Fair student placement

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Abstract We revisit the concept of fairness in the Student Placement framework. We declare an allocation as α-equitable if no agent can propose an alternative allocation that nobody else might argue to be inequitable. It turns out that α-equity is compatible with efficiency. Our analysis fills a gap in the literature by giving normative support to the allocations improving, in terms of efficiency, the Student Optimal Stable allocation.

Keywords Student placement · Fair matching

1 Introduction

This paper explores the trade-off between efficiency and equity in the Student Placement problem and provides a new 'compromise' solution for this family of models that always select, at least, one allocation.

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Student Placement mechanisms were modeled in Balinski and Sönmez (1999), inspired by the two-sided problems introduced by Gale and Shapley (1962). These authors explore how school seats should be distributed among the students. A specific, and thus differential, characteristic of this problem is that schools are modeled so as to capture social conventions on commonly accepted primitives of equity. These social agreements are captured in the description of how schools prioritize the different students.[1] The connection between Student Placement and two-sided matching problems has inspired some authors to propose allocation systems to improve upon the ones that were established in some geographical areas (Abdulkadiroğlu and Sönmez 2003) and/or to identify the problems that current Student Placement systems might exhibit, and thus propose how to avoid them.

One of the main problems faced by the Student Placement systems is the existence of an equity–efficiency trade-off (see, e.g., Example 1 in Abdulkadiroğlu and Sönmez 2003). The persistence of this conflict is reflected in the proposal of two (incentive compatible) procedures to distribute the available seats among the newcomers. The first one, named the Top Trading Cycles mechanism, TTC hereafter, ensures allocative efficiency at the expense of equity. The second procedure, known as the Student Optimal Stable mechanism, seeks to ensure equity at the cost of reducing the efficiency of the allocation. Since Abdulkadiroğlu and Sönmez (2003) the literature has proposed a few ways to reduce the relevance of such a conflict. In parallel with this normative approach, it has become commonly known that any attempt to reduce the efficiency–equity collision conflicts with the design of incentive compatible mechanisms (see, e.g., Kesten 2010, Proposition 1).

In this matter, Kesten (2010) resorts to the idea of 'consent' by some students to avoid the potential presence of some inefficiency. The interpretation of Kesten's consent is very related to the algorithm he describes to reduce the relevance of this trade-off between the two properties above, namely efficiency and equity. Under the Efficiency-Adjusted Deferred Acceptance algorithm, EADAM hereafter, the students may consent to waive her priority over some schools. A waiver by some student holds whenever two conditions are fulfilled. The first one is that, by waiving the priority, she does not hurt herself; the second one is that by waiving, the assignment of other students improves. This idea of consent has been also employed by Tang and Yu (2014) to introduce an algorithm that is computationally simpler than the EADAM. As Kesten (2010) and Tang and Yu (2014) show, their algorithms select efficient allocations that minimize the presence of inequity when all the students consent to waive their priorities. The lack of a rationale behind the consenting process prevents EADAM to substitute the role of a fairness concept overcoming the efficiency–equity trade-off.

Recently, Morrill (2015) has concentrated on procedural equity, rather than allocative equity. His equity notion can be described as follows: imagine that we are employing a specific mechanism. As a consequence, Abel obtains a seat at School 1. Then, Beth argues that it is unfair because she prefers to be allocated at School 1

[1] The main aspects affecting these priorities are (1) the distance between each student's residence and where the school is placed; and (2) the number of siblings attending the school (see, e.g., Abdulkadiroğlu and Sönmez 2003).

rather than being at her actual school. Beth's objection is disregarded whenever there is another student who might obtain the seat Abel was assigned to by misreporting her preferences. Morrill (2015) proves that the TTC is the only mechanism that is strategy-proof, efficient and fulfills his equity notion (see Morrill 2015, Definitions 1 and 2). In Morrill (2015), the conditions that allow an objection—to an allocation—to be admissible is very related to the procedure used to select the allocation. This induces that some of the placements sanctioned as just are hardly identifiable as equitable (see Example 2).

In a framework close to ours, Abdulkadiroğlu and Sönmez (1998) propose an alternative way to circumvent the efficiency–equity trade-off. They exploit the fact that some students declare preferences for schools that are not in their area of residential priority. They propose the following procedure to combine efficiency and equity: first, to ensure efficiency, they allow students to sequentially select their preferred school among those with vacant positions. Then, since this procedure is very dependent on the ordering in which the students make their choices—and thus very inequitable—select the ordering in which the selections are made at random, by drawing a fair raffle among all the possible orderings. This mechanism, known as the Random Serial Dictatorship, combines ex-post efficiency and ex-ante equity. Nevertheless, as pointed out by Bogomolnaia and Moulin (2001), the Random Serial Dictatorship mechanism fails to be ex-ante efficient. Furthermore, Bogomolnaia and Moulin (2001) establish the incompatibility of the two appealing normative properties from an ex-ante perspective.

