Journal of Logic, Language and Information **13:** 241–266, 2004.

Logics for Classes of Boolean Monoids

GERARD ALLWEIN, HILMI DEMIR, and LEE PIKE
Naval Research Laboratory, Code 5543, 4555 Overlook Ave. SW, Washington, DC 20375, U.S.A.
E-mail: allwein@itd.nrl.navy.mil

(Received 12 August 2002; in final form 21 May 2003)

Abstract. This paper presents the algebraic and Kripke model soundness and completeness of a logic over Boolean monoids. An additional axiom added to the logic will cause the resulting monoid models to be representable as monoids of relations. A star operator, interpreted as reflexive, transitive closure, is conservatively added to the logic. The star operator is a *relative modal operator*, i.e., one that is defined in terms of another modal operator. A further example, relative possibility, of this type of operator is given. A separate axiom, antilogism, added to the logic causes the Kripke models to support a collection of abstract topological uniformities which become concrete when the Kripke models are dual to monoids of relations. The machinery for the star operator is shown to be a recasting of Scott-Montague neighborhood models. An interpretation of the Kripke frames and properties thereof is presented in terms of certain CMOS transister networks and some circuit transformation equivalences. The worlds of the Kripke frame are wires and the Kripke relation is a specialized CMOS pass transistor network.

Key words: Algebras of relations, Boolean monoids, CMOS circuits, correspondence theory, Kripke frames, relative modalities

1. Boolean Monoids (BM)

Boolean monoids consist of a Boolean lattice and a monoid operation that is ordered by the lattice. The unit of the monoid is not necessarily the top to the lattice. Boolean monoids are simpler structures than propositional dynamic algebras but slightly more complicated than action algebras (Pratt, 1990a, 1990b). It turns out in both cases, however, Boolean monoids have simpler dual Kripke frames.

Propositional dynamic logic has two sorts, a *relational sort* and a *propositional sort*. The relational sort is to represent bits of program at a fairly high level of generality, while the propositional sort is to express pre and post conditions "between" bits of program. A relation then is thought to be an input-output relation of information, namely the information which makes the pre and post conditions true. In practice, usually only the post conditions are used. The expressions are of the form:

$$[a]p,$$

where $[a]$ is a relation or modality and p is a Boolean proposition or formula in classical propositional logic, depending upon whether the dynamic algebras or dynamic logics are being used. The idea is that $[a]$ represents a modal necessity operator. Of course there is a different modal necessity operator for every unique bit of program. The relational sort has other connectives: a join for choice, a composition, and a star, *, for iterated composition.

One can get to our notion of Boolean monoid by identifying the two sorts. Initially, we leave out the iterated composition and then add it back in a controlled fashion noting some interesting features along the way.

Boolean monoids display a wide range of interesting semantic attributes. When the monoid operator, $\odot$, is residuated with left ($\leftarrow$) and right ($\rightarrow$) entailments, i.e.,

$$a \odot b \leq c \text{ iff } b \leq a \rightarrow c \text{ iff } a \leq c \leftarrow b,$$

the monoids become classical relevance logic monoids lacking permutation, $a \odot b \leq b \odot a$, and contraction, $a \leq a \odot a$. These can be added conservatively. The classical negation can be weakened to form a DeMorgan negation, an order inverting period two connective (it lacks $-a + a = \top$ and $-a \wedge a = \bot$ where $\top$ and $\bot$ are the top and bottom of the lattice, respectively).

A star, *, operator may be added. This is interpreted as reflexive, transitive closure. It is related to, but weaker than, the star operator considered by Ng and Tarski (1977) and Ng (1984). However, in the presence of the left and right residuals for $\odot$, the star operator considered in this paper is equivalent. The star operator is actually an instance of a *relative modal operator*. A relative modal operator is one that requires another modal operator for its definition. This paper gives an example of a relative possibility operator defined relative to a backwards-looking possibility. The former is a weak operator but the latter is an operator that can be managed with Kripke style possible worlds. The star operator, being very weak in some respects, is given a Scott–Montague neighborhood style semantics.

The addition of the axiom $(a \odot b) \wedge (a \odot -b) = \bot$ will cause the algebra to be representable as an algebra of relations. If a further Horn style axiom, antilogism, $a \odot b \leq c$ implies $-c \odot b \leq -a$, is added, then the algebra of relations defines a^* as a topological uniformity on a set.

In addition to all of this, the Kripke style models for basic Boolean monoids admit an interpretation, or better, a realization, in terms of certain CMOS circuits. Propositions are made true or false by wires (worlds), and certain Kripke relation conditions yield circuit transformations such that the circuit before transformation and the circuit after transformation are logically equivalent.

DEFINITION 1.0.1. A *Boolean monoid*, $\mathcal{B} = (B, +, \wedge, -, \top, \bot, \odot, 1)$ is a structure where

- $(B, +, \wedge, -, \top, \bot)$ is a Boolean algebra with a top, $\top$, and bottom, $\bot$;
- $(B, \odot, 1)$ is a monoid;
- $a \odot (b + c) = (a \odot b) + (a \odot c)$ and $(b + c) \odot a = (b \odot a) + (c \odot a)$, hence the monoid operation is order preserving;

- $a \odot \bot = \bot = \bot \odot a$, i.e., the monoid operation is *normal*.

DEFINITION 1.0.2. A *starred Boolean monoid* (BM*) is a Boolean monoid which satisfies the following axioms

- $1 + (a^* \odot a^*) + a \leq a^*$;
- $1 + (b \odot b) + a \leq b$ implies $a^* \leq b$.

LEMMA 1.0.3 (Residuation). *The operators* $\wedge$ *and the defined operator* $\supset$*, i.e.,* $a \supset b = -a + b$*, are residuated in any Boolean algebra, i.e.,* $a \wedge b \leq c$ *iff* $b \leq a \supset c$.

2. Boolean Monoid Logic (BML)

The core axioms have normal Boolean monoids as their algebraic models. These axioms may be extended with a new axiom (Group C below) which forces the resultant models to be representable as algebras of relations. Antilogism may be added to force an interpretation in terms of abstract uniformities, and, with the Group C axiom, concrete uniformities. Axioms may be added to make the logic a contractionless relevance logic without permutation (with a classical negation as opposed to relevance logic's usual DeMorgan negation). The $\supset$ connective (below) is the usual classical entailment that associates to the left. The $\neg$ connective binds the tightest, $\circledcirc$ binds less strongly than $\neg$. And $\supset$, $\wedge$, and $\vee$ bind least tightest of all. The Boolean connectives $\vee$ and $\wedge$ are derived in the usual way from $\supset$ and $\neg$, the constant T is defined as $C \supset C$ for some chosen well-formed formula C, and the constant F defined as $\neg T$. The constant t corresponds to the monoid identity and in general, $t \neq T$. Also, $A \equiv B$ is short hand for $A \supset B$ and $B \supset A$ and is used in Axiom 7 of Group A. The Scheme **A** axioms and rule define BML, the **A** plus the **B** axioms and rules define BML*.

Axiom Scheme: Group A

1. $A \supset (B \supset A)$;
2. $(\neg C \supset \neg B) \supset (\neg C \supset B \supset C)$;
3. $A \circledcirc t \supset A$;
4. $A \circledcirc F \supset F$;
5. $(A \vee B) \circledcirc C \supset ((A \circledcirc C) \vee (B \circledcirc C))$;
6. $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$;
7. $((A \circledcirc B) \circledcirc C) \equiv (A \circledcirc (B \circledcirc C))$;
8. $t \circledcirc A \supset A$;
9. $F \circledcirc A \supset F$;
10. $A \circledcirc (B \vee C) \supset ((A \circledcirc B) \vee (A \circledcirc C))$.

Axiom Scheme: Group B

1. $t \supset A^*$;
2. $A^* \circledcirc A^* \supset A^*$;
3. $A \supset A^*$.

Axiom Scheme: Group C

1. $(A \circledcirc B) \wedge (A \circledcirc \neg B) \supset F$.

Deduction Rules: Group A

$$\frac{A \quad A \supset B}{B} \text{ Modus Ponens.}$$

Deduction Rules: Group B

$$\frac{(t \vee (B \circledcirc B) \vee A) \supset B}{A^* \supset B} \text{ * Introduction.}$$

Deduction Rules: Group D

$$\frac{A \circledcirc B \supset C}{\neg C \circledcirc B \supset \neg A} \text{ Antilogism.}$$

2.1. Algebraic Soundness and Completeness

Algebraic soundness is shown in the usual way. The axioms are proved to be "true" in the class of Boolean monoids which forms the base for the induction via the rules of inference to show that all theorems are "true." Truth here means truth as represented in the algebraic models. In this case, truth is represented by the algebraic element, $\top$.

DEFINITION 2.1.1. Let $\mathcal{B}$ be a Boolean monoid. An interpretation, $[\![\cdots]\!]$, is a mapping of the propositional variables of BML into $\mathcal{B}$ such that $[\![t]\!] = 1$, $[\![T]\!] = \top$, and $[\![F]\!] = \bot$. It can be extended in the obvious way to preserve the connectives, i.e.,

1. $[\![A \supset B]\!] =_{\text{def}} -[\![A]\!] + [\![B]\!]$;
2. $[\![\neg A]\!] =_{\text{def}} -[\![A]\!]$;
3. $[\![A \circledcirc B]\!] =_{\text{def}} [\![A]\!] \odot [\![B]\!]$;
4. $[\![A^*]\!] =_{\text{def}} [\![A]\!]^*$.

