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Research Article

Dynamics, Chaos Control, and Synchronization in a Fractional-Order Samardzija-Greller Population System with Order Lying in (0, 2)

A. Al-khedhairi,[1] S. S. Askar,[1,2] A. E. Matouk ,[3] A. Elsadany,[4] and M. Ghazel[5]

[1] *Department of Statistics and Operations Researches, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia*
[2] *Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt*
[3] *Department of Basic Engineering Sciences, College of Engineering, Majmaah University, Al-Majmaah 11952, Saudi Arabia*
[4] *Mathematics Department, College of Sciences and Humanities Studies Al-Kharj, Prince Sattam Bin Abdulaziz University, Al-Kharj, Saudi Arabia*
[5] *Mathematics Department, Faculty of Science, Hail University, Hail 2440, Saudi Arabia*

Correspondence should be addressed to A. E. Matouk; aematouk@hotmail.com

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This paper demonstrates dynamics, chaos control, and synchronization in Samardzija-Greller population model with fractional order between zero and two. The fractional-order case is shown to exhibit rich variety of nonlinear dynamics. Lyapunov exponents are calculated to confirm the existence of wide range of chaotic dynamics in this system. Chaos control in this model is achieved via a novel linear control technique with the fractional order lying in (1, 2). Moreover, a linear feedback control method is used to control the fractional-order model to its steady states when $0 < \alpha < 2$. In addition, the obtained results illustrate the role of fractional parameter on controlling chaos in this model. Furthermore, nonlinear feedback synchronization scheme is also employed to illustrate that the fractional parameter has a stabilizing role on the synchronization process in this system. The analytical results are confirmed by numerical simulations.

1. Introduction

Dynamic analysis of engineering and biological models has become an important issue for research [1–10]. One of the most fascinating dynamical phenomena is the existence of chaotic attractors. Due to the importance of chaos, it has been investigated in various academic disciplines [11–15]. The sensitivity to initial conditions which characterizes the existence of chaotic attractors was first noticed by Poincaré [16].

According to their potential applications in a wide variety of settings, fractional-order differential equations (FODEs) have received increasing attention in engineering [17–23], physics [24], mathematical biology [25-26], and encryption algorithms [27]. Moreover, FODEs play an important role in the description of memory which is essential in most biological models.

Chaos synchronization and control in dynamical systems are essential applications of chaos theory. They have become focal topics for research since the elegant work of Ott et al. in chaos control [28] and the pioneering work of Pecora and Carroll in chaos synchronization [29]. Chaos control is sometimes needed to refine the behavior of a chaotic model and to remove unexpected performance of power electronics. Synchronization of chaos has also useful applications to biological, chemical, physical systems and secure communications. Furthermore, synchronization and control in chaotic fractional-order dynamical systems have been investigated by authors [30–32].

The integer-order Samardzija-Greller population model is a system of ODEs that generalizes the Lotka-Volterra equations and expresses the behaviors of two species predator-prey population dynamical system. This model was proposed by Samardzija and Greller in 1988 [33]. They had proved the existence of complex oscillatory behavior in this model. In 1999, chaos synchronization had firstly been investigated in this model by Costello [34]. Oancea et al. utilized this system to achieve the pest control in agricultural systems [35]. In 2018, Elsadany et al. proved the existence of generalized Hopf (Bautin) bifurcation in this model [36]. Recently, some works investigating synchronization in the fractional-order Samardzija-Greller model with order less than one have appeared [37-38]. Up to the present day, dynamics, chaos control, and synchronization have not been investigated in the fractional-order Samardzija-Greller system with order lying in (0, 2).

So, our objective in this paper is to investigate the rich dynamics and achieve chaos synchronization and control in the fractional-order Samardzija-Greller model with order lying in (0, 2). Moreover, applying the fractional Routh-Hurwitz criteria [23, 39] enables us to illustrate the role of the fractional parameter on synchronizing and controlling chaos in this model. Motivated by the previous discussion, numerical verifications are performed to show the existence of wide range of chaotic dynamics in this model using the aids of phase portraits and Lyapunov exponents.

2. Mathematical Preliminaries

Here, Caputo definition [17] is adopted as

$$D^{\alpha} f(t) = \mathbf{J}^{l-\alpha} f^{(l)}(t), \tag{1}$$

where $\alpha \in \mathbb{R}^{+}$, $f^{(l)}$ denotes the ordinary lth derivative of $f(t)$, l is an integer that satisfies $l-1 < \alpha \le l$, and the operator $\mathbf{J}^{\beta}$ is defined by

$$\mathbf{J}^{\beta} g(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} g(\tau) d\tau, \quad \beta > 0, \tag{2}$$

where $\Gamma(.)$ is defined as

$$\Gamma(\varsigma) = \int_0^{\infty} t^{\varsigma-1} e^{-t} dt. \tag{3}$$

Therefore, fractional modeling provides more accuracy in both theoretical and experimental results of the ecological model which are naturally related with long range memory behavior which is very important in modeling ecological systems. Thus, increasing the range of the fractional parameter α from the interval (0, 1) that is commonly used in most literatures to the interval (0, 2) provides greater degrees of freedom in modeling the population system. In addition, increasing the interval of fractional parameter increases the complexity in the system since the fractional parameter α is in a close relationship with fractals which are abundant in ecosystems. Furthermore, as α is increased, more adequate description of the whole time domain of the process is achieved and the system may show rich variety of complex behaviors such as sensitive dependence on initial conditions.

Consider the following fractional-order system:

$$D^{\alpha} X = \mathbf{J} X, \tag{4}$$

which represents a linearized form of the nonlinear system:

$$D^{\alpha} X = G(X), \tag{5}$$

where $0 < \alpha < 2$, $X \in \mathbb{R}^n$, $G \in \mathbf{C}(\Psi, \mathbb{R}^n)$. Moreover, the following inequalities [40]

$$|\arg(\lambda_i)| > \frac{\alpha\pi}{2}, \quad i = 1, \dots, n, \tag{6}$$

determine the local stability of a steady state $\bar{X} \in \mathbb{R}^n$ of the fractional-order system (4), where λ_i is any eigenvalue of the matrix $\mathbf{J}$. The stability region for $\alpha \in (0, 1]$ and $\alpha \in (1, 2)$ is shown in Figures 1(a) and 1(b), respectively. Moreover, stability conditions and their applications to nonlinear systems of FODEs were investigated in [23, 41].

3. A Three-Dimensional Samardzija-Greller Model

A three-dimensional Samardzija-Greller model [33] is described as

$$\begin{aligned} \frac{dx}{dt} &= (1-y)x + (c-az)x^2, \\ \frac{dy}{dt} &= (-1+x)y, \\ \frac{dz}{dt} &= \left(-b + ax^2\right)z, \end{aligned} \tag{7}$$

where the prey of the population is denoted by x and the predators of the population are denoted by y and z. The predators y, z do not interact directly with one another but compete for prey x, and a, b, c are nonnegative parameters used to discuss the bifurcation phenomena in the model as stated by Samardzija and Greller in 1988 [33]. Furthermore, the positive value of the parameter a indicates that two different predators y and z consume the prey x. However, if parameter a is vanished, a classical version of Lotka-Volterra model is obtained. By replacing first-order derivatives in system (7) with fractional derivatives of order $0 < \alpha < 2$, we obtain

$$\begin{aligned} D^{\alpha} x &= (1-y)x + (c-az)x^2, \\ D^{\alpha} y &= (-1+x)y, \\ D^{\alpha} z &= \left(-b + ax^2\right)z. \end{aligned} \tag{8}$$

So, a greater degree of accuracy can be obtained in our population model by using the property of evolution (8) due to the existence of fractional derivatives which is essential to the proposed Samardzija-Greller model

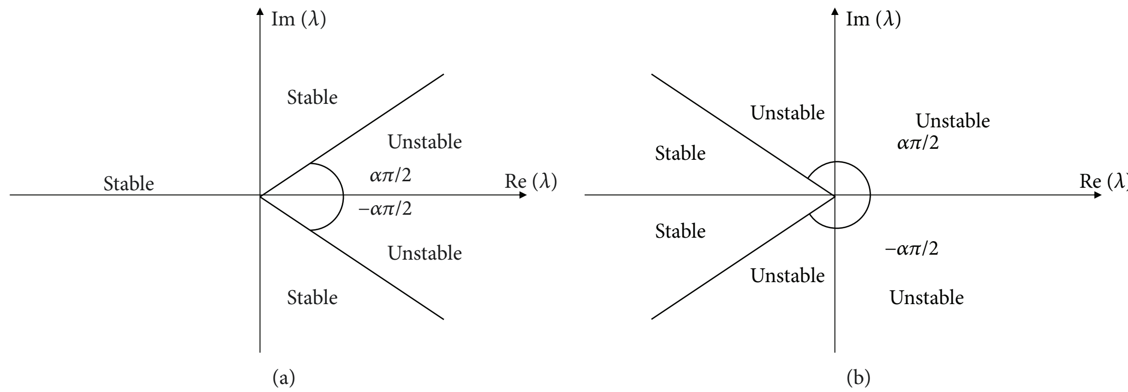


Figure 1: Stability and instability regions of system (4): (a) the case $0 < \alpha < 1$; (b) the case $1 \le \alpha < 2$.

because of its ability to provide a realistic description of the population model which involves processes with memory and hereditary properties. Furthermore, fractional derivatives provide greater degrees of freedom in modeling predator-prey ecosystems and closely related to fractals which are abundant in population models as well as its ability to describe the whole time domain for the process.

