Annals of Pure and Applied Logic 108 (2001) 205–227
ANNALS OF PURE AND APPLIED LOGIC
www.elsevier.com/locate/apal

Conservative translations ☆

Hércules A. Feitosa[a], Itala M. Loffredo D'Ottaviano[b,c,*]
[a]Department of Mathematics, Universidade Estadual Paulista (Unesp), Brazil
[b]Centre for Logic, Epistemology and the History of Science, Brazil
[c]Department of Philosophy, Universidade Estadual de Campinas (Unicamp), C.P. 6133, 13081-970 Campinas, S.P., Brazil

Abstract

In this paper we introduce the concept of conservative translation between logics. We present some necessary and sufficient conditions for a translation to be conservative and study some general properties of logical systems, these properties being characterized by the existence of conservative translations between the systems. We prove that the class constituted by logics and conservative translations between them determines a co-complete subcategory of the bi-complete category constituted by logics and translations. © 2001 Elsevier Science B.V. All rights reserved.

0. Introduction

A historical survey of the use of translations for the study of the inter-relations between logical systems is presented in [6], where the different approaches to the use of the term "translation" are discussed.

Prawitz and Malmnäs [17] constitute the first known paper in the literature in which a general definition for the term translation is introduced. But [22] and [5] are the first works with a general systematical study on translations between logics, studying inter-relations between propositional calculi by the analysis of their mutual translations.

Da Silva et al. [20] propose a more general definition of this concept. Logics are characterized simply as pairs constituted by a set and a consequence operator, and translations between logics are defined as maps that preserve consequence relations. The authors show, among other basic results, that logics together with translations form a bi-complete category of which topological spaces with topological continuous

☆ We would like to thank an anonymous referee for useful suggestions concerning the exposition.
* Correspondence address: Department of Philosophy, Universidade Estadual de Campinas (Unicamp), C.P. 6133, 13081-970 Campinas, S.P., Brazil.
E-mail addresses: haf@bauru.unesp.br (H.A. Feitosa), itala@cle.unicamp.br (I.M.L. D'Ottaviano).

PII: S0168-0072(00)00046-4

functions constitute a full subcategory, and study some connections between translations between logics and uniformly continuous functions between the spaces of their theories.

The aim of this paper is to investigate an important subclass of translations, the conservative translations, which strongly preserve logical consequence and were introduced in [6].

We begin with the basic concepts, properties and results on translations between logics, introduced by da Silva et al. [20], necessary for the development of the work.

In Section 1 we also characterize some logical properties that may be preserved via translations.

In Section 2 we introduce the concept of conservative translation. Before discussing the historical translations of Kolmogoroff, Glivenko, Gödel and Gentzen, we analyse the general definitions of translation between logics of Prawitz, Wójcicki and Epstein, in the terms of our definition of conservative translation.

In the following section we study necessary and sufficient conditions for a translation to be conservative.

We then study some general properties of logical systems, which are characterized by the existence of conservative translations between them.

In Section 5 we prove that the class constituted by logics and conservative translations between them determines a co-complete subcategory of the bi-complete category constituted by logics and translations.

Finally, in the last section, we prove an important necessary and sufficient condition for the existence of a conservative translation between two logics, which was fundamental to obtain several conservative translations we have studied.

This initial segment of a theory of conservative translations was developed with a grant from the "Conselho Nacional de Pesquisas do Brasil" (CNPq) and also constitutes part of the results of a more general research project on the subject *Mathematical and Computational Aspects of Translations between Logics*, which was developed sponsored by the "Fundação de Amparo à Pesquisa do Estado de São Paulo" (FAPESP), Brazil. This project is reported in [1].

1. Translations between logics

In this section we present the definitions of logic and translation between logics introduced in [20] and also some basic results that allow us to characterize the structure of the class constituted by logics and translations.

Logics are characterized, in the most general sense, as sets with consequence relation and translations between logics as consequence relation preserving maps.

1.1. Definition. A *logic* $\mathscr{A}$ is a pair (A, C) such that A is a set, called the *domain* or the *universe* of $\mathscr{A}$, and C is a *consequence operator* in A, that is, $C : \mathscr{P}(A) \to \mathscr{P}(A)$ is a function that satisfies, for $X, Y \subseteq A$, the following conditions:

(i) $X \subseteq C(X)$;
(ii) If $X \subseteq Y$, then $C(X) \subseteq C(Y)$;
(iii) $C(C(X)) \subseteq C(X)$.

It is trivial that, for every $X \subseteq A, C(C(X)) = C(X)$. Thus, a logic is a set A together with an inflationary (i), increasing (ii) and idempotent (iii) operator on 2^A.

In general, we identify a logic $\mathscr{A}$ with its domain A.

It's clear that our definition does not take into consideration the so-called non-monotonic logics.

1.2. Definition. Given a logic (A, C), $X \subseteq A$ is a *theory*, or a *closed set* in A, if $C(X) = X$. And X is *open* if the complement of X is closed.

1.3. Definition. A consequence operator C in A is *finitary* if, for every $X \subseteq A$, there are $X_i \subseteq X$, X_i finite, for $i \in I$, such that

$$C(X) = \bigcup_{i \in I} C(X_i).$$

1.4. Definition. Let C and C^* be consequence operators in A. The operator C is *stronger* than C^*, what is denoted by $C^* < C$, if every closed set according to C is also a closed set according to C^*. In this case we also say that C^* is *weaker* than C.

1.5. Proposition. *Let C and C^* be consequence operators in A. Then C is stronger than C^* if, and only if, $C^*(X) \subseteq C(X)$, for every $X \subseteq A$.*

Proof. If $C^* < C$, then $C^*(C(X)) = C(X)$. As $X \subseteq C(X)$, then $C^*(X) \subseteq C^*(C(X)) = C(X)$.

On the other hand, if X is closed according to C, as $C^*(X) \subseteq C(X) = X$, then $X = C^*(X)$, that is, X is closed, according to C^*. □

1.6. Definition. Let $\mathscr{A} = (A, C)$ be a logic and X a set. Given a map $F : A \to X$, we define the *co-induced consequence operator by F and $\mathscr{A}$* in X, C_X, in the following way: given $Y \subseteq X, Y$ is a closed set in X if $F^{-1}(Y)$ is a closed set in A. In these conditions we say that the *logic* (X, C_X) is *co-induced* by F and A.

Dually, given $G : Y \to A$ we define the *consequence operator induced by G and A*: given $Z \subseteq Y, Z$ is closed in Y if $Z = G^{-1}(W)$, with W a closed set in A.

Since closure is preserved by intersections and intersections by the inverse image of maps, relative to a given consequence operator, Definition 1.6 determines exactly one co-induced and one induced consequence operator.

1.7. Proposition. *Let $\mathscr{A}$ be a logic, B a set, $F: A \to B$ a function and C_B the consequence operator co-induced by F and A in B. Then C_B is the weakest consequence operator that makes F a translation.*

Proof. C_B is a consequence operator and makes F a translation. Let C^* be another consequence operator in B that makes F a translation. If $C \subseteq B$ is closed according to C^*, then $F^{-1}(C)$ is a closed set in A. So, by the definition of C_B, we have that C is a closed set in B relatively to C_B, that is, $C_B < C^*$. □

Dually, C_A is the strongest consequence operator that makes a function $G: A \to B$ a translation, where $\mathscr{A}$ is the logic induced in A by G and the logic B.

1.8. Definition. Let $F: A_1 \to A_2$ be a map between logics. A subset $A \subseteq A_1$ is said to be *saturated* relatively to F if, for $a \in A$ and for every $b \in A_1, F(a) = F(b)$ implies $b \in A$.

