, and $\bar{a}' \not\mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{s}}_C \bar{b}_0\bar{b}_1$ for all $\bar{a}'$ that satisfy $\varphi(\bar{x}, \bar{b}_0\bar{b}_1)$. □

Proposition 28. $\mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{d}}$ *is a preindependence relation.*

Proof. Left to the reader [1, 2]. □

Three extensible preindependence relations

We will call a preindependence relation *extensible* if it also satisfies the following axiom. Note that the empty relation is an extensible preindependence relation.

(extension) If $A \mathop{\smile\!\!\!\!\!\!\;|}_C B$ and $\hat{B} \supseteq B$, then there is $A' \equiv_{BC} A$ such that $A' \mathop{\smile\!\!\!\!\!\!\;|}_C \hat{B}$.

By invariance, it is equivalent to require that there is a set $\hat{B}' \equiv_{BC} \hat{B}$ such that $A \mathop{\smile\!\!\!\!\!\!\;|}_C \hat{B}'$. The relation $\mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{s}}$ is never extensible, because $a \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{s}}_\emptyset a$ holds for every single element a. On the other hand, $\mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{d}}$ is extensible in all simple theories and also in many, though not in all, non-simple theories.

$\mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{u}}$ is even better, but before we can see this we need a few more definitions. If $\bar{x}$ is a tuple of variables, $A^{\bar{x}}$ denotes the tuples of elements in A that are compatible with $\bar{x}$. An *ultrafilter* on a small subset $\mathcal{A} \subseteq \mathbb{M}^{\bar{x}}$ is a subset $\mathcal{U} \subset \mathcal{P}(\mathcal{A})$ with the following properties. Every superset of a member of $\mathcal{U}$ is in $\mathcal{U}$, $\mathcal{U}$ is closed under finite intersections, and for every set $X \subseteq C^{\bar{x}}$ exactly one of X and $\mathcal{A} \setminus X$ is in $\mathcal{U}$. If $\mathcal{A} = A^{\bar{x}}$, we may just say that $\mathcal{U}$ is an ultrafilter on A. If $\mathcal{U}$ is an ultrafilter on $\mathcal{A} \subseteq \mathbb{M}^{\bar{x}}$, then the *average type* of $\mathcal{U}$ over a set B is defined as $\mathrm{Av}(\mathcal{U}/B) = \{\varphi(\bar{x}; \bar{b}) \mid \{\bar{a} \in \mathcal{A} \mid \models \varphi(\bar{a}; \bar{b})\} \in \mathcal{U}\}$.

Proposition 29. *1.* $\mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{u}}$ *is an extensible preindependence relation.*

2. $\bar{a} \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{u}}_C B$ *if and only if* $\mathrm{tp}(\bar{a}/BC) = \mathrm{Av}(\mathcal{U}/BC)$ *for some ultrafilter* $\mathcal{U}$ *on* C*.*

Proof. 1: This is an immediate consequence of 2 and Proposition 26. If $\bar{a} \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{u}}_C B$, then there is an ultrafilter $\mathcal{U}$ on C as in 1. Given any $\hat{B} \supseteq BC$, let $\bar{a}'$ satisfy $\mathrm{Av}(\mathcal{U}/BC)$. Then clearly $\bar{a}' \equiv_{BC} \bar{a}$ and $\bar{a}' \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{u}}_C B$. 2: First suppose $\bar{a} \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{u}}_C B$. Let $\mathcal{F} = \{\mathcal{C} \subseteq C^{\bar{x}} \mid$ *for some formula* $\varphi \in \mathrm{tp}(\bar{a}/BC)$ *we have* $\mathcal{C} \supseteq \varphi[C]\}$. Then it is easy to see that $\mathcal{F}$ contains $C^{\bar{x}}$ and is closed under finite intersections. Hence by the ultrafilter theorem there is an ultrafilter $\mathcal{U} \supseteq \mathcal{F}$ on $C^{\bar{x}}$ which extends $\mathcal{F}$. Since $\varphi(\bar{x}) \in \mathrm{tp}(\bar{a}/BC)$ implies $\{\bar{c} \in C \mid \models \varphi(\bar{c})\} \in \mathcal{F} \subseteq \mathcal{U}$, we have $\mathrm{tp}(\bar{a}/BC) \subseteq \mathrm{Av}(\mathcal{U}/BC)$, hence $\mathrm{tp}(\bar{a}/BC) = \mathrm{Av}(\mathcal{U}/BC)$. Conversely, if $\mathrm{tp}(\bar{a}/BC) = \mathrm{Av}(\mathcal{U}/BC)$, then for every formula $\varphi(\bar{x})$ over BC the set $\{\bar{c} \in C \mid \models \varphi(\bar{c})\}$ is in $\mathcal{U}$, hence not empty. □

If we have an arbitrary preindependence relation $\perp\!\!\!\perp$, we can use it to define a new relation $\perp\!\!\!\perp^{*}$ as follows.

$$A \mathop{\perp\!\!\!\perp^{*}}_{C} B \iff \big(\text{ for all } \hat{B} \supseteq B \text{ there is } A' \equiv_{BC} A \text{ s.t. } A' \mathop{\perp\!\!\!\perp}_{C} \hat{B}\big).$$

It is not surprising that the new relation $\perp\!\!\!\perp^{*}$ will satisfy the extension axiom. But it turns out that the other axioms are preserved under this operation, and so we have the following result.

Proposition 30. *Suppose $\perp\!\!\!\perp$ is a preindependence relation. Then $A \perp\!\!\!\perp^{*}_{C} B$ implies $A \perp\!\!\!\perp_{C} B$, and $\perp\!\!\!\perp^{*}$ is the weakest extensible preindependence relation with this property.*

Proof. Left to the reader [1, 2]. ☐

The idea behind Proposition 30 is by no means new. Recall that a formula $\varphi(\bar{x};\bar{b})$ *forks* (or *implicitly divides*) over a set C if it implies a formula $\bigvee_{i<n} \varphi_i(\bar{x};\bar{b}_i)$ such that each $\varphi_i(\bar{x};\bar{b}_i)$ divides over C. A complete type over a superset of C forks over C if it contains a formula that forks over C [49]. Let us write $A \perp\!\!\!\perp^{\mathrm{f}}_{C} B$ if there is no $\bar{a} \in A$ such that $\mathrm{tp}(\bar{a}/BC)$ forks over C.

Remark 31. $\perp\!\!\!\perp^{\mathrm{f}} = (\perp\!\!\!\perp^{\mathrm{d}})^{*}$. Hence $\perp\!\!\!\perp^{\mathrm{f}}$ is an extensible preindependence relation.

Proof. Left to the reader. ☐

By analogy, if $C \subseteq B$ and $\bar{a} \not\perp\!\!\!\perp^{\mathrm{s}*}_{C} B$, we will say that $\mathrm{tp}(\bar{a}/B)$ *quasiforks* over C.

