Big-Oh Notations, Elections, and Hyperreal Numbers: A Socratic Dialogue

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Abstract

We provide an intuitive motivation for the hyperreal numbers via electoral axioms. We do so in the form of a Socratic dialogue, in which Protagoras suggests replacing big-oh complexity classes by real numbers, and Socrates asks some troubling questions about what would happen if one tried to do that. The dialogue is followed by an appendix containing additional commentary and a more formal proof.

1 Act One: Replacing big-oh notations by real numbers

Protagoras: My dear Socrates, don't you agree mathematicians make everything too complicated?

Socrates: You're right, Protagoras. How would you reform mathematics?

Protagoras: I'd start by removing big-oh notations; $O(n^2)$, $O(n \log n)$, $O(n!)$, ... These notations are an eyesore!

Socrates: You'd get rid of big-oh notations, like $O(n^2)$ and $O(2^n)$? What would you replace them with?

Protagoras: The real numbers. Everyone loves $\mathbb{R}$.

Socrates: I agree $\mathbb{R}$ is less confusing than big-oh notation. I'm slow, though. I need examples. What would you replace $O(n)$ with?

Protagoras: It's not important. Haven't you noticed that degrees and radians both work just fine for measuring angles?

Socrates: Yes. But please humor me. I know it's arbitrary, but exactly which real would you replace $O(n)$ with?

Protagoras: I'd replace $O(n)$ with 1.

Socrates: And $O(n^2)$?

Protagoras: I'd replace $O(n^2)$ with 2.

Socrates: So instead of saying, "The algorithm has complexity $O(n)$," you'd say, "The algorithm has complexity level 1"?

Protagoras: Yes.

Socrates: And instead of, "My algorithm runs in time $O(n^2)$," you'd say, "My algorithm runs in time complexity 2"?

Protagoras: Precisely!

Socrates: So $O(n)$ becomes 1 and $O(n^2)$ becomes 2. I suppose next you'll say $O(n^3)$ becomes 3?

Protagoras: Naturally.

Socrates: And $O(n^k)$ becomes k, for every natural number k?

Protagoras: Now you've got it!

Socrates: What would you replace $O(2^n)$ by?

Protagoras: Let's make $O(2^n)$ be 1000. Any big number would work.

Socrates: So, "Complexity $O(2^n)$" becomes "Complexity level 1000"?

Protagoras: Yes.

Socrates: And, "Complexity $O(n^k)$" becomes "Complexity level k"?

Protagoras: Yes.

Socrates: I feel a lot smarter. See, my slow brain takes $O(2^n)$ time to think about anything. But now I realize $O(2^n)$ is polynomial time!

Protagoras: I don't follow.

Socrates: Isn't n^{1000} a polynomial? Aren't $O(2^n)$ and $O(n^{1000})$ both replaced by 1000?

Protagoras: You trickster! Very well, let's fix that. Let's replace

$$O(1), O(n), O(n^2), \ldots$$

by a strictly increasing sequence

$$r_0 < r_1 < r_2 \ldots$$

of reals in $[0, 500)$. We're replacing $O(2^n)$ with 1000 so we better ensure this sequence doesn't get anywhere near 1000.

Socrates: We have to, if my brain runs slower than polynomial time. So we'll replace $O(1)$, $O(n)$, $O(n^2)$, …by $r_0 < r_1 < r_2 < \dots$ in $[0, 500)$. If the r_i converge, can we assume $r_i \to 500$?

Protagoras: They must converge, by the Monotone Convergence Theorem. Without loss of generality, we can assume $r_i \to 500$.

Socrates: I suppose r_{100} must be around 499.99 then.

Protagoras: r_i must be at least 499.99 for some i. I see no harm in letting $r_{100} = 499.99$.

Socrates: So instead of saying, "My algorithm has complexity level $O(n^{100})$," you'd say, "My algorithm has complexity level 499.99"?

Protagoras: That is what we have decided.