In [22] logics are seen in a more restrictive way as algebras with consequence operators, that is, a logic (A, C) is such that A is a formal language and C is a consequence operator in the free algebra **Form**(A) of the formulas of A. A consequence operator C in **Form**(A) is said to be *structural* if $s(C(\Gamma)) \subseteq (C(s(\Gamma))$, for every $\Gamma \subseteq$ **Form**(A) and every endomorphism s of **Form**(A).

We characterize logical systems as particular cases of logics.

1.9. Definition. A *logical system* defined over L is a pair $\mathscr{L} = (L, C)$, where L is a formal language and C is a structural consequence operator in **Form**(L).

Da Silva et al. [20] introduce the following definition, that captures the essential feature of a logical translation, that is, captures the intuition of a map preserving the consequence relation.

1.10. Definition. A *translation* from the logic $\mathscr{A}_1$ into the logic $\mathscr{A}_2$ is a map $T: A_1 \to A_2$ such that, for every $X \subseteq A_1$:

$$T(C_1(X)) \subseteq C_2(T(X)).$$

If T is a translation, it is obvious that, for every $x \in A_1$:

$$x \in C_1(X) \Rightarrow T(x) \in C_2(T(X)),$$

but the converse does not hold in general.

In the particular case in which $\mathscr{A}_1$ and $\mathscr{A}_2$ are logical systems and their syntactical consequence relations are correct relatively to C_1 and C_2, respectively, that is, for $\Delta \cup \{\alpha\} \subseteq$ **Form**(A_1) and $\Gamma \cup \{\beta\} \subseteq$ **Form**(A_2), $\Delta \vdash_{C_1} \alpha$ if and only if $\alpha \in C_1(\Delta)$ and $\Gamma \vdash_{C_2} \beta$ if and only if $\beta \in C_2(\Gamma)$, one has that T is a translation if, and only if

$$\Delta \vdash_{C_1} \alpha \Rightarrow T(\Delta) \vdash_{C_2} T(\alpha). \tag{1}$$

We observe that, if $\Delta = \emptyset$ then $T(\emptyset) = \emptyset$ and, therefore, every translation between logical systems takes theorems into theorems, that is

$$\vdash_{C_1} \alpha \Rightarrow \vdash_{C_2} T(\alpha).$$

The following theorem, introduced in [20], with a proof similar to the proof of the equivalent topological results concerning continuous functions, is related to some of the results of the next sections.

1.11. Theorem. *The following assertions are equivalent*:
(i) *T is a translation from $\mathscr{A}_1$ into $\mathscr{A}_2$*;
(ii) *For every $X \subseteq A_1, C_2(T(C_1(X)) = C_2(T(X))$*;
(iii) *The inverse image of a closed set is closed*;
(iv) *For every $Y \subseteq B, C_1(T^{-1}(Y)) \subseteq T^{-1}(C_2(Y))$.* □

1.12. Definition. Two logics $\mathscr{A}_1$ and $\mathscr{A}_2$ are said to be *L-homeomorphic* if there is a bijective function $T : A_1 \to A_2$ such that T and T^{-1} are translations. The function T is called a *L-homeomorphism*.

1.13. Proposition. *Given a bijection $T : A_1 \to A_2, T$ is a L-homeomorphism if, and only if, for every $A \subseteq A_1, T(C_1(A)) = C_2(T(A))$.* □

In the same paper by da Silva, D'Ottaviano and Sette, the authors show that the category [1] **Tr** constituted by logics and translations between them is bi-complete, that is, it is both complete and co-complete.

1.14. Theorem. *The category* **Tr** *whose objects are logics and whose morphisms are translations between logics has products and equalisers, and has co-products and co-equalisers.* □

The proof of this theorem is presented in detail in [6].

It is well known that topological spaces can be defined as sets with closure operators, which besides the conditions of Definition 1.1 also satisfy the conditions:
(iv) $C(\emptyset) = \emptyset$;
(v) $C(X \cup Y) = C(X) \cup C(Y)$.

The category of topological spaces with continuous functions is a full subcategory of **Tr**, that is, translations between topological spaces are continuous functions in the topological sense.

The following definition and results are well known in the topological case.

1.15. Definition. A *closed mapping* is a function for which the image of every closed set is a closed set.

[1] In what follows we will be using some fundamental concepts and results of Category Theory, which can be found in any introductory text on the subject.

1.16. Proposition. *Let $T : A_1 \to A_2$ be a translation. Then T is closed if, and only if, for every $A \subseteq A_1, C_2(T(A)) \subseteq T(C_1(A))$.* □

1.17. Corollary. *A function $T : A_1 \to A_2$ is a closed translation if, and only if, for every $A \subseteq A_1, T(C_1(A)) = C_2(T(A))$.* □

1.18. Proposition. *Let $T_1 : A_1 \to A_2$ be a surjective closed translation. Then a function $T_2 : A_2 \to A_3$ is a translation if, and only if, $T_2 \circ T_1 : A_1 \to A_3$ is a translation.* □

In the following results we characterize some logical properties that may be preserved via translations.

1.19. Definition. Let L_1 be a language containing only unary and binary connectives and such that $p_0, p_1, p_2, \ldots$ denote the atomic formulas of L_1. If L_2 is a language, we say that $* : L_1 \to L_2$ is a *schematical mapping* if there are schemata of formulae $A, B_\sharp$ and $C_\perp$ of L_2 such that:

(i) $p^* = A(p)$, for every atomic formula p of L_1;
(ii) $(\sharp\varphi)^* = B_\sharp(\varphi^*)$, for every unary connective $\sharp$ of L_1;
(iii) $(\varphi \perp \psi)^* = C_\perp(\varphi^*, \psi^*)$, for every binary connective $\perp$ of L_1.

A schematical application is a homeomorphism between languages, for it preserves the algebraic structure of the algebra of formulae associated with the respective languages.

1.20. Definition. A schematical mapping $*$ is said to be *literal relatively to a given connective* $\sharp$, or $\perp$, if this connective is mapped by $*$ into itself, that is, $(\sharp\varphi)^* = \sharp\varphi^*$ or $(\varphi \perp \psi)^* = \varphi^* \perp \psi^*$, respectively.

A schematical application $* : L_1 \to L_2$ is *literal* if it is literal relatively to every connective of L_1.

1.21. Definition. A *translation T* between logical systems is *schematical* if it is a schematical mapping.

1.22. Definition. A translation $T : L_1 \to L_2$ is *trivial-invariant* if, for every set $\Delta \subseteq \mathbf{Form}(L_1)$, $T(\Delta)$ is trivial in $\mathbf{Form}(L_2)$.

1.23. Proposition. *Let $T : L_1 \to L_2$ be a translation. The following conditions are equivalent*:

(i) *T is trivial-invariant*;
(ii) *The set $Im(T)$ is trivial*;
(iii) *There is a subset of $Im(T)$ that is trivial.*

Proof. (iii) $\Rightarrow$ (i)) If $\Gamma \subseteq \mathbf{Form}(L_1)$ is trivial, then $Im(T) = T(C_1(\Gamma)) \subseteq C_2(T(\Gamma))$. As there is a trivial set contained in $Im(T)$, by Proposition 3.2.14 $Im(T)$ and $C_2(T(\Gamma))$ are trivial. Hence $C_2(C_2(T(\Gamma)) = C_2(T(\Gamma)) = \mathbf{Form}(L_2)$, that is, $T(\Gamma)$ is trivial. □

1.24. Definition. Given a logical system $\mathscr{L}$, a set $\Delta \subseteq \mathbf{Form}(L)$ is *non-trivial* if $C(\Delta) \neq \mathbf{Form}(L)$ and is *trivial* otherwise.