Exercise 1.3 in Chapter III of Shelah's book contains an example of a theory without the independence property where a type can fork over its domain; so $\perp\!\!\!\perp^{\mathrm{d}}$ need not be extensible in a theory with NIP [49]. But this does not imply that $\perp\!\!\!\perp^{\mathrm{f}}$ is useless in such theories. This is demonstrated by Itay Ben-Yaacov's compact abstract theories, a generalisation of complete first-order theories. He has shown that the natural generalisations of stability and simplicity to this context entail most of the good properties that hold in the classical context, but $\perp\!\!\!\perp^{\mathrm{d}} = \perp\!\!\!\perp^{\mathrm{f}}$ can fail in general [7].

So we now have three extensible preindependence relations: $\perp\!\!\!\perp^{\mathrm{u}}$, $\perp\!\!\!\perp^{\mathrm{f}}$ and $(\perp\!\!\!\perp^{\mathrm{s}})^{*}$. In the next subsection we will compare them.

Strong splitting, quasiforking, and almost invariant types

An automorphism of the monster model is a *Lascar strong automorphism* over C if it is a composition of automorphisms each of which fixes a model $M \supseteq C$ pointwise [26]. The *Lascar strong type* $\mathrm{lstp}(\bar{a}/C)$ is the orbit of $\bar{a}$ under Lascar strong automorphisms over C or, equivalently, the equivalence class of $\bar{a}$ under $\overset{\mathrm{lstp}}{\equiv}_C$, the transitive closure of having the same type over a model $M \supseteq C$.

Lemma 32. *If $\bar{a} \equiv_M \bar{a}'$ for a model M, then there is a sequence $(\bar{a}_i)_{i<\omega}$ of M-indiscernibles such that both $\bar{a}^\frown(\bar{a}_i)_{i<\omega}$ and $\bar{a}'^\frown(\bar{a}_i)_{i<\omega}$ are M-indiscernible. On the other hand, every sequence $(\bar{a}_i)_{i<\omega}$ indiscernible over a set C is also indiscernible over a model $M \supseteq C$.*

Proof. Let $p(\bar{x})$ be a global type which extends $\mathrm{tp}(\bar{a}/M)$ and which is finitely satisfiable in M. (Such a type exists by compactness and is called a global coheir of $\mathrm{tp}(\bar{a}/M)$.) Let $\bar{a}_n$ realise $p{\restriction}\bar{a}\bar{a}'\bar{a}_0\ldots\bar{a}_{n-1}$, for all $n < \omega$. It is not hard to check that the sequences $\bar{a}, \bar{a}_0, \bar{a}_1, \ldots$ and $\bar{a}', \bar{a}_0, \bar{a}_1, \ldots$ are both indiscernible over M. (Such sequences are called coheir sequences.) For the converse choose a model M, extend the sequence sufficiently, and extract an indiscernible sequence over the model. Hence there is an isomorphic model over which the original sequence is indiscernible. ☐

Corollary 33. *Having the same Lascar strong type over a set C is the transitive closure of occurring in a C-indiscernible sequence together.*

A global type $p(\bar{x})$ is *almost invariant* over a set C if it is setwise fixed under every automorphism of the monster model that fixes a model $M \supseteq C$ pointwise. (It is *invariant* over C if it is setwise fixed under every automorphism that fixes C pointwise. We will not need this notion here.) We write $A \mathop{\perp\!\!\!\perp}^{\mathrm{i}}_{C} B$ if for every tuple $\bar{a} \in A$ the type $\mathrm{tp}(\bar{a}/BC)$ has a global extension that is almost invariant over C.

The idea behind the statement of the next proposition is to consider a tuple $\bar{a}$ which is outside the monster model. What we really do is replace the monster model by a sufficiently 'big' model N, so we can stay within the monster model. Recall that a structure is called *strongly κ-homogeneous* if every partial isomorphism between subsets of cardinality $< \kappa$ can be extended to an automorphism. Every theory has κ-saturated, strongly κ-homogeneous models for arbitrary κ [16].

Proposition 34. *If $N \supseteq C$ is a $(|T|+|C|)^+$-saturated, strongly $(|T|+|C|)^+$-homogeneous model, then the following conditions are equivalent.*

1. $\bar{a} \mathop{\perp\!\!\!\perp}^{\mathrm{i}}_{C} N$.
2. $\bar{a} \mathop{\perp\!\!\!\perp}^{\mathrm{s}}_{C} N$.
3. *If a sequence $(\bar{b}_i)_{i<\omega}$ in N is indiscernible over C, then it is also indiscernible over $\bar{a}C$.*
4. *For $\bar{b}_0, \bar{b}_1 \in N$, $\bar{b}_0 \overset{\mathrm{lstp}}{\equiv}_C \bar{b}_1 \implies \bar{b}_0 \overset{\mathrm{lstp}}{\equiv}_{\bar{a}C} \bar{b}_1$.*
5. $\mathrm{tp}(\bar{a}/N)$ *is invariant under all Lascar strong automorphisms of N over C.*
6. $\mathrm{tp}(\bar{a}/N)$ *is invariant under all automorphisms of N that fix a model M (such that $C \subseteq M \subseteq N$) pointwise.*
7. *The type $\mathrm{tp}(\bar{a}/N)$ does not split over any model M s.t. $C \subseteq M \subseteq N$, i.e.:*
   For $\bar{b}_0, \bar{b}_1 \in N$ and every model M (such that $C \subseteq M \subseteq N$), $\bar{b}_0 \equiv_M \bar{b}_1 \implies \bar{b}_0 \equiv_{\bar{a}M} \bar{b}_1$.
8. *For $\bar{b}_0, \bar{b}_1 \in N$ and every model M (such that $C \subseteq M \subseteq N$), $\bar{b}_0 \equiv_M \bar{b}_1 \implies \bar{b}_0 \equiv_{\bar{a}C} \bar{b}_1$.*
9. *The type $\mathrm{tp}(\bar{a}/N)$ does not split strongly[8] over C, i.e.:*
   If $(\bar{b}_i)_{i<\omega}$ is indiscernible over C and $\bar{b}_0, \bar{b}_1 \in N$, then $\bar{b}_0 \equiv_{\bar{a}C} \bar{b}_1$.

Proof. $3 \Rightarrow 4$: Easy using both directions of Corollary 33. $4 \Rightarrow 5 \Rightarrow 6$: Obvious. $6 \Rightarrow 7$: Easy using homogeneity of N. $7 \Rightarrow 8$: Trivial. $8 \Rightarrow 9$: Easy using Corollary 33. $9 \Rightarrow 3$: Suppose 3 holds and $(\bar{b}_i)_{i<\omega}$ is a C-indiscernible sequence in N. Let $i_0 < i_1 < \cdots < i_m < \omega$ and $j_0 < j_1 < \cdots < j_m < \omega$. We need to show that $\bar{b}_{i_0} \ldots \bar{b}_{i_m} \equiv_{\bar{a}C} \bar{b}_{j_0} \ldots \bar{b}_{j_m}$. It suffices to do the case $i_m < j_0$. Apply 9 to an indiscernible sequence starting with $\bar{b}_{i_0} \ldots \bar{b}_{i_m}$ and $\bar{b}_{j_0} \ldots \bar{b}_{j_m}$.