A BML sentence A is true under an interpretation $[\![\cdots]\!]$ iff $\top \leq [\![A]\!]$. A sentence A is valid in $\mathcal{B}$ iff it is true under all interpretations and it is BM-valid iff it is valid in all Boolean monoids $\mathcal{B}$.

Soundness of BML is demonstrated by showing that the axioms of BML are BM-valid and the rules of inference of BML preserve truth. A short proof outline follows:

$A \supset B$ is true in the interpretation $[\![\cdots]\!]$ iff $\top \leq [\![A \supset B]\!]$ iff $\top \leq [\![A]\!] \supset [\![B]\!]$. By the Residuation Lemma 1.0.3, this latter is true iff $\top \wedge [\![A]\!] \leq [\![B]\!]$ iff $[\![A]\!] \leq [\![B]\!]$. Hence $A \supset B$ is true in the interpretation $[\![\cdots]\!]$ iff $[\![A]\!] \leq [\![B]\!]$.

All axioms are true under an interpretation $[\![\cdots]\!]$. The following is an example proving Axiom 6 from Group A is true:

1 $[\![B]\!] \supset [\![C]\!] \leq [\![B]\!] \supset [\![C]\!]$ . . . . identity

2 $[\![B]\!] \wedge [\![B \supset C]\!] \leq [\![C]\!]$ . . . . 1, $\leq$ transitivity

3 $-[\![A]\!] + ([\![B]\!] \wedge [\![B \supset C]\!]) \leq$
$-[\![A]\!] + [\![C]\!]$ . . . . 2, monotonicity of $+$

4 $(-[\![A]\!] + [\![B]\!]) \wedge (-[\![A]\!] + [\![B \supset C]\!]) \leq$
$-[\![A]\!] + [\![C]\!]$ . . . . distribution

5 $([\![A]\!] \supset [\![B]\!]) \wedge ([\![A]\!] \supset [\![B \supset C]\!]) \leq$
$[\![A]\!] \supset [\![C]\!]$ . . . . def. of $\supset$

6 $[\![(A \supset (B \supset C))]\!] \leq [\![((A \supset B) \supset (A \supset C))]\!]$ . . 5, residuation.

The demonstration that each of the rules of inference preserves truth is also routine. The * Introduction rule is an example. Assume $1 + [\![B]\!] \odot [\![B]\!] + [\![A]\!] \leq [\![B]\!]$, then in any BM*, $[\![A]\!]^* \leq [\![B]\!]$ automatically follows.

Thus if a formula is provable in BML (BML*), it is BM (BM*)-valid; this is stated formally as:

THEOREM 2.1.2. *BML (BML*) Logic is sound with respect to Boolean monoids (starred Boolean monoids).*

The following lemma allows the transfer of provability of a sentence A in the logic to the condition $\top \leq [\![A]\!]$ for an arbitrary interpretation in a Boolean monoid:

LEMMA 2.1.3. $\vdash A$ *iff* $\vdash T \supset A$.

Proof. Assume $\vdash A$. From the axioms, $\vdash A \supset (T \supset A)$. From the assumption $\vdash A$, and modus ponens, $\vdash T \supset A$. For the other half, assume $\vdash T \supset A$, then $\vdash T$ is provable (i.e., $C \supset C$ for some wff C), and so $\vdash A$ follows. □

From Birkhoff (1967), a *variety of algebras* is any class of algebras which is closed under homomorphic images, subalgebras and products. Varieties always have free algebras and these free algebras reside in the very same varietal class. The free algebra can be formed by the usual Lindenbaum–Tarski construction on the logic for which the class provides the algebraic models. That is, the set of well-formed formulas is "divided" by bi-implication yielding a carrier set of equivalence classes, and operations on those classes are defined via the equivalence class representatives.

A *quasi-variety of algebras* (see Graetzer, 1979) is any class of algebras which is closed under isomorphic images, subalgebras, products, and ultraproducts. And the same theorem applies, namely that quasi-varieties have free algebras in the same class. From Birkhoff, any class of algebras closed under subalgebras and products has free algebras isomorphic to the algebra produced by the usual free algebra construction. Dunn (2001) points out that (in connection to this construction used in the proof of Birkhoff's theorem on the existence of free algebras) the resulting free algebra need not be in the original class of algebras, but it is isomorphic to one that is. Since quasi-varieties are closed under isomorphisms, the construction actually does yield a free algebra in the original class.

Quasi-varieties satisfy the usual algebraic identity types of axioms and also universal Horn style axioms, i.e., implicational axioms which are considered as universally closed. Boolean monoids with a * operator form a quasi-variety, and the free algebra is formed using the same Lindenbaum–Tarski construction familiar from logics which have algebraic varieties as their models.

THEOREM 2.1.4. *BML (BML*) is complete with respect to Boolean monoids (starred Boolean monoids).*

Proof. The construction of the Lindenbaum–Tarski algebra (LTA) for BML (BML*) is as follows: for formulas A and B, say A *is equivalent to* B and write $A \equiv B$ iff both $\vdash A \supset B$ and also $\vdash B \supset A$. The relation $\equiv$ is an equivalence relation and let $[A]$ refer to the equivalence class of A under $\equiv$. Define $[A] \leq [B]$ iff $\vdash A \supset B$. Define $[A] \supset [B] = [A \supset B]$; $[\neg A] = -[A]$; $[A] \odot [B] = [A \circledcirc B]$; and if BML*, $[A]^* = [A^*]$. One can show that $[A] \leq [B]$ iff $[A] + [B] = [B]$ and it is well known that the LTA forms a Boolean lattice with top element $\top$ and bottom element $\bot$. That it forms a monoid is clear from the axioms. It needs to be shown that the LTA for BML* is a starred Boolean monoid.

* is a well-defined operation on the set of equivalence classes; i.e., if $A \equiv B$, then $A^* \equiv B^*$. This follows directly from the deduction rule * Introduction. Assume $\vdash A \supset B$, then $\vdash B \supset B^*$, hence $\vdash A \supset B^*$ is derivable. From $\vdash t \supset B^*$ and $\vdash (B^* \circledcirc B^*) \supset B^*$, $\vdash (t \vee (B^* \circledcirc B^*) \vee A) \supset B^*$ is derivable. Using * Introduction immediately yields $\vdash A^* \supset B^*$. Since the proof is symmetric in A and B, $\vdash B^* \supset A^*$ is derivable also. Hence $A^* \equiv B^*$.

It must be verified that the set of equivalence classes of BML satisfies the definition of an BM. This is routine. Just as easily, the equivalence classes of BML* must satisfy the conditions of a starred Boolean monoid. The axiom holds in virtue of the definition of the LTA. The story is similar for the rule, i.e., assume $1 + ([B] \odot [B]) + [A] \leq [B]$ then $\vdash (t \vee (B \circledcirc B) \vee A) \supset B$. Using * Introduction, $\vdash A^* \supset B$ and hence $[A]^* \leq [B]$.

Consequently, $[\ldots]$ is an interpretation. Completeness follows via a contraposition argument: assume $\nvdash A$, then by the definition of $\leq$ in the LTA, $\top \not\leq [A]$. Therefore, for all BM (BM*) and for all interpretations, $[\![\ldots]\!]$, $\top \leq [\![A]\!]$ implies $\vdash A$. □

2.2. BOOLEAN MONOID FRAMES

The Kripke frames for Boolean monoids are exactly what one would expect given frames for Boolean algebras and frames for relevance logics. That is, they are collections of points which are maximal filters when the frames arise from a Boolean monoid. And there is a single three place relation which is used in evaluating the monoid operation. Also, there is a set of "zero worlds" (the terminology is from relevance logic) used in evaluating the unit of the monoid.

At the risk of creating confusion with notation, this paper will assume that a BM frame will be denoted as $\mathcal{X} = (X, \mathcal{X}, \mathbb{X})$ where $\mathcal{X} \subseteq X \times X \times X$ is the three place relation on worlds and $\mathbb{X} \subseteq X$ is the collection of zero worlds. The symbol $\mathcal{X}$ is overloaded but since the three place relation is so central to the frame, the reader is asked to overlook this and accept the simplicity it gives to the notation. Context will distinguish the two uses of "$\mathcal{X}$." For X and $\mathbb{X}$, this same letter in the two different fonts means different things, but both are related to the same structure.

DEFINITION 2.2.1. A *Boolean monoid frame*, $\mathcal{X} = (X, \mathcal{X}, \mathbb{X})$, is a structure where X is a set of points, $\mathcal{X} \subseteq X \times X \times X$, and $\mathbb{X} \subseteq X$ and $\mathbb{X} \neq \emptyset$. The following axioms apply:

- $\mathcal{X}^2uvyz$ iff $\mathcal{X}^2u(vy)z$; where

$$\mathcal{X}^2uvyz \text{ iff } \exists x(\mathcal{X}uvx \text{ and } \mathcal{X}xyz),$$

$$\mathcal{X}^2u(vy)z \text{ iff } \exists w(\mathcal{X}uwz \text{ and } \mathcal{X}vyw).$$

- There is some $z \in \mathbb{X}$ such that $\mathcal{X}xzx$ and $\mathcal{X}zxx$.
- For all $y \in \mathbb{X}$, ($\mathcal{X}xyz$ or $\mathcal{X}yxz$) implies $x = z$.