Our approach in the present paper departs from the above-mentioned analysis. Our aim is not to propose a systematic way to select efficient allocations that are not questionable in terms of equity.[2] Our objective is to find a weakening of equity that turns out to be compatible with efficiency.

To describe how we reach our target it might be relevant to go back to the origins of this literature. As we pointed out, the growth of the literature on Student Placement is related to the analysis of two-sided matching mechanisms. The idea of justified envy, as defined by Balinski and Sönmez (1999), is borrowed from the notion of pairwise stability introduced by Gale and Shapley (1962). This connection invites to explore how some classical ideas of stability, weaker than that of the core, might be re-interpreted in terms of weak equity in the Student Placement framework to elude the disconnection between efficiency and (weak) equity. Therefore, we discuss how to state whether an individual's objection is admissible.[3]

In this paper, we propose a notion of weak equity that follows the idea of absence of envy introduced by Foley (1967).[4] To illustrate whether an allocation is weakly equitable or not, let us assume that this allocation has been proposed. A student can

[2] For completeness we will describe, in the Appendix, one of such procedures.

[3] This approach, introduced by Aumann and Maschler (1964), is in the origin of a wide literature studying different weak notions of stability in co-operative games. In particular, the logic process described in Zhou (1994) to define its Bargaining Set is the closest to our notion of equity. Related to stability notions in marriage problems, Klijn and Massó (2003) propose a weak stability notion also inspired by the Zhou's Bargaining Set.

[4] Remark 1 discusses the main differences between justified envy, as defined by Balinski and Sönmez (1999), and our weak equity.

object to this allocation by proposing an alternative allocation assigning her to a school where she has priority (with respect to some of the students previously assigned to that school). Then, the society should evaluate whether the objection is accepted or not. The proposal is defeated, and thus the objection is not accepted, whenever some student can present a new objection to the previous one formulated in the same terms. Otherwise, the objection is admissible. We consider that an allocation is almost-equitable, or α-equitable henceforth, whenever no student is able to object to it, by proposing an alternative allocation which cannot be objected in the same terms. We prove, Theorem 1, the existence of efficient, α-equitable allocations. Recall that under equity, as defined by Balinski and Sönmez (1999), such an existence cannot be granted. A further question that we deal with is how strong our equity property is. In Theorem 2 we show that α-equity is a weaker condition of equity, which overcomes the trade-off with efficiency: whenever we restrict to efficient allocations, α-equity and equity coincide unless no equitable allocation exists.

The remaining of the paper is organized as follows: Sect. 2 introduces the basic model. α-equity is defined in Sect. 3, which also contains our main results. Finally, Sect. 4 concludes. The proofs are gathered in the Appendix.

2 The framework

A group of n students has to be distributed among m different schools. The set $I = \{1, \ldots, i, \ldots, n\}$ stands for the set of students, whereas $S = \{s_1, \ldots, s_j, \ldots, s_m\}$ describes the set of schools. We consider the existence of an outside option s_o, which is interpreted by each student as the possibility of not attending any school in S.

Each student, say i, orders the schools according her own preferences $\succ_i$ which describes a linear preordering in $S \cup \{s_o\}$. $\succsim_i$ represents i's weak preferences, i.e., $s_j \succsim_i s_h$ denotes that either s_j is preferred to s_h under i's preferences or $s_j = s_h$.

Each school, say $s_j \in S$, has a fixed capacity $q_j \geq 1$ denoting its number of available seats, and interpreted as the maximum of students it can enroll. To guide potential admissions procedures, each school determines how to prioritize students through a linear preordering in I. P_j denotes the priority list by s_j.[5]

A Student Placement problem, or simply a problem, can be synthetically described as $\mathbb{P} \equiv \{(I, \succ); (S, P, Q)\}$, where $\succ \equiv (\succ_i)_{i \in I}$ is the preferences profile; $P \equiv (P_j)_{s_j \in S}$ is the vector describing each school priority list; and $Q \equiv (q_j)_{s_j \in S}$ summarizes the capacities for each school.

A solution to our problem, or matching, is a distribution of the seats among the students. It is formally described through a correspondence $\mu \colon I \cup S \twoheadrightarrow I \cup S \cup \{s_o\}$ such that

(a) for each $i \in I$, $\mu(i) \in S \cup \{s_o\}$;
(b) for each $s_j \in S$, $\mu(s_j) \subseteq I$, and $|\mu(s_j)| \leq q_j$; and
(c) for each $i \in I$ and $s_j \in S$, $\mu(i) = s_j$ if, and only if, $i \in \mu(s_j)$.

[5] For completeness, we consider that the number of seats that the outside option has is $q_o \geq n$. Since s_o can enroll all the students, there is no need of describing any particular priority list.

The central normative properties, related to a matching, in the Student Placement framework are efficiency and equity, as described below.