To obtain the steady states of system (8), we set $D^\alpha x = 0$, $D^\alpha y = 0, D^\alpha z = 0$, then the system has three nonnegative equilibria given as $E_0 = (0,0,0), E_1 = (1, 1 + c, 0)$, and $E_2 = (m, 0, (1 + mc)/ma)$, where $m = \sqrt{b/a}$.

4. Stability of System (8) with α Lying in (0, 2)

Assume that system (8) is written in the following form:

$$\begin{aligned} D^\alpha x_1(t) &= f_1(x_1, x_2, x_3), \\ D^\alpha x_2(t) &= f_2(x_1, x_2, x_3), \\ D^\alpha x_3(t) &= f_3(x_1, x_2, x_3), \end{aligned} \tag{9}$$

$\alpha \in (0, 2)$. The characteristic polynomial of the steady state X^* of system (9) is given as

$$\mathbf{P}(\lambda) = \lambda^3 + r_1\lambda^2 + r_2\lambda + r_3 = 0. \tag{10}$$

The discriminant of $\mathbf{P}(\lambda)$ is given by

$$D(\mathbf{P}) = 18r_1r_2r_3 + r_1^2r_2^2 - 4r_3r_1^3 - 4r_2^3 - 27r_3^2. \tag{11}$$

If all roots of (10) satisfy the inequalities (6), then the steady state of the linearized form of system (9) is locally asymptotically stable (LAS). To discuss the local stability of X^*, we prove the following theorem:

Theorem 1. *The steady state X^* of system (9) is LAS if* $|\arg(\lambda_i(\mathbf{J}(X^*)))| > \alpha\pi/2$, $i = 1,2,3$, $\alpha \in (0, 2)$, *where* $\mathbf{J}$ *is the Jacobian matrix computed at the steady state* X^*.

Proof 1. Let s_i be a very small perturbation defined as $s_i = x_i - x_i^*$, where $i = 1,2,3$. Therefore, (9), with derivatives in the Caputo sense, can be written as

$$D^\alpha s_i(t) = f_i(s_i + x_i^*), \tag{12}$$

with initial values $s_i(0) = x_i(0) - x_i^*$. A linearization of the above equation, based on the Taylor expansion, can take the form

$$D^\alpha s = \mathbf{J}s, \tag{13}$$

where $\alpha \in (0, 2)$, $\mathbf{J}$ is computed at the steady state $X^* = (x_1^*, x_2^*, x_3^*)$, and $s = [s_1\ s_2\ s_3]^T$. Hence,

$$\begin{aligned} D^\alpha s &= \left(\mathbf{P}Q\mathbf{P}^{-1}\right)s, \\ D^\alpha\left(\mathbf{P}^{-1}s\right) &= Q\left(\mathbf{P}^{-1}s\right), \end{aligned} \tag{14}$$

where $Q = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$, λ_i is an eigenvalue of $\mathbf{J}$, and $\mathbf{P}$ is the corresponding eigenvector. Suppose that $\gamma = \mathbf{P}^{-1}s = [\gamma_1 \quad \gamma_2 \quad \gamma_3]^T$, then it follows that

$$\begin{aligned} D^\alpha \gamma &= Q\gamma, \\ D^\alpha \gamma_i &= \lambda_i\gamma_i. \end{aligned} \tag{15}$$

The above system can easily be solved by the aid of Mittag-Leffler functions as follows

$$\gamma_i(t) = \sum_{k=0}^{\infty} \frac{t^{\alpha k}\lambda_i^k}{\Gamma(1 + \alpha k)}\gamma_i(0), \quad i = 1,2,3. \tag{16}$$

If the eigenvalue λ_i satisfies the condition $|\arg(\lambda_i(\mathbf{J}(X^*)))| > \alpha\pi/2$, then the perturbations $s_i(t)$ are decreasing which implies that X^* is LAS.

Moreover, the fractional Routh-Hurwitz (FRH) stability criterion is provided by the following lemma:

Lemma 1 (see [39]).

(i) The steady state X^* is LAS iff

$$r_1 > 0,$$
$$r_3 > 0, \quad (17)$$
$$r_1 r_2 > r_3,$$

provided that the discriminant of (10) is positive.

(ii) If the discriminant of (10) is negative, and $r_1 \geq 0$, $r_2 \geq 0, r_3 > 0$, then the steady state X^* is LAS when the fractional parameter α is less than 2/3, while if fractional parameter α is greater than 2/3, and $r_1 < 0, r_2 < 0$, then X^* is not LAS.

(iii) If the discriminant of (10) is negative, and $r_1 > 0$, $r_2 > 0, r_1 r_2 = r_3$, then X^* is LAS when $0 < \alpha < 1$.

(iv) Point X^* is LAS only if $r_3 > 0$.

Hence, the following lemmas are provided.

Lemma 2 (see [23, 39]). *If the discriminant of (10) is negative and $r_1 r_2 = r_3$, $r_1 > 0, r_2 > 0$, then the steady state X^* is LAS just when $0 < \alpha < 1$.*

Lemma 3 (see [42]). *Assume that system (5) is written in the form*

$$D^{\alpha} X(t) = \mathbf{C} X(t) + \mathbf{h}(X(t)), \quad (18)$$

where $\alpha \in (0, 2)$, $\mathbf{C} \in \mathbb{R}^{n\times n}$ is a constant matrix, and $\mathbf{h}(X(t))$ is a nonlinear vector function such that

$$\lim_{\|X(t)\|\to 0} \frac{\|\mathbf{h}(X(t))\|}{\|X(t)\|} = 0, \quad (19)$$

where $\|.\|$ represents the l_2-norm, then the zero steady state of system (5) is LAS provided that $|\arg(\lambda_i(\mathbf{C}))| > \alpha\pi/2$, $i = 1, \ldots, n$.

4.1. Stability Conditions for the Steady State E_0. The characteristic equation of the steady state E_0 is described as

$$\mathbf{P}(\lambda) = \lambda^3 + b\lambda^2 - \lambda - b = 0. \quad (20)$$

It is easy to check that $r_3 = -b < 0$, so by applying the stability condition (iv), we conclude that the steady state E_0 of system (8) is unstable.

4.2. Stability Conditions for the Steady State E_1. The characteristic equation of the steady state E_1 is described as

$$\begin{aligned}\mathbf{P}(\lambda) = {} & \lambda^3 + (b-a-c)\lambda^2 + (1+ac-bc+c)\lambda \\ & + (c+1)(b-a) = 0.\end{aligned} \quad (21)$$

So by utilizing the FRH conditions (i)–(iv), we obtain the following results:

(1) If the discriminant of (21) is negative; the steady state E_1 is LAS provided that $a + c < b, b \leq a + 1 + 1/c, \alpha < 2/3$.

However, if $a + c > b, b > a + 1 + 1/c, \alpha > 2/3$, the steady state E_1 is unstable.

(2) The steady state E_1 of system (8) is LAS only if $b > a$.

Remark 1. The stability conditions (i) and (iii) are not satisfied for the steady state E_1.

4.3. Stability Conditions for the Steady State E_2. The characteristic equation of the steady state E_2 is described as

$$\begin{aligned}\mathbf{P}(\lambda) = {} & \lambda^3 + (2-m)\lambda^2 + (2b+1+2mbc-m)\lambda \\ & + 2b(mc+1)(1-m) = 0,\end{aligned} \quad (22)$$

where $m = \sqrt{b/a}$. So by utilizing the FRH conditions (i)–(iv), we obtain the following results:

(1) When the discriminant of (22) is positive, the steady state E_2 of system (8) is LAS iff

$$b < a,$$
$$c > \frac{3ma - 2a - b - 2ab}{2mab}. \quad (23)$$

(2) When the discriminant of (22) is negative, the steady state E_2 is LAS provided that $b < a, c \geq m - 2b - 1/2$ $mb, \alpha < 2/3$. However, the steady state E_2 is unstable if $b > 4a, c < (m - 2b - 1)/2mb, \alpha > 2/3$. Moreover, if the discriminant of (22) is negative, $b < 3a$, then E_2 is LAS for all $0 < \alpha < 1$.