The logical system $\mathscr{L}$ is trivial if $C(\emptyset) = \mathbf{Form}(L)$, that is, $\mathbf{Theo}(L) = \mathbf{Form}(L)$, $\mathbf{Theo}(L)$ being the set of theorems of $\mathscr{L}$.

1.25. Definition. Let $\mathscr{L}$ be a logical system, the language L having as a symbol of negation the symbol $\neg$. A set $\Delta \subseteq \mathbf{Form}(L)$ is *¬-inconsistent* if there is a formula α such that $\alpha \in C(\Delta)$ and $\neg\alpha \in C(\Delta)$. The set Δ is *¬-consistent* if it is not ¬-inconsistent.

The logic L is *¬-consistent* if $\mathbf{Theo}(L)$ is ¬-consistent, that is, it is not the case that $\alpha \in C(\emptyset)$ and $\neg\alpha \in C(\emptyset)$, for every formula α.

1.26. Definition. A logical system $\mathscr{L}$ is a *vacuum logic* if $C(\emptyset) = \emptyset$, that is, if $\mathbf{Theo}(L) = \emptyset$.

We observe that every trivial set is inconsistent, relatively to any negation. Usually, to be trivial is equivalent to being inconsistent, but there are logical systems, as for instance the paraconsistent systems (see [2]), such that to be inconsistent is not the same as to be trivial. Therefore, in these systems, it is possible to have a ¬-inconsistent set without every formula being among its consequences.

1.27. Proposition. *Let $\mathscr{L}_1$ and $\mathscr{L}_2$ be logical systems whose languages have the negation ¬ and let $T : L_1 \to L_2$ be a literal translation relatively to ¬. If $\mathscr{L}_2$ is ¬-consistent, then $\mathscr{L}_1$ is ¬-consistent.* □

1.28. Proposition. *There is no translation from a non-vacuum system into a vacuum system.* □

As is known, two interesting examples of vacuum systems are the three-valued calculi of Kleene $\mathbf{Kl}_1$ and of Bochvar $\mathbf{B}_{3I}$ (see [15]). According to the previous results there is no translation into these logics from, for instance, classical logical, intuitionistic logic, etc.

2. The concept of conservative translation

In the literature, definitions of translations between logics require, in general, that the converse of (1), that appears right after Definition 1.10, also holds. We prefer to addopt the concept as defined in [20] in order to accommodate certain maps that seem to be obvious examples of translations, such as the identity map from intuitionistic into classical logic and the forgetfulness map from modal logic into classical logic, but which would be ruled out if equivalence were substituted for implication in (1).

In this section we define the concept of conservative translation, introduced in [6], for those functions for which the converse of (1) holds. Conservative translations characterize a special subclass of translations.

2.1. Definition. Given two logics $\mathscr{A}_1=(A_1, C_1)$ and $\mathscr{A}_2=(A_2, C_2)$, *a conservative mapping* from $\mathscr{A}_1$ into $\mathscr{A}_2$ is a function $T: A_1 \to A_2$ such that, for every $x \in A_1$:

$$x \in C_1(\emptyset) \Leftrightarrow T(x) \in C_2(\emptyset).$$

2.2. Definition. A *conservative translation* from $\mathscr{A}_1$ into $\mathscr{A}_2$ is a function $T: A_1 \to A_2$ such that, for every set $B \cup \{x\} \subseteq A_1$:

$$x \in C_1(B) \Leftrightarrow T(x) \in C_2(T(B)).$$

If $\mathscr{L}_1=(L_1, C_1)$ and $\mathscr{L}_2=(L_2, C_2)$ are logical systems, a conservative translation is a function $T: \mathbf{Form}(L_1) \to \mathbf{Form}(L_2)$ such that, for every subset $\Gamma \cup \{\alpha\} \subseteq \mathbf{Form}(L_1)$:

$$\Gamma \vdash_{C_1} \alpha \Leftrightarrow T(\Gamma) \vdash_{C_2} T(\alpha).$$

The term conservative, for these translations which strongly preserve consequence relations, was chosen by analogy to the term conservative extension of a theory.

2.3. Translations for Prawitz and Malmnäs. According to the general definition introduced in [17], a translation from a logical system S_1 into a logical system S_2 is a function t such that, for every formula α of S_1:

$$\vdash_{S_1} \alpha \Leftrightarrow \vdash_{S_2} t(\alpha).$$

If such a function exists, S_1 is said to be interpretable in S_2.

If, for every set $\Gamma \cup \{\alpha\}$ of formulas in S_1:

$$\Gamma \vdash_{S_1} \alpha \Leftrightarrow t(\Gamma) \vdash_{S_2} t(\alpha),$$

where $t(\Gamma)=\{t(\beta) \mid \beta \in \Gamma\}$, S_1 is said to be interpretable in S_2 by t with respect to derivability.

Prawitz's definition of translation coincides with our definition of conservative mapping, constituting in fact a particular case of our conservative translations. But, in general, a translation according to Prawitz is neither conservative, nor even a translation in our sense.

When a system is interpretable into another by a translation with respect to derivability, then this Prawitz's translation is a conservative translation according to our definitions.

Prawitz also introduces the concept of schematical translation, for those translations that are defined by schemata of formulae of S_2.

2.4. Translations for Wójcicki. For [22], given two propositional languages S_1 and S_2 with the same set of variables, a mapping t from S_1 into S_2 is said to be a translation if, and only if, two conditions are satisfied:

(i) there is a formula $\varphi(p_0)$ in S_2 in one variable p_0 such that, for each variable p, $t(p) = \varphi(p)$;

(ii) for each connective η_i of S_1 there is a formula φ_i in S_2 in the variables $p_1, \dots, p_k$, such that, for all $\alpha_1, \dots, \alpha_k$ in S_1, k being the arity of η_i, we have that

$$t(\eta_i(\alpha_1, \dots, \alpha_k)) = \varphi_i(t\alpha_1 \mid p_1, \dots, t\,\alpha_k \mid p_k).$$

A propositional calculus is defined as a pair $\mathscr{C} = (S, C)$, where C is a consequence operation in the language S. If for the propositional calculi $\mathscr{C}_1 = (S_1, C_1)$ and $\mathscr{C}_2 = (S_2, C_2)$ there is a translation t from S_1 into S_2, such that for every $X \subseteq S_1$ and every $\alpha \in S_1$,

$$\alpha \in C_1(X) \Leftrightarrow t(\alpha) \in C_2(t(X)),$$

Wójcicki says that the calculus $\mathscr{C}_1$ has a translation into the calculus $\mathscr{C}_2$.

According to these definitions, translations between logical systems for Wójcicki are strict cases of conservative translations in our sense, that is, they are derivability preserving schematical translations in Prawitz's sense.

2.5. Translations for Epstein and Krajewski. In [5], Epstein and Krajewski define a validity mapping of a propositional logic L into a propositional logic M as a map t from the language of L into the language of M such that, for every formula α:

$$\models_L \alpha \Leftrightarrow \models_M t(\alpha).$$

A translation is a validity mapping t such that, for every set Γ of formulas and every formula α of L:

$$\Gamma \models_L \alpha \Leftrightarrow t(\Gamma) \models_M t(\alpha).$$

As Epstein's book deals with strongly complete propositional calculi, his validity mappings coincide with our conservative mappings and with Prawitz's translations; and his translations are our conservative translations.