Given a problem $\mathbb{P}$ and two different allocations for it, say μ and μ', we say that μ dominates μ' whenever $\mu(i) \succ_i \mu'(i)$ for each student i whose allocation differs. μ is *efficient* if there is no matching dominating it.

In the Student Placement framework, the notion of equity follows the original description by Foley (1967), in the sense that it is required that no student is envied by anyone else. The existence of justifiable envy requires the coincidence of two facts. The first one is the presence of some student expounding her envy. The second one is the consent of the school proposed by this student.

We say that student i envies i' at matching μ whenever there is a school, say $s_j \in S$, such that $s_j = \mu(i')$ and

$$s_j \succ_i \mu(i). \tag{1}$$

This envy is justifiable whenever Condition (1) is fulfilled and

$$i \; P_j \; i'. \tag{2}$$

Matching μ is *equitable* if no student is justifiably envied by someone else.

Finally, we say that matching μ is *fair* whenever it is both efficient and equitable. The set of fair allocations for problem $\mathbb{P}$ is denoted as $\mathscr{F}(\mathbb{P})$.

Remark 1 The notion of equity we introduce below differs from that proposed by Balinski and Sönmez (1999). These authors also consider two additional sources of inequity of allocation μ:

(i) The absence of individual rationality from some student's point of view, i.e. the presence of some i such that $s_o \succ_i \mu(i)$; and
(ii) The existence of an unoccupied seat at some desired school, i.e. the existence of some $s_j \in S$ and $i \in I$ such that $|\mu(s_j)| < q_j$ and $s_j \succ_i \mu(i)$.

Note that the two possibilities above are related to the lack of efficiency rather than to the presence of envy between students. Nevertheless, given that we are interested in allocations combining efficiency and equity, our conception of fairness coincides with that of Balinski and Sönmez (1999).

The existence of efficient or equitable allocations is easily granted. In particular, the TTC algorithm (Abdulkadiroğlu and Sönmez 2003) always produces efficient allocations and the Deferred Acceptance algorithm (Gale and Shapley 1962) associates an equitable matching, known as the Student Optimal Stable matching and denoted as μ^{SO}, to each Student Placement problem.

Unfortunately, there are instances where efficiency and equity become incompatible, as pointed out in the next example.

Example 1 Consider the following problem involving three students and three schools. $I = \{1, 2, 3\}$, $S = \{a, b, c\}$. Each school has one available seat, i.e. $q_j = 1$ for each school. Preferences and priorities are described in the next table.

$\succ_1$	$\succ_2$	$\succ_3$
a	c	c
c	a	a
b	b	b

P_a	P_b	P_c
3	2	1
2	1	3
1	3	2

This problem has four efficient matchings:

	$\succ_1$	$\succ_2$	$\succ_3$
μ^A	a	b	c
μ^B	a	c	b
μ^C	b	a	c
μ^D	b	c	a

Note that none of these matchings is equitable. This is because,

(a) at matching μ^A student 2 justifiably envies 1;
(b) at matching μ^B student 3 justifiably envies 1;
(c) at matching μ^C student 1 justifiably envies 3; and
(d) at matching μ^D student 1 justifiably envies 2.

Therefore, this problem has no fair allocation.

3 Almost-equitable allocations

As we mention in Example 1 above, the problem we considered has no fair allocation. In particular matching μ^A fails to be equitable. The reason is that student 2 justifiably envies 1. We can interpret 2's objection to μ^A as the proposal of an alternative allocation, say μ^E, fulfilling two properties. First, 2 prefers the school assigned to her under μ^E rather than the one she obtains under μ^A. Second, the aspiration by 2 is supported by the school assigned under μ^E as described below. This objective can be reached by describing μ^E as $\mu^E(1) = b$, $\mu^E(2) = a$ and $\mu^E(3) = c$.

Therefore, we can reconsider how students challenge an allocation to be implemented. In such a case an objection of a student to a matching requires the proposal of an alternative allocation. This new proposal must be supported by the school. Then we say that the student objects to the initial allocation through the alternative one. We only consider the objections that are based on the lack of equity. This allows us to redefine equity as the absence of ε-objections, namely objections justified by the lack of equity.

Given a problem $\mathbb{P}$ and a matching μ we say that student i objects in terms of equity to μ through μ' whenever there is a school $s_j \in S$ such that (1) $\mu'(i) = s_j$, (2) $s_j \succ_i \mu(i)$ and (3) $i \ P_j \ i'$ for some student $i' \in \mu(s_j)$. In such a case we say that $(i; \mu')$ constitutes an ε-objection to μ by student i.

Now, we are reconsidering the arguments above, related to allocation μ^A in Example 1. This matching can be taken, in the absence of admissible ε-objections, as a default. The question that we deal with is how to determine whether an ε-objection qualifies as admissible or not. According to Zhou (1994) we say that a student's objection is admissible whenever no one else wishes to ε-object to this student's proposal.