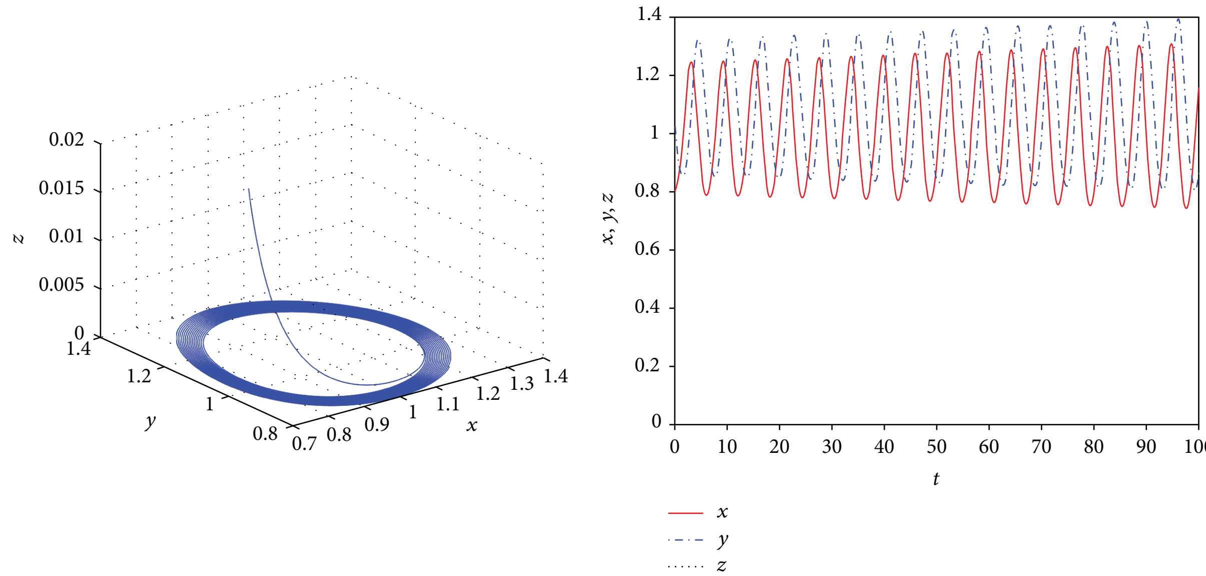
(3) The steady state E_2 of system (8) is LAS only if $b < a$.

5. Hopf Bifurcation in Samardzija-Greller Model (8)

Although exact periodic solutions do not exist in autonomous fractional systems [43], some asymptotically periodic signals converge to limit cycles have been observed by numerical simulations in many fractional systems. These limit cycles attract the nearby solutions of such systems. Therefore, Hopf bifurcation (HB) in autonomous fractional systems is expected to occur around an equilibrium point of such systems if it has a pair of complex conjugate eigenvalues and at least one negative eigenvalue. However, determining the precise bifurcation type is difficult.

In Samardzija-Greller system (8), the occurrence of HB is expected around $E_1 = (1, 1 + c, 0)$ at the critical parameter

$$c^* = \frac{2 + 2\sqrt{2 + \tan^2(\alpha\pi/2)}}{1 + \tan^2(\alpha\pi/2)}, \quad (24)$$



(a) (b)

FIGURE 2: The trajectory of system (8) with the parameter set $(a, b, c) = (0.8,3,0.07)$ and fractional parameter $\alpha = 0.98$ is attracted by a limit cycle: (a) view in xyz-space; (b) the solution versus time.

at which the transversality condition holds and $2(1-\sqrt{2}) < c < 2(1+\sqrt{2})$, $a < b$. For $\alpha = 0.98$, $a = 0.8$, $b = 3$, and $c = 0.07$, system (8) converges to a limit cycle as shown in Figure 2.

6. Chaos in Samardzija-Greller Model (8)

Based on the algorithm given in [44] and using the parameter set $(a, b, c) = (8/10, 3, 3)$, the maximum values of Lyapunov exponents (MLEs) of system (8) are computed. The calculations of the MLEs give the values 0.21315, 0.038468, 0.0089, and 0.1562 when $\alpha = 1.1$, $\alpha = 1.05$, $\alpha = 1.00$, and $\alpha = 0.99$, respectively. In Figure 3, it is shown that the chaotic attractor of system (8) related to the fractional-order case is more complicated than the attractor related to the integer-order case.

Another parameter set $(a, b, c) = (3, 4, 10)$ is used to explore a variety of complex dynamics and new chaotic regions in the Samardzija-Greller model (7) and its corresponding fractional-order form. The results are depicted in Figure 4 which shows that the complex dynamics of the system exist in a wide range of the fractional parameter α. These foundations are confirmed by the calculation of maximal Lyapunov exponents which are depicted in Figure 5.

7. Chaos Control in System (8)

In the following, we will control chaos in system (8) when $1 < \alpha < 2$ using linear control technique and also we will discuss the role of fractional parameter α on controlling chaos in Samardzija-Greller model using a linear feedback control technique.

7.1. Chaos Control of Samardzija-Greller System (8) with α Lying in (1, 2) via Linear Control. Assume that the controlled Samardzija-Greller system is described by

$$\begin{aligned} D^{\alpha}x &= (1-y)x + (c-az)x^2 - u_1, \\ D^{\alpha}y &= y(x-1) - u_2, \\ D^{\alpha}z &= \left(-b + ax^2\right)z - u_3. \end{aligned} \tag{25}$$

Thus, system (25) is rewritten as

$$D^{\alpha}X(t) = \mathbf{C}X(t) + \mathbf{h}(X(t)) - U, \tag{26}$$

where $X(t) = [x \quad y \quad z]^T$,

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -b \end{bmatrix} \tag{27}$$

is a constant parameter matrix, and the linear controller U is written as

$$U = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = MKX, \tag{28}$$

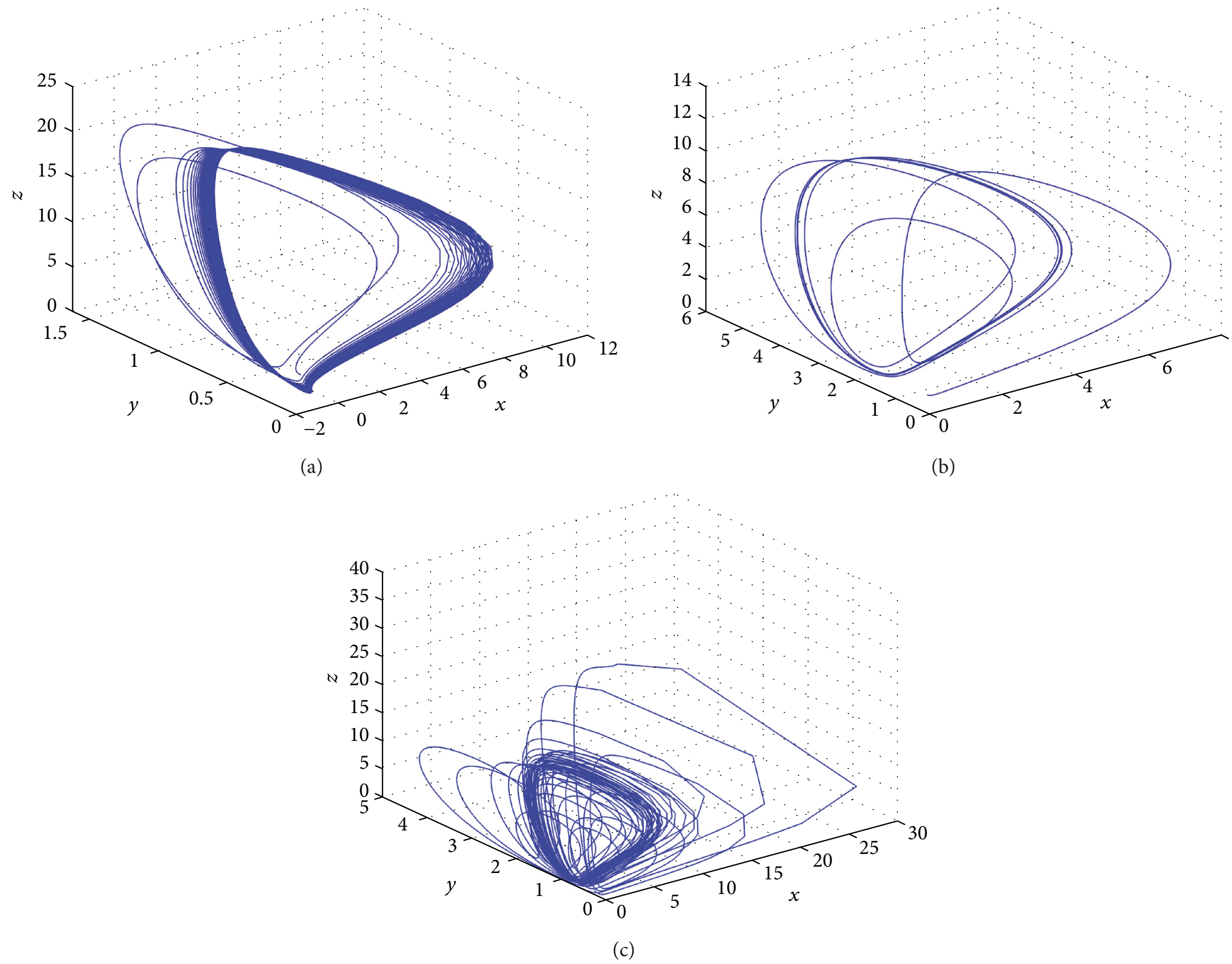


Figure 3: Phase diagrams of Samardzija-Greller model (8) with $a = 8/10$, $b = c = 3$, and α equals: (a) 1.1, (b) 1.00, and (c) 0.99.