There are, of course, implicit universal quantifications given to the free variables in the above conditions.

2.3. SOUNDNESS

A Boolean monoid of sets can be derived from a BM frame by taking the powerset of the set of worlds to be the carrier set. The operators will be defined below. A BM model is a BM frame together with a valuation $[\![\ldots]\!]$ which assigns a set of worlds to each propositional variable; $[\![\ldots]\!]$ is then extended to assign a set of worlds to each formula. Thus models for both the algebraic semantics and Kripke semantics of BML rely on a valuation of the atomic propositions. For a Boolean monoid of sets, the valuation chooses elements of the carrier set. For a BM model, the valuation chooses a collection of worlds where a proposition is to be *true*. After that, both kinds of valuations are extended via induction on the language structure to cover all the expressions in the language. Hence, the initial valuations for both

Table I. Derivation of the Kripke semantics.

$x \models A$	iff	$x \in [\![A]\!]$ for A a propositional variable		
$x \models T$	iff	$x \in [\![T]\!]$	iff	$x \in X$
$x \models t$	iff	$x \in [\![t]\!]$	iff	$x \in \mathbb{X}$
$x \models \neg A$	iff	$x \in [\![\neg A]\!]$	iff	$x \notin [\![A]\!]$
			iff	$x \not\models A$
$x \models A \supset B$	iff	$x \in [\![A \supset B]\!]$	iff	$x \in -[\![A]\!] \cup [\![B]\!]$
			iff	$x \not\models A$ or $x \models B$
$x \models A \circledcirc B$	iff	$x \in [\![A \circledcirc B]\!]$	iff	$x \in [\![A]\!] \odot [\![B]\!]$
			iff	$\exists y, z(Ryzx$ and $y \in [\![A]\!]$ and $z \in [\![B]\!])$
			iff	$\exists y, z(Ryzx$ and $y \models [\![A]\!]$ and $z \models [\![B]\!])$

kinds of models are essentially the same as long as an algebraic structure is chosen that derives from the BM frame by the framework just described. The algebraic set theoretic model can then be "unwound" into a BM model.

Let $[\![\ldots]\!]$ be a valuation of BML on all the propositional variables. The double turnstile $\models$ for the BM evaluation will mean the set theoretic "element of." Hence $x \models A$ is defined as $x \in [\![A]\!]$ where $[\![\ldots]\!]$ is the interpretation resulting from the usual extension of the valuation of the propositional variables to the connectives. Table I then gives the BM evaluation scheme taken from an interpretation defined on the Boolean monoid of sets that the BM frame provides.

Notice that the following chain of iffs is valid: $\vdash A$ iff $\vdash T \supset A$ iff $[\![T]\!] \subseteq [\![A]\!]$ iff for all $x \in X$, $x \models T$. As a consequence, the following remark can be made:

Remark 2.3.1. All that needs to be shown for soundness with respect to a Kripke semantics, once it is known that BML is sound with respect Boolean monoids, is that every BM frame yields a Boolean monoid of sets and that the BM interpretation conditions arise directly from the definitions yielding this Boolean monoid of sets.

THEOREM 2.3.2. *Let $\mathcal{X} = (X, \mathcal{X}, \mathbb{X})$ be a Boolean monoid frame. Let $\mathcal{X}^\circ = (X^\circ, \cup, \cap, -, X, \emptyset, \hat{\odot}, \mathbb{X})$ where X° is the powerset of X, $(\cup, \cap, -)$ are union, intersection, and relative difference over X°, X and $\emptyset$ are the respective top and bottom of the powerset lattice of sets, $\hat{\odot}$ is defined as*

$$C \mathbin{\hat{\odot}} D = \{z \mid \exists x, y(x \in C \text{ and } y \in D \text{ and } \mathcal{X}xyz)\},$$

and $\mathbb{X}$ is the unit of the monoid operation, $\hat{\odot}$. Then $\mathcal{X}^\circ$ so defined is a Boolean monoid.

Proof. This follows from Stone (1936) and Jónsson and Tarski (1951–1952). See also Sahlqvist (1975), van Benthem (1984), Blackburn et al. (2001). □

2.4. COMPLETENESS

Completeness follows via a contraposition argument using the LTA provided by BML and the fact that a representation theorem can be shown. The representation map, β, takes a Boolean monoid into a Boolean monoid of sets via a BM frame that is generated directly from the LTA. This representation map is shown to be a 1-1 homomorphism. Then one argues as follows: suppose that $\nvdash C$, then $\top \not\leq [C]$ where $[\ldots]$ denotes the equivalence class of C in the LTA. Since the representation map, β, is 1-1 and a homomorphism, then $\beta\top \not\subseteq \beta[C]$. By construction, the map $\beta \circ [\ldots]$ is itself a valuation and hence by the valuation conditions for BM models, there is some world x such that $x \not\models C$. Contraposing the argument yields

$$(\text{for all } x \in X, x \models C) \text{ implies } \vdash C.$$

The work in this section shows that from any Boolean monoid, a *canonical* BM frame can be constructed. In the process, a 1-1 representation homomorphism is constructed into the Boolean monoid of sets derived from the canonical frame. The following theorem is a recap of similar theorems in Jónsson and Tarski (1951–1952) and Dunn (2001).

DEFINITION 2.4.1. Let $\mathcal{B}$ be a Boolean monoid. The *canonical frame* is $\mathcal{B}_\circ = (Y, \mathcal{Y}, \mathbb{Y})$ where Y is the collection of all proper, maximal filters, $\mathcal{Y}$ is defined with

$$\mathcal{Y}xyz \text{ iff } [(a \in x \text{ and } b \in y) \text{ implies } a \odot b \in z],$$

and $\mathbb{Y}$ is defined with

$$z \in \mathbb{Y} \text{ iff } 1 \in z.$$

THEOREM 2.4.2. *Let $\mathcal{B}$ be a Boolean monoid, the canonical frame $\mathcal{B}_\circ$ is a Boolean monoid frame.*

Proof. This follows from Sahlqvist (1975) and van Benthem (1984). See also Blackburn et al. (2001). □

The following theorem (or at least one close to the following) can also be found in Jónsson and Tarski (1951–1952), Dunn (1990), and Routley and Meyer (1973).

THEOREM 2.4.3. *The function $\beta : B \longrightarrow (B_\circ)^\circ$ defined by*

$$\beta a = \{x \mid a \in x \text{ and } x \textit{ is a maximal filter}\}$$

is a 1-1, homomorphism.

3. An Algebra of Relations

Consider the Stone representation theorem for Boolean lattices. Every point in the dual space is a maximal filter. In frames for many substructural logics, there is a partial or quasi-order relation, $\leq$, on points of the frame. For Boolean monoid frames, $\leq$ is the identity relation. This allows for a very close connection between frames and certain binary relations.

In sympathy with the connection between frames and these binary relations, it is possible to construct a relationship between the algebra of sets and an *algebra of relations*. One hesitates to call it a *relation algebra* since it lacks the converse operation.

Assume $\mathcal{X} = (X, \mathcal{X}, \mathbb{X})$ is a Boolean monoid frame. The algebra of relations will be constructed via the definition:

$$\langle A \rangle = \{\langle x, z \rangle \mid \exists y \in A \text{ and } \mathcal{X}xyz\}.$$

This definition begs the question: is it possible to construct a Boolean monoid frame of pairs, and then show that the algebra of relations can be extracted directly from that? To see the issue more clearly, consider the following commutative diagram and recall that commutative diagrams express equalities. Let

$$\mathcal{X} = (X, \mathcal{X}, \mathbb{X}),$$

$$\langle \mathcal{X} \rangle = (X, \langle \mathcal{X} \rangle),$$

$$\mathcal{X}^\circ = (X^\circ, \cup, \cap, -, X, \emptyset, \hat{\circ}, \mathbb{X}),$$

$$\langle \mathcal{X} \rangle^\circ = (\langle X^\circ \rangle, \cup, \cap, -, \langle X \rangle, \langle \emptyset \rangle, \langle \circ \rangle, \langle \mathbb{X} \rangle).$$

$(X, \langle \mathcal{X} \rangle)$ is meant to be a binary relation $\langle \mathcal{X} \rangle$ over the ambient set X. Then the following object diagram should commute

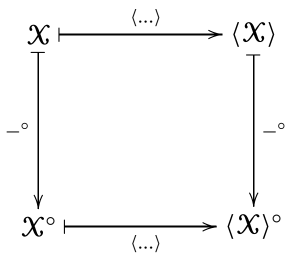


Diagram 1.

To prevent clutter, the $\langle \ldots \rangle$ and the $^\circ$ arrows are parameterized with the category of their input. That is, the $\langle \ldots \rangle$ operating on algebras is certainly different from the $\langle \ldots \rangle$ operating on frames. Likewise, the $^\circ$ operating on frames is different from the $^\circ$ operating on a mere binary relation. This section will show how to fill in all these arrows. In fact, it turns out that in order to fill in the bottom horizontal arrow, the top arrow will need to be invertible. Also, the diagram will not commute as is. The right hand arrow will need to include a closure condition.

3.1. Binary Frames

DEFINITION 3.1.1. A *binary, Boolean monoid frame*, or just *binary frame* is a reflexive, transitive relation.