The functions called grammatical translations by Epstein are particular cases of Prawitz's schematical translations with respect to derivability and coincide with our schematical translations.

Epstein proves the following result, concerning the non-classical logics he studies in his book.

Theorem. *If $\mathscr{L}$ is a non-classical propositional system in whose language the symbols $\neg$ and $\rightarrow$ occur as primitive connectives, then there is no schematical translation from $\mathscr{L}$ into the classical propositional calculus.*

Epstein also introduces the concept of semantically faithful translations, but we will not deal with such a concept in this paper.

2.6. Kolmogoroff's translation. Kolmogoroff's [14], is apparently the first paper on the "translability" of classical logic into intuitionistic logic. Kolmogoroff introduces the system $\mathscr{B}$ of the *General Logic of Judgments* and the system $\mathscr{H}$ of the *Special Logic of Judgments*.

Kolmogoroff's system $\mathscr{B}$, which constitutes in fact the first axiomatization for intuitionistic logic introduced in the literature, preceeding Heyting's system H of 1930, is constructed on a formal language with two primitive connectives, $\neg$ for the negation and $\rightarrow$ for the conditional, and is characterized by the following axioms and rules:

Axiom $K_1 : \varphi \rightarrow (\psi \rightarrow \varphi)$,

Axiom $K_2 : (\varphi \rightarrow (\varphi \rightarrow \psi)) \rightarrow (\varphi \rightarrow \psi)$,

Axiom $K_3 : (\varphi \rightarrow (\psi \rightarrow \sigma)) \rightarrow (\psi \rightarrow (\varphi \rightarrow \sigma))$,

Axiom $K_4 : (\psi \rightarrow \sigma) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \sigma))$,

Axiom $K_5 : (\varphi \rightarrow \psi) \rightarrow ((\varphi \rightarrow \neg\psi) \rightarrow \neg\varphi)$,

R_1 (MP) $: \alpha, \alpha \rightarrow \beta / \beta$,

R_2 (Substitution) $: \vdash \alpha(p) / \vdash \alpha(p \mid \beta))$.

The first four axioms coincide with the axioms introduced by Hilbert in 1923 in order to formalize the classical propositional calculus. Nowadays, we know that Kolmogoroff's system $\mathscr{B}$ is equivalent to the minimal logic J introduced in [13].

The system $\mathscr{B}$ is extended to the formal system $\mathscr{H}$ by adding

Axiom $K_6 : \neg\neg\varphi \rightarrow \varphi$

and it is proved that $\mathscr{H}$ is equivalent to the classical propositional calculus presented in [12].

A function is inductively defined that, to each formula φ of $\mathscr{H}$ associates a formula φ^K of $\mathscr{B}$ by adding a double negation to each subformula of φ:

$K : \mathscr{H} \rightarrow \mathscr{B}$,

$(p)^K =_{df} \neg\neg p$,

$(\neg\varphi)^K =_{df} \neg\neg(\neg\varphi^K)$,

$(\varphi \rightarrow \psi)^K =_{df} \neg\neg(\varphi^K \rightarrow \psi^K)$.

The following result is proved:

Theorem. *Let* $\mu = \{\mu_1, \ldots, \mu_n\}$ *be a set of axioms and* $\mu^K = \{\mu_1^K, \ldots, \mu_n^K\}$. *If* $\mu \vdash_{\mathscr{H}} \varphi$, *then* $\mu^K \vdash_{\mathscr{B}} \varphi^K$.

Kolmogoroff extends his systems $\mathscr{B}$ and $\mathscr{H}$ to the systems of quantification theory $\mathscr{B}\mathscr{L}$ and $\mathscr{H}\mathscr{L}$, respectively. He suggests that the above theorem is provable relative to these systems and we think that it is reasonable to assert that he does foresee that the system of classical number theory is translatable into intuitionistic theory and therefore is intuitionistically consistent. But Kolmogoroff's paper was known only when the results of Gödel and Gentzen on the relative consistency of classical arithmetic with respect to intuitionistic arithmetic were already well known.

Prawitz and Malmnäs [17] prove that, for each set of formulae $\Gamma \cup \{\varphi\}$:

$$\Gamma \vdash_{\mathscr{H}\mathscr{L}} \varphi \Leftrightarrow K(\Gamma) \vdash_{\mathscr{B}\mathscr{L}} K(\alpha).$$

Hence, according to our definitions, K is an example of a schematical conservative translation from classical into intuitionistic first-order predicate logic; and so it is a translation according to the general definitions in the literature that we have mentioned.

2.7. Glivenko's translation. The main result of Glivenko [8], which was used by Gödel [9], is the following:

Theorem. *If φ is a theorem of the classical propositional calculus, then $\neg\neg\varphi$ is a theorem of the intuitionistic propositional calculus.*

We are considering here a function G which maps each formula of the classical calculus into its double negation in the intuitionistic calculus (that is, $G(\varphi) = \neg\neg\varphi$) and preserves theorems.

The following corollary follows immediately.

Corollary. *A formula $\neg\varphi$ is a theorem of the classical propositional calculus if, and only if, $\neg\varphi$ is a theorem of the intuitionistic propositional calculus.*

Feitosa [6] shows by using algebraic semantics that, besides being a translation, Glivenko's function G is also a conservative translation. This result is not presented in this paper.

2.8. Gödel's interpretations. Gödel knew [8], but apparently he did not know [14].

Gödel [9] extends the result he presented in 1932 in Vienna. The paper introduces a function Gd_1 from classical propositional logic CL into the Heyting intuitionistic propositional logic H:

$$Gd_1 : CL \to H,$$

$$(p)^{Gd_1} =_{df} p,$$

$$(\neg\,\varphi)^{Gd_1} =_{df} \neg\varphi^{Gd_1},$$

$$(\varphi \wedge \psi)^{Gd_1} =_{df} \varphi^{Gd_1} \wedge \psi^{Gd_1},$$

$(\varphi \vee \psi)^{Gd_1} =_{df} \neg(\neg\, \varphi^{Gd_1} \wedge \neg\, \psi^{Gd_1})$,
$(\varphi \to \psi)^{Gd_1} =_{df} \neg(\varphi^{Gd_1} \wedge \neg\, \psi^{Gd_1})$.

This interpretation is extended to a function from classical arithmetic *PA* into a modified version H' of Heyting's intuitionistic arithmetic and the following result is proved.

Theorem. *If* $\vdash_{PA} \varphi$, *then* $\vdash_{H'} \varphi^{Gd_1}$.

In fact, it can be proved that

$$\vdash_{PA} \varphi \Leftrightarrow \vdash_{H'} \varphi^{Gd_1},$$

as conjectured by Gödel in his paper. But it is easy to see that Gödel's "translation" does not preserve derivability, even in the propositional case. For, as

$$\neg\neg p \vdash_{CL} p,$$

if the interpretation Gd_1 preserves derivability, we have that

$$Gd_1(\neg\neg\, p) \vdash_H Gd_1(p)$$

$$\neg\neg Gd_1(p) \vdash_H Gd_1(p)$$

$$\neg\neg p \vdash_H p$$

$$\vdash_H \neg\neg\, p \to p,$$

which is false.

Hence, Gödel's interpretation is a translation in the sense of Prawitz, but is not a translation according to our definition.

In 1933 Gödel also introduced an interpretation from intuitionistic propositional logic into his "modal" system $\mathscr{G}$.

In [10] the system $\mathscr{G}$ has the classical connectives $\sim, \wedge, \vee, \to$ and the connective B, such that $B\alpha$ is read "α is provable" but not necessarily in a certain formal system.