For a given problem $\mathbb{P}$ and matching μ we say that an ε-objection $(i; \mu')$ is admissible whenever there is no student $i' \neq i$ and matching μ'' such that i' objects in terms of equity to μ' through μ''. Otherwise, we say that $(i'; \mu'')$ constitutes an ε-counter-objection to $(i; \mu')$.

We can now give a formal definition of what an almost equitable allocation is.

Given a problem $\mathbb{P}$, we say that matching μ is α-*equitable* if there is no (i, μ') constituting an admissible ε-objection to μ.

An efficient allocation that is also α-equitable is said α-*fair*. $\mathscr{F}^{\alpha}(\mathbb{P})$ denotes the set of α-fair allocations for problem $\mathbb{P}$.

Remark 2 Note that, for a given problem $\mathbb{P}$ each equitable allocation is also α-equitable. The opposite is, in general, not true. For instance, matching μ^A in Example 1 is α-equitable but it fails to be equitable. Note that the unique student that can exert an ε-objection to μ^A is 2, who exhibits justifiable envy to 1. Therefore, when objecting to μ, 2 must propose a matching μ' such that $\mu'(2) = a$. Now, we consider the different possibilities related to μ'.

1. $\mu'(3) \neq c$. Then, at μ', 3 justifiably envies 2. Therefore, for any μ'' such that $\mu''(3) = a$, $(3; \mu'')$ constitutes an ε-counter-objection to $(2; \mu')$.
2. $\mu'(3) = c$. Then, at μ', 1 justifiably envies 3. Then, for any μ'' with $\mu''(1) = c$ we have that $\left(1; \mu''\right)$ constitutes an ε-counter-objection to $(2; \mu')$.

Observe that Remark 2 above also suggest that the efficiency of an allocation might be compatible with its α-equity. This is asserted in Theorem 1 below.

Theorem 1 *Each problem $\mathbb{P}$ has at least one α-fair allocation.*

The proof of the result above is addressed in the Appendix. It is built in a constructive way that exhibits some similarities with Varian's proof of the existence of a fair allocation in distributive economies (see Varian 1974). Our constructive proof can be synthesized as follows: first, compute an equitable—and thus α-equitable—allocation. Select the Student Optimal Stable allocation. Once each student is allocated a seat, this can be interpreted as her initial endowment in an exchange economy. Since money does not play an active role in our model, this economy is reinterpreted in terms of a housing market (Shapley and Scarf 1974). By identifying an equilibrium in this market, we obtain an efficient allocation. This is done through the application of the Gale's Algorithm.[6] We finally show that this efficient matching is also α-equitable.

It is well known that for any given problem $\mathbb{P}$ the set of its fair allocations is either a singleton or the empty set. As Theorem 1 reports, there is always an allocation fulfilling α-equity. A way to measure how much the equity requirement has been relaxed, when adopting α-equity instead of equity, comes from comparing the sizes of the two sets of fair and α-fair allocations, when the former is non-empty. This comparison is straightforwardly derived from our Theorem 2 below.

[6] This algorithm was introduced in Shapley and Scarf (1974) under the name of Top Trading Cycle algorithm. Since Abdulkadiroğlu and Sönmez (2003) refers a similar, but different algorithm by using the same name, we prefer to call it Gale's algorithm to avoid confusion.

To be precise, Theorem 2 allows describing the set of α-fair allocations that each problem exhibits as the efficient matchings dominating the Student Optimal Stable matching, except when such an allocation is efficient itself.

Theorem 2 *Let $\mathbb{P}$ be a problem. Matching μ is an α-fair allocation for $\mathbb{P}$ if, and only if, μ is efficient and for each student, say i, $\mu(i) \succsim_i \mu^{SO}(i)$.*

The proof of the above result is relegated to the Appendix.

We conclude this section by introducing two direct implications from Theorem 2 above. The first one, Corollary 1, establishes that the Student Optimal Stable matching is the unique allocation (if any) combining efficiency and equity. The second one, Corollary 2, reports that when a fair matching exists, the sets of fair and α-fair allocations coincide. Notice that this implies, in particular, that even though α-fairness is weaker than fairness, it coincides with standard fairness on a restricted domain (i.e. the domain where the standard equity and efficiency properties are compatible).

Corollary 1 *For each problem $\mathbb{P}$, $\mathscr{F}(\mathbb{P}) \subseteq \{\mu^{SO}(\mathbb{P})\}$.*[7]

Corollary 2 *For each problem $\mathbb{P}$ such that $\mu^{SO}(\mathbb{P})$ is efficient, $\mathscr{F}(\mathbb{P}) = \mathscr{F}^{\alpha}(\mathbb{P})$.*

4 Concluding remarks

In this paper, we explored the compatibility of weak notions of equity with allocative efficiency in the Student Placement problem. Our first concern is to adapt the classical proposal by Foley (1967) that identifies equity of an allocation with the absence of envy, i.e. no student envies someone else's seat.