where U is to be designed later, and $\mathbf{h}(X(t))$ is also defined as

$$\mathbf{h}(X(t)) = \begin{bmatrix} -xy + cx^2 - ax^2 z \\ xy \\ ax^2 z \end{bmatrix}. \tag{29}$$

Obviously,

$$\begin{aligned} &\lim_{\|X(t)\|\to 0} \frac{\|\mathbf{h}(X(t))\|}{\|X(t)\|} \\ &\quad = \lim_{\|X(t)\|\to 0} \frac{\sqrt{(-xy + cx^2 - ax^2 z)^2 + x^2 y^2 + a^2 x^4 z^2}}{\sqrt{x^2 + y^2 + z^2}} \\ &\quad \le \lim_{\|X(t)\|\to 0} \frac{\sqrt{x^2 \left[(-y + cx - axz)^2 + y^2 + a^2 x^2 z^2\right]}}{\sqrt{x^2}} \\ &\quad = \lim_{\|X(t)\|\to 0} \sqrt{(-y + cx - axz)^2 + y^2 + a^2 x^2 z^2} = 0. \end{aligned} \tag{30}$$

So, according to Lemma 3, the zero steady state $(x - x^*, y - y^*, z - z^*)$ of the controlled system (26) is LAS if the linear control matrix is chosen such that $|\arg(\lambda_i(\mathbf{C} - U))| > \alpha\pi/2$, $i = 1,2,3$.

The controlled Samardzija-Greller model (25) is numerically integrated as $a = 3$, $b = 4$, $c = 10$, and $\alpha = 1.1$ and linear control functions

$$\begin{aligned} u_1 &= k_1 (x - x^*), \\ u_2 &= k_2 (y - y^*), \\ u_3 &= k_3 (z - z^*) + \varepsilon (x - x^*), \end{aligned} \tag{31}$$

where $k_1, k_2, k_3 \ge 0$ and ε is a real constant. Thus, the selection $k_1 > 1, k_2 > 0, k_3 > 0$ ensures that the conditions $|\arg(\lambda_i(\mathbf{C} - U))| > \alpha\pi/2$, $i = 1,2,3$, hold. Figures 6(a)–6(c) show that system (26) is controlled to the steady states E_0, E_1, E_2 as using ($k_1 = 100$, $k_2 = 150$, $k_3 = 100$, and $\varepsilon = 100$), ($k_1 = 100$, $k_2 = 150$, $k_3 = 150$, and $\varepsilon = 0.3$), ($k_1 = 100$, $k_2 = 100$, $k_3 = 150$, and $\varepsilon = 0.3$), respectively.

7.2. Chaos Control of Samardzija-Greller System (8) via the FRH Criterion. The FRH criterion is employed to control chaos in system (8) using linear feedback control technique which is more easy of implementation, cheap in cost, and

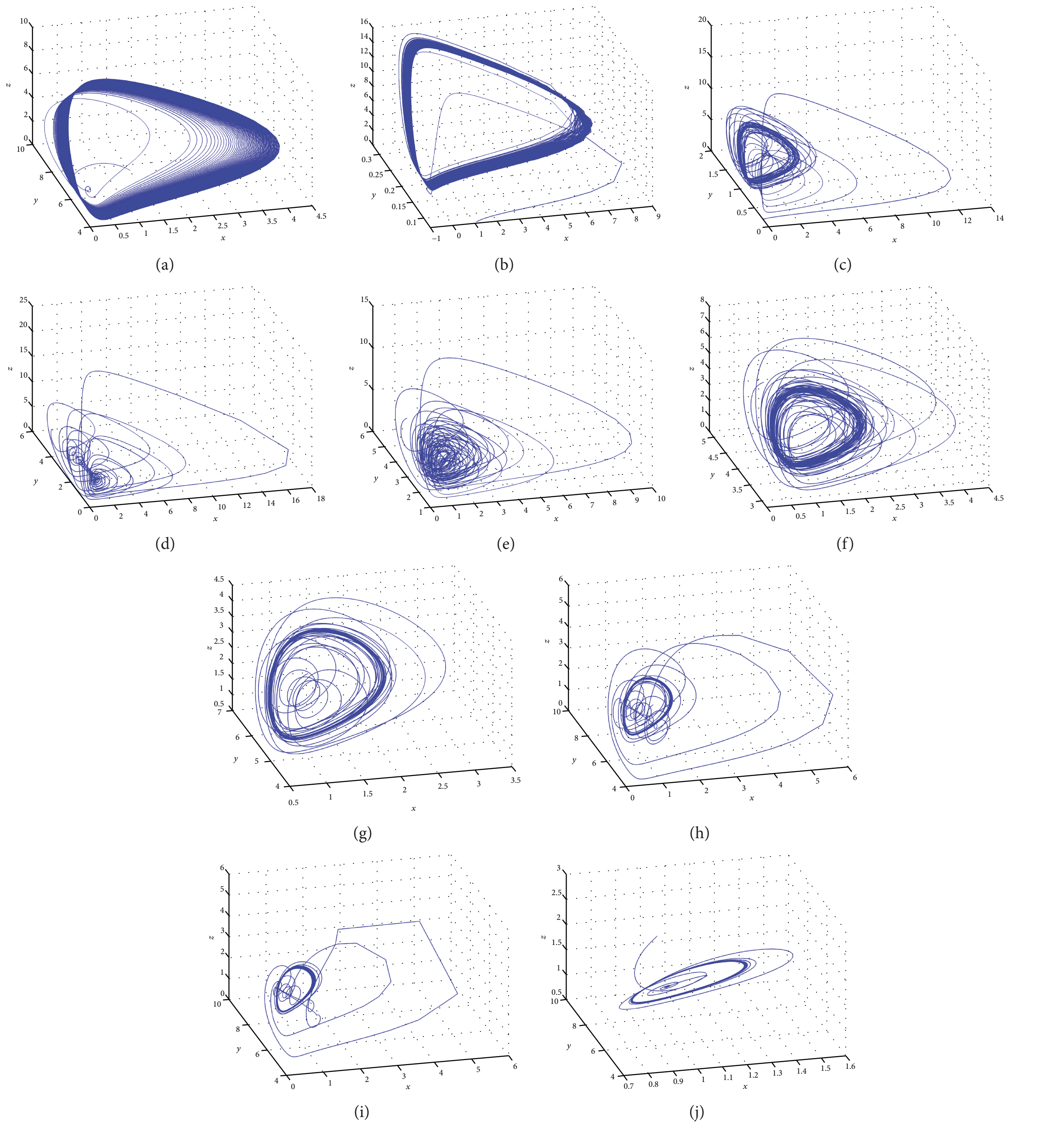


Figure 4: Phase diagrams of Samardzija-Greller model (8) with $\alpha = 3$, $b = 4$, $c = 10$, and α equals: (a) 1.1, (b) 1.05, (c) 1.00, (d) 0.98, (e) 0.95, (f) 0.90, (g) 0.80, (h) 0.70, (i) 0.60, and (j) 0.45.

more appropriate of being designed in real-world applications. Moreover, we will illustrate the role of fractional parameter α on controlling chaos in system (8). Consider the following system:

$$D^{\alpha}X = G(X), \tag{32}$$

where $\alpha \in (0, 2)$ and $X = (x, y, z)$. System (32) has the controlled form

$$D^{\alpha}X = G(X) - K(X - X^{*}), \tag{33}$$

where $X^{*} = (x^{*}, y^{*}, z^{*})$ is a steady state of (32) and $K = \text{diag}(k_1, k_2, k_3), k_1, k_2, k_3 \geq 0$ are the feedback control gains

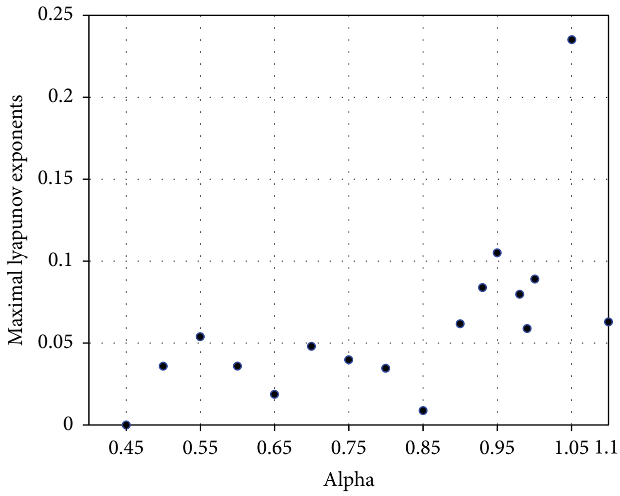


Figure 5: The MLEs of system (8) for different values of the parameter α and $(a, b, c) = (3, 4, 10)$.