Besides the classical propositional axioms, $\mathscr{G}$ has the following axioms and rule:

Axiom $G_1 : B\varphi \to \varphi$,
Axiom $G_2 : B\varphi \to (B(\varphi \to \psi) \to B\psi)$,
Axiom $G_3 : B\varphi \to BB\varphi$,
Rule: $\varphi / B\varphi$.

The system $\mathscr{G}$ is equivalent to the Lewis' system of strict implication, to which the Becker axiom ($Np < NNp$) is added, with Bp interpreted by "necessary p" (Np).

The Gödel interpretation Gd_2 is defined by

$$Gd_2 : H \to \mathscr{G}$$

$$(p)^{Gd_2} =_{df} p,$$

$$(\neg\varphi)^{Gd_2} =_{df} \neg B\varphi^{Gd_2},$$
$$(\varphi\wedge\psi)^{Gd_2} =_{df} \varphi^{Gd_2}\wedge\psi^{Gd_2},$$
$$(\varphi\vee\psi)^{Gd_2} =_{df} B\varphi^{Gd_2}\vee B\psi^{Gd_2},$$
$$(\varphi\to\psi)^{Gd_2} =_{df} B\varphi^{Gd_2}\to B\psi^{Gd_2}.$$

Theorem. $\vdash_H \varphi \Rightarrow \vdash_{\mathscr{G}} \varphi^{Gd_2}$.

Answering affirmatively a conjecture of Gödel, McKinsey and Tarski [16] prove that the converse of the above theorem also holds, and Rasiowa and Sikorski [18] prove the extension of both results to the predicate calculi HP and $\mathscr{GP}$, that is

$$\vdash_{HP} \varphi \Leftrightarrow \vdash_{\mathscr{GP}} \varphi^{Gd_2}.$$

Feitosa [6] studies several other papers related to this Gödel's interpretation.

Prawitz and Malmnäs [17] show that the interpretation Gd_2 does not preserve derivability. Hence, Gödel's function Gd_2 is not a translation in our sense.

2.9. Gentzen's translation. The aim of [7] is to show that the applications of the law of double negation in proofs of classical arithmetic can in many instances be eliminated. As an important consequence of this fact, Gentzen presents a constructive proof of the consistency of pure classical logic and elementary arithmetic with respect to the corresponding intuitionistic theories, which is obtained from the definition of an adequate translation between their languages and consequently relation preserving.

Gentzen defines an interpretation from Peano classical arithmetics (PA) into intuitionistic arithmetics (IA), very similar to the one introduced in [9]. The difference between the two functions is in the interpretation of the atomic formulas and of the implication:

$$Gz: PA \to IA,$$

$$(p)^{Gz} =_{df} \neg\neg p,$$
$$(\neg\varphi)^{Gz} =_{df} \neg\varphi^{Gz},$$
$$(\varphi\wedge\psi)^{Gz} =_{df} \varphi^{Gz}\wedge\psi^{Gz},$$
$$(\varphi\vee\psi)^{Gz} =_{df} \neg(\neg\varphi^{Gz}\wedge\neg\psi^{Gz}),$$
$$(\varphi\to\psi)^{Gz} =_{df} (\varphi^{Gz}\to\psi^{Gz}),$$
$$(\forall x\varphi)^{Gz} =_{df} \forall x\varphi^{Gz},$$
$$(\exists x\varphi)^{Gz} =_{df} \neg(\forall x\neg\varphi^{Gz}).$$

The other formulas are translated into themselves.

In Gentzen's paper the proofs are carried out intuitionistically. The consistency of classical arithmetic relatively to intuitionistic arithmetic is proved as an immediate consequence of the following theorem.

Theorem. *If α is classically provable from Γ, then α^{Gz} is intuitionistically provable from Γ^{Gz}. That is*

$$\Gamma \vdash_{CL} \varphi \Leftrightarrow \Gamma^{Gz} \vdash_{H} \varphi^{Gz}.$$

Hence, in spite of Gödel's interpretation not being a translation, the Gentzen interpretation is a conservative translation.

Ending this section, we observe that among the historical "translations" of the literature, developed mainly in order to study inter-relations between logics, only the two Gödel interpretations are not conservative translations, for in fact they are not even translations according to our definition, since they simply preserve theoremhood.

3. Basic results on conservative translations

The following results introduce two necessary and two sufficient conditions for a translation to be conservative.

3.1. Proposition. *Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be logics and $T:\mathscr{A}_1 \to \mathscr{A}_2$ a map and let every closed subset $A \subseteq A_1$ be saturated relative to T. If $T(C_1(A)) = C_2(T(A))$, then T is a conservative translation.*

Proof. Since $T(C_1(A)) \subseteq C_2(T(A))$, it follows that T is a translation. If $T(x) \in C_2(T(A)) = T(C_1(A))$, then there is a $y \in C_1(A)$ such that $T(x) = T(y)$. As $C_1(A)$ is closed, by hypothesis it is saturated and therefore $x \in C_1(A)$. □

3.2. Proposition. *Let $T : A_1 \to A_2$ be an injective mapping between the logics $\mathscr{A}_1$ and $\mathscr{A}_2$. If, for every $A \subseteq A_1, T(C_1(A)) = C_2(T(A))$, then T is a conservative translation.*

Proof. If T is injective, then every subset of A_1 is saturated relatively to T. □

3.3. Proposition. *Let $T : A_1 \to A_2$ be a map such that, for every $A \subseteq A_1, C_2(T(A)) \subseteq$ $\mathbf{Im}(T)$. If T is a conservative translation, then $T(C_1(A)) = C_2(T(A))$.*

Proof. Since T is a translation, we have $T(C_1(A)) \subseteq C_2(T(A))$. On the other hand, if $y \in C_2(T(A))$, then $y \in$ $\mathbf{Im}(T)$ and so there is an $x \in A_1$ such that $T(x) = y$; as T is conservative and $T(x) \in C_2(T(A))$, we have that $x \in C_1(A)$ and therefore $T(x) = y \in T(C_1(A))$, that is, $C_2(T(A)) \subseteq T(C_1(A))$. □

3.4. Corollary. *Let $T : A_1 \to A_2$ be a surjective mapping. If T is a conservative translation, then $T(C_1(A)) = C_2(T(A))$, for every $A \subseteq A_1$.* □

3.5. Corollary. *Let $T : A_1 \to A_2$ be a bijective mapping. Then T is a conservative translation if, and only if, $T(C_1(A)) = C_2(T(A))$, for every $A \subseteq A_1$.* □

The last corollary give us a necessary and sufficient condition for a mapping between logics to be a conservative translation, but only in the case of the function being bijective.

We also observe that, according to Proposition 1.13, every L-homeomorphism is a conservative translation, but it is not the case that every conservative translation is a L-homeomorphism.

The following result introduces a necessary and sufficient condition for a translation being conservative, in the case of the consequence relations of the logics being finitary.

3.6. Theorem. *Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be logics with finitary consequence operators and $T: A_1 \to A_2$. The function T is a conservative translation if, and only if, for every finite $A \cup \{x\} \subseteq A_1$, $x \in C_1(A)$ is equivalent to $T(x) \in C_2(T(A))$.* □

In the case when $\mathscr{A}_1$ and $\mathscr{A}_2$ are strongly complete logical systems, the result above corresponds to the compactness of the systems.

The next theorem supplies a general necessary and sufficient condition for a translation to be conservative, which is very useful in the study of conservative translations.

