On the Classical Limit of Quantum Mechanics

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Abstract

Contrary to the widespread belief, the problem of the emergence of classical mechanics from quantum mechanics is still open. In spite of many results on the $\hbar \to 0$ asymptotics, it is not yet clear how to explain within standard quantum mechanics the classical motion of macroscopic bodies. In this paper we shall analyze special cases of classical behavior in the framework of a precise formulation of quantum mechanics, Bohmian mechanics, which contains in its own structure the possibility of describing real objects in an observer-independent way.

1 Introduction

According to the general wisdom there shouldn't be any problem with the classical limit of quantum mechanics. In fact, in any textbook of quantum mechanics one can easily find a section where the solution of this problem is explained (see, e.g., [13]) through Ehrenfest theorem, WKB approximation or simply the observation that the canonical commutation relations become Poisson brackets. Indeed, one might easily get the impression that it is only a matter of putting all known results into order for obtaining a rigorous derivation of classical

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mechanics from quantum mechanics. Consider, for example, the standard argument based on the Ehrenfest theorem: if the initial wave function is a narrow wave packet, the packet moves approximately according to Newtonian law $F = ma$. This argument entails that classicality is somehow associated with the formation and preservation of narrow wave packets.

There are, however, several problems connected with this. The most serious one is that a wave packet typically spreads and there is a definite time after which the classical approximation will break down. Even if it might appear that, for massive bodies, the wave packet will remain narrow for very long times, it can easily be shown that interactions will typically generate very spread out wave functions, even for massive bodies (e.g, for the center of mass of an asteroid undergoing chaotic motion). But which mechanism should prevent the wave function from spreading?

Recently, it has been suggested that *decoherence*, due to the interaction of the system with its environment could provide the desired mechanism: The environment, constantly interacting with the body, could somehow act as a measuring device of the macroscopic variables of the body—say, its center of mass—producing in this way a narrow wave function in the macroscopic directions of its configuration space (see [8] and references therein). But the composite system formed by the system of interest and its environment is itself a closed system and Schrödinger's evolution of this enlarged system tends to produce spreading of the total wave function over the total configuration space. Thus, decoherence alone is not sufficient to explain the emergence of the classical world from standard quantum mechanics: there is still the necessity to add to Schrödinger's equation the collapse of the wave function. Within standard quantum mechanics, this is the only way to guarantee the formation of a narrow wave function in the macroscopic directions of the configuration space of the system. However, Schrödinger's evolution plus the collapse is not a precise microscopic theory: the division between microscopic and macroscopic world (where the collapse takes place) is not part of the theory. Thus, as Bell has suggested [5], we need to go beyond standard quantum mechanics: or the wave function doesn't provide the whole description of the state of a system or Schrödinger's equation needs to be modified (for a proposal of this kind, see [7]).

In this paper we shall move the first steps towards a complete derivation of the classical limit within the framework of Bohmian mechanics, a theory about particles moving in physical space and which accounts for all quantum phenomena. While we refer to future work for a complete and thorough analysis (see [1] and [3]) we shall show here how Bohmian mechanics allows to go much beyond the standard approach and to explain the emergence of classicality even for spread out wave functions. We shall formulate the classical limit as a scaling limit in terms of an adimensional parameter ϵ given by the ratio between two relevant length scales, namely, the "wave length" of the particle, and the "scale of variation" of the potential L. We shall show how, in some special cases, the emergence of classical behaviour is associated with the limit $\epsilon \to 0$.

2 Classical Limit in Bohmian Mechanics

Bohmian mechanics is a theory in which the world is described by particles whose configurations follow trajectories determined by a law of motion. The state of a system is described by the couple (X, ψ), where $X = (X_1, ..., X_N)$ are the configurations of the particles composing the system and ψ is the wave function evolving according to Schrödinger's equation. For a review of Bohmian mechanics, see the other contribution to the same volume and references therein [2].