Our approach distinguishes between objections in terms of Pareto improvements and those that are justified because of merely inequality aspects. This distinction simplifies a proper normative analysis based on efficiency and/or equity from an allocative perspective.

Since efficiency and equity are two appealing properties, but incompatible in our framework, we investigate how the equity notion might be relaxed to circumvent such incompatibility. By adapting the idea of 'resentment' (see Rawls 1971, pg. 533), we propose a notion of 'almost'-equity that can be justified as follows: when a student claims that the allocation fails to be equitable, she must propose an alternative allocation. The new proposal must match two natural properties. The first one is that the claimant should benefit when adopting her proposal. Otherwise, what she is proposing, if accepted, harms her own interests. Clearly, it is hard to expect a student to object an allocation, when such a demand opposes her well-being. The second restriction is that the new proposal overcomes the students' possible claims in terms of equity. Note that, otherwise the proposing student can be criticized because what she suggests is to proceed unfairly. Note that, when a student objects because of equity reasons, the Golden Rule (or ethic reciprocity) 'do not do unto others what you would not want done to yourself' applies.

[7] We employ here, as well as in Corollary 2, the expression $\mu^{SO}(\mathbb{P})$ rather than the usual μ^{SO} just to highlight that this matching is referred to the problem $\mathbb{P}$.

We find that our notion of α-equity is compatible with the efficiency requirement. More than that, we demonstrate that when efficiency and equity are compatible, α-equity and equity are equivalent across the set of efficient allocations. As a consequence of that, our results complement the analysis of other authors worried about the equity–efficiency pairing,

(a) for any problem $\mathbb{P}$, the outcome of the EADAM (Kesten 2010) is α-fair when all the students consent;
(b) for any problem satisfying the acyclicity condition (Ergin 2002), there is a unique α-fair matching. It is its Student Optimal Stable matching.

As we mentioned in the Introduction, this is the first paper to suggest a weakening of equity that is compatible with efficiency on the whole domain of student placement problems. Morrill (2015) analyses a kind of procedural equity. His definition of a just allocation does not embody our idea of fairness. As can be seen in Example 2, the TTC algorithm fails to select either fair or α-fair allocations.

Example 2 Consider problem $\mathbb{P}$ involving three students, $I = \{1, 2, 3\}$ and three schools, $S = \{a, b, c\}$, having a vacant seat each. The next table summarizes students' preferences and school priorities.

$\succ_1$	$\succ_2$	$\succ_3$
c	a	a
b	b	b
a	c	c

P_a	P_b	P_c
1	2	3
2	1	2
3	3	1

When applying the TTC to this problem we obtain matching μ^{TTC}, with $\mu^{\text{TTC}}(1) = c$; $\mu^{\text{TTC}}(2) = b$; and $\mu^{\text{TTC}}(3) = a$.

Note that any efficient matching must allocate 1 at c. This is because c is the best school for 1 and the worst for the remaining students. Therefore, the conflict in which students 2 and 3 incur—because both of them want to be allocated at school a- is elucidated in accordance with P_a. As a result, a is assigned to 2. Notice that this rationing yields to describe the unique fair—and thus α-fair—matching, μ^{SO}, such that $\mu^{\text{SO}}(1) = c$; $\mu^{\text{SO}}(2) = a$; and $\mu^{\text{SO}}(3) = b$. It can be also verified that $(2; \mu^{\text{SO}})$ constitutes an admissible ε-objection to μ^{TTC}, pointing out the lack of equity exhibited by the latter matching.

To conclude, we want to suggest an open question related to the design of mechanisms implementing α-fair allocations. There are some authors pointing out that no strategy-proof mechanism dominates the Student Optimal Stable mechanism (see, e.g. Abdulkadiroğlu et al. 2009; Kesten 2010; Kesten and Kurino 2016). Therefore, our Theorem 2 allows determining that no incentive compatible mechanism selects α-fair allocations. In this line, Alcalde and Romero-Medina (2015) explore the existence of restrictions on students' preferences where such mechanisms exist. Their findings are not much encouraging. If no condition is imposed on school priorities, manipulability of mechanism selecting α-fair allocations is guaranteed unless the students' admissible preferences satisfy the (restrictive) β-Condition. Hence, as a natural complement to our results, it might be interesting to ask about the existence of mechanisms whose expected outcome—given that students should behave strategically—is α-fair.

Appendix

Appendix A: Existence of α-fair allocations

As we anticipated in Sect. 3 we proceed to prove Theorem 1 in a constructive way. The process can be described as follows: consider a given Student Placement problem $\mathbb{P}$, and a matching μ. By interpreting μ as an initial endowment for each student, we can understand that $(\mathbb{P}; \mu)$ constitutes a 'seating market' where students are allowed to exchange the seats they have been allocated. This market exhibits some similarities with the 'housing market' introduced by Shapley and Scarf (1974). Therefore, the tools usually employed to solve housing problems might be useful to explore how exchanges are conducted in the seating markets.