3.7. Theorem. *A translation $T: A_1 \to A_2$ is conservative if, and only if, for every $A \subseteq A_1$, $T^{-1}(C_2(T(A))) \subseteq C_1(A)$.*

Proof. Let T be conservative. For every $x \in T^{-1}(C_2(T(A)))$ we have that $T(x) \in T \circ T^{-1}(C_2(T(A))) \subseteq C_2(T(A))$ and so $T(x) \in C_2(T(A))$. As T is conservative, $x \in C_1(A)$.

On the other hand, if $T(x) \in C_2(T(A))$, as $T^{-1}(C_2(T(A))) \subseteq C_1(A)$, then $T^{-1}(T(x)) \subseteq C_1(A)$. Since $x \in T^{-1}(T(x))$, we have that $x \in C_1(A)$. □

4. Conservative translations between logical systems

In this section we study some general properties of logical systems, which are characterized by the existence of translations between the systems.

4.1. Theorem. *If there is a recursive and conservative translation from a logical system $\mathscr{L}_1$ into a decidable logical system $\mathscr{L}_2$, then $\mathscr{L}_1$ is decidable.*

Proof. Let $T: \mathscr{L}_1 \to \mathscr{L}_2$ be a recursive conservative translation. Given $\alpha \in \mathbf{Form}(L_1)$, as T is recursive, we can determine $T(\alpha)$. As $\mathscr{L}_2$ is decidable, it is possible to verify if $T(\alpha)$ is or is not a theorem of $\mathscr{L}_2$. Hence, $\vdash_{C_2} T(\alpha)$ implies $\vdash_{C_1} \alpha$ and $\nvdash_{C_2} T(\alpha)$ implies $\nvdash_{C_1} \alpha$. □

4.2. Corollary. *There is no recursive conservative translation from the first-order logic $L_{\omega\omega}$ into the classical propositional calculus CP.* □

Though there is no such translation, it is easy to verify that the forgetfulness function $F : L_{\omega\omega} \to CP$, that maps every formula α of $L_{\omega\omega}$ into a formula of CP, simply erasing all the quantifiers and the respective parentheses and variables of α, is a recursive translation.

In fact, in the previous theorem it is not necessary that T should be a translation, it is sufficient that T be a conservative mapping.

4.3. Theorem. *If $\mathscr{L}_1$ is a decidable logical system and if there is a surjective and conservative translation $T : \mathscr{L}_1 \to \mathscr{L}_2$, then $\mathscr{L}_2$ is decidable.* □

4.4. Proposition. *Let $\mathscr{L}_1$ be a logical system with an axiomatics Λ. If there is a surjective and conservative translation $T : \mathscr{L}_1 \to \mathscr{L}_2$, then $T(\Lambda)$ is an axiomatics for $\mathscr{L}_2$.* □

4.5. Theorem. *Let $T : \mathscr{L}_1 \to \mathscr{L}_2$ be a conservative application. If $\mathscr{L}_1$ is non-trivial, then $\mathscr{L}_2$ is non-trivial.* □

Whenever non-triviality is equivalent to consistency, the function T preserves consistency.

4.6. Theorem. *Let $T : \mathscr{L}_1 \to \mathscr{L}_2$ be a surjective and conservative map. If $\mathscr{L}_2$ is non-trivial, then $\mathscr{L}_1$ is non-trivial.* □

4.7. Theorem. *Let $\mathscr{L}_1$ and $\mathscr{L}_2$ be logical systems, with languages in which the conditional symbol $\to$ occurs as a primitive connective or may be defined. If $T : \mathscr{L}_1 \to \mathscr{L}_2$ is a conservative translation, is literal relatively to $\to$ and $\mathscr{L}_2$ admits a deduction theorem, then $\mathscr{L}_1$ also admits such a theorem.* □

4.8. Theorem. *In the conditions of the previous theorem, if $T : \mathscr{L}_1 \to \mathscr{L}_2$ is a surjective conservative translation, is literal relatively to $\to$ and $\mathscr{L}_1$ admits a deduction theorem, then $\mathscr{L}_2$ also admits such a theorem.* □

5. The category TrCon

The class constituted by logics and conservative translations between them determines a category, denoted by **TrCon**, which is in fact a subcategory of **Tr**, the bi-complete category constituted by logics and translations.

In this section we show that **TrCon** is a co-complete subcategory relatively to the co-product of **Tr**.

The following result, which asserts that the class **TrCon** of logics and conservative translations is a subcategory of **Tr**, is trivial.

5.1. Proposition. *The composition of conservative translations is a conservative translation. The identity map is a conservative translation. The composition of conservative translations is associative. The identity is a unity for composition.* □

The product and co-product in **Tr** are defined in the following way.

5.2. Definition. Given a family $\{\mathscr{A}_i = (A_i, C_i)\}_{i \in I}$ of logics, the *product logic* of the $\mathscr{A}_i$ is the logic $\mathscr{P} = (P, C_p)$, also denoted by $\mathscr{P} = \pi_{i \in I} \mathscr{A}_i$, such that
(i) $P = \pi_{i \in I} A_i$;
(ii) C_p is the strongest consequence operator in P that makes every projection into every A_i a translation.

5.3. Definition. We say that $\mathscr{S} = (S, C_S)$ is the *sum logic* or *co-product logic* of a family $\{\mathscr{A}_i = (A_i, C_i)\}_{i \in I}$ of logics, when the sets A_i are pairwise disjoint, $S = \amalg_{i \in I} A_i$ is the direct sum of the A_i and C_S is the weakest consequence operator that makes all the inclusions $q_i : A_i \to S$ translations.

5.4. Example. Let $\mathscr{P}$ be the product logic of a family $\{A_i\}_{i \in I}$ of logics. It is trivial that the projections $p_i : P \to A_i$ are not necessarily conservative translations, for every $i \in I$. In fact, let

$\mathscr{P} = \mathscr{A}_1 \times \mathscr{A}_2$ with

$A_1 = \{x_1\}, \quad C_1(\emptyset) = \{x_1\}$ and $C_1(\{x_1\}) = \{x_1\}$,

$A_2 = \{x_2\}, \quad C_2(\emptyset) = \emptyset$ and $C_2(\{x_2\}) = \{x_2\}$,

$A_1 \times A_2 = \{(x_1, x_2)\}, \quad C_p(\emptyset) = \emptyset$ and $C_p(\{x_1, x_2\}) = \{(x_1, x_2)\}$.

In this case we have that, for $(x_1, x_2) \in P$, $p_1(x_1, x_2) = x_1 \in C_1(\emptyset)$, but $(x_1, x_2) \notin C_P(\emptyset)$. By Example 5.4, we conclude that the product in **Tr** is not a product in **TrCon**.

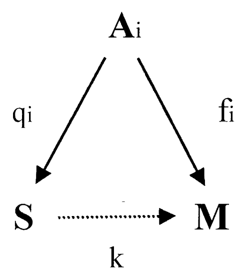
5.5. Proposition. *Let S be the sum of a family $\{\mathscr{A}_i\}_{i \in I}$ of logics. Every inclusion $q_i : A_i \to S$ is a conservative translation.*

Proof. Let $A \cup \{x\} \subseteq A_i$. If $q_i(x) \in C_S(q_i(A))$, as $q_j(y) = y$, for every $y \in A_j$, then $x \in C_S(A)$. Since the A_i are pairwise disjoint, then $C_S(A) \cap A_i = C_i(A)$. Therefore $x \in C_i(A)$. □

5.6. Proposition. *The category* **TrCon** *has co-products.*

Proof. Let $\{\mathscr{A}_i = (A_i, C_i)\}_{i \in I}$ be a family of logics such that, for $i \neq j$, $A_i \cap A_j = \emptyset$; and let $S = \amalg_{i \in I} A_i$, that is, S is the direct sum of the A_i.