Socrates: Would you object if I proposed that we should next declare that $O(n^{200})$ should be replaced by 499.99001?

Protagoras: Yes, I would certainly object!

Socrates: Why?

Protagoras: Because $O(n^{200})$ is far bigger than $O(n^{100})$, Socrates. But 499.99001 is hardly any bigger than 499.99 at all.

Socrates: I see. Well then, what should we replace $O(n^{200})$ by, if 499.99001 is too close to 499.99, which is $O(n^{100})$?

Protagoras: I see where you're going with this. No matter which number I choose for $O(n^{200})$, whether it be 499.999 or even 499.9999, you'll say that it's still barely any larger than 499.99, which is $O(n^{100})$. And even if you accept that 499.9999 is big enough compared to 499.99, you'll just go right on and ask me what I'd replace $O(n^{500})$ by, and then I'll be stuck even worse. Socrates, I'm starting to think you're deliberately trying to make me look foolish.

Socrates: I am guilty as charged.

2 Act Two: The Republic

Protagoras: It seems whatever reals I choose, you'll catch me in one of your infamous traps. How would you replace big-oh notation, then?

Socrates: I doubt I'm wise enough, Protagoras. But if you like, we can try to reason it out together. Consider a Republic…

Protagoras: You and your Republics!

Socrates: Isn't a Republic a big group of people making decisions together?

Protagoras: Yes, but I don't see how that's relevant.

Socrates: Isn't big-oh notation all about comparing growth rates?

Protagoras: Yes, I suppose so...

Socrates: Do the numbers have a king ruling them?

Protagoras: Certainly not.

Socrates: Then if the natural numbers lived together in a Republic, how would they decide, given $f, g : \mathbb{N} \to \mathbb{R}$, which function grows faster?

Protagoras: They have no king, so they would have to call a vote.

Socrates: Good idea. Let the natural numbers vote whether f outgrows g, or whether g outgrows f, or whether they grow at the same rate. How does natural number n vote?

Protagoras: Hmmm...I suppose that...

- If $f(n) > g(n)$ then n votes that f outgrows g.
- If $f(n) < g(n)$ then n votes that g outgrows f.
- If $f(n) = g(n)$ then n votes that they grow at the same rate.

Socrates: So, if $f(75) > g(75)$, then 75 votes that f outgrows g? And if $f(30) = g(30)$, then 30 votes that f and g have equal growth rate?

Protagoras: Yes.

Socrates: Now I see why Democritus called you a math genius.

Protagoras: But how can we define the outcome of infinitely many votes?

Socrates: Call $S \subseteq \mathbb{N}$ a *winning bloc* if S's votes alone already guarantee electoral victory. What axioms can you think of for the collection of all winning blocs?

Protagoras: Well, let's see...

- (Properness) You lose if no one votes for you: $\emptyset$ is not a winning bloc.
- (Monotonicity) More votes can't hurt: If S is a winning bloc, then every superset of S is a winning bloc.
- (Maximality) Someone wins: for any finite partition $\mathbb{N} = S_1 \cup \cdots \cup S_k$, one of the S_i must be a winning bloc.

Socrates: I don't understand that Maximality axiom. Can you explain it to me?

Protagoras: Well, isn't the point of an election to determine a winner?

Socrates: Yes.

Protagoras: Wouldn't it be a scandal, then, if the votes were collected and then there was no winner determined by them?

Socrates: Quite so.

Protagoras: So there you have it. If the natural numbers vote between k different candidates, that's a partition of $\mathbb{N}$ into k different pieces.

Socrates: Ahh I see. Someone must win in that case, ergo, one of those k different pieces is a winning bloc.

Protagoras: Precisely.

Socrates: If voters decide that f outgrows g, and that g outgrows h, don't you think they'd better also decide that f outgrows h?

Protagoras: Hmm... I think we could force that by requiring:

- (Closure Under Intersections) If S and T are winning blocs, then so is $S \cap T$.

Socrates: And didn't we also agree that the natural numbers have no king?

