Brit. J. Phil. Sci. 42 (1991), 187–194 *Printed in Great Britain*

On Some Paradoxes of the Infinite

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ABSTRACT

In the paper below the authors describe three super-tasks. They show that although the abstract notion of a super-task may be, as Benacerraf suggested, a conceptual mismatch, the completion of the three super-tasks involved can be defined rather naturally, without leading to inconsistency, by means of a particular kinematical interpretation combined with a principle of continuity.

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1 ROSS' PARADOX

In an introductory textbook on probability theory, Ross mentions an interesting paradox of the infinite which he describes as follows (see Ross [1988], p. 46). Suppose that we possess an infinitely large urn and an infinite collection of balls labeled 1, 2, 3, and so on. There are exactly as many balls as there are natural (standard) numbers. Consider the following thought experiment. At 1 minute to 12 p.m. balls numbered 1 through 10 are placed in the urn, and ball number 1 is withdrawn; at $\frac{1}{2}$ minute to 12 p.m., balls numbered 11 through 20 are placed in the urn, and ball number 2 is withdrawn; at $\frac{1}{4}$ minute to 12 p.m., balls numbered 21 through 30 are placed in the urn and ball number 3 is withdrawn; at $\frac{1}{8}$ minute to 12 p.m. balls numbered 31 through 40 are placed in the urn, and ball number 4 is withdrawn, and so on. How many balls are in the urn at 12 p.m. and what are their labels?

Ross concludes that the urn is empty at 12 p.m. arguing as follows. Consider any ball, say ball number *n*, and that at some time prior to 12 p.m. [in fact, at

$(\frac{1}{2})^{n-1}$ minutes to 12 p.m.] this ball is withdrawn from the urn. This means that for each n, ball number n is not in the urn at 12 p.m.; the urn must therefore be empty at that time.

The answer is surprising because in the course of the experiment the number of balls in the urn grows above all bounds [to be precise: for each n, at $(\frac{1}{2})^{n-1}$ minutes to 12 p.m., there are $9n$ balls in the urn]. Yet at 12 p.m. the urn is suddenly empty! This is what we call *Ross' paradox.*

It is obvious that Ross abstracts from the moving mechanism causing the movement of the balls in the defined manner. Such an abstraction is quite acceptable from a mathematical point of view.

2 A SECOND THOUGHT EXPERIMENT

At the authors' university, generations of mathematics and computer science students and a considerable number of staff have been exposed to Ross' paradox without questioning the reasoning involved. There are, however, good reasons to be worried about the argument leading to the paradox. In order to see this clearly we change the experiment.

Let us suppose that at 1 minute to 12 p.m. balls numbered 1 through 9 are placed in the urn, and instead of withdrawing a ball we add a zero to the label of ball number 1 so that its label becomes 10; at $\frac{1}{2}$ minute to 12 p.m., balls numbered 11 through 19 are placed in the urn, and we added a zero to the label of ball number 2 so that it becomes ball number 20; at $\frac{1}{4}$ minute to 12 p.m., balls numbered 21 through 29 are placed in the urn and ball number 3 becomes number 30; at $\frac{1}{8}$ minute to 12 p.m. balls numbered 31 through 39 are placed in the urn, and ball number 4 becomes 40, and so on. At each instant, instead of withdrawing the ball with the smallest label, we add a zero to its label so that its number is multiplied by 10. How many balls are in the urn at 12 p.m. and what are their labels?

A careful consideration of what happens during the course of the experiment seems to yield the following result. At 12 p.m. the urn contains infinitely many balls and the label of each ball is a natural number followed by infinitely many zeros [an omega-sequence of zeros, mathematicians would say]. Consider, for example, the ball that enters the urn with the label 1. That ball never leaves the urn. Immediately after it enters the urn, its label becomes 10; as soon as 10 is the smallest label inside the urn it becomes 100; as soon as 100 is the smallest label, it becomes 1000; and so on. At 12 p.m. there will be a 1 on the ball followed by infinitely many zeros.

This is remarkable! Why? If we compare Ross' experiment with this experiment, we notice that for each n, at $(\frac{1}{2})^{n-1}$ minutes to 12 p.m. the contents of the urn in the two experiments are the same. In both experiments we have, at 1 minute to 12 p.m., balls labeled 2 through 10 in the urn; in both experiments we have, at $\frac{1}{2}$ minute to 12 p.m., balls labeled 3 through 20 in the urn, and so on. The two experiments seem to be equivalent as regards the contents of the urn: at each instant before 12 p.m. they bring about precisely the same situation in the urn. Yet in the first experiment the urn is empty at 12 p.m., while in the second experiment the urn contains infinitely many balls at 12 p.m. This sounds paradoxical.