$7 \Rightarrow 1$: Fix a model M, $C \subseteq M \subseteq N$, of cardinality $|M| \leq |T| + |C|$. By 7 we have an obvious map associating to every type $q(\bar{y}) \in S(M)$ a type $r_q(\bar{x}, \bar{y}) \in S(M)$. The global type $p(\bar{x}) = \bigcup_{\bar{b} \in \mathbb{M}} r_{\mathrm{tp}(\bar{b}/M)}(\bar{x}, \bar{b})$ extends $\mathrm{tp}(\bar{a}/N)$ and is clearly complete, consistent, and invariant over M. It is not hard to see that it is also invariant over every other model containing C. Thus it is almost invariant over C. $1 \Rightarrow 7$: Easy using homogeneity of the monster model. $3 \Rightarrow 2$: Obvious using saturation of N. $2 \Rightarrow 9$: Obvious. □

By the equivalence of 1 and 2, a complete type quasiforks over a subset of its domain if and only if it has no global extension that is almost over the subset.

Corollary 35. $\mathop{\perp\!\!\!\perp}^{\mathrm{i}} = (\mathop{\perp\!\!\!\perp}^{\mathrm{s}})^*$ *is an extensible preindependence relation.*

Proof. Both relations satisfy the extension axiom, so by the equivalence of 3 and 2 in Proposition 34 they agree. $(\mathop{\perp\!\!\!\perp}^{\mathrm{s}})^*$ is an extensible preindependence relation by Propositions 27 and 30. □

Corollary 36. $A \mathop{\perp\!\!\!\perp}^{\mathrm{u}}_{C} B \implies A \mathop{\perp\!\!\!\perp}^{\mathrm{i}}_{C} B \implies A \mathop{\perp\!\!\!\perp}^{\mathrm{f}}_{C} B$.

[8] For the definitions of splitting and strong splitting see, e. g., Shelah's book [49].

Proof. Since $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.7em}}^{\mathrm{u}}$ and $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.7em}}^{\mathrm{i}}$ are extensible, for both implications we may assume that $B = N \supseteq C$ is a model as in Proposition 34. If $\bar{a} \not\mathop{\smile\hspace{-0.9em}\mid\hspace{0.7em}}^{\mathrm{i}}_C B$, and hence condition 6 of Proposition 34 does not hold, then there are tuples $\bar{b}_0 \equiv_C \bar{b}_1$ such that $\bar{b}_0 \not\equiv_{\bar{a}C} \bar{b}_1$. Let $\varphi(\bar{x}, \bar{y})$ be such that $\models \varphi(\bar{a}, \bar{b}_0) \wedge \neg\varphi(\bar{a}, \bar{b}_1)$. Then $\varphi(\bar{a}, \bar{b}_0) \wedge \neg\varphi(\bar{a}, \bar{b}_1)$ is clearly not satisfied in C, so $\bar{a} \not\mathop{\smile\hspace{-0.9em}\mid\hspace{0.7em}}^{\mathrm{u}}_C B$. The second implication is easy using Remark 25.3 and condition 4 of Proposition 34. □

6 Bounded forking

We say that a preindependence relation $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.7em}}$ is *bounded* if it satisfies the following axiom. Moreover, if f is as in the axiom we say that $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.7em}}$ is bounded by f.

(bound) There is a function f such that for every type $p(\bar{x}) \in S(C)$ ($\bar{x}$ a finite tuple) and every model $M \supseteq C$ the set $\left\{ \mathrm{tp}(\bar{a}/M) \supseteq p(\bar{x}) \;\middle|\; \bar{a} \mathop{\smile\hspace{-0.9em}\mid\hspace{0.7em}}_C M \right\}$ has cardinality at most $f(|T| + |C|)$.

Proposition 37. *1. $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.7em}}^{\mathrm{i}}$ is bounded by $f(\kappa) = 2^{2^{\kappa}}$.*

2. $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.7em}}^{\mathrm{i}}$ is the weakest bounded extensible preindependence relation.

Proof. 1: A global type $p(\bar{x})$ that is almost invariant over a set C can be described by the map associating to every formula $\varphi(\bar{x}, \bar{y})$ the set $\{\mathrm{lstp}(\bar{b}/C) \mid \varphi(\bar{x}, \bar{y}) \in p(\bar{x})\}$. Let $M \supseteq C$ be a model of size $|T| + |C|$. Then the number of Lascar strong types over C is at most the number of types over M, which is at most $2^{|T|+|C|}$. Thus there can be at most $2^{2^{|T|+|C|}}$ such sets. 2: If $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.7em}}$ is an extensible preindependence relation which is not stronger than $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.7em}}^{\mathrm{i}}$, then for some model $N \supseteq C$ as in Proposition 34 condition 9 is violated. It easily follows that there is a formula $\varphi(\bar{a}; \bar{y})$ and a sequence of indiscernibles $(\bar{b}_j)_{j\in\mathbb{Z}}$ such that either $\models \varphi(\bar{a}; \bar{b}_j)$ if and only if $j = 0$, or $\models \varphi(\bar{a}; \bar{b}_j)$ if and only if $j \geq 0$. Now for every extension $(\bar{b}_j)_{j\in J}$ of the sequence which is still indiscernible over C, using strong finite character of $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.7em}}$ and compactness we can build $|J|$ different $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.7em}}$-independent extensions of $\mathrm{tp}(\bar{a}/C)$ to $C \cup \{\bar{b}_j \mid j \in J\}$. □

Corollary 38. *The following conditions are equivalent.*

1. *$\mathop{\smile\hspace{-0.9em}\mid\hspace{0.7em}}^{\mathrm{f}}$ is bounded.*
2. *$\mathop{\smile\hspace{-0.9em}\mid\hspace{0.7em}}^{\mathrm{f}}$ is bounded by $f(\kappa) = 2^{2^{\kappa}}$.*
3. *$\mathop{\smile\hspace{-0.9em}\mid\hspace{0.7em}}^{\mathrm{f}} = \mathop{\smile\hspace{-0.9em}\mid\hspace{0.7em}}^{\mathrm{i}}$.*

Proof. 3 ⇒ 2 by Proposition 37.1. 2 ⇒ 1 is trivial. 1 ⇒ 3: By Corollary 36 $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.7em}}^{\mathrm{f}}$ is always weaker than $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.7em}}^{\mathrm{i}}$. If $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.7em}}^{\mathrm{f}}$ is bounded, then by Proposition 37.2 $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.7em}}^{\mathrm{f}}$ is also stronger than $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.7em}}^{\mathrm{i}}$. □

Remark 39. In a theory with NIP the equivalent conditions of Corollary 38 are satisfied.