As every inclusion $q_i : A_i \to S$ is a conservative translation, we have to show that if $(M, \{f_i\}_{i \in I})$ is such that M is a logic and f_i is a conservative translation, for every $i \in I$, then, according to the following diagram:



there is a unique conservative translation k that makes the diagram commutative.

By Theorem 1.14, there is a unique translation k in the conditions above. So, we only have to prove that such a k is conservative.

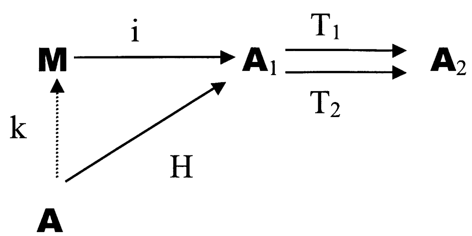
Let $B \cup \{x\} \subseteq S$ and $k(x) \in C_M(k(B))$. As $k(x) = f_j(x) \in C_M(k(B))$, there is one $j \in I$ such that $x \in A_j$. Since every f_j is conservative, then $x \in C_j(A_j) \subseteq C_j(B) \subseteq C_S(B)$. Hence, k is conservative. □

5.7. Proposition. *The category* **TrCon** *has equalizers.*

Proof. We have to prove that, for every pair of logics $\mathscr{A}_1, \mathscr{A}_2$, there is an equalizer of every pair of morphisms between A_1 and A_2.

Given two conservative translations $T_1, T_2 : A_1 \to A_2$ of **TrCon**, let $M = \{x \in A_1 \mid T_1(x) = T_2(x)\}$ and let $i : M \to A_1$ be the inclusion function, with $\mathscr{M} = (M, C_M)$ the logic induced by A_1 and i.

We will prove that the pair $(\mathscr{M}, i)$ is the equalizer of T_1 and T_2, that is, we will prove that, if $H : A \to A_1$ is a conservative translation such that $T_1 \circ H = T_2 \circ H$ then there is only one morphism k of **TrCon** that makes the following diagram commutative.



As T_1, T_2 and H are translations then, by Theorem 1.14, there is only one translation k that makes the diagram commutative. This function $k : A \to M$ is well defined by $k(x) = H(x)$ and is such that $H = i \circ k$.

Now, as T_1, T_2 and H are conservative, let's see that i and k are also conservative translations.

For every $B \subseteq M$, if $i(x) \in C_1(i(B))$ then $x \in C_1(B) \cap M$, that is, $x \in C_M(B)$. Therefore i is conservative.

For $D \subseteq A$, let $k(x) \in C_M(k(D))$. Since $k(x) = H(x) \in C_1(H(D))$, as H is conservative, we have that $x \in C_A(D)$. Then k is conservative.

The uniqueness of k is already guaranteed by Theorem 1.14. □

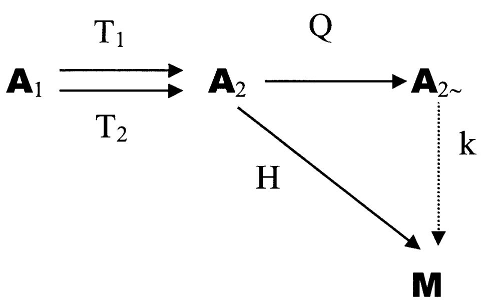
5.8. Definition. Let $\mathscr{A}$ be a logic and $\equiv$ an equivalence relation on A. The function $Q : A \to A_{/\equiv}$, given by $Q(x) = [x]$, is said to be the *quotient mapping* of to the relation $\equiv$. If $C_\equiv$ is the consequence operator co-induced by A and Q, then the pair $\mathscr{A}_\equiv = (A_{/\equiv}, C_\equiv)$ is the logic co-induced by $\mathscr{A}$ and Q.

5.9. Proposition. *The category* **TrCon** *has co-equalizers.*

Proof. We have to show that every pair T_1, T_2 of morphisms between every two logics $\mathscr{A}_1$ and $\mathscr{A}_2$ has co-equalizer.

Given two conservative translations $T_1, T_2 : A_1 \to A_2$ of **TrCon**, let $N = \{(T_1(x), T_2(x) \mid x \in A_1\}$ and let $\sim$ be the smallest equivalence relation in A_2 that contains N. Considering $Q : A_2 \to A_{2/\sim}$, let $\mathscr{A}_{2\sim} = (A_{2/\sim}, C_\sim)$ be the logic co-induced by A_2 and Q.

We will prove that the pair $(A_{2\sim}, Q)$ is the co-equalizer of T_1 and T_2, that is, we will prove that, if $H : A_2 \to M$ is a conservative translation such that $H \circ T_1 = H \circ T_2$ then there is only one morphism k of **TrCon** that makes the following diagram commutative.



As T_1, T_2 and H are translations then, by Theorem 1.14, there is only one translation k that makes the diagram commutative. This function $k : A_{2\sim} \to M$ is well defined by $k([x]) = H(x)$ and is such that $H = k \circ Q$.

Now, as T_1, T_2 and H are conservative, let us see that Q and k are conservative.

Q is a translation. For every $B \subseteq A_2$, if $Q(B) = \{[x] \mid x \in B\}$ and $Q(x) \in C_\sim(Q(B))$ then, for some $y \in [x]$, we have that $y \in C_2(B)$. As $C_2(y) = C_2(x)$, $x \in C_2(y)$ and therefore $x \in C_2(B)$, that is, Q is conservative.

Now, let $[y] \cup [D] \subseteq A_{2\sim}$. If $k([y]) \in C_M(k([D]))$, as $H(y) = k([y])$, then $H(y) \in C_M(k([D])) = C_M(H(D))$. As H is conservative, then $y \in C_2(D)$. Therefore, by $Q, [y] \in C_\sim([B])$ and hence k is conservative.

The uniqueness of k follows from the proof of Theorem 1.14. □

5.10. Theorem. *The category* **TrCon** *is a co-complete subcategory of* **Tr**, *with the co-product of* **Tr**. □

The results of this section characterize the subcategory **TrCon** of **Tr** as a category of special interest. **Tr** is bi-complete, but **TrCon** is not, for, despite having equalizers, it does not have product.

6. A fundamental result

In this section we prove a very important result.

The last theorem will give us a necessary and sufficient condition for the existence of a conservative translation between two logics. This condition, that corresponds to the existence of a conservative translation between specific quotient structures associated to the logics, was fundamental to obtain several conservative translations we have studied.

6.1. Definition. If, A and B are sets, we say that $F : A \to B$ is *compatible* with an equivalence relation $\equiv$ on A, when $x_1 \equiv x_2$ implies $F(x_1) = F(x_2)$.

6.2. Proposition. *Let A_1 and A_2 be logics and $T : A_1 \to A_2$ a translation. If T is compatible with an equivalence relation $\equiv$ on A_1, then there is a unique function $T^* : A_{1\equiv} \to A_2$ such that $T^* \circ Q = T$, where Q is the quotient mapping on A_1. The function T^* is a translation.*

Proof. We define $T^* : A_{1\equiv} \to A_2$ by $T^*([x]) = T(x)$.

The function T^* is well defined, given the compatibility of T. Besides, $T^* \circ Q(x) = T^*([x])$, that is, $T^* \circ Q = T$.

Now, let $F : A_{1\equiv} \to A_2$ be another map such that $T = T^* \circ Q = F \circ Q$. As Q is surjective, then it admits an inverse to the right and so $T^* = T^* \circ (Q \circ Q^{-1}) = F \circ (Q \circ Q^{-1}) = F$.