In order to formulate the classical limit in Bohmian mechanics, it can be useful to write the wave function ψ in the polar form $\psi = Re^{\frac{i}{\hbar}S}$. From Schrödinger's equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\sum_{k=1}^{N}\frac{\hbar^2}{2m_k}\nabla_k^2\psi + V\psi\,,$$

one obtains the continuity equation for R^2,

$$\frac{\partial R^2}{\partial t} + \mathrm{div}\left[\left(\frac{\nabla_k S}{m}\right)R^2\right] = 0, \tag{1}$$

and the modified Hamilton-Jacobi equation for S

$$\frac{\partial S}{\partial t} + \frac{(\nabla_k S)^2}{2m} + V - \sum_k \frac{\hbar^2}{2m_k}\frac{\nabla_k^2 R}{R} = 0. \tag{2}$$

Equation (1) suggests that $\rho = R^2$ can be interpreted as a probability density. Note that equation (2) is the usual classical Hamilton-Jacobi equation with an additional term

$$U \equiv -\sum_k \frac{\hbar^2}{2m_k}\frac{\nabla_k^2 R}{R}, \tag{3}$$

called the quantum potential. One then sees that the (size of the) quantum potential provides a rough measure of the deviation of Bohmian mechanics from its classical approximation.

In this way, it might seem that the classical limit is something trivial: ensure the quantum potential is somehow small and then classical mechanics arises from Bohmian mechanics. What is not trivial at all is to understand what are the physical conditions corresponding to the smallness of the quantum potential. In the next sections we shall discuss a simple model of a macroscopic body moving in an external potential and we shall see that there exists a precise limit in which the time evolution of the center of mass of the body is approximately classical.

2.1 Motion in an External Potential

Consider a macroscopic body composed of N particles, with positions $(X_1, ..., X_N)$, and masses $m_1, \ldots, m_N$, subjected only to internal forces. The center of mass of the body is

$$X = \frac{\sum_i m_i X_i}{\sum_i m_i}.$$

The Hamiltonian of the body can be written as

$$H = \frac{\hbar^2}{2m}\nabla^2 + H_{\mathrm{rel}}, \tag{4}$$

where $m \equiv \sum_i m_i$, and H_{rel} is depending only on the relative coordinates $y_i = x_i - x$ (and their derivatives). Therefore, if the wave function of the body $\Psi = \Psi(x, y)$ (at some "initial" time) factorizes as a product of the wave function $\psi = \psi(x)$ of the center of mass and the wave function $\phi = \phi(y)$ of the internal degrees of freedom, this product form will be preserved by the dynamics and the state (X, ψ) will evolve autonomously: ψ satisfies to Schrödinger's equation with Hamiltonian (4) and X evolves according to

$$\frac{dX_t}{dt} = \frac{\hbar}{m}\mathrm{Im}\left[\frac{\nabla\psi(X_t, t)}{\psi(X_t, t)}\right]. \tag{5}$$

If there is an external potential $\sum_i V_i(q_i)$, the Hamiltonian will assume the form

$$H = \frac{\hbar^2}{2m}\nabla^2 + V(x) + H^{(x,y)} \tag{6}$$

where $V(x) = \sum_i V_i(x)$, $H^{(x,y)} = H_{\text{rel}} + H_{\text{int}}$, with H_{int} describing the interaction between the center of mass and the relative coordinates. If V_i are slowly varying on the size of the body, H_{int} can be treated as a small perturbation, and, in first approximation, neglected. Thus, if $\Psi = \Psi(x, y) = \psi(x)\phi(y)$, at some time, this product form will be preserved by the dynamics. In this way we end up again with a very simple one body problem: ψ evolves according to Schrödinger's equation with Hamiltonian

$$H = \frac{\hbar^2}{2m}\nabla^2 + V(x)$$

and the position X of the center of mass of the body evolves according to (5).

2.2 Classical Limit as a Scaling Limit

Usually, the classical limit is associated with the limit $\hbar \to 0$, meaning by this $\hbar \ll A_0$, where A_0 is *some* characteristic action of the corresponding classical motion (see, e.g., [11], [13], [6]). Note that, while ψ doesn't have any limit as $\hbar$ goes to zero, the couple (R, S), defined by the $\hbar$-dependent change of variables $\psi = Re^{\frac{i}{\hbar}S}$, does have a limit. Formally, this limit can be read by setting $\hbar \equiv 0$ in equations (1) and (2) and thus the couple (R, S) becomes the pair (R^0, S^0), where R^0 satisfies the classical continuity equation and S^0 the classical Hamilton-Jacobi equation.