(FCGs) that are selected according to the FRH criterion in such a way that

$$\lim_{t\to\infty} \|X - X^*\| = 0. \tag{34}$$

So, controlled Samardzija-Greller system (8) is given as

$$\begin{aligned} D^\alpha x &= (1-y)x + (c-az)x^2 - k_1(x-x^*),\\ D^\alpha y &= (-1+x)y - k_2(y-y^*),\\ D^\alpha z &= \left(-b+ax^2-k_3\right)z + k_3 z^*, \end{aligned} \tag{35}$$

where $0<\alpha<2$ and $X^* = (x^*, y^*, z^*)$ is a steady state of system (8). Now, we are going to control chaotic system (8) to the steady state E_1. Hence, system (35) becomes

$$\begin{aligned} D^\alpha x &= (1-y)x + (c-az)x^2 - k_1(x-1),\\ D^\alpha y &= (-1+x)y - k_2(y-1-c),\\ D^\alpha z &= \left(-b+ax^2-k_3\right)z. \end{aligned} \tag{36}$$

Then, system (36) has the following characteristic equation that is evaluated at the steady state E_1:

$$\mathbf{P}(\lambda) = \lambda^3 + r_1\lambda^2 + r_2\lambda + r_3 = 0, \tag{37}$$

where

$$\begin{aligned} r_1 &= k_1 + b - (a+c),\\ r_2 &= 1 + c + ac - ak_1 + bk_1 - bc,\\ r_3 &= (1+c)(b-a). \end{aligned} \tag{38}$$

System (36) is numerically integrated with $a=3$, $b=4$, $c=10$, $\alpha=0.95$, and $k_1=8.5, k_2=1.5, k_3=50$. Consequently, the fractional Routh-Hurwitz condition (iii) holds, that is, the discriminant of (37) is negative, r_1, r_2 are positive and $r_3 = r_1 r_2$. So according to Lemma 2, system (36) is stabilized to the steady state $E_1 = (1{,}11{,}0)$ with the fractional parameter α only in the interval $(0,1)$. Figure 7(a) shows that the states of Samardzija-Greller model (36) approach the steady state $E_1 = (1{,}11{,}0)$. However, Figures 7(b) and 7(c) show that the states of system (36) are not stabilized to the steady state $E_1 = (1{,}11{,}0)$ when $\alpha=1$ and $\alpha=1.1$, respectively. These results illustrate the role of the parameter α on suppressing chaos in Samardzija-Greller model (8).

Obviously, E_0 satisfies the fractional Routh-Hurwitz condition (i) as using the above selection of parameters and FCGs $k_1>1, k_2=1.5, k_3=50$ and $k_1=8.5, k_2>0, k_3=50$. So, system (35) is controlled to E_0 for $\alpha=0.95, k_1=8.5, k_2=1.5, k_3=50$ and for $\alpha=1.1, k_1=8.5, k_2=250, k_3=50$. The results are depicted in Figure 8.

Also, $E_2 = (1.154700539{,}0{,}3.622008471)$ satisfies the fractional Routh-Hurwitz condition (i) as using the above selection of parameters and FCGs $k_1>0, k_2=1.5, k_3=50$ and $k_1=8.5, k_2>0.1547005382, k_3=50$. Hence, system (35) is controlled to $E_2 = (1.154700539{,}0{,}3.622008471)$ for $\alpha = 0.95, k_1=8.5, k_2=1.5, k_3=50$ and for $\alpha=1.1, k_1=8.5, k_2=250, k_3=50$. The results are illustrated in Figure 9.

8. Chaos Synchronization of Samardzija-Greller System (8) via Nonlinear Feedback Control

The drive system is introduced as

$$\begin{aligned} D^\alpha x_1 &= (1-y_1)x_1 + (c-az_1)x_1^2,\\ D^\alpha y_1 &= (-1+x_1)y_1,\\ D^\alpha z_1 &= \left(-b+ax_1^2\right)z_1, \end{aligned} \tag{39}$$

and the response system is presented as

$$\begin{aligned} D^\alpha x_2 &= (1-y_2)x_2 + (c-az_2)x_2^2 + v_1,\\ D^\alpha y_2 &= (-1+x_2)y_2 + v_2,\\ D^\alpha z_2 &= \left(-b+ax_2^2\right)z_2 + v_3, \end{aligned} \tag{40}$$

where $\alpha\in(0,2)$ and v_1, v_2, v_3 are the nonlinear feedback controllers. Then, we suppose that

$$\begin{aligned} e_1 &= x_2 - x_1,\\ e_2 &= y_2 - y_1,\\ e_3 &= z_2 - z_1. \end{aligned} \tag{41}$$

By subtracting (39) from (40) and using (41), we get

$$\begin{aligned} D^\alpha e_1 &= e_1 - (x_2e_2 + y_1e_1) + c(x_2+x_1)e_1\\ &\quad - a\left(x_2^2e_3 + z_1(x_2+x_1)e_1\right) + v_1(e_1,e_2,e_3),\\ D^\alpha e_2 &= -e_2 + x_2e_2 + y_1e_1 + v_2(e_1,e_2,e_3),\\ D^\alpha e_3 &= -be_3 + a\left(x_2^2e_3 + z_1(x_2+x_1)e_1\right) + v_3(e_1,e_2,e_3). \end{aligned} \tag{42}$$