Of course, now a theorem must be proven to show that $\langle \ldots \rangle$ operating on a Boolean monoid frame really does yield a reflexive, transitive relation.

THEOREM 3.1.2. *Let $\mathcal{X} = (X, \mathcal{X}, \mathbb{X})$ be a Boolean monoid frame, then let $\langle \ldots \rangle$ be defined by*

$$\langle \mathcal{X} \rangle = (X, \{\langle x, z\rangle \mid \exists y(\mathcal{X}xyz)\}).$$

Then $\langle \mathcal{X} \rangle$ is a reflexive, transitive relation.

Proof. All pairs $\langle x, x\rangle \in \langle \mathcal{X} \rangle$ since for every $x \in X$, there is some $z \in \mathbb{X}$ such that $\mathcal{X}xzx$. So $\langle \mathcal{X} \rangle$ is reflexive. Assume $\langle u, x\rangle, \langle x, z\rangle \in \langle \mathcal{X} \rangle$, then there is some v, y such that $\mathcal{X}uvx$ and $\mathcal{X}xyz$, therefore $\mathcal{X}^2uvyz$ holds. From associativity on the Boolean monoid frame, $\mathcal{X}^2u(vy)z$ holds, so there is some w such that $\mathcal{X}uwz$ and $\mathcal{X}vyw$. By definition $\langle u, z\rangle \in \langle \mathcal{X} \rangle$ and $\langle \mathcal{X} \rangle$ is transitive. □

An arrow in the other direction, $\langle\!\cdot \ldots \cdot\!\rangle$, can be defined using the following definition:

DEFINITION 3.1.3. Let $\mathcal{M} = (\mathsf{M}, \mathcal{M})$ be a reflexive, transitive relation and hence $\mathcal{M} \subseteq \mathsf{M} \times \mathsf{M}$. Define $\langle\!\cdot\, \mathcal{M} \,\cdot\!\rangle = (X, \mathcal{X}, \mathbb{X})$ where

$$X = \mathsf{M} \text{ and } \mathbb{X} = \{\langle \mathsf{a}, \mathsf{a}\rangle \mid \langle \mathsf{a}, \mathsf{a}\rangle \in \mathcal{M}\}.$$

And define $\mathcal{X}$ with

$$\mathcal{X}xyz \text{ iff } xy = z$$

where the elided operation between x and y is a partial operation defined by

$$xy = z \text{ iff } x = \langle \mathsf{a}, \mathsf{b}\rangle \text{ and } y = \langle \mathsf{b}, \mathsf{c}\rangle \text{ and } z = \langle \mathsf{a}, \mathsf{c}\rangle.$$

THEOREM 3.1.4. *Let $\langle\!\cdot\, \mathcal{M} \,\cdot\!\rangle = (X, \mathcal{X}, \mathbb{X})$ be defined as above, then $\langle\!\cdot\, \mathcal{M} \,\cdot\!\rangle$ is a Boolean monoid frame.*

Proof. $\underline{\mathcal{X}^2uvyz \text{ iff } \mathcal{X}^2u(vy)z}$: Assume $\mathcal{X}^2uvyz$, then there exists some x such that $\mathcal{X}uvx$ and $\mathcal{X}xyz$. Since elements in the relation are pairs, and using the definition of $\mathcal{X}$, let

$$u = \langle \mathsf{a}, \mathsf{b}\rangle, \quad v = \langle \mathsf{b}, \mathsf{c}\rangle, \quad x = \langle \mathsf{a}, \mathsf{c}\rangle, \quad y = \langle \mathsf{c}, \mathsf{d}\rangle, \quad z = \langle \mathsf{a}, \mathsf{d}\rangle.$$

Since $\mathcal{M}$ is transitive, there is a w such that

$$w = \langle \mathsf{b}, \mathsf{d}\rangle = \langle \mathsf{b}, \mathsf{c}\rangle\langle \mathsf{c}, \mathsf{d}\rangle = vy, \quad z = \langle \mathsf{a}, \mathsf{d}\rangle = \langle \mathsf{a}, \mathsf{b}\rangle\langle \mathsf{b}, \mathsf{d}\rangle = uw.$$

Therefore, there exists some w such that $\mathcal{X}uwz$ and $\mathcal{X}vyw$, i.e., $\mathcal{X}^2u(vy)z$. The other direction is similar.

There is some $z \in \mathbb{X}$ such that $\mathcal{X}xzx$ and $\mathcal{X}zxx$: Let $x = \langle \mathfrak{a}, \mathfrak{b} \rangle$, then, since $\mathcal{M}$ is reflexive, choose $z = \langle \mathfrak{b}, \mathfrak{b} \rangle$. By definition, $xz = x$, hence $\mathcal{X}xyz$. Now choose $z = \langle \mathfrak{a}, \mathfrak{a} \rangle$. By definition, $zx = x$ hence $\mathcal{X}zxx$.

For all $y \in \mathbb{X}$, ($\mathcal{X}xyz$ or $\mathcal{X}yxz$) implies $x = z$: Assume $\mathcal{X}xyz$ and $x = \langle \mathfrak{a}, \mathfrak{b} \rangle$. Since $y \in \mathbb{X}$, and $\mathcal{X}xyz$, $y = \langle \mathfrak{b}, \mathfrak{b} \rangle$ and $z = xy = \langle \mathfrak{a}, \mathfrak{b} \rangle = x$. Assume $\mathcal{X}yxz$ and $x = \langle \mathfrak{a}, \mathfrak{b} \rangle$. Since $y \in \mathbb{X}$, and $\mathcal{X}yxz$, $y = \langle \mathfrak{a}, \mathfrak{a} \rangle$ and $z = yx = \langle \mathfrak{a}, \mathfrak{b} \rangle = x$. □

The arrow $\langle \ldots \rangle$ on frames has a left-inverse just when $\langle \ldots \rangle$ is injective. To define a left inverse will require that an extra condition be added to the frame. This extra condition turns out to be similar to an operation Pratt (1990b) chose for showing "simple representable Boolean monoids" are isomorphic to a subalgebra of a Boolean monoid. However, Pratt is working with the first order theory (essentially) of the Boolean monoid frames. The theory and language he uses are not considered by him to be expressing the first order theory of frames for Boolean monoids. But it is surprising and mutually justifying that the different approaches yield essentially the same conclusion.

To make $\langle \ldots \rangle$ injective, clearly it must be that

$$(\mathcal{X}xyz \text{ and } \mathcal{X}xy'z) \text{ implies } y = y'.$$

Consider two objects, $\mathcal{X} = (X, \mathcal{X}, \mathbb{X})$ and $\mathcal{Y} = (Y, \mathcal{Y}, \mathbb{Y})$. For $\langle \ldots \rangle$ to yield the same object, it is necessary that $X = Y$ since if they differed on some x, then (without loss of generality) $\langle x, x \rangle$ would be in $\langle \mathcal{X} \rangle$ and $\langle x, x \rangle$ would not be in $\langle \mathcal{Y} \rangle$. So, assume $X = Y$. For $\langle \mathcal{X} \rangle = \langle \mathcal{Y} \rangle$ as structures, it is necessary that $\langle \mathcal{X} \rangle = \langle \mathcal{Y} \rangle$ as relations. The above condition imposed on the frames $\mathcal{X}$ and $\mathcal{Y}$ assures that $\mathcal{X}xyz$ iff $\langle \mathcal{X} \rangle xz$ iff $\langle \mathcal{Y} \rangle xz$ iff $\mathcal{Y}xyz$.

It must also be shown that $\mathbb{X} = \mathbb{Y}$. Let $y \in \mathbb{X}$. Since $X = Y$, there is some $z' \in \mathbb{Y}$ such that $\mathcal{Y}z'yy$. And since $\mathcal{X} = \mathcal{Y}$, $\mathcal{X}z'yy$. However, $y \in \mathbb{X}$ and hence $z' = y$ and therefore $y \in \mathbb{Y}$. Since the argument is symmetric, $\mathbb{Y} \subseteq \mathbb{X}$ also.

Incidentally, this leads to $\mathcal{X}yyy$ for all $y \in \mathbb{X}$ and thus that $\mathcal{X}yzy$ for any z implies $z = y$.

3.2. Extracting an Algebra of Relations

THEOREM 3.2.1. *Let $\mathcal{X} = (X, \mathcal{X}, \mathbb{X})$ be a Boolean monoid frame with the additional axiom*

$$\mathcal{X}xyz \text{ and } \mathcal{X}xy'z \text{ implies } y = y'.$$

Then $\langle \ldots \rangle$ is a homomorphism from the Boolean monoid of sets to a Boolean monoid of relations.

Proof. $\underline{\langle A \cup B \rangle = \langle A \rangle \cup \langle B \rangle}$: $\langle x, z \rangle \in \langle A \cup B \rangle$; iff there is some y such that $y \in A \cup B$ and $\mathcal{X}xyz$; iff ($y \in A$ or $y \in B$) and $\mathcal{X}xyz$; iff ($y \in A$ and $\mathcal{X}xyz$) or ($y \in B$ and $\mathcal{X}xyz$); iff $\langle x, z \rangle \in \langle A \rangle \cup \langle B \rangle$.

$\underline{\langle A \cap B \rangle = \langle A \rangle \cap \langle B \rangle}$: The proof is similar to the previous case.