Proof. By extensibility it is sufficient to show that $\bar{a} \mathop{\smile\hspace{-0.9em}\mid\hspace{0.7em}}^{\mathrm{f}}_C N \implies \bar{a} \mathop{\smile\hspace{-0.9em}\mid\hspace{0.7em}}^{\mathrm{i}}_C N$ for $N \supseteq C$ as in Proposition 34. We check condition 3 of the proposition. If $(\bar{b}_i)_{i<\omega}$ is C-indiscernible and $\bar{b}_0, \bar{b}_1 \in N$, hence $\bar{a} \mathop{\smile\hspace{-0.9em}\mid\hspace{0.7em}}^{\mathrm{d}}_C \bar{b}_0\bar{b}_1$, then $\bar{b}_0 \equiv_{\bar{a}C} \bar{b}_1$ since otherwise by applying Remark 25.3 we would get a violation of finite alternation rank. □

The fact that in NIP theories $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.7em}}^{\mathrm{f}} = \mathop{\smile\hspace{-0.9em}\mid\hspace{0.7em}}^{\mathrm{i}}$ (and therefore $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.7em}}^{\mathrm{f}}$ is bounded) is from Observation 5.4 in a recent preprint of Shelah's paper Sh783 [53]. It explains to a large extent Alf Dolich's observation that forking seems to play an important role in o-minimal theories [9]. The converse to Remark 39 looks much too strong to be true, but I could not find a counterexample.

The study of splitting in unstable theories by Ivanov and Macpherson is related to this section, since similar results should hold for quasidividing or strong splitting [20, 19].

Special indiscernible sequences

Recall that the type of an infinite sequence of C-indiscernibles $(\bar{a}_i)_{i\in I}$ is uniquely determined by its order type (the isomorphism class of the linear order I) together with its Ehrenfeucht-Mostowski set over C (which is essentially the sequence of types $p(\bar{x}_0 \dots \bar{x}_{n-1}) = \mathrm{tp}(\bar{a}_{i_0} \dots \bar{a}_{i_{n-1}}/C)$, where $i_0 < \dots < i_{n-1}$). We will call an Ehrenfeucht-Mostowski set over a model M *special* if for any system of sequences $(\bar{a}^j_i)_{i\in I^j}$ (where j ranges over some set J) that realise it there is a tuple $\bar{a}_\infty = \bar{a}^j_\infty$ (for all $j \in J$) such that the concatenated sequences $(\bar{a}^j_i)_{i\in I^j}{}^\frown\bar{a}_\infty = (\bar{a}^j_i)_{i\in I^j\cup\{\infty\}}$ are all indiscernible over M.[9] A special sequence over M is an infinite sequence of M-indiscernibles whose Ehrenfeucht-Mostowski set is special over M.

Remark 40. Suppose T is NIP. Let $\varphi(\bar{x};\bar{b})$ be a formula with parameters. Every sequence of C-indiscernibles has an extension $(\bar{a}_i)_{i\in I}$ which has the following property: For all $\bar{a}$ such that $(\bar{a}_i)_{i\in I}{}^\frown\bar{a}$ is indiscernible over C, the formula $\varphi(\bar{a};\bar{b})$ has the same truth value. Moreover, if $C = M$ is a model and we have two realisations of an Ehrenfeucht-Mostowski set that is special over M, and if both have the above property, then this 'eventual' truth value is the same for both of them.

Proof. Let $(\bar{a}_i)_{i\in I}$ realise the Ehrenfeucht-Mostowski set. Suppose the truth value of $\varphi(\bar{a}_i;\bar{b})$ changes n times. As long as the sequence does not have the desired property, we can extend it and get another realisation for which the truth value changes more than n times. Since $\varphi(\bar{x};\bar{y})$ has finite alternation rank, this process must stop, at a realisation that has the desired property. Now suppose that, moreover, the Ehrenfeucht-Mostowski set is special over $M = C$. If we have two realisations with the desired property, then we can choose as $\bar{a}$ a common continuation of both realisations. □

In a theory without the independence property, we define the *eventual type* $\mathrm{Ev}((\bar{a}_i)_{i\in I}/B) \in S(B)$ over B of a special sequence over M as the set of all formulas $\varphi(\bar{x};\bar{b})$ with parameters in B which are eventually true in the sense of the last remark. The eventual type was used (implicitly) by Poizat, who proved the following [43].

Lemma 41. *Let $p(\bar{x})$ be a global type which is (almost) invariant over a model M. If an infinite sequence $(\bar{a}_i)_{i\in I}$ is such that for all $i \in I$, $\bar{a}_i$ realises $p(\bar{x})\restriction(M\bar{a}_{<i})$, then $(\bar{a}_i)_{i\in I}$ is a special sequence over M. Conversely, if T is NIP and $(\bar{a}_i)_{i\in I}$ is a special sequence over M, then* $\mathrm{Ev}((\bar{a}_i)_{i\in I}/\mathbb{M})$ *does not split over M, and $\bar{a}_i$ realises* $\mathrm{Ev}((\bar{a}_i)_{i\in I}/M\bar{a}_{<i})$ *for all $i \in I$.*

Thus in a theory with NIP there is a bijection between global types that are (almost) invariant over M and special Ehrenfeucht-Mostowski sets over M.

Proof. Suppose $(\bar{a}_i)_{i\in I}$ is such that each $\bar{a}_i$ realises the type $p(\bar{x})\restriction M\bar{a}_{<i}$. The sequence is clearly 1-indiscernible over M. We show that if it is n-indiscernible over M, then it is also $(n+1)$-indiscernible over M. So suppose $i_0 < \dots < i_n$ and $j_0 < \dots < j_n$. For this purpose we may assume that $i_n > j_{n-1}$ and $j_n > i_n - 1$. By induction hypothesis we have $\bar{a}_{i_0} \dots \bar{a}_{i_{n-1}} \equiv_M \bar{a}_{j_0} \dots \bar{a}_{j_{n-1}}$. Now $\bar{a}_{i_n}$ realises $p\restriction M\bar{a}_{i_0} \dots \bar{a}_{i_{n-1}}$ and $\bar{a}_{j_n}$ realises $p\restriction M\bar{a}_{j_0} \dots \bar{a}_{j_{n-1}}$, and moreover p does not split over M. Therefore $\bar{a}_{i_0} \dots \bar{a}_{i_n} \equiv_M \bar{a}_{j_0} \dots \bar{a}_{j_n}$. Thus the sequence is indiscernible over M. It is also special over M: Given any sequences $(\bar{a}^j_i)_{i\in I}$ of the same type over M, let $\bar{a}$ realise $p\restriction M\bar{a}^J_I$. Then clearly each of the sequences $(\bar{a}^j_i)^\frown_{i\in I}\bar{a}$ is M-indiscernible.