Theorem 2. *The drive chaotic system (39) and the response*

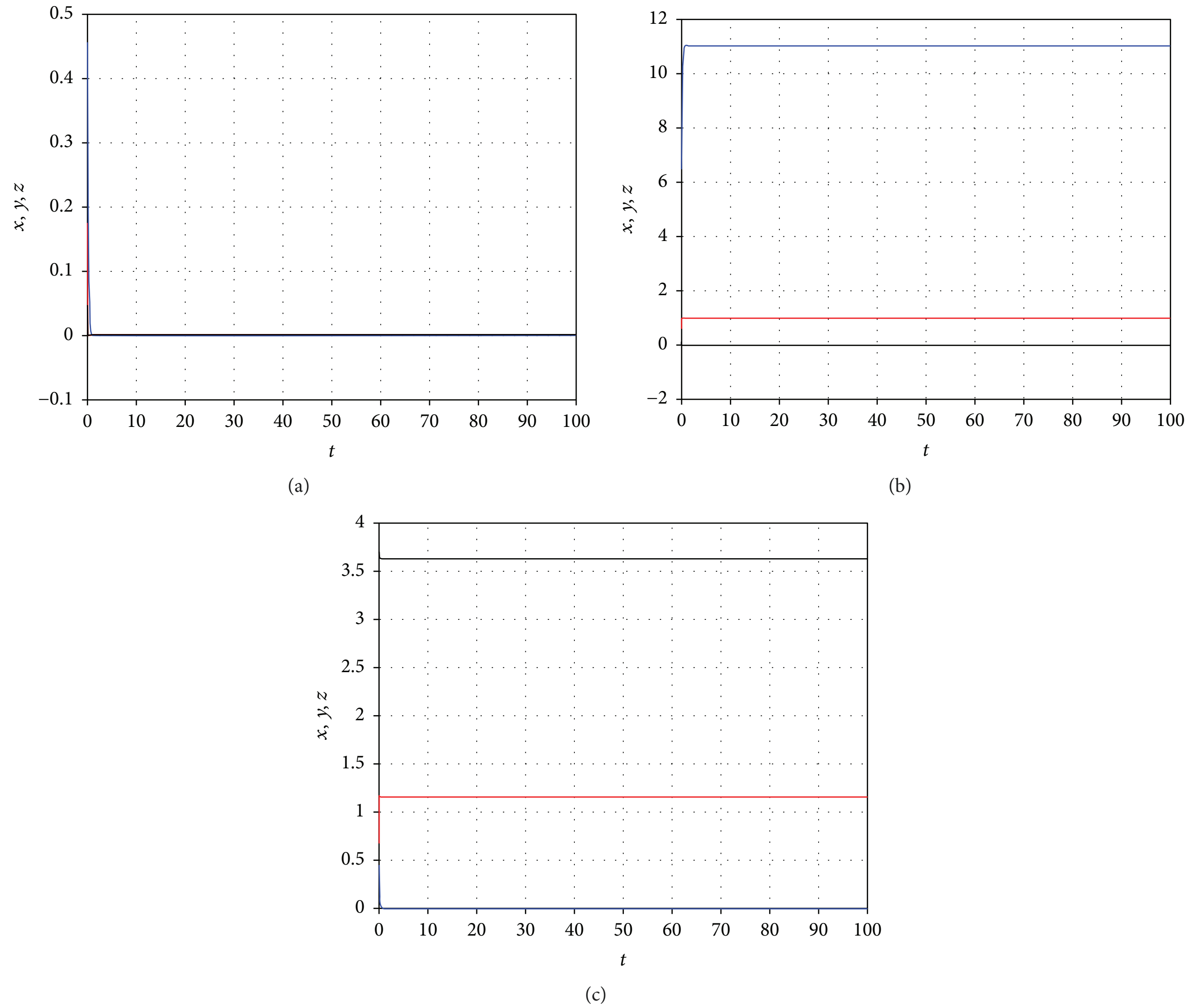


Figure 6: The states of the controlled Samardzija-Greller model (25) converge to the steady state: (a) $E_0 = (0, 0, 0)$, (b) $E_1 = (1, 11, 0)$, and (c) $E_2 = (1.154700539, 0, 3.622008471)$, when $a = 3$, $b = 4$, $c = 10$, and $\alpha = 1.1$ and using the linear controllers (31).

chaotic system (40) are synchronized for $0 < \alpha < 2$, *if the control functions are chosen as*

$$\begin{aligned} v_1(e_1, e_2, e_3) &= u - cv + aw - k_1 e_1, \\ v_2(e_1, e_2, e_3) &= -u - k_2 e_2, \\ v_3(e_1, e_2, e_3) &= -aw - k_3 e_3, \end{aligned} \tag{43}$$

where

$$\begin{aligned} u &= x_2 e_2 + y_1 e_1, \\ v &= x_2 e_1 + x_1 e_1, \\ w &= x_2^2 e_3 + x_2 z_1 e_1 + x_1 z_1 e_1, \end{aligned} \tag{44}$$

and the FCGs must satisfy

$$\begin{aligned} k_1 &> 1, \\ k_2 &> 0, \\ k_3 &> 0. \end{aligned} \tag{45}$$

Proof 2. The Jacobian matrix of system (42) with the controllers (43) evaluated at the origin steady state is described by

$$A(0,0,0) = \begin{bmatrix} 1 - k_1 & 0 & 0 \\ 0 & -1 - k_2 & 0 \\ 0 & 0 & -b - k_3 \end{bmatrix}. \tag{46}$$

So, if we select $k_1 > 1$, $k_2 > 0$, $k_3 > 0$, then all the eigenvalues of the Jacobian matrix (46) have negative signs. Thus, according to condition (6), the origin steady state of system (42) is LAS for $0 < \alpha < 2$. Consequently, the proof is completed.

The fractional-order systems (39) and (40) are numerically integrated using the system's parameters $a = 8/10$, $b = 3$, and $c = 3$, the controllers (43) with $k_1 = 20$, $k_2 = 20$, $k_3 = 20$, and the fractional parameters $\alpha = 0.99$ and $\alpha = 1.1$, respectively. The results are depicted in Figure 10.

8.1. The Role of Fractional Parameter on Synchronizing Chaos in System (8) via Nonlinear Feedback Control Method. To

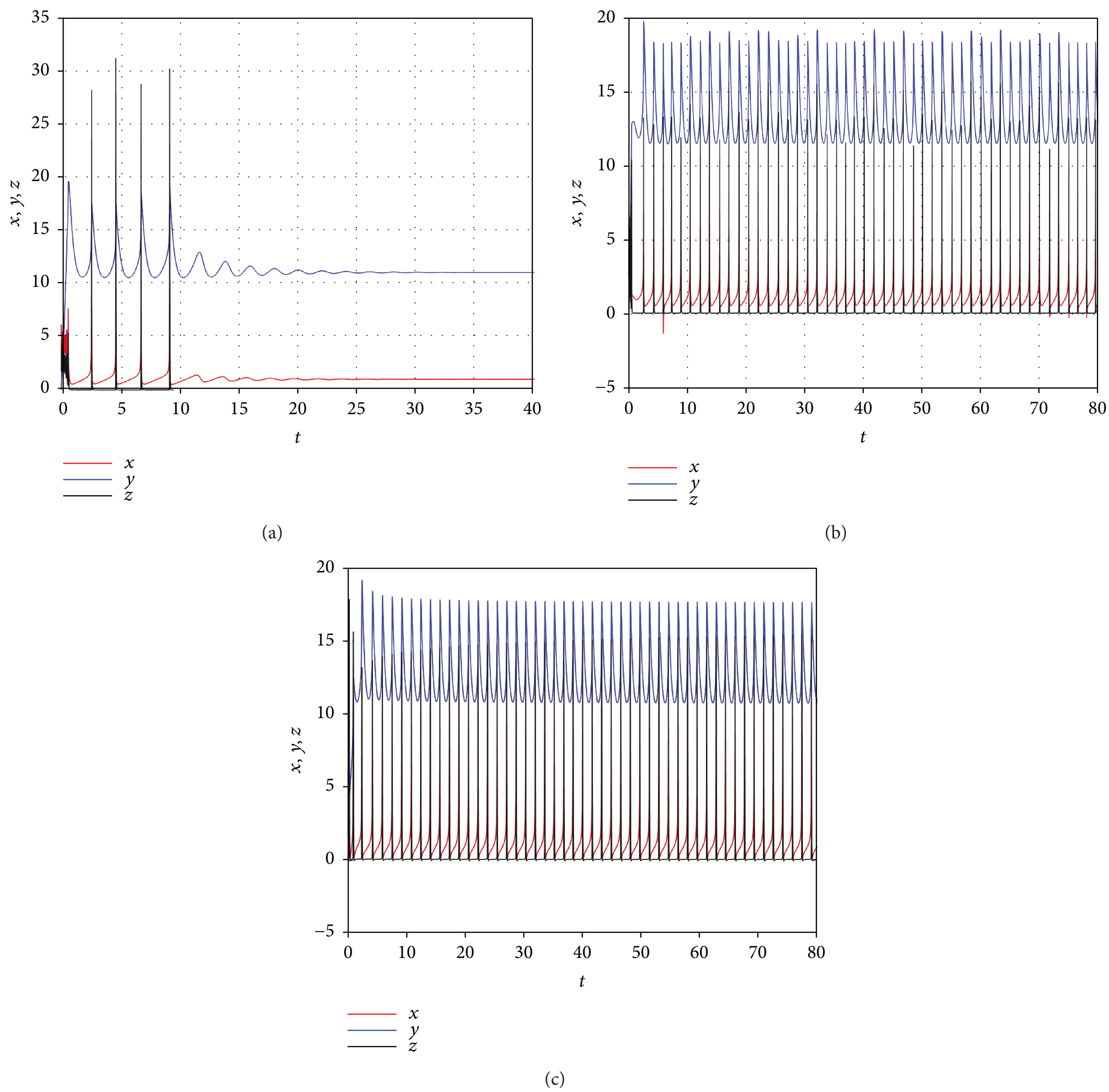


Figure 7: The states of the controlled Samardzija-Greller model (36) (a) converge to the steady state $E_1 = (1, 11, 0)$ when $\alpha = 0.95$, (b) are not stabilized to the steady state $E_1 = (1, 11, 0)$ when $\alpha = 1$, (c) are not stabilized to the steady state $E_1 = (1, 11, 0)$ when $\alpha = 1.1$, using the parameter values $a = 3$, $b = 4$, and $c = 10$ and FCGs $k_1 = 8.5$, $k_2 = 1.5$, and $k_3 = 50$.

explain the role of fractional parameter α on the synchronization process of this model, we present the following theorem:

Theorem 3. *The trajectories of the drive chaotic Samardzija-Greller model (39) asymptotically approach the trajectories of the response chaotic Samardzija-Greller model (40) just when the fractional parameter α is less than one, if the control functions are chosen as*

$$\begin{aligned} v_1(e_1, e_2, e_3) &= (b - 1 - k_1)e_1 - e_2 - ae_3 + u - cv + aw, \\ v_2(e_1, e_2, e_3) &= (1 + c)e_1 + (1 - k_2)e_2 - u, \\ v_3(e_1, e_2, e_3) &= -aw - k_3e_3, \end{aligned} \tag{47}$$

where $a = 3$, $b = 4$, $c = 10$, $k_1 = 3.9$, $k_2 = 0.1$, and $k_3 = 10$.