Protagoras: Oh, right!

- (Non-Dictatorialness) There is no $n \in \mathbb{N}$ such that $\{n\}$ is a winning bloc. (Such an n would be a *dictator*.)

Socrates: But I fear we're getting nowhere. Surely there's no way to satisfy all these axioms simultaneously, is there?

Protagoras: You're thinking of Arrow's Impossibility Theorem. But Socrates, Arrow's Theorem assumes there are finitely many voters; $\mathbb{N}$ has infinitely many voters. Arrow's Theorem isn't applicable. Let's see... yes! Using Zorn's Lemma, I'm quite sure our axioms are consistent!

Socrates: Slow down. Can you state some definitions?

Protagoras: Definition:

- A set $\mathcal{U}$ of subsets of $\mathbb{N}$ (called *winning blocs*) is an *ultrafilter* if it satisfies Properness, Monotonicity, Maximality, and Closure Under Intersections.
- $\mathcal{U}$ is *free* if it also satisfies Non-Dictatorialness.

Theorem: Free ultrafilters exist.

So when the natural numbers vote, we can decide the outcome. But Socrates, how does this help us replace big-oh notations?

3 Act Three: The Hyperreals

Socrates: Tell me, why are $1/2$ and $2/4$ considered the same number?

Protagoras: Because $(1, 2)$ and $(2, 4)$ are in the same equivalence class modulo a certain equivalence relation.

Socrates: So the rational numbers are equivalence classes of pairs?

Protagoras: Yes. The whole point of the rationals is to compare proportions (you could even say, "growth rates," in some sense) between pairs.

Socrates: Could we adapt the construction of the rationals to get numbers for comparing growth rates of functions like n^3 and 2^n?

Protagoras: Oh, I see. Using our "voters"! Okay, fix a free ultrafilter $\mathcal{U}$...

- Definition: If $f, g : \mathbb{N} \to \mathbb{R}$, declare $f \sim g$ if the naturals vote that f and g have the same growth rate (as decided by $\mathcal{U}$).
- Lemma: $\sim$ is an equivalence relation.
- Definition: The $\sim$-equivalence classes are called *hyperreal numbers*.

Socrates: Do these "hyperreal numbers" have any structure?

Protagoras: Yes! Let $[f]$ be f's equivalence class.

- Definition: For all $f, g : \mathbb{N} \to \mathbb{R}$, we define $[f] + [g] = [f + g]$, $[f][g] = [fg]$, and $[f] < [g]$ if and only if the naturals vote that g outgrows f (using $\mathcal{U}$).
- Theorem: This makes the hyperreals an ordered field extension of $\mathbb{R}$.

Socrates: Which number should replace $O(f(n))$?

Protagoras: The hyperreal number $[f]$. Or an appropriate neighborhood thereof, if we must respect that $O(f(n)) = O(C \cdot f(n))$ for any positive real number C.

Socrates: Aren't you worried I'll find some f and g such that $O(f(n))$ is far bigger than $O(g(n))$ and yet $[f] \approx [g]$?

Protagoras: Not any more, Socrates. I'm protected from your tricks now by a whole Republic of voters!

4 Conclusion

If we try replacing big-oh complexity classes by real numbers, we paint ourselves into a corner. But comparing growth rates by letting natural numbers vote leads to ultrafilters and hyperreal numbers via electoral axioms (this was previously observed in [2]). We can then replace big-oh complexity classes by (classes of) hyperreal numbers without painting ourselves into a corner.

This suggests the hyperreals could potentially be quite familiar to computer scientists. They've (almost) been using them all along!

