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Research Article

On Hofstadter Heart Sequences

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The Hofstadter Q-sequence and the Hofstadter-Conway \$10000 sequence are perhaps the two best known examples of meta-Fibonacci sequences. In this paper, we explore an unexpected connection between them. When the Q-sequence is subtracted from the Conway sequence, a chaotic pattern of heart-shaped figures emerges. We use techniques of Pinn and Tanny et al. to explore this sequence. Then, we introduce and analyze an apparent relative of the Q-sequence and illustrate how it also generates heart patterns when subtracted from the Conway sequence.

1. Introduction

The Hofstadter Q-sequence is recursively defined by the nested recurrence relation $Q(n) = Q(n - Q(n-1)) + Q(n - Q(n-2))$ and initial values $Q(1) = Q(2) = 1$. This is sequence A005185 in Sloane's On-Line Encyclopedia of Integer Sequences (OEIS) [1], and its first few terms are

$$1, 1, 2, 3, 3, 4, 5, 5, 6, 6, 6, 8, 8, 8, 10, 9, 10, 11, 11, 12, \ldots$$

In his book *Gödel, Escher, Bach: An Eternal Golden Braid* [2], Hofstadter introduced this sequence as a small mystery. He noted that, despite the simplicity of the sequence's definition, the sequence values appear to be largely unpredictable. But there appears to be some sort of overarching structure (see Figure 1), a proof of which has thus far evaded mathematicians [3]. Even more frustratingly, it is unknown whether $Q(n)$ even *exists* for all n. If it happens that $Q(n-1) \geq n$, then $Q(n)$ would refer to a nonpositive index and fail to exist. In the event of such happenstance, we say that the Q-sequence *dies*. Based on the pattern in Figure 1 (which continues beyond the depicted terms), it is widely believed that the Q-sequence does not die. We know that it exists for at least $12 * 10^9$ terms [4].

Golomb describes the interestingly erratic behavior of Q-sequence as "wildly chaotic," though he proved that if $\lim_{n\to\infty}(Q(n)/n)$ exists, then it must be one half [5]. But, a priori, there is no reason this limit should exist; Golomb himself describes another sequence satisfying the Q-recurrence (A244477) for which the limit does not exist. This conditional limit value is essentially the only rigorous result proved about the Q-sequence [6]. For this reason, studies on the Q-sequence have primarily been experimental.

Observations on Q-sequence clearly suggest that the beginning points of the apparent block structures are close to 3 times consecutive powers of 2. In order to explain the fractal-like behavior of Q-sequence, the first extensive study was carried out by Pinn [7]. In his work, $Q(n)$ is described as a child of its "mother" $Q(n - Q(n-1))$ and "father" $Q(n - Q(n-2))$. With this methodology, Pinn partitions the values of n into interval "generations" that correspond to the "sausages" in Figure 1 and he noted that the first eleven generations have well-defined starting points, where the difference between $Q(n)$ and $n/2$ suddenly jumps. Thereafter, a detailed statistical analysis indicates that the subsequent generations begin around odd powers of $\sqrt{2}$. More recently, Dalton et al. instead used a recursive method to detect generations [8]. They defined a function $M_p(n)$, called the generation sequence for $Q(n)$ based on spot p, by the recurrence $M_p(n) = M_p(n - Q(n-1)) + 1$ and initial conditions $M_p(1) = M_p(2) = 1$. This function is used to determine the beginning points of the generations of Q-sequence. More precisely, the least value of t such that $M_p(t)$ is equal to k gives $g(k)$ that is defined as the start point of the kth generation of Q-sequence. Table 1 shows the values of $g(k)$ (the starting

Table 1: The values of $g(k)$ that refer to the start points of maternal generations of the Q-sequence.

	m				
	1	2	3	4	5
$g(m+11)$	3031	6043	12056	24086	48043
$g(m+16)$	95286	189268	376996	750285	1497135
$g(m+21)$	2977109	5942404	11823550	23585708	47059762

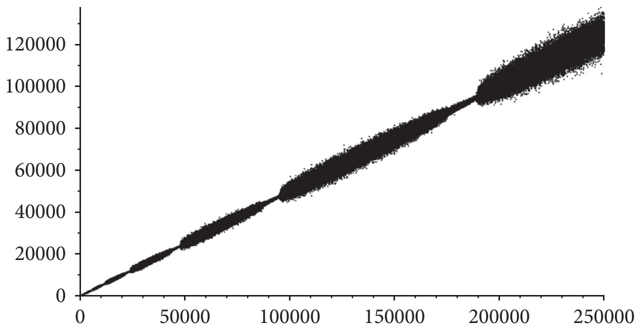

Figure 1: Hofstadter Q-sequence.

points of the first eleven generations agree, regardless of which method is used; for generation numbers $k > 11$, the two approaches give different points for starts of generations [8]) in the range we will consider for our sequences in this paper.