For the converse, suppose $\bar{b} \equiv_M \bar{b}'$ and consider a formula $\varphi(\bar{x},\bar{y})$. Let $(\bar{a}_i)_{i\in J}$ be an extension of $(\bar{a}_i)_{i\in I}$ such that whenever $(\bar{a}_i)_{i\in J}{}^\frown\bar{a}$ is indiscernible, then $\varphi(\bar{a},\bar{b})$ has the same truth value. Let $(\bar{a}'_i)_{i\in J}$ be such that $(\bar{a}'_i)_{i\in J}\bar{b}' \equiv_M (\bar{a}_i)_{i\in J}\bar{b}$. Then clearly $(\bar{a}'_i)_{i\in J}$ has the analogous property with respect to $\varphi(\bar{x},\bar{b}')$, and with the same truth value. Therefore $\mathrm{Ev}((\bar{a}_i)_{i\in I}/\mathbb{M})$ does not split over M. Finally we have to show that each $\bar{a}_i$ realises $\mathrm{Ev}((\bar{a}_i)_{i\in I}/M\bar{a}_{<i})$. For $i_0 < i_1 < \dots < i_{n-1} < i$ and a formula $\varphi(\bar{x}_0,\dots,\bar{x}_n)$ over M let $(\bar{a}_i)_{i\in J}$ be a M-indiscernible extension of $(\bar{a}_i)_{i\in I}$ such that whenever $(\bar{a}_i)_{i\in J}{}^\frown\bar{a}$ is M-indiscernible $\varphi(\bar{a}_{i_0},\dots,\bar{a}_{i_{n-1}},\bar{a})$ has the same truth value. But then by M-indiscernibility $\varphi(\bar{a}_{i_0},\dots,\bar{a}_{i_{n-1}},\bar{a}_{i_n})$ also has the same truth value. □

[9] By compactness it is sufficient to check this condition for one particular order type, e. g. $I = I' = \omega$ (and finite J). Hence our definition is equivalent to Poizat's. Moreover, in the definition we could require $|J| = 2$ without changing any of the arguments below. This shows that in NIP theories the resulting weaker notion is actually equivalent.

Theorem 42. *The following conditions are equivalent.*

1. *T is NIP.*

2. *$\mathop{\smile\!\!\!\!\!\!|}^{f}$ is bounded by $f(\kappa) = 2^\kappa$, i. e., for every parameter set A, there are at most $2^{|T|+|A|}$ global types that do not fork over A.*

3. *There is a cardinal $\lambda \geq |T|$ such that every 1-type over a model of size λ has strictly less than 2^{2^λ} global coheirs.*

Proof. 2 ⇒ 3: Choose any $\lambda \geq |T|$. Take any model M of cardinality λ and any $p(x) \in S_1(M)$. The global coheirs of p are finitely satisfiable in M, so they do not fork over M. By 2 there are at most 2^λ global types that do not fork over M, which is in fact strictly less than 2^{2^λ}. 1 ⇒ 2: By Löwenheim-Skolem we may assume that $A = M$ is a model. By Remark 39 it is sufficient to show that there are at most $2^{|T|+|M|}$ global types that are special over M. By Lemma 41 the global types that are special over M are in bijection with the Ehrenfeucht-Mostowski sets that are special over M. Clearly there are at most $2^{|T|+|M|}$ Ehrenfeucht-Mostowski sets over M. 3 ⇒ 1: If T has the independence property, then there is a formula $\varphi(x;\bar{y})$ such that $\mathrm{alt}(\varphi(x;\bar{y})) = \infty$, and an indiscernible sequence $(a_i)_{i<\lambda}$ which witnesses this. Let $M \supseteq \{a_i \mid i < \lambda\}$ be a model of size $|M| = \lambda$. By Hausdorff's Theorem on ultrafilters there are 2^{2^λ} ultrafilters $\mathcal{U}$ on the set $\{a_i \mid i < \lambda\}$ [45, Theorem 8.11]. For each of them let $p_\mathcal{U} = \mathrm{Av}(\mathcal{U}/\mathbb{M})$ be its global average type. It is a coheir of its restriction to M. For two distinct ultrafilters $\mathcal{U}, \mathcal{U}'$ there is a set $A \subset \{a_i \mid i < \lambda\}$ such that $A \in \mathcal{U}$ and $A \notin \mathcal{U}'$. By compactness there is a tuple $\bar{b}$ such that $\models \varphi(a_i;\bar{b}) \iff a_i \in A$. Hence $\varphi(x;\bar{b}) \in p_\mathcal{U}$ and $\varphi(x;\bar{b}) \notin p_{\mathcal{U}'}$. □

Except for a minor improvement in the last theorem ($\mathop{\smile\!\!\!\!\!\!|}^{f}$ instead of $\mathop{\smile\!\!\!\!\!\!|}^{i}$ in condition 2), everything in this section is due to Poizat [42, 43].

7 Stability

A formula $\varphi(\bar{x};\bar{y})$ is said to have the *strict order property* if for every $n < \omega$ there are tuples $\bar{b}_0, \ldots, \bar{b}_{n-1}$ such that $\varphi(\bar{x};\bar{b}_0)^\mathbb{M} \subsetneq \varphi(\bar{x};\bar{b}_1)^\mathbb{M} \subsetneq \ldots \varphi(\bar{x};\bar{b}_{n-1})^\mathbb{M}$.[10] The following result was already in Shelah's paper that introduced the independence property [48].

Lemma 43. *A formula $\varphi(\bar{x};\bar{y})$ has the order property if and only if $\varphi(\bar{x};\bar{y})$ has the independence property or there is a formula, with parameters, $\chi(\bar{y})$ such that $\varphi(\bar{x};\bar{y}) \wedge \chi(\bar{y})$ has the strict order property.*

Proof. It is easy to check that if $\varphi(\bar{x};\bar{y})$ has the independence property, or if some $\varphi(\bar{x};\bar{y}) \wedge \chi(\bar{y})$ has the strict order property, then $\varphi(\bar{x};\bar{y})$ has the order property.

Now suppose $\varphi(\bar{x};\bar{y})$ has the order property but not the independence property. We are looking for a formula $\chi(\bar{x})$ with parameters such that $\varphi(\bar{x};\bar{y}) \wedge \chi(\bar{x})$ has the strict order property. Since there are arbitrarily long sequences witnessing that φ has the order property, we can extract an indiscernible sequence that also exhibits the order property. So let $(\bar{a}_i\bar{b}_i)_{i<\omega}$ be an indiscernible sequence such that $\models \varphi(\bar{a}_i;\bar{b}_j)$ holds if and only if $i < j$. Since φ does not have the independence property, φ has finite alternation rank, and so there is a number $n < \omega$ such that the formula

$$\varphi(\bar{a}_0;\bar{y}) \wedge \neg\varphi(\bar{a}_1;\bar{y}) \quad \wedge \quad \varphi(\bar{a}_2;\bar{y}) \wedge \neg\varphi(\bar{a}_3;\bar{y}) \quad \wedge \quad \ldots \quad \wedge \quad \varphi(\bar{a}_{2n-2};\bar{y}) \wedge \neg\varphi(\bar{a}_{2n-1};\bar{y})$$

[10]It is customary to abbreviate the strict order property as SOP. This has become problematic since Shelah introduced the (slightly weaker) *strong order property*, which he also abbreviates as SOP, and various weaker variants of it that he called SOP_3, SOP_4, SOP_5, etc. [51].