3 A THIRD THOUGHT EXPERIMENT

The situation becomes even more paradoxical when we consider the following third thought experiment. All we need is a slight modification of the second experiment. First of all we stipulate that inside the urn there are as many places as there are natural numbers. The places are labeled 1, 2, 3, and so on. Secondly, the experiment is executed in the same manner as the second experiment but with one change: we prescribe that a ball shall always occupy the place with the corresponding label, *i.e.* a ball with label n must necessarily be put in place number n. This means that as soon as a ball gets another label it must be moved to another place. As soon as ball number 1 gets label 10 it is moved to place number 10. As soon as its number becomes 100 it is moved to place number 100, etc. What is the situation at 12 p.m. in this thought experiment? The amount of balls inside the urn obviously grows throughout the experiment [to be precise: for each n, at $(\frac{1}{2})^{n-1}$ minutes to 12 p.m., there are $9n$ balls in the urn]. The balls are also occupying places with larger and larger numbers [to be precise: for each n, at $(\frac{1}{2})^{n-1}$ minutes to 12 p.m., the places with numbers $n+1$ through $10n$ are occupied]. The following conclusions can be drawn: that infinitely many balls enter the urn, that no ball ever leaves the urn, and yet all places inside the urn are at empty 12 p.m.!

4 SUPER-TASKS

The thought-experiments described involve an infinite sequence of acts. The execution of such an infinite sequence of acts was aptly called a *super-task* by Thomson in his [1954]. Thomson discussed among others the following super-task. A lamp is turned on and off infinitely many times. Is the lamp on or off after the task has been executed? This example of Thomson's lamp clearly brings to light some of the problems concerning super-tasks.

From the point of view of classical mathematics[1] there are no a priori objections against considering infinite sequences of well-defined acts, or the corresponding infinite sequences of well-defined intermediate states. Yet the notion of the *complete subsequent execution of all acts involved* goes further than that.

Let us take the Thomson lamp example and consider the infinite sequence of

[1] For constructivists in the philosophy of mathematics or others who reject the actual infinite, super-tasks, obviously, are not a acceptable object of investigation.

acts and the infinite sequence of intermediate states. *Logically speaking*, such an infinite sequence of acts, or the infinite sequence of intermediate states of affairs, implies *nothing whatsoever* concerning the state of affairs at 12 p.m. Something which is true before 12 p.m. need not be true at 12 p.m. Take the sentence 'It's before midnight' and the fact that it stops being a true sentence when midnight strikes.

As soon as one becomes aware of this, the easiest way to get rid of the strange thought experiments and their paradoxical outcome is to say that the notion of the complete subsequent execution of all infinitely many acts involved does not make much sense. This view was expressed by Benacerraf, who suggested in his [1962] that the notion of a super-task is a 'conceptual mismatch', a compound consisting of two components ('infinite series' and 'completion of a series of acts') connected with such totally different areas of applicability that the criteria normally employed fail to apply. As long as one discusses super-tasks in general, Benacerraf seems to be right. In the case of our urn experiments, however, this attitude is unsatisfactory. It does not explain why, at first sight, the reasoning leading to the paradoxical conclusions is so convincing.

We will restrict ourselves to the three urn experiments which we considered in the first three sections of this paper, and we will interpret them in such a way that more or less natural conclusions on the situation at 12 p.m. can be drawn. The idea behind it is very simple. In the case of the three urn experiments, the reasoning concerning the situation at 12 p.m. must be based upon *an implicit interpretation with one or more implicit assumptions*, concerning what completion of the task under that interpretation implies. We will try to make such an interpretation and the implicit assumptions explicit and use them to define the notion 'completion of an infinite series of acts' with that interpretation. Then we will reconsider the experiments and look at the reasoning behind them more carefully. We hope to show that with the interpretation in question, the reasoning in the case of the three urn experiments is correct, roughly speaking, and that the outcome of the experiments is strange, without, however, leading to inconsistency.