We have noted that the Q-sequence appears to have elements of chaos and structure. There are also many sequences that have similar definitions to the Q-sequences that are not chaotic [4–6, 9–12]. Some of these sequences have combinatorial interpretations involving counting leaves in infinite trees [10]. But, for us, the most important well-behaved meta-Fibonacci sequence is the *Hofstadter-Conway $10000 sequence* (A004001, shortened to the *Conway sequence*), given by recurrence $C(n) = C(C(n-1)) + C(n - C(n-1))$ and initial conditions $C(1) = C(2) = 1$. The name of this sequence comes from a prize of $10000 that John H. Conway offered for the discovery of a value of n such that $|C(k)/k - 1/2| < 1/20$ for all $k > n$. Mallows provided such an analysis just a few weeks after and he showed that corresponding $n = 1489$ [13]. The sequence $C(n)$ is monotone increasing with successive differences either 0 or 1, and $\lim_{n\to\infty}(C(n)/n) = 1/2$. Furthermore, $C(n) \geq n/2$ for all n, with equality if and only if n is a power of 2. Figure 2 is a plot of $C(n) - n/2$ that is symmetric in its zeros [11], which illustrates this sequence's fractal-like structure. Mallows conjectured that graphs of these structures converge to a special curve form that can be parametrized in terms of the Gaussian distribution, and this is proved by Kubo and Vakil [11]. The Conway sequence, like the Q-sequence, can be partitioned into generations. In this case, the generation boundaries are the powers of 2. More precisely, the beginning point of the kth maternal generation is $2^{k-1} + 1$ for $k > 1$ [8].

This paper is structured as follows. In Section 2, we introduce and analyze the Hofstadter Chaotic Heart sequence, which is constructed as a difference of the Q-sequence and the Conway sequence. Then, in Section 3, we carry out a similar construction with a new sequence in place of the Q-sequence and we report a variety of interesting observations. Finally, we offer some concluding remarks in Section 4.

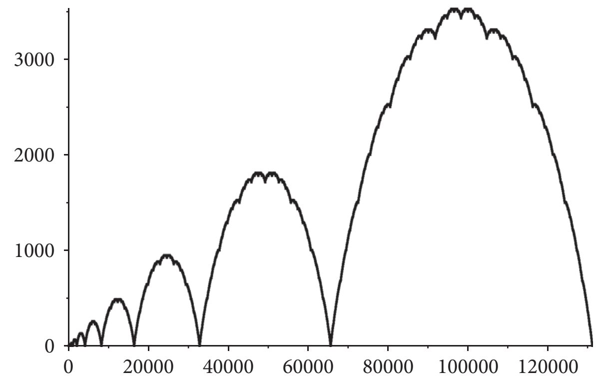

Figure 2: Hofstadter-Conway $10000 sequence, minus $n/2$.

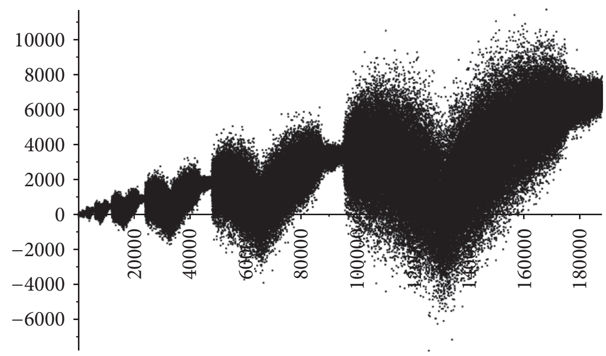

Figure 3: Hofstadter Chaotic Heart sequence.

2. The Hofstadter Chaotic Heart Sequence

We wish to consider a difference of the Q-sequence and the Conway sequence. The Q-sequence seems to hover around $n/2$, whereas the Conway sequence never dips below $n/2$. For this reason, it is more natural to consider the Conway sequence minus the Q-sequence.

Definition 1. Let $H(n) = C(n) - Q(n)$, where $Q(n)$ denotes the nth term in the Q-sequence and $C(n)$ denotes the nth term in the Conway sequence.

We call the sequence $(H(n))_{n\geq 1}$ the Hofstadter Chaotic Heart sequence (A284019). The reason for this name should be clear from a glance at its plot in Figure 3. At this point, it would be nice to observe appearance process of erratic heart shapes with increasing numbers of generations. For a plot of successive generations of the Hofstadter Chaotic Heart sequence, see Figure 4.

The sequence has a fractal-like structure of ever-growing hearts. The heart shapes can be described by considering the following ingredients:

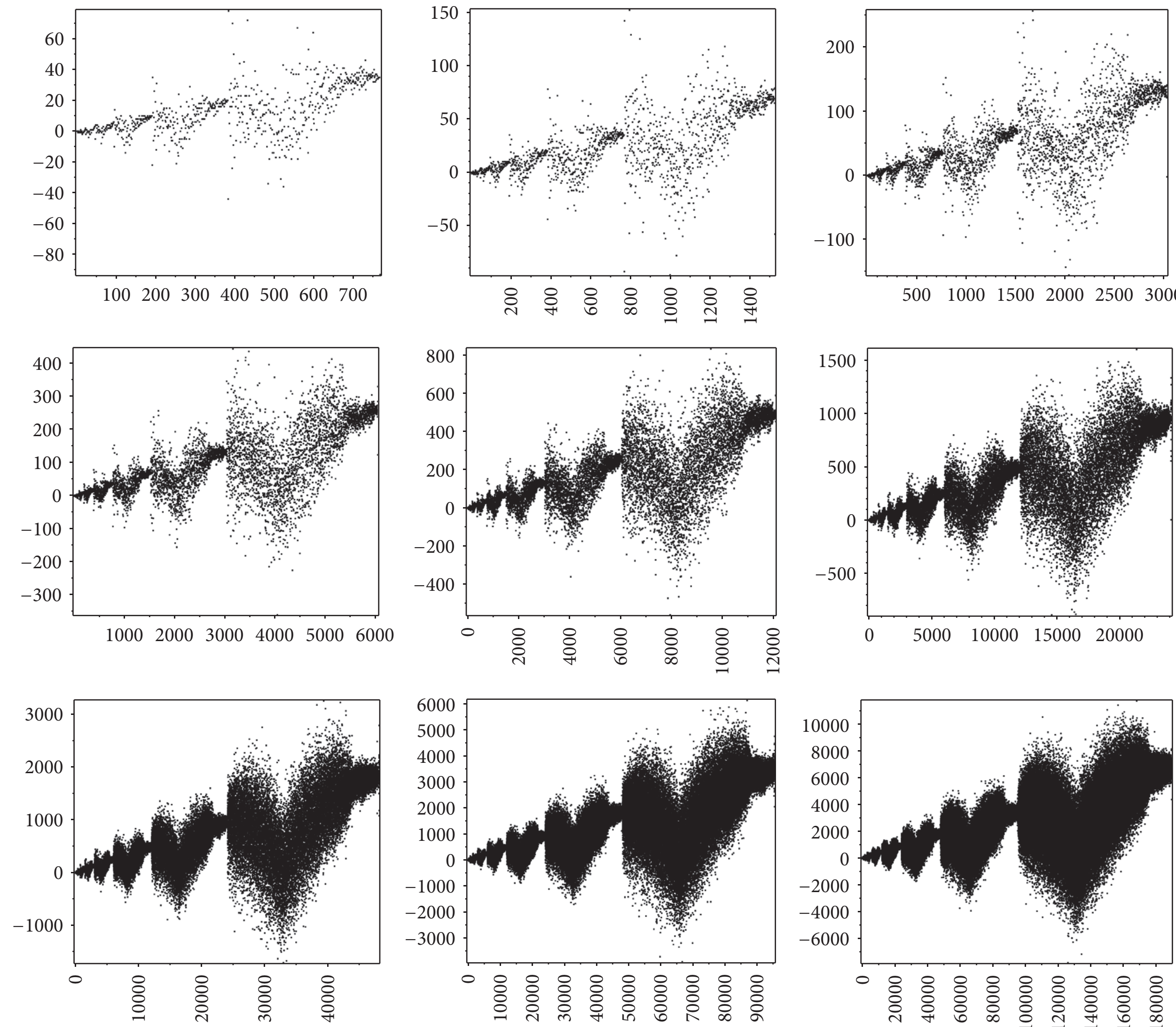

Figure 4: Hofstadter Chaotic Heart sequence for increasing values of k (generation number).

(i) The Q-sequence is built out of apparently self-similar "sausages."

(ii) The $Q(n) - n/2$ is approximately symmetric within each generation.

(iii) The Q-sequence appears to grow approximately like $n/2$.

(iv) The Q-sequence has generational divides around 3 times powers of 2, which sit halfway between consecutive powers of 2.

(v) The Conway sequence grows like $n/2$.

(vi) The Conway sequence has self-similar generations divided at powers of 2.

(vii) $C(n) - n/2$ is symmetric about 3 times the power of 2 that started the preceding generation.

(viii) $C(n) - n/2 \geq 0$.

All of these points together imply that the bottom of each heart should be near a power of 2, and this is precisely what is observed. This is all summarized in Figure 5. We also marked the initial values of $g(k)$ in order to show the determinative role of generational structure of Q-sequence.

We shall now analyze the Hofstadter Chaotic Heart sequence using Pinn's statistical technique and the Maternal Spot Generation method (we also study the same analysis with Pinn's generational method that estimates whether the integer part of $2^{k-1/2}$ is the beginning point of the kth generation of Q-sequence for $k \geq 12$; results observed are mainly similar to the values of Table 2). As in Pinn's work, let $B(n) = C(n) - n/2$, and let $S(n) = Q(n) - n/2$. In our analysis, we will not use the integral part of $n/2$ unlike Pinn's work [7], in order to prevent information loss in statistical analysis. In other words, $B(n)$ and $S(n)$ give noninteger terms for odd values of n. For a given sequence $F(n)$, let us define, as Pinn does, $M_k(F(n))^2$ (see below), a variance of sorts for $F(n)$,

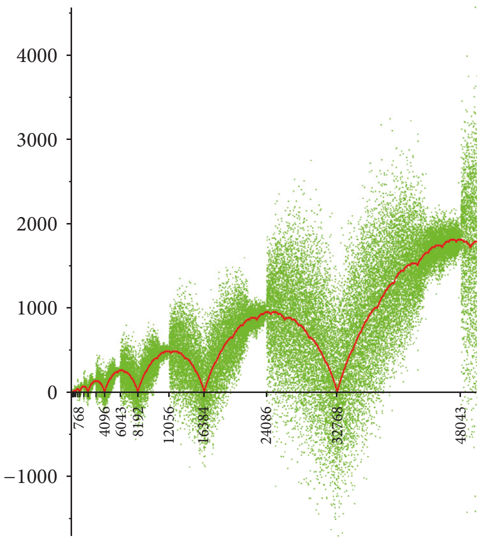

Figure 5: Hofstadter Chaotic Heart sequence with $C(n) - n/2$ overlaid.