Proof 3. The Jacobian matrix of system (42) with the controllers (47) evaluated at the origin steady state is described by

$$A(0,0,0) = \begin{bmatrix} b - k_1 & -1 & -a \\ 1 + c & -k_2 & 0 \\ 0 & 0 & -b - k_3 \end{bmatrix} \tag{48}$$

and has the following characteristic equation:

$$\begin{aligned} \mathbf{P}(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 &= 0, \\ a_1 &= k_1 + k_2 + k_3, \\ a_2 = -b^2 + b(k_1 - k_3) + (c + 1) + k_1k_2 &+ k_2k_3 + k_1k_3, \\ a_3 = (b + k_3)(c + 1) - b^2k_2 + bk_2(k_1 - k_3) &+ k_1k_2k_3. \end{aligned} \tag{49}$$

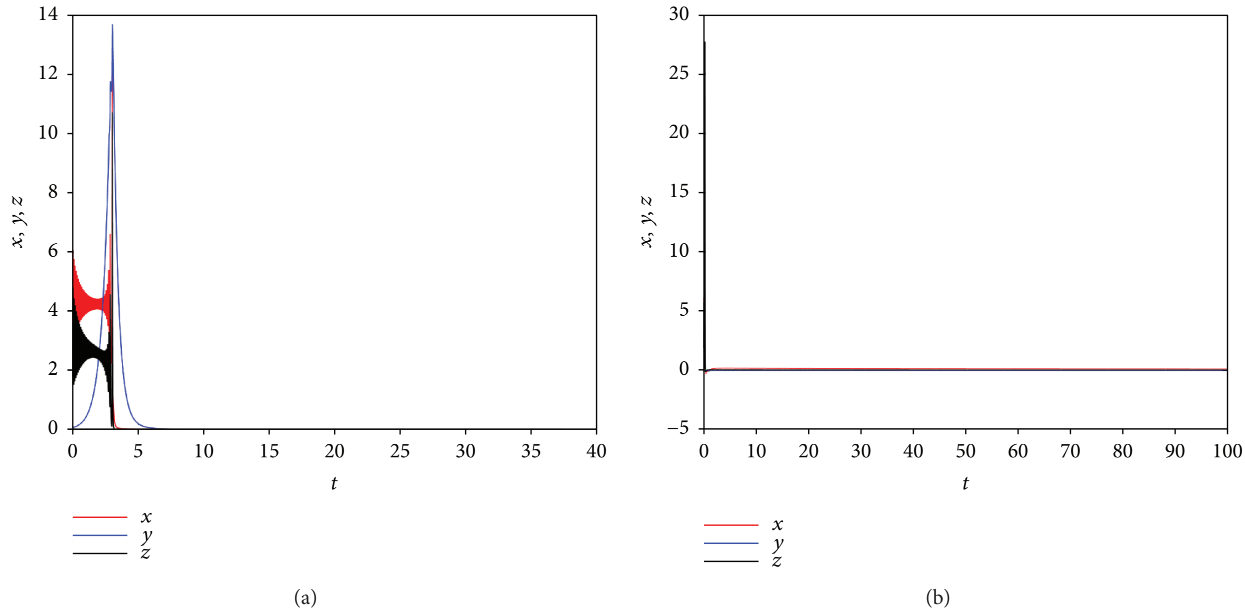


FIGURE 8: The trajectories of Samardzija-Greller model (35) with $a = 3$, $b = 4$, and $c = 10$ are controlled to $E_0 = (0, 0, 0)$ as (a) $\alpha = 0.95$, $k_1 = 8.5$, $k_2 = 1.5$, and $k_3 = 50$; (b) $\alpha = 1.1$, $k_1 = 8.5$, $k_2 = 250$, and $k_3 = 50$.

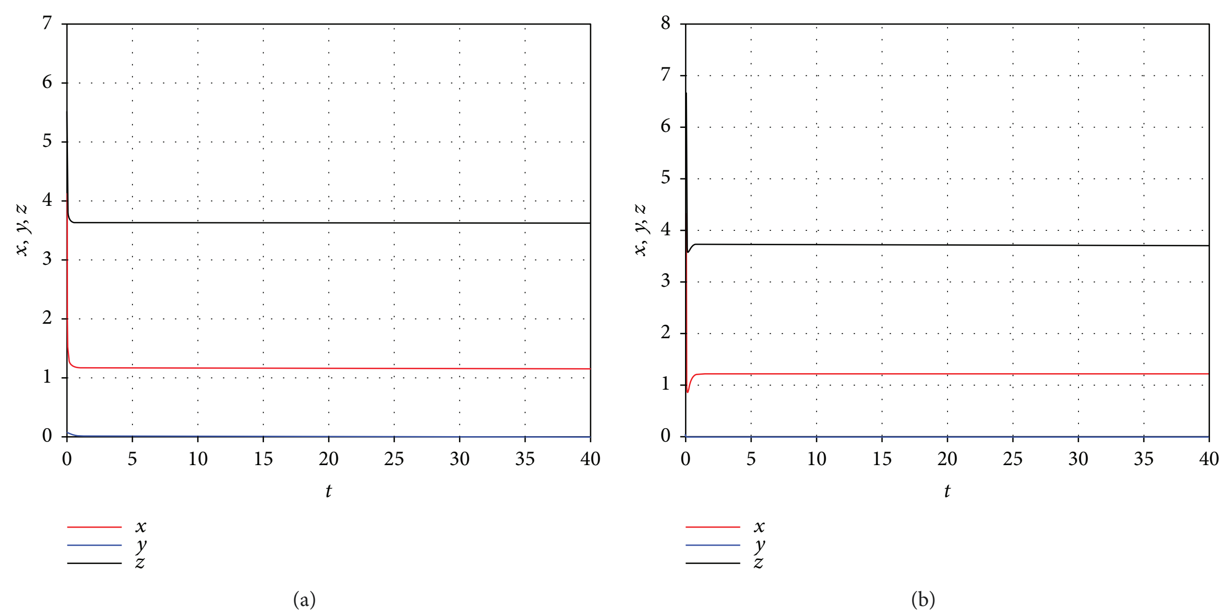


FIGURE 9: The trajectories of Samardzija-Greller model (35) with $a = 3$, $b = 4$, and $c = 10$ are controlled to $E_2 = (1.154700539, 0, 3.622008471)$ as (a) $\alpha = 0.95$, $k_1 = 8.5$, $k_2 = 1.5$, and $k_3 = 50$; (b) $\alpha = 1.1$, $k_1 = 8.5$, $k_2 = 250$, and $k_3 = 50$.

Using the parameters $a = 3$, $b = 4$, and $c = 10$ and the controllers (47) along with the selection $k_1 = 3.9$, $k_2 = 0.1$, $k_3 = 10$, the characteristic equation (49) satisfies the fractional Routh-Hurwitz condition (iii), since its discriminant is negative, $a_1 \in \mathbb{R}^+$, $a_2 \in \mathbb{R}^+$, and $a_3 = a_1 a_2$. So according to Lemma 2, the origin steady state of system (42) is LAS only for all $\alpha \in (0, 1)$. Consequently, the states of Samardzija-Greller model (39) asymptotically approach the states of Samardzija-Greller model (40) as the controllers (47) are employed. Since the characteristic equation (49) has a pair of purely imaginary eigenvalues, the condition of asymptotic stability of the origin steady state (0,0,0) of (42) is not satisfied as $1 \leq \alpha < 2$. Thus, the theorem is now proved.

Therefore, as the parameter α is decreased, the nonlinear feedback control technique has a stabilizing effect on the synchronization process in the chaotic Samardzija-Greller systems. These results are illustrated in Figure 11.

On the other hand, the origin steady state (0,0,0) of (42) satisfies the fractional Routh-Hurwitz condition (i) as using $a = 3$, $b = 4$, and $c = 10$ and FCGs $k_1 > 3.9$, $k_2 = 0.1$, $k_3 = 10$.