$\underline{\langle X - A \rangle = \langle X \rangle - \langle A \rangle}$: Assume $\langle x, z \rangle \in \langle X \rangle - \langle A \rangle$. Then there is some y such that $\mathcal{X}xyz$ and $\langle x, z \rangle \notin \langle A \rangle$. Therefore $y \notin A$, and hence $y \in X - A$, so $\langle x, z \rangle \in \langle X - A \rangle$.

Assume $\langle x, z \rangle \in \langle X - A \rangle$, then there is at least one y such that $\mathcal{X}xyz$ and $y \notin A$. Also, $\langle x, z \rangle \notin \langle A \rangle$ since otherwise there would be some $y' \in A$ and $\mathcal{X}xy'z$. Given the additional axiom on the frame, $y = y'$ and that would be a contradiction. Hence $\langle x, z \rangle \in \langle X \rangle$ and $\langle x, z \rangle \notin \langle A \rangle$, yielding $\langle x, z \rangle \in \langle X \rangle - \langle A \rangle$.

$\underline{\langle A \mathbin{\hat{\odot}} B \rangle = \langle A \rangle \langle \circ \rangle \langle B \rangle}$: Let $\langle u, z \rangle \in \langle A \rangle \langle \circ \rangle \langle B \rangle$, then there is some x such that $\langle u, x \rangle \in \langle A \rangle$ and $\langle x, z \rangle \in \langle B \rangle$. Hence there is some $v \in A$ and $y \in B$ such that $\mathcal{X}uvx$ and $\mathcal{X}xyz$. Using the definitions for $\mathcal{X}^2$, this yields $\mathcal{X}^2uvyz$. And associativity is assumed, hence $\mathcal{X}^2u(vy)z$ holds. Unpacking this yields there is some w such that $\mathcal{X}uwz$ and $\mathcal{X}vyw$. Combining conditions together, $\mathcal{X}vyw$ and $v \in A$ and $y \in B$. Therefore $w \in A \odot B$. Combining this last fact with $\mathcal{X}uwz$ yields $\langle u, z \rangle \in \langle A \mathbin{\hat{\odot}} B \rangle$.

The other direction is similar using the other direction for the associativity condition on $\mathcal{X}$. Let $\langle u, z \rangle \in \langle A \mathbin{\hat{\odot}} B \rangle$. Then there is some $w \in A \mathbin{\hat{\odot}} B$ such that $\mathcal{X}uwz$. Since $w \in A \mathbin{\hat{\odot}} B$ there is some $v \in A$ and $y \in B$ such that $\mathcal{X}vyw$. Hence $\mathcal{X}^2u(vy)z$ holds, and from associativity, $\mathcal{X}^2uvyz$. Therefore, there is some x such that $\mathcal{X}uvx$ and $\mathcal{X}xyz$. Hence $\langle u, x \rangle \in \langle A \rangle$ and $\langle x, z \rangle \in \langle B \rangle$, and $\langle u, z \rangle \in \langle A \rangle \langle \circ \rangle \langle B \rangle$.

$\underline{\langle \mathbb{X} \rangle = \{\langle x, x \rangle\}}$: Since for all x, $\mathcal{X}xzx$ holds for at least one $z \in \mathbb{X}$, the diagonal relation is at least contained in $\langle \mathbb{X} \rangle$. In the other direction, if $\mathcal{X}xyz$ and $y \in \mathbb{X}$, then $x = z$ and so $\langle \mathbb{X} \rangle \subseteq \{\langle x, x \rangle\}$. □

Notice that the top of this lattice of relations, $\langle X \rangle$, is not all of $X \times X$ but only the elements $\langle x, z \rangle$ of $X \times X$ for which there exist some y and $\mathcal{X}xyz$.

Next, the arrow $-^\circ$ on binary frames must be defined.

DEFINITION 3.2.2. Let $(\mathsf{M}, \mathcal{M})$ be a reflexive, transitive relation. Define

$$(\mathsf{M}, \mathcal{M})^\circ =_{\text{def}} (\mathcal{M}^\circ, \cup, \cap, -, \mathcal{M}, \emptyset, \mathbb{M}, \hat{\odot})$$

where $\mathcal{M}^\circ = \{A \mid A \subseteq \mathcal{M}\}$. The operations $\cup$, $\cap$, and $-$ are the usual Boolean set operations over $\mathcal{M}^\circ$. $\mathcal{M}$ and $\emptyset$ are the top and the bottom of this lattice of sets, $\mathbb{M} = \{\langle x, x \rangle \mid \langle x, x \rangle \in \mathcal{M}\}$, and $\hat{\odot}$ is the relative product of the relations over $\mathcal{M}^\circ$.

Let $\mathcal{X} = (X, \mathcal{X}, \mathbb{X})$ be a Boolean monoid frame and let $(\mathsf{M}, \mathcal{M}) = (X, \langle \mathcal{X} \rangle)$, then one would like to show $\langle X^\circ \rangle = (\mathsf{M}, \mathcal{M})^\circ$. However, this is not the case. The reason is that information is lost going from $(X, \mathcal{X}, \mathbb{X})$ to $(X, \langle\mathcal{X}\rangle)$. In particular, let C be some set of elements of $\langle\mathcal{X}\rangle$, then for any $\langle x, z\rangle \in C$, there is some y such that $\mathcal{X}xyz$. There can easily be another pair $\langle x', z'\rangle$ such that $\mathcal{X}x'yz'$, but $\langle x', z'\rangle \notin C$. Consequently, C need not be a set in $\langle X^\circ\rangle$.

The upshot is that to cause the square Diagram 1 to commute, sets in the lattice constructed from $(X, \langle\mathcal{X}\rangle)$ must be closed up under $\mathcal{X}$.

$$\overline{C} = \{\langle x', z'\rangle \mid \exists x, y, z(\langle x, z\rangle \in C \text{ and } \mathcal{X}xyz)\}.$$

THEOREM 3.2.3. *Diagram 1 commutes as long as $\mathcal{X}$ closed sets are used for the right hand arrow.*

Proof. Let $\overline{C} \in \mathcal{M}^\circ$, and let

$$B = \{y \mid \exists x, y, z((\langle x, z\rangle \in C \text{ and } \mathcal{X}xyz)\}$$

is a set in X°, and $\langle B\rangle = \overline{C}$.

Let $\langle B\rangle \in \langle X^\circ\rangle$, then clearly $\langle B\rangle = \overline{\langle B\rangle}$, and hence $\langle B\rangle \in \mathcal{M}^\circ$. Therefore, the two carrier sets agree, and hence will agree on their set operations. It is obvious the two set algebras also agree on $\langle\mathbb{X}\rangle$. □

Canonically, an extra axiom on Boolean monoids is needed to complete the correspondence theory between properties of Boolean monoids and conditions on their frames.

THEOREM 3.2.4. *Let $\mathcal{B}$ be a Boolean monoid with the axiom*

$$(a \odot b) \wedge (a \odot -b) = \bot,$$

and let $\mathcal{B}_\circ = (Y, \mathcal{Y}, \mathbb{Y})$ be the canonical frame. Then

$$\mathcal{X}xyz \text{ and } \mathcal{X}xy'z \text{ implies } y = y'$$

is satisfied.

Proof. All the worlds in the canonical frame are maximal worlds, hence if y and y' differ, they must at least differ on some element b such that $b \in y$ and $-b \in y'$. Since x cannot be empty, there is some $a \in x$ such that $a \odot b \in z$ and $a \odot -b \in z$. Since z is a filter, $(a \odot b) \wedge (a \odot -b) \in z$. From the axiom, $\bot \in z$ and therefore z is not a proper filter which is a contradiction to the requirement that all filters in the canonical frame be proper. □

4. Adding Reflexive, Transitive Closure

The star, *, operator considered in this section is somewhat weaker than the Kleene * operator which has been considered in the literature (Pratt, 1990b; Kozen, 1990). However, in the presence of the residuals, $\leftarrow$ and $\rightarrow$, it turns out to be equivalent.

4.1. Relative Modal Operators

The reason there is no simple first order condition on the frames that allows the definition of the * operator is that while the * operator is monotone with respect to the order relation on the lattice, the * operator does not distribute over either of the lattice operators $\wedge$ and $+$. This has the effect of forcing a neighborhood construction. The situation is the same in modal logics when, say, the $\Diamond$ does not distributive over $+$. Both directions of the distribution can be denied, i.e.,

$$\Diamond a + \Diamond b \leq \Diamond(a + b), \qquad \Diamond(a + b) \leq \Diamond a + \Diamond b,$$

to get a weaker logic (Chellas, 1980).

The weaker frames lack the two-place Kripke relation and have in its place a function. Let (X, N) be a modal frame where X is a collection of worlds and N is a function $N : X \longrightarrow \mathcal{P}(\mathcal{P}X)$ where $\mathcal{P}$ is the powerset operator. Then, N returns a collection of subsets of X for each point. There are at least two notions to consider here which make the resulting structure weaker than a typical modal frame. First, it is not necessary that N_x (N evaluated at x) must return sets such that x is actually a member of any of them. That condition (below on right) requires the following modal condition (on the left) in the correspondence theory of algebraic conditions and their frame counterparts:

$$c \leq \Diamond c, \qquad C \in N_x \text{ implies } x \in C.$$

The second notion is that the range of N need not include all of the sets in the dual algebra of sets. The above axiom and condition actually only work when $\Diamond c = -\Box - c$. And in fact, the set N_x is really closely tied to the $\Box$ operator. The operator $\Box$ on the algebra of sets is defined as:

Q1: $\Box C = \{x \mid C \in N_x\}$.