Table 2: The values of $\alpha(k, S(n))$, $\alpha(k, B(n))$, $\alpha(k, H(n))$, and $\beta(k, B(n), S(n))$ for Maternal Spot Generation intervals on the Q-sequence.

k	$\alpha(k, S(n))$	$\alpha(k, B(n))$	$\alpha(k, H(n))$	$\beta(k, B(n), S(n))$
10	0.848	0.889	0.871	0.965
11	0.762	0.908	0.822	0.994
12	0.875	0.907	0.886	0.993
13	0.878	0.933	0.904	0.994
14	0.879	0.925	0.900	0.996
15	0.882	0.944	0.917	0.991
16	0.887	0.919	0.903	0.976
17	0.885	0.939	0.914	0.988
18	0.880	0.932	0.909	0.994
19	0.887	0.948	0.922	0.989
20	0.884	0.945	0.921	0.998
21	0.889	0.949	0.927	0.984
22	0.881	0.954	0.928	1.000
23	0.888	0.951	0.930	0.986
24	0.884	0.959	0.935	0.998
25	0.885	0.956	0.936	0.995

where the indices n considered are in the kth generation. Also borrowing notation from Pinn, let $\langle F(n)\rangle_k$ denote the average value of $F(n)$ over the kth generation, and define $\alpha(k, F(n))$, $N_k(B(n), S(n))$, and $\beta(k, B(n), S(n))$ as below. In our analysis, kth generation boundaries are taken from Table 1 because of the observation in Figure 5 that clearly suggests that generational boundaries of Q-sequence are also boundaries of the heart patterns. For example, if $k = 13$, then $g(13) = 6043$ is the first index and $g(14) - 1 = 12055$ is the last index for the computation of the following quantities:

(i) $M_k(F(n))^2 = \langle F(n)^2\rangle_k - \langle F(n)\rangle_k^2$

(ii) $\alpha(k, F(n)) = \log_2(M_k(F(n))/M_{k-1}(F(n)))$

(iii) $N_k(B(n), S(n))^2 = \langle B(n) * S(n)\rangle_k - \langle B(n)\rangle_k * \langle S(n)\rangle_k$

(iv) $\beta(k, B(n), S(n)) = \log_2(N_k(B(n), S(n))/N_{k-1}(B(n), S(n)))$

Since $H(n) = B(n) - S(n)$, for the same kth generation boundaries, it is clear that $M_k(H(n))^2 = M_k(B(n))^2 + M_k(S(n))^2 - 2 * N_k(B(n), S(n))^2$, $\alpha(k, H(n)) = \alpha(k, S(n)) + 1/2 * \log_2(\gamma_k(B(n), S(n))/\gamma_{k-1}(B(n), S(n)))$, where $\gamma_k(B(n), S(n)) = 1 + (M_k(B(n))/M_k(S(n)))^2 - 2 * (N_k(B(n), S(n))/M_k(S(n)))^2$. See Table 2 for the values of α for $S(n)$, $B(n)$, and $H(n)$ and the values of $\beta(B(n), S(n))$ in our experimental range.

3. The "Brother" Sequence

Aside from the Q-sequence, there are many other solutions to Hofstadter's Q-recurrence with different initial conditions [6]. Are there any other initial conditions for Hofstadter's Q-recurrence such that $C(n) - Q(n)$ has a fractal-like structure of ever-growing hearts where $C(n)$ is the Conway sequence? We aimed to find a simple answer for this question. Therefore, firstly, we have done experiments on Q-recurrence with three initial conditions. Let us denote the initial conditions $Q(1) = q_1$, $Q(2) = q_2$, and $Q(3) = q_3$ by (q_1, q_2, q_3), where $q_1, q_2, q_3 \geq 1$. In this case, the sequence with the initial conditions $(1, 1, 2)$ is Hofstadter's Q-sequence. Sequences with initial conditions $(1, 2, 3)$, $(1, 3, 1)$, $(2, 3, 3)$, $(a, 1, 1)$, and $(b, 1, 2)$ are essentially the same as Hofstadter's Q-sequence for all $a \geq 1$ and $b > 1$. Sequences with initial conditions $(3, 2, 1)$ (A244477) and $(1, 3, 2)$ are quasi-periodic [5]. Initial conditions $(1, 2, 2)$, $(2, 2, 2)$, $(2, 2, 3)$, $(3, 2, 2)$, and $(5, 2, 1)$ give sequences which are much more chaotic compared to the Q-sequence. All other combinations of (q_1, q_2, q_3) immediately die except $(2, 2, 1)$. The only remaining sequence is the one with initial conditions $(2, 2, 1)$ that has a behavior so closely related to the Q-sequence itself and thus we call it the *Brother sequence*.

Definition 2. Let $Q_b(n)$ be defined by the recurrence $Q_b(n) = Q_b(n - Q_b(n-1)) + Q_b(n - Q_b(n-2))$ for $n > 3$, with the initial conditions $Q_b(1) = Q_b(2) = 2$, $Q_b(3) = 1$.

The first few terms of the Brother sequence (A284644) are

$$2, 2, 1, 3, 5, 3, 5, 6, 4, 6, 10, 5, 7, 9, 9, 10, 11, 11, 12, \ldots.$$

Definition 3. Let $H_b(n) = C(n) - Q_b(n)$, where $Q_b(n)$ denotes the nth term in the Q_b-sequence and $C(n)$ denotes the nth term in the Conway sequence.

Figure 7 shows the plot of the sequence $H_b(n)$. See also Figure 8 for the appearance process of heart patterns with increasing numbers of generations. In fact, the hearts here are more clearly defined, as a result of the more gradual beginnings of the sausage structures in the Brother sequence.