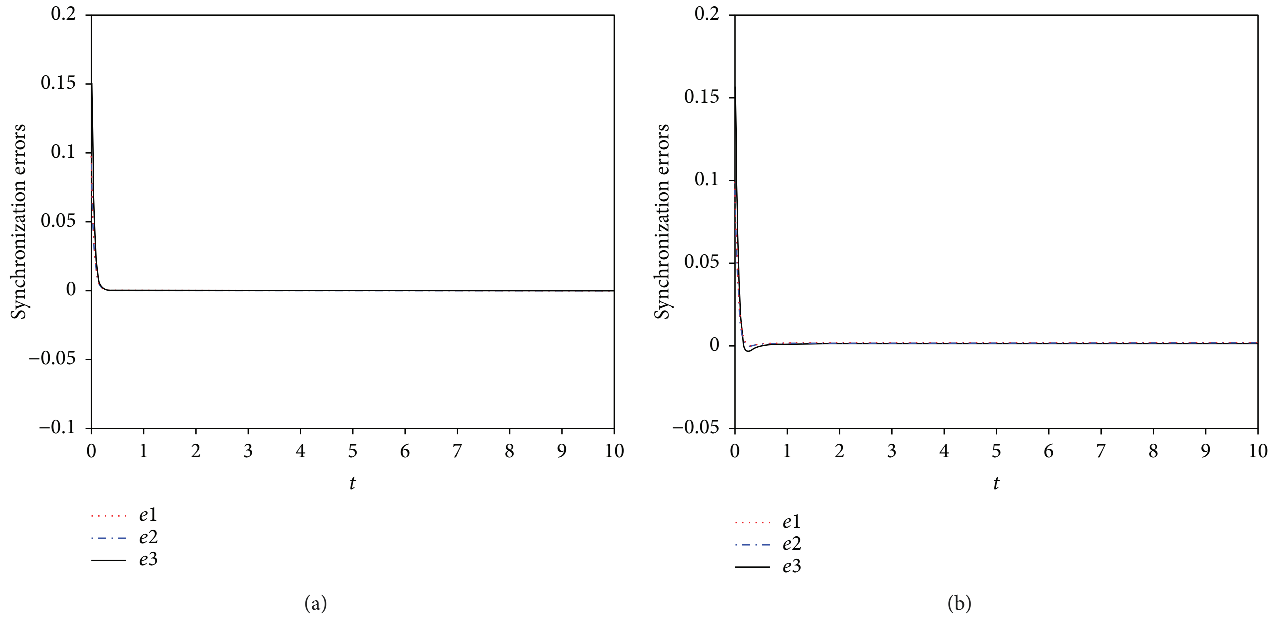


Figure 10: The synchronization errors (41) converge to zero as using the parameter values $a = 0.8$ and $b = c = 3$, the controllers (43), the FCGs $k_1 = k_2 = k_3 = 20$, and fractional order (a) $\alpha = 0.99$ and (b) $\alpha = 1.1$.

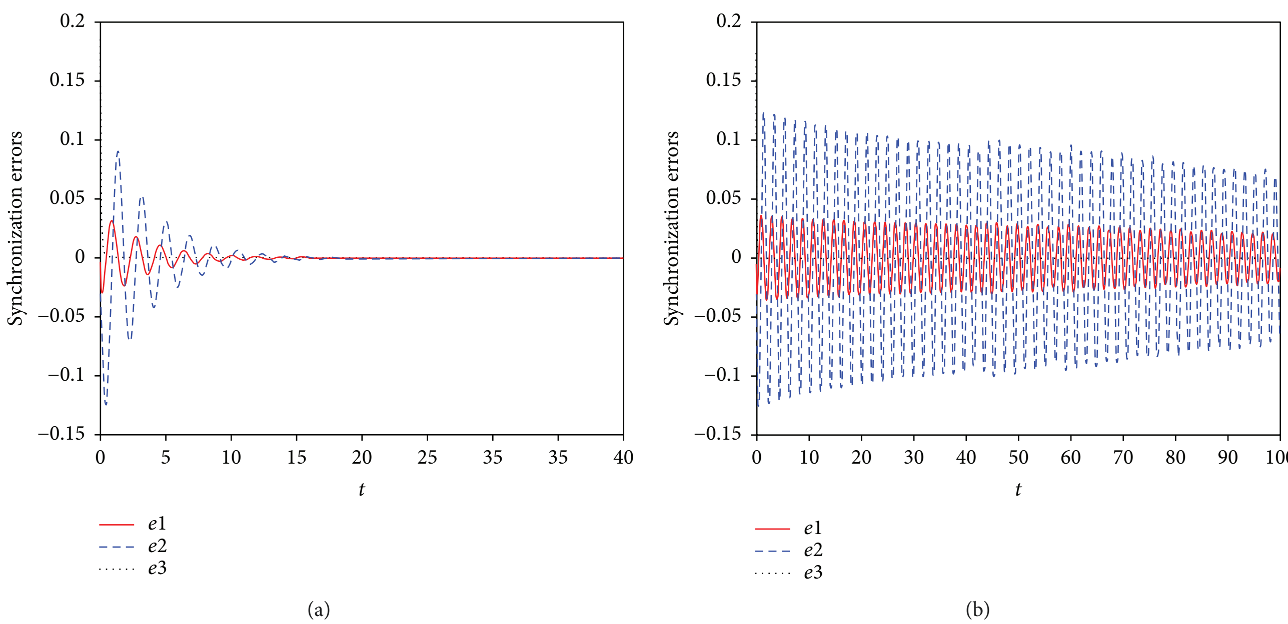


Figure 11: The synchronization errors (41) with $a = 3$, $b = 4$, and $c = 10$ and the controllers (47) with FCGs $k_1 = 3.9$, $k_2 = 0.1$, and $k_3 = 10$ (a) converge to zero when $\alpha = 0.95$ and (b) oscillate about zero when $\alpha = 1.0$.

So, systems (39) and (40) are synchronized for $\alpha = 1.1$, $k_1 = 15$, $k_2 = 0.1$, $k_3 = 10$ as shown in Figure 12.

9. Concluding Remarks

In this paper, nonlinear dynamics, conditions of chaos control, and synchronization are studied in the fractional Samardzija-Greller population model with fractional order between zero and two. To the best of the authors' knowledge, dynamical behaviors and chaos applications in this model have not been investigated elsewhere when the fractional order lies between zero and two. This kind of study has great importance to predator-prey ecosystems since it provides greater degrees of freedom in modeling such systems. In addition, increasing the interval of fractional parameter helps to raise the complexity in the system since the fractional parameter is in a close relationship with fractals which are abundant in ecosystems. Furthermore, as the fractional parameter α is increased, more accurate description of the whole time domain of the process is achieved and the system will show rich dynamics such as the existence of chaos.

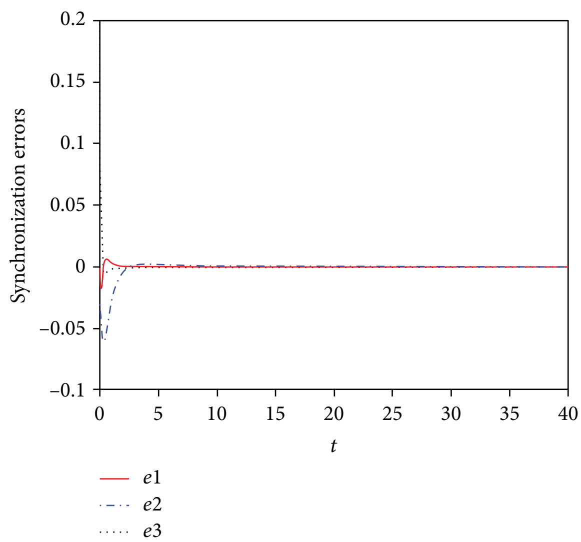


FIGURE 12: The synchronization errors (41) converge to zero as using the parameters $a = 3$, $b = 4$, and $c = 10$, fractional order $\alpha = 1.1$, and controllers (47) with FCGs $k_1 = 15$, $k_2 = 0.1$, and $k_3 = 10$.

By using the fractional Routh-Hurwitz criterion, local stability in this model has been demonstrated as $\alpha \in (0, 2)$. Moreover, the chaoticity of this model has been numerically examined by calculating the maximal Lyapunov exponents using the Wolf's algorithm. The calculations show that chaos exists in this system when the fractional parameter lies in the intervals (0, 1) and (1, 2).

Chaos control in this system has been achieved via a novel linear control technique as $\alpha \in (1, 2)$. Furthermore, the role of fractional parameter α on controlling chaos in this model has also been illustrated using a linear feedback control technique. It has been proved that using the appropriate FCGs, the trajectory of Samardzija-Greller population model is stabilized to the steady state $E_1 = (1, 1 + c, 0)$ as the fractional parameter α lies between zero and one.

Chaos synchronization has been achieved in this system via nonlinear feedback control method. It has also been shown that, under certain choice of nonlinear controllers, the fractional-order Samardzija-Greller model is synchronized only when the fractional parameter α is less than one. Thus, the obtained results illustrate that the parameter α has also a stabilizing role on the synchronization process in this system.

All the analytical results have been verified using numerical simulations.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that this article content has no conflict of interest.

Acknowledgments

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