cannot be satisfied. On the other hand, if we push all negations to the right we get the formula

$$\varphi(\bar{a}_0;\bar{y}) \wedge \varphi(\bar{a}_1;\bar{y}) \wedge \cdots \wedge \varphi(\bar{a}_{n-1};\bar{y}) \quad \wedge \quad \wedge \quad \neg\varphi(\bar{a}_n;\bar{y}) \wedge \cdots \wedge \neg\varphi(\bar{a}_{2n-2};\bar{y}) \wedge \neg\varphi(\bar{a}_{2n-1};\bar{y}),$$

which can be satisfied. (In fact, it is satisfied by $\bar{b}_{n-1}$. This is all we needed the $\bar{b}_i$ for.) We can do the pushing stepwise, in each step going from a formula

$$\alpha(\bar{a}_0 \dots \bar{a}_{k-1};\bar{y}) \quad \wedge \quad \neg\varphi(\bar{a}_k;\bar{y}) \wedge \varphi(\bar{a}_{k+1};\bar{y}) \quad \wedge \quad \beta(\bar{a}_{k+2} \dots \bar{a}_{2n-1};\bar{y})$$

to the formula

$$\alpha(\bar{a}_0 \dots \bar{a}_{k-1};\bar{y}) \quad \wedge \quad \varphi(\bar{a}_k;\bar{y}) \wedge \neg\varphi(\bar{a}_{k+1};\bar{y}) \quad \wedge \quad \beta(\bar{a}_{k+2} \dots \bar{a}_{2n-1};\bar{y}).$$

At one of these steps we must get from a non-satisfiable formula to a satisfiable formula. Let α, β, k be as in such a step, let $\chi(\bar{y})$ be the formula $\alpha(\bar{a}_0 \dots \bar{a}_{k-1};\bar{y}) \wedge \beta(\bar{a}_{k+2} \dots \bar{a}_{2n-1};\bar{y})$, and let $\varphi'(\bar{x};\bar{y})$ be $\varphi(\bar{x};\bar{y}) \wedge \chi(\bar{y})$. Now we have

$$\models \quad \forall\bar{y}(\varphi'(\bar{a}_{k+1};\bar{y}) \to \varphi'(\bar{a}_k;\bar{y})) \quad \wedge \quad \exists\bar{y}(\neg\varphi'(\bar{a}_{k+1};\bar{y}) \wedge \varphi'(\bar{a}_k;\bar{y})).$$

Let $(\bar{a}_q)_{q\in\mathbb{Q}}$ be an indiscernible extension of $(\bar{a}_i)_{i<\omega}$. For $i < \omega$ let $\bar{a}'_i = \bar{a}_{k+i/(i+1)}$, so $\bar{a}'_0 = \bar{a}_k$, and let $\bar{a}'_\omega = \bar{a}_{k+1}$. Then $(\bar{a}'_i)_{i\le\omega}$ is indiscernible over the parameters of φ'. Thus we have

$$\models \quad \forall\bar{y}(\varphi'(\bar{a}'_{i+1};\bar{y}) \to \varphi'(\bar{a}'_i;\bar{y})) \quad \wedge \quad \exists\bar{y}(\neg\varphi'(\bar{a}'_{i+1};\bar{y}) \wedge \varphi'(\bar{a}'_i;\bar{y}))$$

for all $i < \omega$, so φ' has the strict order property. □

Theorem 44. *The following conditions are equivalent for a NIP theory T.*

1. *There is no formula $\varphi(x,y)$ with parameters, with single variables x and y, which defines a partial order with infinite chains on the universe.*
2. *T does not have the strict order property.*
3. *T is simple.*
4. *T is stable*

Proof. It is well known and easy to check that a stable theory is simple, a simple theory cannot have the strict order property, that not having the strict order property is preserved under adding parameters, and that a theory without the strict order property does not have a formula which defines a partial order with infinite chains. Therefore it only remains to show that 1 implies 4. In an unstable theory there is a formula $\varphi(x;\bar{y})$, with x a single variable, which has the order property [49]. Hence there is also a formula $\varphi(\bar{x};y)$, with y a single variable, which has the order property. By the lemma there is a formula $\varphi'(\bar{x};y)$ with parameters which has the strict order property. Hence the formula $\psi(y;y')$ defined as $\forall\bar{x}(\varphi'(\bar{x};y) \to \varphi'(\bar{x};y'))$, which is clearly reflexive and transitive, has infinite chains. In a final step we can define a formula $\psi'(y;y')$ in which we make incomparable any two elements that are equivalent according to ψ: $\psi(y;y') \ \wedge \ \neg\psi(y';y)$. This formula clearly defines an irreflexive partial order with infinite chains. □

The observation that in an unstable NIP theory there is a partial order with infinite chains on the *single elements* is apparently due to Shelah. The rest of the theorem is, of course, also due to him. Finally, stability is characterised by 'nice' behaviour of $\mathop{\smile\!\!\!\!\!\!\,|}^{\text{f}}$ in the same way as simplicity is characterised by 'nice' behaviour of non-forking:

Theorem 45. *The following conditions are equivalent:*

1. *T is stable.*

2. *T is simple and* $\mathop{\smile\!\!\!\!\mid}^{\mathrm{i}} = \mathop{\smile\!\!\!\!\mid}^{\mathrm{f}}$.

3. $\mathop{\smile\!\!\!\!\mid}^{\mathrm{i}}$ *satisfies the local character axiom.*

4. $\mathop{\smile\!\!\!\!\mid}^{\mathrm{i}}$ *is an independence relation (see [1]).*

5. $\mathop{\smile\!\!\!\!\mid}^{\mathrm{i}}$ *is symmetric.*

Proof. 1 implies 2: If T is stable, then T is simple and NIP. 2 implies 1: If T is simple unstable, then T has the independence property and so there is a formula $\varphi(\bar{x};\bar{y})$ with infinite alternation number. Thus there is a tuple $\bar{b}$ and an indiscernible sequence $(\bar{a}_i)_{i<\omega}$ such that $\not\models \varphi(\bar{a}_i;\bar{b}) \leftrightarrow \varphi(\bar{a}_{i+1};\bar{b})$ for all $i<\omega$. We may assume that the sequence of pairs $(\bar{a}_{2i}a_{2i+1})_{i<\omega}$ is indiscernible over $\bar{b}$, and we can extend to a sequence $(\bar{a}_i)_{i<\omega+\omega}$ such that the pairs are still indiscernible over $\bar{b}$ and the above condition holds for all $i<\omega+\omega$. Then $\bar{a}_{<\omega+\omega} \mathop{\smile\!\!\!\!\mid}^{\mathrm{f}}_{\bar{a}_{<\omega}} \bar{b}$ by Corollary 36, hence $\bar{b} \mathop{\smile\!\!\!\!\mid}^{\mathrm{f}}_{\bar{a}_{<\omega}} \bar{a}_{<\omega+\omega}$ by forking symmetry. But clearly $\bar{b} \not\mathop{\smile\!\!\!\!\mid}^{\mathrm{s}}_{\bar{a}_{<\omega}} \bar{a}_{<\omega+\omega}$. 2 implies 3: If T is simple, $\mathop{\smile\!\!\!\!\mid}^{\mathrm{f}}$ satisfies the local character axiom. 3 implies 2: If $\mathop{\smile\!\!\!\!\mid}^{\mathrm{i}}$ satisfies the local character axiom, then so does $\mathop{\smile\!\!\!\!\mid}^{\mathrm{f}}$, and therefore T is simple. Moreover, $\mathop{\smile\!\!\!\!\mid}^{\mathrm{i}}$ is a preindependence relation satisfying the local character axiom, hence an independence relation. Since $\mathop{\smile\!\!\!\!\mid}^{\mathrm{f}}$ is the strongest independence relation [1], $A \mathop{\smile\!\!\!\!\mid}^{\mathrm{f}}_C B$ implies $A \mathop{\smile\!\!\!\!\mid}^{\mathrm{i}}_C B$, so $\mathop{\smile\!\!\!\!\mid}^{\mathrm{i}} = \mathop{\smile\!\!\!\!\mid}^{\mathrm{f}}$. 3 implies 4 by definition, because $\mathop{\smile\!\!\!\!\mid}^{\mathrm{i}}$ is an extensible preindependence relation. 4 implies 5 by Kim's symmetry argument [25, 1]. 5 implies 3 because $\mathop{\smile\!\!\!\!\mid}^{\mathrm{u}}$ satisfies the dual of local character (left and right sides reversed) [2] and $A \mathop{\smile\!\!\!\!\mid}^{\mathrm{u}}_C B \implies A \mathop{\smile\!\!\!\!\mid}^{\mathrm{i}}_C B$. □