Figure 9 depicts the Brother sequence. Like the Q-sequence, it appears to be composed of self-similar chaotic sausages staying near $n/2$. Also, like the Q-sequence, it is easy to show that if $\lim_{n\to\infty}(Q_b(n)/n)$ exists, it must be equal to $1/2$. Interestingly, the junctions between the sausages of the Brother sequences themselves appear to consist of smaller sausages (see Figure 11).

Table 3: The values of $P(n)$ sequence: these points will be used as start points of generations in order to compute statistical quantities of $Q_b(n)$.

	m				
	1	2	3	4	5
$P(m+0)$	1	3	8	19	41
$P(m+5)$	85	173	349	701	1405
$P(m+10)$	2800	5576	11128	22221	44342
$P(m+15)$	88422	176507	352062	702831	1403235
$P(m+20)$	2802382	5598862	11185734	22353592	44674558

We now study the generational structure of $Q_b(n)$. Our purpose is to compute $\alpha(k, S_b(n))$, where $S_b(n) = Q_b(n) - n/2$. Note that we defined $\alpha(k, F(n))$ and $\langle F(n)\rangle_k$ for a corresponding $F(n)$ in the previous section. Keeping those definitions in mind, we need to determine intervals that contain each main block of the Brother sequence in our measurement range. We will call these *main blocks* because of the observation, shown in Figure 11, that there are also smaller block structures. To determine the starting points of main blocks, we employ the following method. When the minimum of the father $(n - Q_b(n-2))$ and mother $(n - Q_b(n-1))$ spots is greater than or equal to the first member of the kth generation, we define n as the first member of the $(k+1)$st generation. In the range of our experiments, this methodology partitions the data cleanly into generations. To quantify the purity of our generations, we define $\epsilon(k, S_b(n))$ to be the proportion of terms in the kth generation whose father or mother spot is located in the $(k-2)$nd generation. If ϵ is close to 0, then the generations are close to pure.

In our case, $Q_b(1)$ will be the beginning of the first generation, and $Q_b(3)$ will be the beginning of the second generation. We now define some auxiliary sequences (similar auxiliary sequences can also be defined for the Conway sequence; let $c_1(n)$ be the least m such that mother $(m - C(m-1))$ spot is equal to n and $c_2(n) = c_1(c_2(n-1))$ for $n > 2$ with $c_2(1) = 1$ and $c_2(2) = 3$; in this case, $c_2(k)$ is the start point of the kth maternal generation which is given in [8]) for the approximation that is mentioned above.

Definition 4. Let $W(n)$ be the least m such that the minimum of father $(m - Q_b(m-2))$ and mother $(m - Q_b(m-1))$ spots is greater than or equal to n.

Definition 5. Let $P(n) = W(P(n-1))$ for $n > 1$, with $P(1) = 1$.

See Table 3 for the first 25 values of $P(n)$. We can confirm that the terms of $P(n)$ can be limits of intervals that contain each main block in our experimental range. See Figure 10 for an example, that is, $P(17)$.

So, we will use them as the beginning of our generations in order to compute statistical quantities that we mentioned above and in the previous section. Table 4 shows the main statistical quantities with this partition.

Let us now elucidate Figure 11 in more depth. To this end, we need the following definition.

Table 4: $\log_2(M_k(S_b(n)))$, $\alpha(k, S_b(n))$, $\beta(k, S_b(n), S(n))$, and $\epsilon(k, S_b(n))$ values for generations that $P(n)$ sequence determines.

k	$\log_2(M_k(S_b(n)))$	$\alpha(k, S_b(n))$	$\beta(k, S_b(n), S(n))$	$\epsilon(k, S_b(n))$
10	6.129	0.764	0.985	0.049
11	6.960	0.831	0.994	0.050
12	7.774	0.815	1.000	0.042
13	8.637	0.863	0.999	0.033
14	9.506	0.869	0.996	0.029
15	10.384	0.877	0.995	0.026
16	11.250	0.866	0.999	0.023
17	12.128	0.878	0.995	0.019
18	13.004	0.877	0.999	0.017
19	13.885	0.881	0.998	0.014
20	14.766	0.881	0.998	0.012
21	15.647	0.881	0.999	0.009
22	16.528	0.881	0.998	0.007
23	17.409	0.881	0.999	0.006
24	18.292	0.883	0.999	0.005

Definition 6. Let $G(n)$ be the sequence of numbers t such that $Q_b(t) = Q_b(t-1) = Q_b(t-2) = Q_b(t-3)$.

We have computed the first few terms of G(A287118): 84, 172, 348, 700, 1404, 2720, 2754, 5448, 10904, 21816, 43640, and 87288.

The G numbers are significative for the Brother sequence in order to detect the junctions of small elliptic-like sausages up to 10^5. For example, 87288 is the index of the juncture of small sausages in Figure 11. Detailed analysis of generational characteristics of smaller block structures in the connection of main blocks can be a future work. However, we will mention some properties of the G sequence that exhibits some curious patterns. For example, $G(n) = P(n+5) - 1$ for $0 < n < 6$. Additionally, there is an interesting order that $Q_b(n)$ sequence has for $n < 10^5$. Table 5 simply summarizes this. Results show that 2754 is an exception in terms of quantity of $(Q_b(G(n)+1) - Q_b(G(n)))/Q_b(G(n))$. This observation has significance since different studies have searched order signs in highly chaotic nature of Q-sequence [7, 8]. In here, $Q_b(n)$ that is an apparent relative of Q-sequence exhibits an unexpected pattern on the juncture of small elliptic-like sausages in short scale.