5 Appendix

The electoral axioms used by our fictional Protagoras to define an ultrafilter translate easily into the usual axioms of an ultrafilter. Those axioms and the additional results cited by Protagoras can be found in [4]. The idea of using electoral axioms to motivate ultrafilters was suggested in [2]. It was known since the 1970s that free ultrafilters provide infinite-voter counterexamples to Arrow's Impossibility Theorem [5]. Alexander observed in [1] that the *real-life* Protagoras made a certain claim (reported by Plato) with non-Archimedean implications similar to those of our fictional dialogue:

> "The very day you start [as my student], you will go home a better man, and the same thing will happen the day after. Every day, day after day, you will get better and better" [6]

(so that if Protagoras and his student live forever, and if Protagoras's goodness level does not change, and if students do not excel their teachers, and if "better" means "significantly better" (ruling out diminishing returns), then Protagoras's goodness is implied to exceed all real numbers).

Critics might claim Act 1 of this dialogue is trivial because the set of big-oh complexity classes has larger cardinality than $\mathbb{R}$. But Socrates' argument suggests that even the *countable* subset $\{O(n), O(n^2), \ldots\} \cup \{O(2^n)\}$ is already non-embeddable in $\mathbb{R}$ in some sense.

A more formal proof that the big-oh complexity classes cannot be meaningfully embedded into $\mathbb{R}$ can be accomplished by a diagonalization argument, as follows.

Let $\mathcal{O}$ be the set of big-oh complexity classes. Note that each complexity class is an equivalence class of functions $f : \mathbb{N} \to \mathbb{R}$; for instance, the complexity class $O(n^2)$ contains not only $f(n) = n^2$, but also $g(n) = 14n^2 + 3n - 14$ and $k(n) = 0.0001n^2 + \log n$. Also, there is an order $<$ on the set of complexity classes.

Suppose there is a correspondence $h : \mathcal{O} \to \mathbb{R}$ that is strictly increasing. Suppose further that h is unbounded.

Construct a sequence (a_n) as follows. For $n = 1$, there exists a complexity class C_1 such that $h(C_1) \geq 1$. Choose a function $f_1 \in C_1$. Let $a_1 = f_1(1)$. For $n = 2$, there exists a complexity class C_2 such that $h(C_2) \geq 2$ and $C_2 > C_1$. Choose a function $f_2 \in C_2$ such that $f_2(n) > f_1(n)$ for all n. Let $a_2 = f_2(2)$. Continue in this manner; for each $k \geq 2$, there exists a complexity class C_k such that $C_k > C_{k-1}$ and $h(C_k) \geq k$. Choose a function $f_k \in C_k$ such that $f_k(n) > f_{k-1}(n)$ for all $n \in \mathbb{N}$. Let $a_k = f_k(k)$. Call the resulting sequence (a_n) a *complexity diagonal* sequence.

Now, consider an algorithm requiring a_n steps for n inputs. Let C be the complexity class for this algorithm. Let $r = h(C)$ and let $M = \lceil r \rceil$. Note that for all $n \geq M + 2$, $a_n = f_n(n) > f_{M+1}(n)$, hence our algorithm has complexity level L for some $L \geq C_{M+1} > C$. This is a contradiction. Hence there can be no such correspondence h.

If h is bounded, modify the argument by letting $u = \sup\{h(C) \mid C \in \mathcal{O}\}$ and replacing $h(C_k) \geq k$ with $h(C_k) \geq u - \frac{1}{k}$. There is no complexity class C such that $h(C) = u$ (since there is no largest complexity class), and with $r = h(C)$, let $M \in \mathbb{N}$ such that $\frac{1}{M} < u - r$. Then for all $n \geq M + 2$, $a_n = f_n(n) > f_{M+1}(n)$, hence our algorithm has complexity level higher than C_{M+1}. Since $h(C_{M+1}) \geq u - \frac{1}{M+1} > u - \frac{1}{M} > r$, the complexity level is greater than C, which is a contradiction. Again, there can be no such correspondence.

The above construction of (a_n) only makes use of countably many elements of $\mathcal{O}$, so again, the argument is not trivialized by the fact that $|\mathcal{O}| > |\mathbb{R}|$.

For a different description of an embedding of big-oh complexity classes in the hyperreals, see section 5.10 of [3].

References

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