5 KINEMATICAL INTERPRETATION AND A PRINCIPLE OF CONTINUITY

The reasoning in all three cases is based, from our point of view, on a *kinematical interpretation and a principle of continuity* concerning the situation at 12 p.m. We interpret the super-tasks as taking place in a Euclidean space containing many (possibly infinitely) rigid (material) elements. In order to keep things simple, we will restrict ourselves to such a Euclidean plane. In the case of all three experiments the numbered balls are rigid elements moving within the plane. We think of them as circular discs that assume a position in the plane. In the second and third experiments, zeros are added to the labels of balls. Here again, in order to keep things simple, we will assume that we can abstract from the precise way in which the zeros are added. In our interpretation the urn is simply a non-empty point set in the plane. A ball is considered to be inside the urn if the center of the circular disc that represents it is an element of this point set. The ball is outside the urn if its center is not an element of the point set.

In the course of the execution of the super-task the balls are being moved. The principle of continuity on which the reasoning in the sections one, two and three is based says: *If at some moment before 12 p.m. a ball comes to rest at a particular position, which it does not leave till 12 p.m., then it is still at that position at 12 p.m.*

In other words in our interpretation the principle says that if the vector (**p**,**q**) at some time before 12 p.m. becomes a constant vector and remains so until 12 p.m., then the element involved is still at the same position at 12 p.m.

Although we have not succeeded, we have tried to give a definition of the principle of continuity that is less tied to a kinematical interpretation. It is tempting to try to define the principle as follows: as soon as at a certain moment before 12 p.m. the truth value of a sentence P_k, expressing a possible state of affairs, becomes and remains constant, then the truth value at 12 p.m. of that sentence has the same value. This is not very satisfactory. Not only must one obviously exclude a sentence like 'It is now before midnight', but also a seemingly harmless sentence like 'There is at least one ball in the urn'. Otherwise Ross' paradox turns into inconsistency at 12 p.m.

6 THE PRINCIPLE OF CONTINUITY APPLIED

Let us reconsider the super-task that we started with under the interpretation we introduced in Section 5. In order to apply the principle of continuity it is necessary that we define precisely where the balls are at each moment before 12 p.m. Moreover, the principle of continuity only allows us to draw conclusions of a particular type about the situation at 12 p.m. The principle implies statements of the form 'Ball x is at 12 p.m. at position p' and whatever follows logically from such statements. In order to draw a conclusion such as 'The urn is empty at 12 p.m.', that statement must be by definition, equivalent to, for example, 'At 12 p.m. all balls involved assume known positions outside the urn'. In order to be able to draw this particular conclusion on the basis of the principle of continuity for the first urn experiment, we must stipulate that in the course of the super-task, once a ball is removed from the urn and put somewhere outside the urn, it stays where it is.

In this way Ross' paradox appears to be a genuine paradox of the infinite. On the one hand we can draw the conclusion that at 12 p.m. the urn is empty. On the other hand, although in the course of the execution of the super-task the number of balls inside the urn grows above all bounds, we cannot draw the conclusion that the urn contains infinitely many balls at 12 p.m. We have a paradox, but no inconsistency.

Now that we have defined the principle of continuity, it is clear that the second super-task was not defined precisely enough when one considers the third super-task. The two experiments show the statement 'Ball x is inside the urn' to be insufficiently precise. We must know with certainty whether ball x comes to rest inside the urn at a particular moment before 12 p.m. or whether it continues to move. If we stipulate, in the case of the second experiment, that a ball is no longer moved once it is put in the urn, in our interpretation the conclusions that we drew above in the case of the second experiment follow on from the principle of continuity. It is then obviously true that at 12 p.m. the urn contains infinitely many balls and that the label of each ball consists of a natural number followed by infinitely many zeros.

The principle of continuity in its present form cannot be applied to balls that do not come to rest before 12 p.m. In the case of the third super-task, the principle does not allow us to draw conclusions on the positions, at 12 p.m., of the balls that enter the urn during the course of the experiment. The conclusion that no ball leaves the urn during the course of the experiment is correct, but the conclusion that the urn is empty at 12 p.m. is rather bold.

7 A GENERALIZED PRINCIPLE OF CONTINUITY

Our primary aim so far has been to clarify the situation concerning Ross' paradox and related paradoxes. We feel that we succeeded in this by means of our kinematical interpretation of the super-tasks involved and the principle of continuity. Concerning the indeterminacies at 12 p.m. in the case of the third super task, one could ask questions such as 'Is a further definition of the situation after the execution of a super-task possible in the given kinematical interpretation?' and 'Is the execution of a super-task always possible in the given interpretation?' Such questions can lead to the following line of thought.