8 Further research

Shelah showed that unstable theories have the maximal number of models in all cardinalities $\kappa \geq |T|$. Poizat gave a much simpler proof showing that theories with the independence property have the maximal number of models (even λ-resplendent models) in all cardinalities $\kappa = 2^\lambda \geq |T|$ [44].

Pillay recently gave a new proof of a remarkable theorem by Shelah which says that part of the definability of types in stable theories can be salvaged in the NIP context [53, 39]. Many of Shelah's more technical results around NIP also did not find their place in this paper [49, 52, 53, 56, 58].

Keisler and his student Siu-Ah Ng studied a notion of non-forking for certain measures that generalise types, in a theory without the independence property [23, 24, 32, 33, 34, 35]. So far *Keisler measures* have not become part of the mainstream, but this is about to change, since they were used recently in the proofs of Pillay's o-minimal group conjecture and the o-minimal case of the compact domination conjecture [17, 18]. One of the results of this line of work was that every type-definable group in a NIP theory with an invariant measure has a minimal type-definable subgroup of bounded index. Shelah then proved, without using measures, that this is true for all NIP theories [55]. Ben-Yaacov introduced the *randomisation of a theory.* The randomisation of a NIP theory is again NIP, and its elements may turn out to play the role of new "imaginaries" that can "realise" measures[8].

In his initial paper on simple theories, Shelah introduced the *tree property of the second kind,* which was already implicit ('$\kappa r_{\mathrm{inp}} < \infty$') in his book [50, 49]. The tree property of the second kind implies the tree property and the independence property. Hence theories without the tree property of the second kind are a common generalisation of simple theories and of NIP theories. So far they have not been examined systematically.

In a sense, the independence property is about 'weak interpretations' of a random bipartite graph. In a similar way we can look for 'weak interpretations' of '$n+1$-partite $n+1$-hypergraphs'. As a result we get the notion of being *n-dependent*, for $1 \leq n < \omega$. Shelah recently became interested in this hierarchy of generalisations of NIP [54, 57].

As to strengthenings of NIP, Shelah has been looking for a notion of 'super-dependent' that would be related to NIP roughly as superstability is to stability. The most fruitful approximation so far seems to be the class of *strongly dependent* theories, which are precisely those in which Theorem 20 holds with $\kappa = \aleph_0$ if B is finite. There are also other variants, most notably the stronger notion called *strongly⁺ dependent*. There are strict implications from being superstable to being *strongly⁺ stable* (i. e. stable and strongly⁺ dependent), to being *strongly stable* (i. e. stable and strongly dependent), to just being stable [53, 54]. Using certain ranks which are defined in strongly dependent theories, Shelah achieved the very fundamental result that every sufficiently long sequence in such a theory has a subsequence which is indiscernible [54]. A stable theory is strongly dependent if and only if all types have finite weight [3].

Even stronger is the notion of *dp-minimality*, which generalises strong minimality, o-minimality and p-minimality [38]. Apparently this notion was first defined in a still unpublished paper by Firstenberg and Shelah. A stable theory is dp-minimal if and only if every 1-type has weight ≤ 1.

One could say that the main point about NIP is that it is a natural common generalisation of stability and o-minimality. It appears that stability theory takes much of its strength from the coincidence of a nice 'combinatorial' machinery (indiscernibles and the definition of forking; bounded multiplicity of types; notions such as superstability) and nice 'geometric' notions (an independence relation whose properties include symmetry; notions such as triviality and being one-based). O-minimal theories behave nicely from both points of view, but the combinatorial and geometric notions do not coincide. NIP generalises the nice combinatorial aspects that are common to stable and o-minimal theories.

A notion weaker than forking was developed by Onshuus, Scanlon and Ealy and called *thorn-forking*. It is the key to generalising the geometric aspects of stability and o-minimality. A theory is called *rosy* if thorn-forking is symmetric (is an independence relation). Simple theories and o-minimal theories are rosy. A large part of geometric simplicity theory can be extended to rosy theories, and as one would expect, o-minimal theories are in fact *superrosy* of finite $U^{\text{þ}}$-rank [36, 37, 11, 1].

A priori, NIP and rosiness do not have much to do with each other. Nevertheless, in algebraic contexts there are already encouraging results from combining the two with a group-theoretic condition called *finitely satisfiable generics*. In particular, a superrosy NIP group with finitely satisfiable generics must be abelian-by-finite if it is of $U^{\text{þ}}$-rank 1, and solvable-by-finite if it is of $U^{\text{þ}}$-rank 2 [17, 10].

Acknowledgements

This paper grew out of my talks in the model theory seminar in Barcelona. I thank John Baldwin, Itay Ben-Yaacov, Enrique Casanovas, Artem Chernikov, Davide Penazzi, Anand Pillay, Martin Ziegler, and the stability theory groups in Barcelona and Berlin for many helpful discussions and comments. I thank the Institut de Matemática de la Universitat de Barcelona for their hospitality.

References

[1] Adler, H.: A geometrical introduction to forking and thorn-forking. Preprint (2007).

[2] Adler, H.: Thorn-forking as local forking. Preprint (2007).

[3] Adler, H.: Strong theories, burden and weight. Preprint (2007).

[4] Baldwin, J.T., Benedikt, M.: Stability theory, permutations of indiscernibles, and embedded finite models. Trans. Amer. Math. Soc. **352**, 4937–4969 (2000).

[5] Belegradek, O., Peterzil, Y., Wagner, F.O.: Quasi o-minimal structures. J. Symbolic Logic **65**, 1115–1132 (2000).

[6] Ben-Yaacov, I.: Positive model theory and compact abstract theories. J. Math. Log. **3**, 85–118 (2003).

[7] Ben-Yaacov, I.: Simplicity in compact abstract theories. J. Math. Log. **3**, 163–191 (2003).

[8] Ben-Yaacov, I.: Continuous and random Vapnik-Chervonenkis classes. arXiv:0802.0068v1 (2008).

[9] Dolich, A.: Forking and independence in o-minimal theories. J. Symbolic Logic **69**, 215–240 (2004).

[10] Ealy, C., Krupiński, K., Pillay, A.: Superrosy dependent groups having finitely satisfiable generics. Ann. Pure Appl. Logic **151**, 1–21 (2008).