In anticipation of the semantics for the * operator, assume that the $\Box$ operator comes with the following two axioms (on the left) and their related neighborhood conditions (on the right):

Q2: $\Box c \leq c$ $\qquad$ $C \in N_x$ implies $x \in C$
Q3: $c \leq b$ implies $\Box c \leq \Box b$ $\qquad$ ($C \in N_x$ and $C \subseteq B$) implies $B \in N_x$.

Given these two axioms, there is another way to characterize $\Box C$.

LEMMA 4.1.1. *Let $\mathcal{N}$ be a collection of sets such that $\emptyset \in \mathcal{N}$ and make the following identification:*

$$C \in N_x \text{ iff } \exists B\ (B \in \mathcal{N} \text{ and } x \in B \text{ and } B \subseteq C).$$

The $\Box$ operator in the algebra of sets is also characterized by

Q4: $\Box C = \bigcup\{B \mid B \subseteq C \text{ and } B \in \mathcal{N}\}$.

Proof. Assume $\Box C$ is characterized by Q1–Q3, and let $x \in \Box C$. Then $C \in N_x$ and therefore $x \in C$. Hence, letting B be C, there is some $B \in \mathcal{N}$ such that $x \in B$ and $B \subseteq C$. Therefore $x \in \Box C$ as characterized by Q4. To go the other way, assume $\Box C$ is characterized by Q4, and let $x \in \Box C$. Therefore there is some $B \in \mathcal{N}$ such that $x \in B$ and $B \subseteq C$. In this case, $C \in N_x$. Also, if $C \in N_x$ and $C \subseteq D$, then there is some $B \in \mathcal{N}$ such that $x \in B$ and $B \subseteq C$. But in this case, $B \subseteq D$, and hence $D \in N_x$ also, so N_x is closed upward under $\subseteq$. □

Notice that $\emptyset \in \mathcal{N}$ is a seemingly innocuous specification to $\mathcal{N}$. However, it comes in handy for $\Diamond$. Assume that $\Diamond c = -\Box - c$ and that there are the following two axioms:

- $c \leq \Diamond c$,
- $c \leq b$ implies $\Diamond c \leq \Diamond b$.

The condition on the neighborhoods, expressed in terms of neighborhood sets in $\mathcal{N}$, then becomes

$$\Diamond C =_{\text{def}} \bigcap\{B \mid C \subseteq B \text{ and } X - B \in \mathcal{N}\}.$$

Having $\emptyset \in \mathcal{N}$ guarantees that $\Diamond C$ will be greater than C. Clearly one can show that

$$C \subseteq \Diamond C, \text{ and } C \subseteq D \text{ implies } \Diamond C \subseteq \Diamond D.$$

The latter holds because

$$\{B \mid D \subseteq B \text{ and } X - B \in \mathcal{N}\} \subseteq \{B \mid C \subseteq B \text{ and } X - B \in \mathcal{N}\}.$$

Notice there is no restriction on what the set of neighborhoods contains, only that they be sets from the algebra of sets. $\mathcal{N}$ need not include all the sets in this algebra. Clearly, the neighborhoods to consider for the $\Diamond$ operator are neighborhoods included in "closed sets" of a topology as opposed to neighborhoods including "open sets." It is now clear that the link between $\Box$ and $\Diamond$ can be severed and all that is needed is some collection of "closed" sets, $\mathcal{N}^c$.

Suppose that $c \leq b$ implies $\Diamond c \leq \Diamond b$ is replaced by

$$c \leq \Diamond b \text{ implies } \Diamond c \leq \Diamond b.$$

This says that $\Diamond c$ is the smallest $\Diamond$ element between c and $\Diamond b$; for if $c \leq \Diamond a \leq \Diamond b$, then $\Diamond c \leq \Diamond a$. This Horn style axiom also implies the monotonicity of $\Diamond$ with respect to the lattice order:

$$c \leq b \text{ implies } c \leq \Diamond b \text{ implies } \Diamond c \leq \Diamond b.$$

Curiously, the definition of $\Diamond C$ remains the same:

$$\Diamond C =_{\text{def}} \bigcap \{B \mid C \subseteq B, B \in \mathcal{N}^c\},$$

where $\mathcal{N}^c$ just refers to a collection of sets (including X) from which to take sets used in the intersection.

Further, assume the following axiom (on the left) and the associated definition in the algebra of sets (on the right):

$$\Diamond\Diamond c \leq \Diamond c, \qquad C \in N_x \text{ implies } \Box C \in N_x.$$

This requires that the set $\mathcal{N}^c$ be closed under the operator $\Diamond$. That is, it is required to show that $\Diamond\Diamond C \subseteq \Diamond C$. Assume $x \in \Diamond\Diamond C$, then for all $B \in \mathcal{N}^c$, $\Diamond C \subseteq B$ implies $x \in B$. Since $\mathcal{N}^c$ is closed under $\Diamond$ and $\Diamond C \subseteq \Diamond C$, then $x \in \Diamond C$.

One might think to define

$$\Diamond C =_{\text{def}} \bigcap \{B \mid C \subseteq B, \Diamond B \subseteq B\},$$

in order to pick out of all the sets in the algebra of sets those that are $\Diamond$ closed, but this is an impredicative definition since $\Diamond$ is appearing on both sides. However, if closure could be defined with respect to another operator whose definition was already assured, then all would be well.

There is an operator, $\Diamonddot$, called *backwards possibility*, which can be residuated with the necessity operator, $\Box$, of modal algebras. That is, $\Diamonddot$ satisfies

$$\Diamonddot a \leq b \text{ iff } a \leq \Box b.$$

Just as $\Box$ distributes over meets, $\Diamonddot$ will distribute over joins.

The Kripke frame relation has two equivalent definitions,

$$Rxy \text{ iff } (\Box a \in x \text{ implies } a \in y) \text{ iff } (a \in x \text{ implies } \Diamonddot a \in y),$$

where R is the usual two place Kripke relation. The operator in the algebra of sets is constructed via

$$\Diamonddot C = \{y \mid \exists x \; Rxy \text{ and } x \in C\}.$$

For the rest of this subsection, only the $\Diamonddot$ operator will be assumed.

Considering the axioms for the * operator, notice that they do not involve negation. In effect, distributive lattices would work just as well to host the * operator (and weaker algebras yet do also, i.e., action algebras; see Pratt, 1990a). The $\Diamond$ acts as a weak closure operator. Let us assume that there is a distributive lattice and that a very weak *relative modal operator*, $\Diamond$, is desired. Relative modality means that $\Diamond$ is a modality relative to $\Diamonddot$. It should satisfy the following two axioms:

- $c + \Diamonddot\Diamond c \leq \Diamond c$,
- $c + \Diamonddot b \leq b$ implies $\Diamond c \leq b$.

The first says that $\Diamond c$ is bigger than c and that $\Diamond$ is left $\hat{\Diamond}$-closed. The second says that $\Diamond c$ is the smallest left $\hat{\Diamond}$-closed element between c and any left $\hat{\Diamond}$-closed element b. In the algebra of sets, this leads to the definition

$$\Diamond C =_{\text{def}} \bigcap \{B \mid C \subseteq B \text{ and } \hat{\Diamond} B \subseteq B\}.$$

Because there is the known operator $\hat{\Diamond}$, it is possible to sift the algebra of sets for the left $\hat{\Diamond}$-closed ones. One need not start with some arbitrary collection of neighborhoods. This condition for computing $\Diamond C$ says that not all neighborhoods, i.e., not all sets in the algebra of sets will be considered, only ones which are left $\hat{\Diamond}$-closed and in which C is included.

So it is relatively clear that the semantics in terms of neighborhoods presented here is merely a new twist on the established method of using neighborhoods for weak modal operators. In fact, the new notion is that of *relative modal operators*.

Incidentally, from the axioms on the modal algebras listed above, one can easily prove that

$$c \leq b \text{ implies } \Diamond c \leq \Diamond b.$$

The point of all this is that a neighborhood semantics for a weak $\Diamond$ operator can be captured by using a collection of sets and taking intersections. The main difference between the $\Diamond$ as a weak modal operator and $\Diamond$ as a relative modal operator is that the semantics for the latter can be captured by sifting through all the sets in the lattice of sets looking for $\hat{\Diamond}$ closed ones whereas the former required that a set of sets be somehow given externally for the given structure.

4.2. THE * AXIOMS

To the Boolean monoid, the following axioms for a reflexive, transitive closure operation, *, may be added:

1. $1 + (a^* \odot a^*) + a \leq a^*$;
2. $1 + (b \odot b) + a \leq b$ implies $a^* \leq b$.

The second axiom is not algebraic but is instead the Horn sentence taken from action algebras (Pratt, 1990a). It allows one to say that a^* is the smallest b such that b is reflexively and transitively closed and such that a is "contained" in b. The lattice is simply too impoverished to express reflexive, transitive closure as an infinite join.