Consider a fixed problem $\mathbb{P}$ and compute its Student Optimal Stable matching, μ^{SO}. It can be obtained by applying the Deferred Acceptance algorithm (Gale and Shapley 1962). Associated with the pair $(\mathbb{P}, \mu^{\mathrm{SO}})$ we describe, for each student, her preferences for exchanging, denoted E_i as the linear ordering on I defined as follows:

(a) For each two students h and k such that $\mu^{\mathrm{SO}}(h) \neq \mu^{\mathrm{SO}}(k)$, $h\ E_i\ k$ if, and only if, $\mu^{\mathrm{SO}}(h) \succ_i \mu^{\mathrm{SO}}(k)$; and
(b) For each two distinct students h and k such that $\mu^{\mathrm{SO}}(h) = \mu^{\mathrm{SO}}(k) = s_j \in S \cup \{s_o\}$, $h\ E_i\ k$ if, and only if, $h\ P_j\ k$.[8]

Given the preferences for exchanging associated with each student, $E = (E_i)_{i \in I}$ describes a profile of preferences for exchanging.

Note that the pair $(I; E)$ accommodates the structure of a housing market (Shapley and Scarf 1974). We now describe how to Gale's algorithm is applied to this problem so to calculate the unique core allocation for this market:

Step 1. Build a directed graph whose nodes are the agents in I. This graph has n arcs connecting each student with her preferred 'mate for exchanging,' i.e., for each $i \in I$, there is an arc from i, pointing the maximal on I according E_i. Since there is exactly one arc starting at each of the n nodes, this graph must have at least one cycle.[9] Moreover, no student is involved in two different cycles. Then, each student belonging to a cycle is definitively assigned the seat she is pointing in the cycle,[10] and leaves the market. Let $\mu^{\varepsilon}(i)$ denote the school assigned to i, when she is in a cycle.
Let I^1 be the set of students not belonging to a cycle. Then, if I^1 is empty,

[8] The outcome that we will obtain does not depend on how the outside school s_o prioritizes the different students. Nevertheless, and for the sake of completeness, we consider that school s_o prioritizes the students according their labels; i.e. $i\ P_o\ h$ whenever $i < h$.

[9] A cycle is a set of students, $i_1, \ldots, i_k, \ldots, i_r$, such that for each k, $1 \leq k < r$, there is an arc connecting i_k to i_{k+1}; and i_r is connected to i_1. Note that a cycle might involve a unique student.

[10] Note that we identify each student with the seat she obtains at μ^{SO}. Therefore, when i is pointing h we interpret that i wants to obtain a seat at school $\mu^{\mathrm{SO}}(h)$.

the algorithm terminates producing matching μ^{ε}, previously described. Otherwise, go to step 2.

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Step t. A graph involving the students in I^{t-1} is generated. The nodes coincide with these students. There is an arc connecting each student in I^{t-1} to her preferred mate for exchanging, among the ones that are still in the market, according her preferences for exchanging E_i. As in the previous step, this graph has at least one cycle. Each student involved in a cycle is assigned the seat she is pointing to and leaves the market. Let I^t be the set of students in I^{t-1} being not in a cycle in the graph built in this step. If I^t is empty, the algorithm terminates producing matching μ^{ε}, described throughout steps 1 to t. Otherwise, go to step $t+1$.

Note that, since for each t such that $I^t \neq \emptyset$, $I^t \subsetneq I^{t-1}$, the algorithm ends in finite steps.

Previous to demonstrate that matching μ^{ε} is α-fair, we will illustrate how to compute it through an example:

Example 3 Consider problem $\mathbb{P}$ involving 8 students, $I = \{1, \ldots, i, \ldots, 8\}$ and 4 schools, $S = \{a, b, c, d\}$, having 2 vacant seats each. The students' preferences are

$\succ_1$	$\succ_2$	$\succ_3$	$\succ_4$	$\succ_5$	$\succ_6$	$\succ_7$	$\succ_8$
b	c	c	d	a	a	a	b
a	a	b	b	c	b	b	a
c	d	a	c	d	c	d	c
d	b	d	a	b	d	c	d

The priorities of the schools are

P_a	P_b	P_c	P_d
4	3	7	5
1	2	5	6
2	7	6	3
6	6	8	8
8	4	2	2
7	1	3	4
5	8	1	7
3	5	4	1

We first compute the Student Optimal Stable matching, μ^{SO}. It is calculated by applying the Deferred Acceptance algorithm. At each step, each student applies for her preferred school—among the ones not having rejected her previously—and each school rejects the less prioritized students among those sending it an application, so to keep all its seats filled. The next table describes, for each step, the applications that each school receives, and which are accepted by the school.