[11] Ealy, C., Onshuus, A.: Characterizing rosy theories. J. Symbolic Logic **72**, 919–940 (2007).

[12] Grohe, M., Turán, Gy.: Learnability and definability in trees and similar structures. Theory of Comput. Systems **37**, 193–220 (2004).

[13] Haskell, D., Hrushovski, E., Macpherson D.: Stable domination and independence in algebraically closed valued fields. Lecture Notes in Logic **30**. Cambridge University Press (2008).

[14] Haskell, D., Macpherson, D.: A version of *o*-minimality for the *p*-adics. J. Symbolic Logic **62**, 1075–1092 (1997).

[15] Hausdorff, F.: Grundzüge einer Theorie der geordneten Mengen. Math. Ann. **65**, 435–505 (1908).

[16] Hodges, W.: Model Theory. Encyclopedia of Mathematics and its Applications **42**. Cambridge University Press (1993).

[17] Hrushovski, E., Peterzil, Y., Pillay, A.: Groups, measures, and the NIP. J. Amer. Math. Soc. **21** , 563–596 (2008).

[18] Hrushovski, E., Pillay, A.: On NIP and invariant measures. arXiv:0710.2330v1 (2007).

[19] Ivanov, A.: Strongly determined types and G-compactness. Fund. Math. **191**, 227–247 (2006).

[20] Ivanov, A., Macpherson, D.: Strongly determined types. Ann. Pure Appl. Logic **99**, 197–230 (1999).

[21] Keisler, H.J.: Six classes of theories. J. Australian Math. Soc. (A) **21**, 257–266 (1976).

[22] Keisler, H.J.: The stability function of a theory. J. Symbolic Logic **43**, 481–486 (1978).

[23] Keisler, H.J.: Measures and forking. Ann. Pure Appl. Logic **34**, 119–169 (1987).

[24] Keisler, H.J.: Choosing elements in a saturated model. In: Classification Theory (J.T. Baldwin, editor), Lecture Notes in Math. **1292**, 165–181. Springer, Berlin (1987).

[25] Kim, B.: Forking in simple unstable theories. J. London Math. Soc. (2) **57**, 257–267 (1998).

[26] Lascar, D.: On the category of models of a complete theory. J. Symbolic Logic **47**, 249–266 (1982)

[27] Laskowski, M.C.: Vapnik-Chervonenkis classes of definable sets. J. London Math. Soc. (2) **45**, 377–384 (1992)

[28] Macpherson, D.: Notes on o-minimality and variations. In: Model Theory, Algebra, and Geometry, Math. Sci. Res. Inst. Publ. **39**, 97–130. Cambridge University Press (2000).

[29] Macpherson, D., Marker, D., Steinhorn, C.: Weakly o-minimal structures and real-closed fields. Trans. Amer. Math. Soc. **352**, 5435–5483 (2000).

[30] Macpherson, D., Steinhorn, C.: On variants of *o*-minimality. Ann. Pure Appl. Logic **79**, 165–209 (1996).

[31] Mitchell, William: Aronszajn trees and the independence of the transfer property. Ann. Math. Logic **5**, 21–46 (1972).

[32] Ng, Siu-Ah: A generalization of forking. J. Symbolic Logic **56**, 813–822 (1991).

[33] Ng, Siu-Ah: Definable FN bases. J. Symbolic Logic **56**, 823–831 (1991).

[34] Ng, Siu-Ah: A remark on locally pure measures. J. Symbolic Logic **58**, 1165–1170 (1993).

[35] Ng, Siu-Ah: Loeb extensions and ultrapowers of measures on fragments. Ann. Pure Appl. Logic **60**, 179–189 (1993).

[36] Onshuus, A.: Properties and consequences of thorn-independence. J. Symbolic Logic **71**, 1–21 (2006).

[37] Onshuus, A.: Th-forking, algebraic independence and examples of rosy theories. arXiv:math/0306003v1 (2003).

[38] Onshuus, A., Usvyatsov, A.: On dp-minimal stable theories. Preprint (2007).

[39] Pillay, A.: On externally definable sets and a theorem of Shelah. Preprint (2006).

[40] Pillay, A., Steinhorn, C.: Definable sets in ordered structures. I. Trans. Amer. Math. Soc. **295**, 565–592 (1986).

[41] Plotkin, J.M. (ed.): Hausdorff on ordered sets. History of Mathematics **25**. American Mathematical Society (2005).

[42] Poizat, B.: Théories instables. J. Symbolic Logic **46**, 513–522 (1981).

[43] Poizat, B.: Post-scriptum à "Théories instables". J. Symbolic Logic **48**, 60–62 (1983).

[44] Poizat, B.: Beaucoup de modèles à peu de frais. Groupe d'étude de théories stables **3**(9), 1–7 (1980–1982).

[45] Poizat, B.: Cours de Théorie des Modèles. Nur al-Mantiq wal-Ma'rifah, Villeurbanne (1985).

[46] Poizat, B.: A course in model theory. An introduction to contemporary mathematical logic. Universitext. Springer, New York (2000). Translation of [45].

[47] Rosenstein, J.G.: Linear orderings. Academic Press, New York and London (1982).

[48] Shelah, S.: Stability, the f.c.p. and superstability [Sh10]. Ann. Math. Logic **3**, 271–362 (1971).

[49] Shelah, S.: Classification Theory and the Number of Non-Isomorphic Models, first edn. Studies in Logic and the Foundations of Mathematics 92. North-Holland, Amsterdam (1978).

[50] Shelah, S.: Simple unstable theories [Sh93]. Ann. Math. Logic **19**, 177–203 (1980).

[51] Shelah, S.: Toward classifying unstable theories [Sh500]. Ann. Pure Appl. Logic **80**, 229–255 (1996).

[52] Shelah, S.: Classification theory for elementary classes with the dependence property—a modest beginning [Sh715]. Sci. Math. Jpn. **59**, 265–316 (2004). Early preprint without the font problems at arXiv:math/009056v1 (2000).

[53] Shelah, S.: Dependent first order theories, continued [Sh783]. Israel J. Mathematics, to appear. arXiv:math/0406440v1 or http://shelah.logic.at/files/783.pdf (2006).

[54] Shelah, S.: Strongly dependent theories [Sh863]. Israel J. Mathematics, submitted. arXiv:math/0504197v3 (2007).

[55] Shelah, S.: Minimal bounded index subgroup for dependent theories [Sh876]. Proc. Amer. Math. Soc. **136**, 1087–1091 (2008).

[56] Shelah, S.: Dependent T and existence of limit models [Sh877]. arXiv:math/0609636v1 (2006).

[57] Shelah, S.: Definable groups and 2-dependent theories [Sh886]. arXiv:math/0703045v2 (2007).

[58] Shelah, S.: Dependent theories and the generic pairs conjecture [Sh900]. arXiv:math/0702292v1 (2007).

[59] Vapnik, V.N., Chervonenkis, A.Y.: On the uniform convergence of relative frequencies of events to their probabilities. Theory Probab. Appl. **16**, 264–280 (1971).