We want to confirm the validity of numerical results in Table 4. To this aim, we can choose two different constants and do a careful examination. Between 2^{10} and 2^{24}, experimental observations indicate that $[2^{\sqrt{2}-1+k}, 2^{\sqrt{2}+k}]$ and $[2^{\sin e+k}, 2^{\sin e+k+1}]$ can be limits of intervals that contain each main block. With both limits for successive main blocks, computations show that the value 0.88 appears. Note that both constants are determined by observation on a limited range, so both constants that are $2^{\sqrt{2}-1}$ and $2^{\sin e}$ should be seen as a tool for partitioning the sequence into similar main blocks in order to compute and confirm $\alpha(k, S_b(n))$.

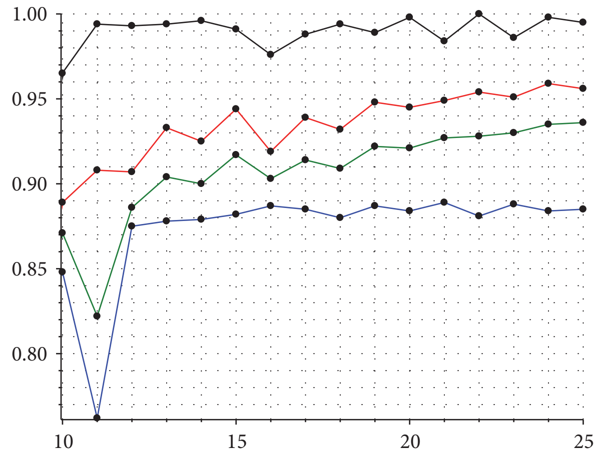

Figure 6: Graph of Table 2 (blue: $\alpha(k, S(n))$, red: $\alpha(k, B(n))$, green: $\alpha(k, H(n))$, black: $\beta(k, B(n), S(n))$). In our experimental range, $\beta(k, B(n), S(n))$ is oscillating as an upper bound and $\alpha(k, H(n))$ is increasing with decreasing fluctuations thanks to domination of $\alpha(k, B(n))$, while $\alpha(k, S(n))$ confirms Pinn's conjecture.

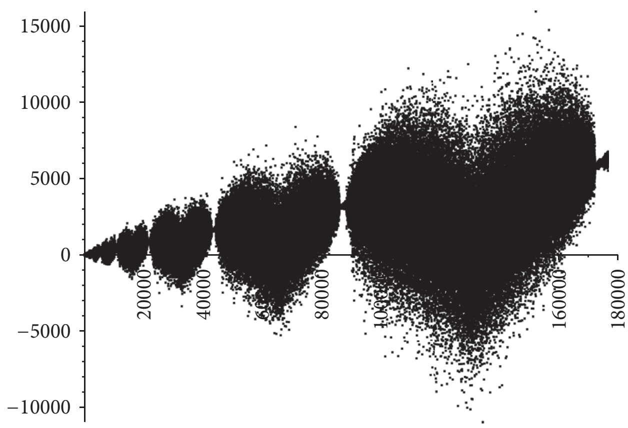

Figure 7: Heart sequence $H_b(n) = C(n) - Q_b(n)$.

The results of computation of $\alpha(k, S_b(n))$ have significance because 0.88 is the constant that Pinn conjectures in his analysis of the Q-sequence (we also confirmed that $\alpha(k, S_b(n) - S(n))$ oscillates around 0.88 for the generational boundaries that are determined by Table 3 in our experimental range) [7], and he obtains the same constant for a similar analysis of a different sequence [14]. It seems that $\alpha(k, S_b(n))$ fluctuates around 0.88 with more orderly behavior. Additionally, $\beta(k, S_b(n), S(n))$ seems fascinatingly stable around 1.

Since we know the certain generational properties of the Brother sequence now, we are ready to analyze the sequence $H_b(n)$ in our experimental range. Our previous analysis suggests $B(n) - H_b(n) \simeq n^{\alpha(k, S_b(n))}$ for sufficiently large k. We carry out analysis similar to the previous section. We use the generational boundaries of the Brother sequence that Table 3 represents. See Table 6 for the results that we obtained. Comparison between Figures 6 and 12 suggests that the signs of order are more evident in here. In this case, $\beta(k, B(n), S_b(n))$ seems really more stable than $\beta(k, B(n), S(n))$ and $\alpha(k, H_b(n))$ is smaller than $\alpha(k, H(n))$ for all $10 \leq k \leq 24$ except $k = 11$.