Step	a	b	c	d	s_o	$\mu^t(a)$	$\mu^t(b)$	$\mu^t(c)$	$\mu^t(d)$	$\mu^t(s_o)$
1	5, 6, 7	1, 8	2, 3	4	–	6, 7	1, 8	2, 3	4	5
2	6, 7	1, 8	2, 3, 5	4	–	6, 7	1, 8	2, 3	4	3
3	6, 7	1, 3, 8	2, 5	4	–	6, 7	1, 3	2, 5	4	8
4	6, 7, 8	1, 3	2, 5	4	–	6, 8	1, 3	2, 5	4	7
5	6, 8	1, 3, 7	2, 5	4	–	6, 8	3, 7	2, 5	4	1
6	1, 6, 8	3, 7	2, 5	4	–	1, 6	3, 7	2, 5	4	8
7	1, 6	3, 7	2, 5, 8	4	–	1, 6	3, 7	5, 8	4	2
8	1, 2, 6	3, 7	5, 8	4	–	1, 2	3, 7	5, 8	4	6
9	1, 2	3, 6, 7	5, 8	4	–	1, 2	3, 7	5, 8	4	6
10	1, 2	3, 7	5, 6, 8	4	–	1, 2	3, 7	5, 6	4	8
11	1, 2	3, 7	5, 6	4, 8	–	1, 2	3, 7	5, 6	4, 8	–

Therefore, μ^{SO} is such that $\mu^{SO}(a) = \{1, 2\}$, $\mu^{SO}(b) = \{3, 7\}$, $\mu^{SO}(c) = \{5, 6\}$ and $\mu^{SO}(d) = \{4, 8\}$. Now, we describe the students' preferences for exchanging. Recall that each student orders I, according the seat each student is allowed under μ^{SO}—for instance, since $a \succ_5 b$, $\mu^{SO}(1) = a$ and $\mu^{SO}(3) = b$, we have that $1\ E_5\ 3$—and ties are broken in accordance with the school priorities, for instance, since $\mu^{SO}(1) = \mu^{SO}(2) = a$, and $1\ P_a\ 2$, each student i should prefer to exchange her seat to 1 rather than to 2. Summarizing, the preferences for exchange are gathered in the following table:

E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8
3	5	5	8	1	1	1	3
7	6	6	4	2	2	2	7
1	1	3	3	5	3	3	1
2	2	7	7	6	7	7	1
5	8	1	5	8	5	8	5
6	4	2	6	4	6	4	6
8	3	8	1	3	8	5	8
4	7	4	2	7	4	6	4

Now, we can run Gale's algorithm. At the first step, each student points her preferred student for exchanging. As Fig. 1 shows, this graph has one cycle involving students 1, 3 and 5. Therefore, each of these students obtains a seat at the school that the student she pointed got at μ^{SO}. In other words, $\mu^{\varepsilon}(1) = \mu^{SO}(3) = b$; $\mu^{\varepsilon}(3) = \mu^{SO}(5) = c$; and $\mu^{\varepsilon}(5) = \mu^{SO}(1) = a$.

Once the first step is concluded, and some students leave the market, the second step is similar to the previous one, taking into account that no (remaining) student can select any student that left the allocative process. Now, as showed in Fig. 2, students 2 and 6 are involved in a cycle. This implies that $\mu^{\varepsilon}(2) = \mu^{SO}(6) = c$; and $\mu^{\varepsilon}(6) = \mu^{SO}(2) = a$.

Fig. 1 Gale's algorithm, first step

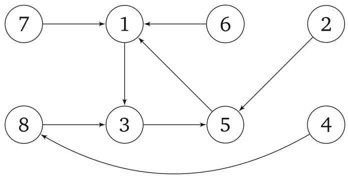

Fig. 2 Gale's algorithm, second step

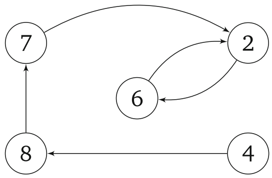

Fig. 3 Gale's algorithm, third step

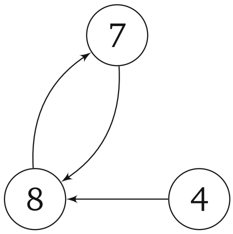

Figure 3 captures the graph constructed at the third step. Now, students 7 and 8 exchange the seats they have been allocated at μ^{SO}.

Once step 3 is concluded, the unique remaining student is 4. The application of Gale's algorithm indicates that this student must keep the seat that μ^{SO} assigned to her.

Note that, as illustrated in Example 3, when running Gale's algorithm, all the students involved in a cycle at the first step obtain a seat at their preferred school. Similarly, all the students involved in a cycle at the second step are allocated a seat at their preferred school, provided that some seats are still unavailable because they have been already allocated at the previous step, and so on. This implies that μ^{ε} is an efficient matching.

We can now formally prove Theorem 1.