DEFINITION 4.2.1. Let $\mathcal{X} = (X, \mathcal{X}, \mathbb{X})$ be a Boolean monoid frame and A a set of points of X, then

$$A^* =_{\text{def}} \bigcap \{B \mid \mathbb{X} \subseteq B,\ B \mathbin{\hat{\odot}} B \subseteq B,\ A \subseteq B\}.$$

This turns out to be correct definition, and soundness and completeness of the axioms can be shown for it.

THEOREM 4.2.2. *Let* $\mathcal{X} = (X, \mathcal{X}, \mathbb{X})$ *be a Boolean monoid frame, then* A^* *satisfies the axioms.*

Proof. Let

$$U_A = \{B \mid \mathbb{X} \subseteq B, B \hat{\odot} B \subseteq B, A \subseteq B\},$$

then $A^* = \bigcap U_A$. By definition, $\mathbb{X}, A \subseteq B$, for all $B \in U_A$, hence $\mathbb{X}, A \subseteq A^*$. Let $z \in A^* \hat{\odot} A^*$, then there is some $x, y \in A^*$ such that $\mathcal{X}xyz$. Since $x, y \in A^*$, $x, y \in B$ for all $B \in U_A$, therefore $z \in B \hat{\odot} B$ and so $z \in B$. Consequently, $z \in A^*$. This shows that

$$\mathbb{X} \cup (A^* \hat{\odot} A^*) \cup A \subseteq A^*.$$

For the second axiom, assume

$$\mathbb{X} \subseteq C \text{ and } C \hat{\odot} C \subseteq C \text{ and } A \subseteq C.$$

Then $C \in U_A$ and hence $A^* = \bigcap U_A \subseteq C$. □

THEOREM 4.2.3. $\beta a^* = (\beta a)^*$.

Proof. Let $x \in \beta a^*$, then $a^* \in x$. Assume that $\beta b \in U_{\beta a}$. From the definition of $U_{\beta a}$,

$$\beta 1 \cup (\beta b \hat{\odot} \beta b) \cup \beta a \subseteq \beta b.$$

Since β has already been shown to be a homomorphism, this is equivalent to $\beta(1 + (b \odot b) + a) \subseteq \beta b$. Since $\cup$ is a lattice operation, $\beta(1 + (b \odot b) + a) \cup \beta b = \beta b$ and hence $\beta(1 + (b \odot b) + a + b) = \beta b$. The homomorphism β is 1-1, hence $1 + (b \odot b) + a + b = b$ and therefore $1 + (b \odot b) + a \leq b$. Given the Horn axiom, $a^* \leq b$ and therefore $\beta a^* \subseteq \beta b$. Hence, $x \in \beta a^*$ implies $x \in \beta b$ for all b such that $\beta b \in U_{\beta a}$, therefore $x \in A^* = \bigcap U_{\beta a}$.

To go the other way, observe from the first axiom that $\beta a^* \in U_{\beta a}$. Hence $(\beta a)^* \subseteq \beta a^*$. □

Consider now the algebra of relations:

DEFINITION 4.2.4. Let $\mathcal{X} = (X, \mathcal{X}, \mathbb{X})$ be a Boolean monoid frame, then

$$\langle U \rangle_{\langle A \rangle} =_{\text{def}} \{\langle B \rangle \mid \langle \mathbb{X} \rangle \subseteq \langle B \rangle, \langle B \rangle \hat{\odot} \langle B \rangle \subseteq \langle B \rangle, \langle A \rangle \subseteq \langle B \rangle\},$$

and

$$\langle A \rangle^* =_{\text{def}} \bigcap \langle U \rangle_{\langle A \rangle}.$$

This is a common definition for the reflexive, transitive closure of relations. Since * is an extra operator, it remains to show:

THEOREM 4.2.5.

$$\langle A^* \rangle = \langle A \rangle^*.$$

Proof. Since $\langle \ldots \rangle$ is a homomorphism, $\langle U_A \rangle = \langle U \rangle_{\langle A \rangle}$ where $\langle U_A \rangle$ is the map $\langle \ldots \rangle$ applied pointwise (setwise) to the elements of U_A.

Let $\langle x, z \rangle \in \langle A^* \rangle$. There is some $y \in A^*$ such that $\mathcal{X}xyz$. Then $y \in B$ for all $B \in U_A$, and consequently, $\langle x, z \rangle \in \langle B \rangle$. Since $\langle B \rangle \in \langle U_A \rangle$, then $\langle B \rangle \in \langle U \rangle_{\langle A \rangle}$ and hence $\langle x, z \rangle \in \langle A \rangle^*$.

To go the other way, notice that $A^* \in U_A$. Hence $\langle A^* \rangle \in \langle U \rangle_{\langle A \rangle}$, and therefore $\langle A \rangle^* \subseteq \langle A^* \rangle$. □

4.3. UNIFORMITIES

There is a connection between the * operator and uniformities (James, 1987) in topology. In order to state the connection of this definition to uniformities, let us first consider an addition to Boolean monoids, antilogism. Antilogism is necessary in order to get the symmetry condition of uniformities to work out.

It will become clear that the axioms for the * operator in the unaltered algebras are essentially abstract forms for quasi-uniformities (Fletcher and Lindgren, 1982) which are weak uniformities in that the symmetry condition is missing. As such, these sorts of definitions are very much in line with the use of neighborhood semantics for relative modal operators. As before, the * operator will be the relative modal operator defined with respect to $\odot$.

DEFINITION 4.3.1. The following property is called *antilogism*:

$$a \odot b \leq c \text{ iff } -c \odot b \leq -a.$$

and corresponds to the logic rule

$$\frac{A \circledcirc B \supset C}{\neg C \circledcirc B \supset \neg A} \text{ rule antilogism}$$

THEOREM 4.3.2. *Let* $\mathcal{X} = (X, \mathcal{X}, \mathbb{X})$ *be a Boolean monoid frame, then the* antilogism frame condition*,*

$$\mathcal{X}xyz \text{ implies } \mathcal{X}zyx,$$

holds in a canonical frame by antilogism when the original Boolean monoid satisfies antilogism.

Proof. Assume $\mathcal{X}xyz$ and that $b \in z$ and $a \in y$. Towards a reductio, assume $-(b \odot a) \in x$, then $-(b \odot a) \odot a \in z$. However, $b \odot a \leq b \odot a$, and from antilogism, $-(b \odot a) \odot a \leq -b$. Since z is a maximal filter, $-b \in z$ which contradicts $b \in z$. Therefore $b \odot a \in x$. □

This is enough to verify completeness of the logic with rule antilogism added. The following theorem verifies soundness.

THEOREM 4.3.3. *Let $\mathcal{X} = (X, \mathcal{X}, \mathbb{X})$ be a Boolean monoid frame satisfying the antilogism frame condition, then $A \hat{\odot} B \subseteq C$ implies $-C \hat{\odot} B \subseteq -A$ where A, B, and C are sets of points of the frame.*

Proof. Assume $A \hat{\odot} B \subseteq C$ and that $z \in -C \hat{\odot} B$. Then there exists x, y such that $\mathcal{X}xyz$, $x \in -C$, and $y \in B$. From the antilogism frame condition, $\mathcal{X}zyx$ holds. Further assume (for reductio) that $z \in A$. By definition, $x \in A \odot B$, and from the first assumption, $x \in C$ which is a contradiction. Hence $z \in -A$ as desired. □

The import of antilogism is that every $\langle A \rangle$ in the algebra of relations is symmetric. This sets the stage for the consideration of uniformities* as a form of neighborhood system on binary frames in sympathy with the neighborhood systems considered earlier. Then the definition of A^* from a Boolean monoid frame can be seen as a weaker, abstract form of a neighborhood system involving uniformities.

Frames in this section will now be assumed to be endowed with this extra condition.

DEFINITION 4.3.4. A *uniformity* on a set X is a filter on $X \times X$ consisting of *entourages*, i.e., relations, such that

1. each entourage D contains the diagonal,
2. if D is an entourage, then for some entourage E, $E \subseteq D^{-1}$,
3. if D is an entourage, then for some entourage D', $D' \circ D' \subseteq D$,

where D^{-1} refers to the relational converse of D and $\circ$ stands for the relational composition operator.

DEFINITION 4.3.5. A *uniformity base* on a set X is any family of subsets of $X \times X$ such that the family satisfies the three conditions to be a uniformity and is a filter base, i.e., closed under finite intersections.

Now consider the property that $\mathcal{X}xyz$ implies $\mathcal{X}zyx$. This will force $\langle A \rangle = \langle A \rangle^{-1}$ where the latter is the reversal of $\langle A \rangle$.

The definition for $\langle A \rangle^*$ can now be seen to define a uniformity base on the set X.

* The notion that uniformities might have something to do with relevance logics is due to J. Michael Dunn who mentioned it once to one of the authors years ago.

THEOREM 4.3.6. *The collection of sets*

$$\langle U_A\rangle = \{\langle B\rangle \mid \langle \mathbb{X}\rangle \subseteq \langle B\rangle, \langle B\rangle \mathbin{\hat{\odot}} \langle B\rangle \subseteq \langle B\rangle, \langle A\rangle \subseteq \langle B\rangle\}$$

satisfies the definition of a uniformity base when the condition

$$\mathcal{X}xyz \text{ implies } \mathcal{X}zyx$$

is imposed on a frame.

Proof.

