# Addendum to "Einstein's "Zur Electrodynamik..." (1905) Revisited, with some Consequences"[1] by S. D. Agashe

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An omission in[1] may cause ambiguity in some of the statements made there. The paper referred to, but did not quote from, Menger[2]. Menger gives the following two theorems (with a different notation) which are relevant to the problem there.

"Satz 8: For (n+1) points $p_1, \ldots, p_{n+1}$ to be embeddable in $\mathbb{R}^n$ with preservation of distance, it is necessary and sufficient that every set of $n$ points be embeddable in $\mathbb{R}^{n-1}$ and that sign $D(p_1, \ldots, p_{n+1}) \neq$ sign $(-1)^n$."

Here, $D$ is the following $(n+2) \times (n+2)$ determinant:

$$D(p_1, p_2, \ldots, p_{n+1}) = \begin{vmatrix} 0 & 1 & 1 & \ldots & 1 \\ 1 & 0 & (p_1\, p_2)^2 & \ldots & (p_1\, p_{n+1})^2 \\ 1 & (p_2\, p_1)^2 & 0 & \ldots & (p_2\, p_{n+1})^2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & (p_{n+1}\, p_1)^2 & (p_{n+1}\, p_2)^2 & \ldots & 0 \end{vmatrix},$$

$(p_i\, p_j)$ denotes the distance between points $p_i$ and $p_j$, $D \geq 0$ if sign $D \neq$ sign $(-1)$, and $D \leq 0$ if sign $D \neq$ sign $(+1)$.

"Theorem II: For a semi-metric space $R$ to be embeddable with preservation of distance in $\mathbb{R}^n$ $(n \geq 0)$, it is necessary and sufficient that the following conditions hold:

$B^n_{n+2}$: For every $(n+2)$ points of $R$, $D(p_1, \ldots, p_{n+2}) = 0$;

$B_k$: For every $k \leq (n+2)$, and for every $k$ points of $R$,sign $D(p_1, \ldots, p_k) \neq$ sign $(-1)^{k-1}$."

Menger uses a formula given by R. H. Schouten for the $n$-dimensional volume $V$ of an "$n+1$

point simplex":

$$V^2 = \frac{(-1)^{n+1}}{2^n(n!)^2}D.$$

In the 3-dimensional case, it can be reduced to Cayley's formula. Whittaker[4] says in passing (without citing any reference): "$\cdots$ by suitably disposing the paticles we can arrange for the six distances to take any values we please, subject only to some inequalities which we shall not trouble about".

In Section 3.4.2 of [1], to obtain the counterpart of I.22 in 3-dimensional geometry, by Satz 8, in addition to the triangle inequalities, one needs the following inequality to be satisfied by the 6 lengths AB,AC,AD,BC,BD,CD (base ABC, vertex D):

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & \mathrm{AB}^2 & \mathrm{AC}^2 & \mathrm{AD}^2 \\ 1 & \mathrm{AB}^2 & 0 & \mathrm{BC}^2 & \mathrm{BD}^2 \\ 1 & \mathrm{AC}^2 & \mathrm{BC}^2 & 0 & \mathrm{CD}^2 \\ 1 & \mathrm{AD}^2 & \mathrm{BD}^2 & \mathrm{CD}^2 & 0 \end{vmatrix} \geq 0. \tag{1}$$

(A visualizable statement equivalent to this inequality is given at the end of this Addendum.)

In Section 4 of [1] also, therefore, a corresponding assumption needs to be made regarding the 6 inter-station distances of the 4 stations $s_0, s_1, s_2, s_3$ of the system S, namely, $d_1, d_2, d_3, d_{12}, d_{23}, d_{31}$.

In Section 5.3 of [1], fortunately, the assumption that the stations $\sigma_0, \sigma_1, \sigma_2, \sigma_3$ of the system $\Sigma$ form a non-degenerate tetrahedron in S with distances in S, and, therefore, satisfy the above determinantal inequality, implies that they satisfy the above determinantal inequality with distances in $\Sigma$, because of the relations between the distances in the two systems.

Incidentally, the problem of determining a relation between the "clocks" of the two systems S and $\Sigma$ remains even if $\Sigma$ is stationary relative to S.

Theorem II could be used to obtain necessary and sufficient conditions for representability of additional points in Section 3.4.3 and in Section 4 of [1]. But, in Section 4, the vector $\boldsymbol{p}$ representing the position of a distant event was to be computed, which is possible if these conditions are satisfied.

In order to give a visualizable statement equivalent to the inequality (1) above, some terminology and notation is necessary. Let $a, b, c$ denote the three edges of one triangular face, say, the base of the tetrahedron, and let $a', b', c'$ denote the other three edges opposite to, or skew with, or non-intersecting with, $a, b, c$ respectively. The tetrahedron has four triangular faces, formed by edge triples $\{a, b, c\}$, $\{a', b', c\}$, $\{a', b, c'\}$, and $\{a, b', c'\}$, three skew-edge pairs $\{a, a'\}$, $\{b, b'\}$, $\{c, c'\}$, four co-terminal edge triples $\{a', b', c'\}$, $\{a, b, c'\}$, $\{a, b', c\}$, $\{a', b, c\}$, and twelve non-triangular non-co-terminal triples. For each of the triangular faces, take the product of the squares of the three edges;

let these be called triangular products. For each of the twelve non-triangular non-co-terminal edge triples, take products of the squares of the three edges; let these be called the non-triangular non-co-terminal products. Finally, for each of the skew-edge pairs, take the product of the squares of the two skew edges with the sum of their squares; let these be called the skew products. Here, then, is a visualizable statement:

A necessary and sufficient condition that the tetrahedron be embeddable in $\mathbb{R}^3$ is that the sum of the three skew products and the four triangular products is less than or equal to the sum of the twelve non-triangular non-co-terminal products.

Another, slightly better, and also visualizable, necessary and sufficient condition is, in symbols:

$$(abc + a'b'c + a'bc' + ab'c')^2 \leq (aa' + bb' + cc') \times$$
$$\times \{(c^2 + c'^2)(aa' + bb' - cc') + (a^2 + a'^2)(bb' + cc' - aa') + (b^2 + b'^2)(cc' + aa' - bb')\}.$$

The inequalities

$$(aa' + bb' - cc') \geq 0, \quad (bb' + cc' - aa') \geq 0, \quad (cc' + aa' - bb') \geq 0,$$

are sometimes referred to as "tetrahedron inequalities"[3]. They are necessary, but not sufficient. Incidentally, the above two forms can also be used to give a formula for the volume of a tetrahedron.

## REFERENCES


[1] S. D. Agashe, "Einstein's "Zur Electrodynamik..." (1905) revisited, with some consequences," *Found. Phys.* **36**, 955-1011 (2006).

[2] Karl Menger, "Untersuchungen über allgemeine Metrik," *Math. Ann.* **100**, 75-163 (1928).

[3] J. A. Scott, "On the tetrahedron inequality," *Math. Gaz.* **89**, 96-99 (2005).

[4] Edmund Whittaker, *From Euclid to Eddington, a Study of Conceptions of the External World* (Cambridge University Press, Cambridge, U.K., 1949), p. 7.