Proof Let $\mathbb{P}$ be a problem and μ^{SO} its Student Optimal Stable matching. Note that, when describing each student's preferences for exchanging we have that for any two students i and h, $h\ E_i\ i$ whenever

1. $\mu^{SO}(h) \succ_i \mu^{SO}(i)$; or
2. $\mu^{SO}(h) = \mu^{SO}(i) = s_j \in S \cup \{s_o\}$ and $h\ P_j\ i$.

In particular, this implies that if $\mu^{SO}(h) = \mu^{SO}(i) = s_j \in S \cup \{s_o\}$ and $h\ P_j\ i$, for each student, say k, $h\ E_k\ i$. Therefore, two agents having a seat at the same school under μ^{SO} will not obtain a definitive seat, when running Gale's algorithm, at the same step.

Moreover, since no student leaves the market at some step if she does not belong to a cycle, and each student points her best 'student for exchanging' according her preferences, we have that, for each i,

$$\mu^{\varepsilon}(i) \succsim_i \mu^{SO}(i).$$

Now, assume that μ^{ε} is not α-fair. Since, as previously reported, it is efficient, it must fail to be α-equitable. Then, there should be a student i and a matching μ' such that $(i; \mu')$ constitutes an admissible ε-objection to μ^{ε}. This implies that there is $s_j = \mu'(i) \succ_i \mu^{\varepsilon}(i) \succsim_i \mu^{SO}(i)$, and thus $s_j \in S$. Therefore, by the Blocking Lemma (Martínez et al. 2010, Theorem 1), matching μ fails to be equitable, which contradicts that ε-objection $(i; \mu')$ to μ^{ε} was admissible. □

Appendix B: Identifying the set of α-fair allocations

We now deal with proving Theorem 2. It establishes that, for a given problem $\mathbb{P}$, matching μ is α-fair if, and only if, it is efficient and, for each student i,

$$\mu(i) \succsim_i \mu^{SO}(i). \tag{3}$$

Note that, since α-equity implies efficiency by definition, we only need to concentrate on the fulfillment of Condition (3) above.

Proof For a given problem $\mathbb{P}$, let μ^{SO} be its Student Optimal Stable matching. Let μ be an α-fair matching. Therefore, it is efficient. Assume μ does not fulfill Condition (3) above. Then, there should be a student i such that

$$\mu^{SO}(i) \succ_i \mu(i). \tag{4}$$

Note that this implies that there is some $s_j \in S$ such that $s_j = \mu^{SO}(i)$. Otherwise, μ is dominated by matching μ' such that $\mu'(i) = s_o$ and, for each student $h \neq i$, $\mu'(h) = \mu(h)$.

Efficiency also implies that $|\mu(s_j)| = q_j$. Otherwise, μ is dominated by matching μ'' such that $\mu''(i) = s_j$ and, for each student $h \neq i$, $\mu''(h) = \mu(h)$. Moreover, α-equity of μ implies that for each $h \in \mu(s_j)$, $h\ P_j\ i$. Note that, otherwise, $(i; \mu^{SO})$ constitutes an admissible ε-objection to μ.

Since $i \in \mu^{SO}(s_j) \backslash \mu(s_j)$ and $|\mu(s_j)| = q_j$, there should be a student $h \in \mu(s_j) \backslash \mu^{SO}(s_j)$. Since $h\ P_j\ i$ and μ^{SO} is equitable, it must be the case that $\mu^{SO}(h) \succ_h \mu(h)$, i.e. Condition (4) is also fulfilled for agent h.

By applying an iterative reasoning, and taking into account that the number of schools is finite, there is an ordered set of students $\{i_t\}_{t=1}^{T}$ and schools $\{s^t\}_{t=1}^{T}$, with $T \leq m$, such that for each t (modulo T),

(a) $\mu(i_t) = s^t$;
(b) $\mu^{SO}(i_t) = s^{t+1}$; and
(c) $s^{t+1} \succ_{i_t} s^t$.

Note that the above implies that μ fails to be efficient. In fact, it is dominated by matching μ' such that for each $i \in \{i_t\}_{t=1}^{T}$, $\mu'(i) = \mu^{SO}(i)$ and, for each $h \notin \{i_t\}_{t=1}^{T}$, $\mu'(h) = \mu(h)$. A contradiction.

Now, consider μ, an efficient matching that satisfies Condition (3). Assume that μ fails to be α-fair. Then, there should be a student i and matching μ' constituting an admissible ε-objection to μ. This implies that

$$\mu'(i) \succ_i \mu(i) \succsim_i \mu^{\text{SO}}(i). \tag{5}$$

Note that, in particular, Condition (5) implies that there is some $s_j \in S$ such that $s_j = \mu'(i)$. Therefore, by the Blocking Lemma (Martínez et al. 2010), μ' fails to be equitable. Therefore, $(i; \mu')$ fails to be admissible as an ε-objection to μ. A contradiction. □

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