Table 5: Hidden order that $G(n)$ sequence shows on $Q_b(n)$ sequence.

n	$Q_b(G(n))$	$Q_b(G(n)+1)$	$(Q_b(G(n)+1) - Q_b(G(n)))/Q_b(G(n))$
1	$11 * 2^2$	$3 * 2^4$	$\frac{1}{11}$
2	$11 * 2^3$	$3 * 2^5$	$\frac{1}{11}$
3	$11 * 2^4$	$3 * 2^6$	$\frac{1}{11}$
4	$11 * 2^5$	$3 * 2^7$	$\frac{1}{11}$
5	$11 * 2^6$	$3 * 2^8$	$\frac{1}{11}$
6	$11 * 31 * 2^2$	$19 * 3^2 * 2^3$	$\frac{1}{(11 * 31)}$
7	$3 * 5 * 23 * 2^2$	$347 * 2^2$	$\frac{2}{(3 * 5 * 23)}$
8	$11 * 31 * 2^3$	$19 * 3^2 * 2^4$	$\frac{1}{(11 * 31)}$
9	$11 * 31 * 2^4$	$19 * 3^2 * 2^5$	$\frac{1}{(11 * 31)}$
10	$11 * 31 * 2^5$	$19 * 3^2 * 2^6$	$\frac{1}{(11 * 31)}$
11	$11 * 31 * 2^6$	$19 * 3^2 * 2^7$	$\frac{1}{(11 * 31)}$
12	$11 * 31 * 2^7$	$19 * 3^2 * 2^8$	$\frac{1}{(11 * 31)}$

4. Conclusion

This study aims to offer a different perspective on meta-Fibonacci sequence. We aim to suggest that there are undiscovered and interesting facts behind two famous meta-Fibonacci sequences and the relations between them. We have carried out a variety of experiments in order to understand the nature of the Hofstadter heart sequences that we introduce. While we explore the Hofstadter Chaotic Heart sequence, we also introduce and study the Brother sequence,

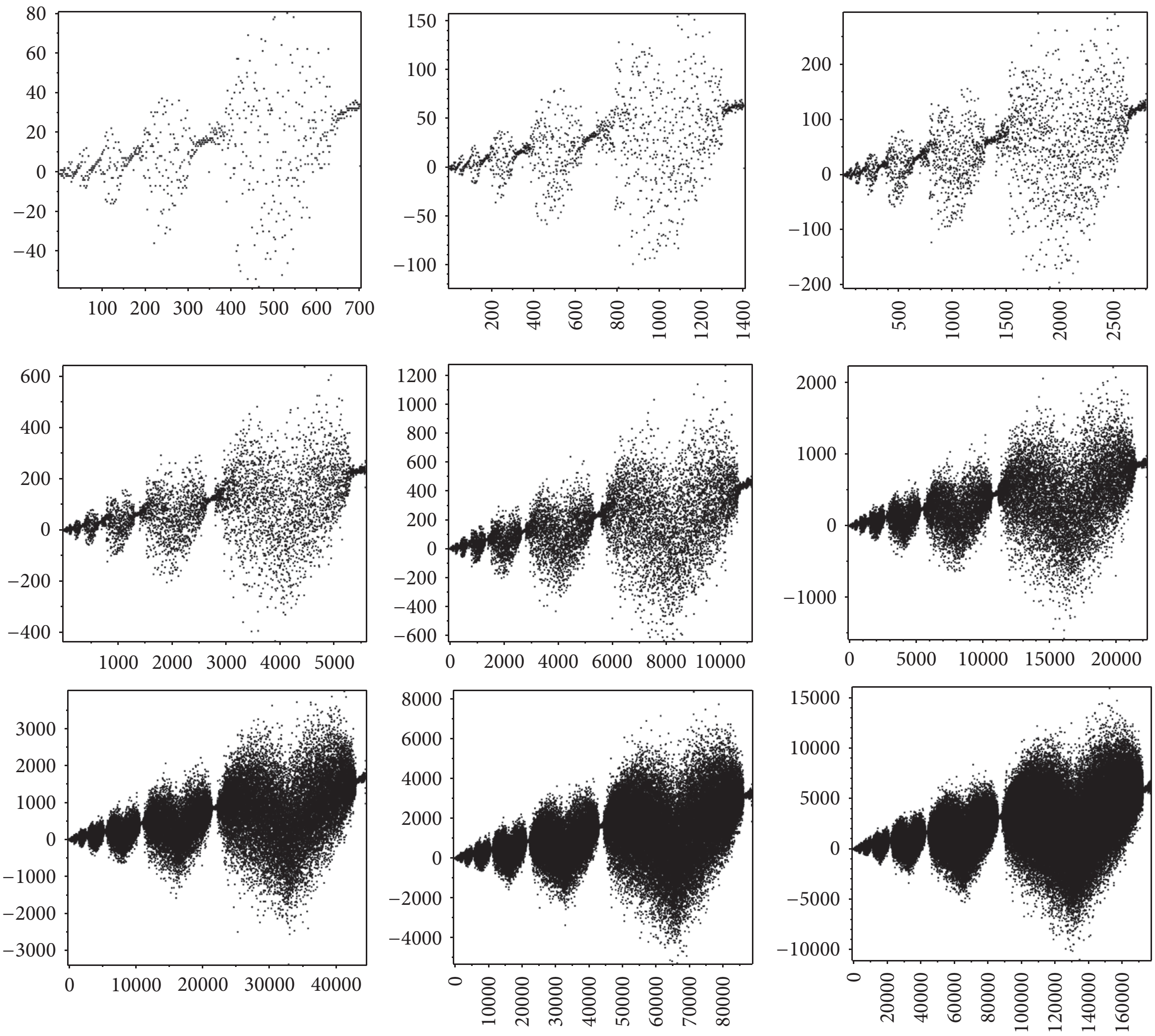

Figure 8: Heart sequence $C(n) - Q_b(n)$, successive generations.