1. $\langle Z\rangle$ is the diagonal and is a subset of every set $\langle B\rangle$ in $\langle U_A\rangle$.
2. If $\langle B\rangle$ is an entourage, then $\langle B\rangle^{-1} = \langle B\rangle$, hence there exists some entourage E such that $E \subseteq \langle B\rangle^{-1}$.
3. Let $\langle B\rangle$ be an entourage, then since $\langle B\rangle \mathbin{\hat{\odot}} \langle B\rangle \subseteq \langle B\rangle$, set $D' = \langle B\rangle$ and the $\circ$ operator to be the $\odot$ operator.
4. $\langle U_A\rangle$ is a uniformity base: clearly, $\langle \mathbb{X}\rangle \in \langle B\rangle, \langle C\rangle$ and hence $\langle \mathbb{X}\rangle \in \langle B\rangle \cap \langle C\rangle$.

   $\langle x, z\rangle \in \langle B\rangle, \langle C\rangle$ iff $\langle x, z\rangle \in \langle B\rangle \cap \langle C\rangle$ iff $\langle z, x\rangle \in (\langle B\rangle^{-1} \cap \langle C\rangle^{-1})$ iff $\langle z, x\rangle \in (\langle B\rangle \cap \langle C\rangle)^{-1}$.

   Let $\langle x, z\rangle \in ((\langle B\rangle \cap \langle C\rangle) \mathbin{\hat{\odot}} (\langle B\rangle \cap \langle C\rangle))$, then there is some y such that $\langle x, y\rangle, \langle y, z\rangle \in \langle B\rangle \cap \langle C\rangle$. Hence $\langle x, y\rangle, \langle y, z\rangle \in \langle B\rangle, \langle C\rangle$. Since $\langle B\rangle, \langle C\rangle \in U_A$, $\langle x, z\rangle \in \langle B\rangle, \langle C\rangle$ and hence $\langle x, z\rangle \in \langle B\rangle \cap \langle C\rangle$. □

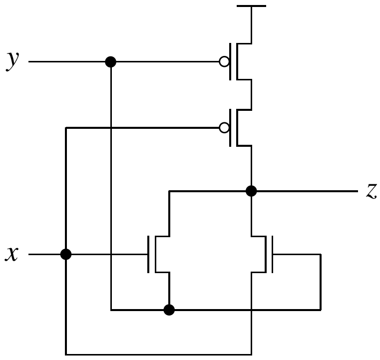
To get a filter, one only needs to close $\langle U_A\rangle$ upward under the subset order. Let this upward closure be indicated by $[\langle U_A\rangle)$.

COROLLARY 4.3.7.

$$\bigcap \langle U_A\rangle = \bigcap [\langle U_A\rangle).$$

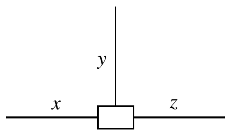
5. An Interpretation in Terms of CMOS Gates

Consider the three place relation and the following interpretation as an array of transistor gates for $\mathcal{X}xyz$ using two pMOS gates (top gates) and two nMOS gates (bottom gates) where the dash on the top most vertical line indicates connection to a constant 1, i.e., a power source:



pMOS gates route their input to their output when the gate charge is 0, and nMOS gates route their input to their output when their gate charge is 1. The CMOS switching circuit above implements the truth table of an exclusive nor below. It will be assumed that all gate switches have this particular configuration and will be denoted by the diagram on the right:

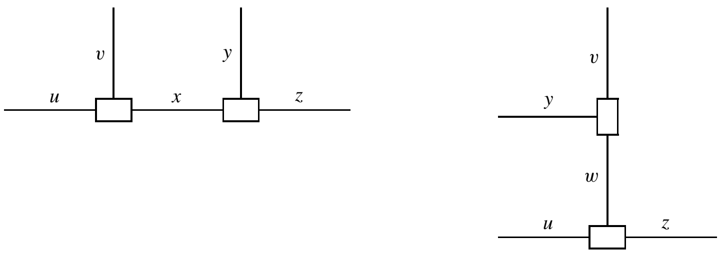
y	x	z
0	0	1
0	1	0
1	0	0
1	1	1



However, viewing the truth table as that of an exclusive nor is not the way to look at the table. Instead, notice that $z = \bar{x}$ (i.e., the complement of x) when $y = 0$, and $z = x$ when y is 1. In words, y controls z's behavior in terms of x. Of course, the diagram says nothing about what else z may be connected to. Now consider the axiom $\mathcal{X}^2uvyz$ iff $\mathcal{X}^2u(vy)z$ written in its expanded form:

$$\exists x\ (\mathcal{X}uvx \text{ and } \mathcal{X}xyz) \text{ iff } \exists w\ (\mathcal{X}uwz \text{ and } \mathcal{X}vyw).$$

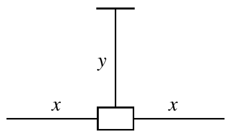
This says that the two diagrams below are equivalent:

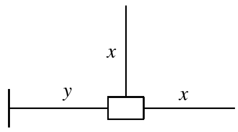


These two circuits are equivalent in that in order to connect u and z, both v and y must be charged. The first diagram uses v to control a gate, while the second uses v

as an "input" to the circuit. In both cases, the existential variable, x in the first case and w in the second, are merely intermediary circuit connections. One can easily verify the two diagrams are functionally equivalent by computing the truth tables for both.

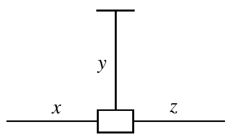
Consider the axioms $\exists y \in \mathbb{X}\ (\mathcal{X}xyx)$ and $\exists y \in \mathbb{X}\ (\mathcal{X}yxx)$. These say we can always draw circuits which correspond to connecting the line x directly to a source of current which is always on:



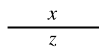


In both cases, the charge on x remains the charge on x. In the left hand case, the power source is always on, so line x is always connected to another line which, for all practical purposes, can also be called x because it will always carry x's charge. The right hand case uses x to switch a line to the power source. The output line x can be considered to be charged with whatever the input line x is charged.

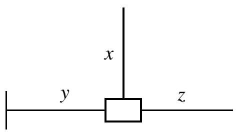
The last axiom, for all $y \in \mathbb{X}$ ($\mathcal{X}xyz$ or $\mathcal{X}yxz$ implies $x = z$) is really two axioms. The first asserts the following:



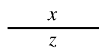
implies



The second axiom asserts

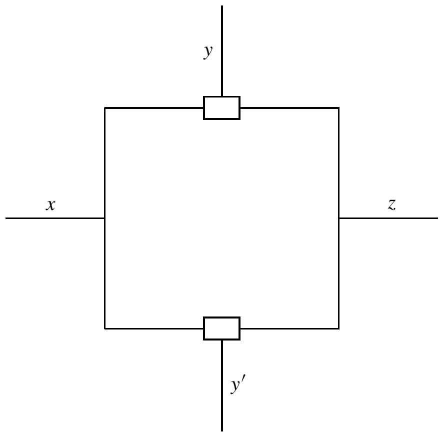


implies

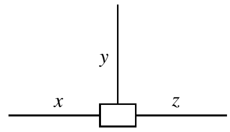


However, if $x = z$, then there is some y such that $\mathcal{X}xyz$ and $\mathcal{X}yxz$ holds. So the implications go both ways. Both are needed in order to treat the rules as re-write rules preserving behavior for circuits.

The Boolean condition, $\mathcal{X}xyz$ and $\mathcal{X}xy'z$ implies $y = y'$, means that every pair of wires $\langle x, z\rangle$ is connected in at most one way. That is, the following diagrammatic (two way) transformation is valid:



iff



Incidently, wires are usually thought of as propositions to be made true or false, and gates as operators. In this scheme, propositions are now seen as things that are made true by wires and gates are relations. This greatly loosens up the formalism since one can say a lot of things about a wire that are not easily expressed as a single proposition.

Consider the set algebra dervived from such an interpretation. The sets are bundles of wires. $z \in A \mathbin{\hat{\odot}} B$ just when z is the output of a switching circuit with a wire in A and a wire in B. Quasi-uniformities derived from these interpretations are also interesting. The relation $\langle A \rangle$ is the collection of wire pairs gated by wires in A. The relation $\langle A \rangle \langle \circ \rangle \langle B \rangle$ is the collection wire pairs such that there is a path from the first wire to the second wire of a pair just when there are gate wires in A and B to make the connection. The relation $\langle A \rangle^*$ the reachability graph (of wires) via the gate wires in A and includes feedback loops. If $\langle A \rangle^+$ is defined as the transitive closure of $\langle A \rangle - \mathbb{X}$, then there is a non-trivial feedback loop just when there is a wire $\langle x, x \rangle \in \langle A \rangle^+$.

The circuits used here are admittedly special purpose. A more fine grained logic should be able to address combinations of pMOS and nMOS gates without needing to group them as is done here. This should be possible, but not with Boolean monoid logic. It might be possible using a positive relevance logic which admits four values for propositions. A world, or wire in this case, either makes a proposition true, false, neither, or both. In terms of canonical worlds, this means that worlds are pairs of prime filters and prime ideals which are allowed to overlap or be such that one is not the set complement of another. In terms of circuits, the neither true nor false value might be associated with a high impedance which is the state a switch is in when there is no current allowed to cross the junction under the gate. A wire could also be driven both true and false given a conflicted circuit. Naturally, this last is to be avoided but the value must still be represented in the logic.

Acknowledgement

The authors thank the anonymous referees for their remarks and corrections.

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