The principle of continuity in its present form actually says: 'position-functions, which become constant before 12 p.m. are continuous at 12 p.m.' A natural generalization is the following principle: If for a certain ball the limit of its position vector $(\mathbf{p},\mathbf{q})$ exists for $t \to 12$ p.m. and that limit equals $(\mathbf{p}_0,\mathbf{q}_0)$, then the position of the element at 12 p.m. is given by $(\mathbf{p}_0,\mathbf{q}_0)$. If we assume, however, that the position functions of the balls are continuous at 12 p.m., then there is no reason to accept discontinuities in the position functions at other moments. In other words we end up with the following *generalized principle of continuity: The position functions of all balls are at all instants* t *continuous functions of time.*

In the first two experiments it, this principle does not influence the conclusions that we drew in Section 6. Before 12 p.m. things have to be arranged in such a way that the position functions of the balls are continuous functions of time. This seems quite possible. All balls, at some moment before 12 p.m., get their definite position which does not change again during the course of the experiment.

In the third experiment is now necessary that it be designed in such a way that for all balls the position vector converges to a limit for $t \rightarrow 12$ p.m. This appears to be possible. Bear in mind that during the course of the experiment ball number 1 becomes ball number 10, and thereafter is number 100, 1000, etc., while all the time the ball must occupy the position in the urn with the corresponding number. It is easy now to design the urn in such a way that the centres of the positions 1, 10, 100, 1000, etc. inside the urn form a converging sequence of points. Depending on the design of the urn, the limit point lies either inside the urn, outside the urn or on the border between the inside and the outside. It is then necessary that for $p \rightarrow \infty$ (and arbitrary fixed n) any sequence of centres of positions n, $10n$, $100n$, $1000n$, . . . $10^p n$ converges to a limit point. Those limit points are the centres of the positions, at 12 p.m., of the balls that enter the urn during the course of the experiment.

8 CONCLUSION

It is clear from the above that the three paradoxical super-tasks with which we started can be interpreted in such a way that there exists quite a natural definition of the outcome of their execution. In our interpretation we assume that the super-tasks are executed in a euclidean space, where a (generalized) principle of continuity holds: the position functions of the balls are continuous functions of time.[2] A special case of this principle of continuity, saying that a position function that becomes constant before 12 p.m. is continuous at 12 p.m., appears to be the basis of our intuitive reasoning with respect to the three supertasks.

[2] In the course of our investigations we did move towards ideas with respect to super-tasks similar to the ones defended by Grünbaum, who studied (from the point of view of classical mechanics) the 'kinematical possibility' of several super-tasks (See his work in Salmon [1970]). Grünbaum considers a super-task kinematically impossible, if for some moving element the limit of the position vector for $t \rightarrow 12$ p.m. does not exist. This is completely in accordance with our interpretation of the urn experiments combined with the generalized principle of continuity. Grünbaum to a certain extent took also moving mechanisms into account and argued *e.g.* that if the super-task corresponding to the Thomson lamp is to be kinematically possible, the switching button must have an amplitude converging to zero. If it would not converge to zero its position function would not be continuous at 12 p.m. From this he and A. Janis inferred that the lamp will in that case be necessarily switched on at 12 p.m.! We have restricted ourselves to the continuity of position functions regardless of their differentiability, but Grünbaum also took velocities into account. He considered supertasks also *kinematically impossible* if their completion requires infinite velocities. This implies that the amplitude of the switching button of the Thomson lamp may not converge too slowly to zero, because, otherwise, certain points on the button would have to cover an infinite distance in a finite time, thus making the execution of the task kinematically impossible.

Our interpretation shows that although the abstract notion of a super-task may be a conceptual mismatch, the notion makes sense in at least one interpretation. It would be interesting to have another, completely different interpretation. We have not been able to find one.

ACKNOWLEDGEMENT

We are grateful to a considerable number of colleagues of the Vrije Universiteit Amsterdam for their continued stimulating interest in the subject of this paper. In particular, however, we would like to thank Bas Jongeling for reading almost all previous versions of the paper and commenting critically on them. We are also grateful to Sally Kuiper for some linguistic improvements.

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