In order to verify that T^* is a translation, let B be a closed set of A_2. As T is a translation, then $T^{-1}(B) = (T^* \circ Q)^{-1}(B) = Q^{-1} \circ (T^*)^{-1}(B)$ is a closed set of A_1. Hence, as Q is the quotient mapping, it follows that $(T^*)^{-1}(B)$ is a closed set of $A_{1\equiv}$. □

6.3. Proposition. *Let A_1, A_2 and A_3 be logics and $T_1 : A_1 \to A_2$ and $T_2 : A_2 \to A_3$ functions*:

(i) *If A_2 is the logic co-induced by A_1 and T_1, then T_2 is a translation if, and only if, $T_2 \circ T_1$ is a translation*;

(ii) *If* A_2 *is the logic induced by* A_3 *and* T_2, *then* T_1 *is a translation if, and only if,* $T_2 \circ T_1$ *is a translation.* □

Given a logic $\mathscr{A} = (A, C)$, let us consider the following relation defined on A:

$$x \sim y \Leftrightarrow C(x) = C(y).$$

It is trivial that $\sim$ is an equivalence relation on A.

By Proposition 1.7, we have that the quotient mapping $Q : A \to A_\sim$, given by $Q(x) = [x] = \{y \mid x \sim y\}$, with $A_\sim$ the logic co-induced by A and Q, is a translation.

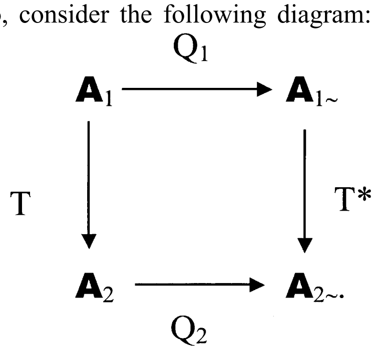
6.4. Proposition. *The map* $Q : \mathscr{A} \to \mathscr{A}_\sim$ *is a conservative translation.*

Proof. If $Q(B) = \{[x] \mid x \in B\}$ and $Q(x) \in C_\sim(Q(B))$ then, for some $y \in [x]$, we have that $y \in C(B)$. As $C(y) = C(x)$, if follows that $x \in C(y)$ and so $x \in C(B)$. □

6.5. Theorem. *Let* $\mathscr{A}_1$ *and* $\mathscr{A}_2$ *be logics, with the domain* A_2 *of* $\mathscr{A}_2$ *denumerable; and let* $\mathscr{A}_{1\sim_1}$ *and* $\mathscr{A}_{2\sim_2}$ *be the logics co-induced by* $\mathscr{A}_1$, Q_1 *and* $\mathscr{A}_2$, Q_2 *respectively, with* $\sim_1$ *and* $\sim_2$ *being equivalence relations defined according to the previous definition on* A_1 *and* A_2, *respectively. In this conditions there is a conservative translation* $T : A_1 \to A_2$ *if, and only if, there is a conservative translation* $T^* : A_{1\sim_1} \to A_{2\sim_2}$.

Proof. Let $T : A_1 \to A_2$ be a conservative translation. As Q_2 is a conservative translation, then $Q_2 \circ T$ is a conservative translation. If $x \sim_1 y$, for $x, y \in A_1$, then $C_1(x) = C_1(y)$. As T is a translation, it follows that $C_2(T(x)) = C_2(T(y))$, so $Q_2(T(x)) = Q_2(T(y))$, and then $Q_2 \circ T$ is compatible with the equivalence relation $\sim_1$. By Proposition 6.2 there is a unique translation $T^* : A_{1\sim_1} \to A_{2\sim_2}$ such that $T^* \circ Q_1 = Q_2 \circ T$.

So, consider the following diagram:



Now, suppose that T^* is not conservative. As Q_1 is surjective, then there is $T^*(Q_1(y)) \in C_{2\sim_2}(T^*(Q_1(B))$, for $B \subseteq A_1$, such that $Q_1(y) \notin C_{1\sim_1}(Q_1(B))$, which implies that $y \notin C_1(B)$ since Q_1 is conservative. As $T^* \circ Q_1 = Q_2 \circ T$, then $Q_2(T(y)) \in C_{2\sim_2}(Q_2(T(A))$ and, since $Q_2 \circ T$ is conservative, it follows that $y \in C_1(B)$, a contradiction. Hence, T^* is conservative.

On the other hand, let T^* be a conservative translation, and let $A_2 = \{y_1, y_2, \ldots\}$ be an enumeration of A_2. We define $T : A_1 \to A_2$ by $T(x) = y$, such that $y \in Q_2^{-1} \circ T^* \circ Q_1(x)$ and y has the smallest index in A_2.

We verify that T is a translation. Let B be a closed set in A_2. As Q_2 is conservative and surjective, by Corollaries 3.4 and 1.17, $Q_2(B)$ is a closed set in $A_{2\sim_2}$. So, since $T^* \circ Q_1$ is a translation, $[Q_2^{-1} \circ T^* \circ Q_1]^{-1}(B) = [T^* \circ Q_1]^{-1}(Q_2(B))$ is a closed set in A_1, and then T is a translation. Besides, $Q_2 \circ T = T^* \circ Q_1$, for $Q_2 \circ Q_2^{-1} = I_{A_{2\sim_2}}$.

Now, suppose that T is not conservative. Then we have $B \cup \{x\} \subseteq A_1$, such that $T(x) \in C_2(T(B))$ and $x \notin C_1(B)$. But, if $T(x) \in C_2(T(B))$ then $Q_2(T(x)) \in C_2(Q_2(T(A)))$ and, as $Q_2 \circ T = T^* \circ Q_1$, it follows that $T^*(Q_1(x)) \in C_2(T^*(Q_1(B)))$. As $T^* \circ Q_1$ is conservative, we have that $x \in C_1(B)$, a contradiction. □

6.6. Corollary. *If the function T^* of the previous theorem exists, then it is injective.*

Proof. If $T^*(Q_1(x)) = T^*(Q_1(y))$, then $Q_2(T(x)) = Q_2(T(y))$ and so $T(x) \sim_2 T(y)$, that is, $C_2(T(x)) = C_2(T(y))$. As T is a conservative translation, then $C_1(x) = C_1(y)$, that is, $x \sim_1 y$. Hence, $Q_1(x) = Q_1(y)$, that is, $[x] = [y]$. □

We observe that the denumerability of A_2 in the hypothesis of Theorem 6.5 is not necessary, if we explicitly use the Axiom of Choice in the proof of the theorem.

Based on Theorem 6.5 and Corollary 6.6, the use of the Lindenbaum algebras associated to logical systems, whenever defined, was shown to be a fundamental method to establish translations or to determine the existence of conservative translations between the systems we have studied.

In [3] we present some conservative translations involving classical logic and the many-valued logics of Lukasiewicz and Post.

In [4] we introduce some conservative translations involving classical logic, Lukasiewicz's three-valued system L_3, the intuitionistic system I^1 introduced by Sette and Carnielli [19] and several paraconsistent logics, as for instance Sette's system P^1, the D'Ottaviano and da Costa system J_3 and da Costa's systems $C_n, 1 \leqslant n \leqslant w$ (see [2]).

Feitosa [6] studies the problem, several times mentioned in the literature, of the existence of conservative translations from intuitionistic logic into classical logic, that will appear in a forthcoming paper.

In most of these results, conservative translations are obtained through the algebraic semantics that correspond to the logical systems under consideration.

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