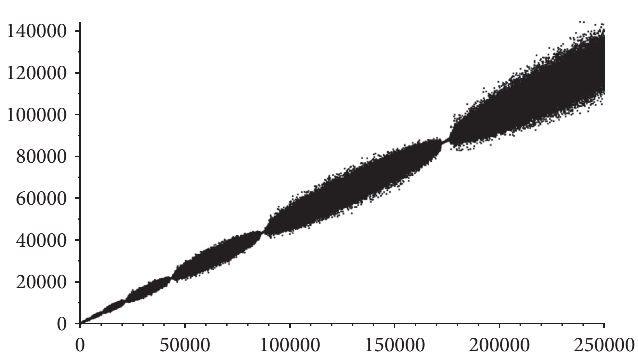

Figure 9: Graph of $Q_b(n)$ for $1 \le n \le 250000$.

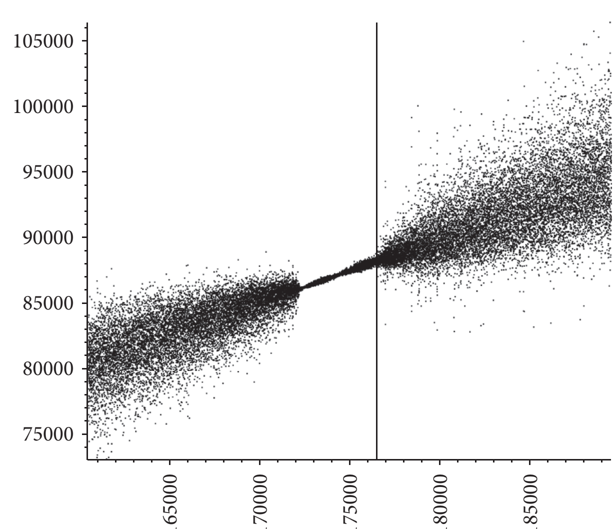

Figure 10: Illustration of $P(17) = 176507$ on the scatterplot of $Q_b(n)$.

which offers meaningful experimental results and intriguing observations in terms of new inferences for Hofstadter's Q-recurrence. Future work could potentially undertake a more

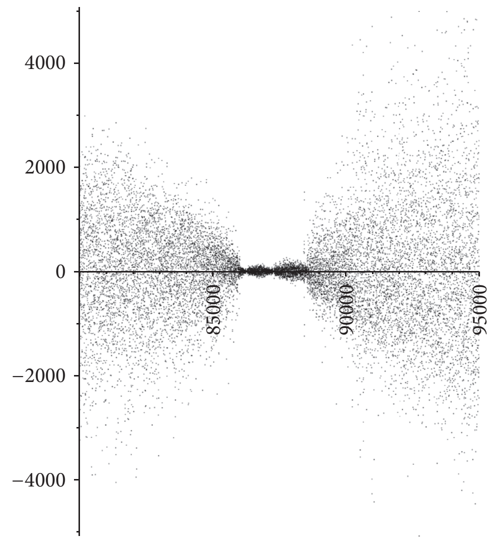

Figure 11: Graph of $S_b(n)$ for $80000 \le n \le 95000$: small elliptic-like sausages appear in the connection of big sausages; more precisely, $G(12) = 87288$ exactly gives the position of juncture of evident small elliptic-like sausages.

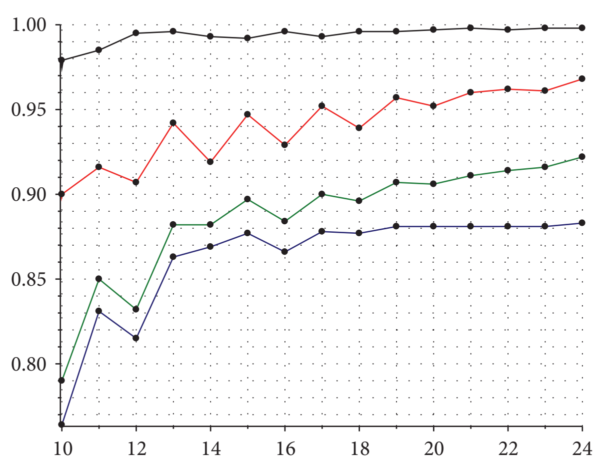

Figure 12: Graph of Table 6 (blue: $\alpha(k, S_b(n))$, red: $\alpha(k, B(n))$, green: $\alpha(k, H_b(n))$, black: $\beta(k, B(n), S_b(n))$).

detailed analysis of its generational structure than we have done here. This study also suggests that there is a mysterious classification of chaotic meta-Fibonacci sequences determined by common characteristics of their respective generational structures.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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Table 6: The values of $\alpha(k, S_b(n))$, $\alpha(k, B(n))$, $\alpha(k, H_b(n))$, and $\beta(k, B(n), S_b(n))$ for the generations of the Q_b-sequence.

k	$\alpha(k, S_b(n))$	$\alpha(k, B(n))$	$\alpha(k, H_b(n))$	$\beta(k, B(n), S_b(n))$
10	0.764	0.900	0.790	0.979
11	0.831	0.916	0.850	0.985
12	0.815	0.907	0.832	0.995
13	0.863	0.942	0.882	0.996
14	0.869	0.919	0.882	0.993
15	0.877	0.947	0.897	0.992
16	0.866	0.929	0.884	0.996
17	0.878	0.952	0.900	0.993
18	0.877	0.939	0.896	0.996
19	0.881	0.957	0.907	0.996
20	0.881	0.952	0.906	0.997
21	0.881	0.960	0.911	0.998
22	0.881	0.962	0.914	0.997
23	0.881	0.961	0.916	0.998
24	0.883	0.968	0.922	0.998

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