The condition $\hbar \ll A_0$ is often regarded as equivalent to another standard condition of classicality which involves some relevant length scales of the motion: if the de Broglie wave length λ is small with respect to the characteristic dimension L determined by the scale of variation of the potential, the behavior of the system should be close to the classical behavior in the same potential (see, e.g., [10]). This condition relates in a completely transparent way a property of the state, namely its wave length, and a properties of the dynamics, namely the scale of variation of the potential.

We shall not enter here into the problem of finding a precise characterization of λ and L (for which we refer to [1] and [3]). For the present purposes it is sufficient to keep in mind that the de Broglie wave length should be regarded as function of the initial wave function, $\lambda = \lambda(\psi_0)$, e.g., given in terms of the mean kinetic energy with respect to ψ_0, and the scale of variation of the potential should be regarded as a suitable function of the potential, $L = L(V)$, for example, for a potential of the form $V(x) = \sin(\frac{2\pi}{a}x)$, $L = a$, for a constant potential (free case) $L = \infty$.

The length scales λ and L allow to define the natural macroscopic scales x' and t' for describing the motion,

$$x' = \frac{x}{L}\,, \qquad t' = \frac{t}{T}. \tag{7}$$

The time scale T is defined as $T = \frac{L}{v}$ and v is the speed defined by λ, $v = \frac{\hbar}{m\lambda}$. The scales L and T tell us what are the fundamental units of measure for the motion: L is the scale in which the potential varies and T is the time necessary to the particle to see its effects. We expect the Bohmian motion on these scales to look classical when the adimensional parameter

$$\epsilon \equiv \frac{\lambda}{L}$$

is getting smaller and smaller. This means in particular that we expect the quantum potential to become negligible (with respect to the kinetic energy) whenever $\epsilon \to 0$.

In the next two sections we shall study some examples of wave functions and potentials for which we can show explicitly that the classical limit arises.

3 Quasi Classical Wave Functions

Consider a family of wave functions depending on $\hbar$ of the *short wave* form

$$\psi_0^\hbar(x) = R_0(x)\mathrm{e}^{\frac{i}{\hbar}S_0(x)}, \tag{8}$$

where $R_0(x)$ and $S_0(x)$ are functions not depending on $\hbar$, and $R(x)$ is not zero only in a limited region of space. The limit $\hbar \to 0$ corresponds to a mathematical trick to simulate the limit $\lambda(\psi_0) \to 0$. This limit, called short wave length limit, is a special case of the limit $\epsilon = \lambda/L \to 0$ in which the wave length is getting small and the external potential is fixed. To see the classical limit arising, one should describe the motion in terms of the macroscopic coordinates $x' = x/L$ and $t' = t/T$. However, since in this case both L and T are constant, there is no substantial difference between the microscopic and macroscopic scale. (Note that the limit $\hbar \to 0$ simulates also the limit of large mass $m \to +\infty$ and for which the potential rescales as $V = m\hat{V}$. In fact, in this case, $1/m$ and $\hbar$ play the same role in the Schrödinger's equation

$$i\frac{\partial\psi}{\partial t} = \left[-\frac{\hbar}{2m}\nabla^2 + \frac{m}{\hbar}\hat{V}\right]\psi.$$

and in the guiding equation (5).)

The approximate solution of Schrödinger's equation for initial condition (8) in the short wave length limit is given by [1] [11]

$$\psi^{(0)}(x,t) = R^{(0)}(x,t)e^{\frac{i}{\hbar}S^{(0)}(x,t)} + O(\hbar), \tag{9}$$

where $S^{(0)}(x,t)$ is the classical action (i.e., the solution of classical Hamilton-Jacobi equation having $S_0(x)$ as initial condition) and $R^{(0)}(x,t) = |dx/dx_0|^{-1/2}R_0(x_0,t)$ (where $x_0 = x_0(x,t)$ is the initial position of the particle that at time t arrives in x transported along the classical path). $R^{(0)}(x,t)$ is the evolution at time t of the initial amplitude $R_0(x,t)$ according to the classical continuity equation (1).

Therefore, in the limit $\hbar \to 0$, the velocity field becomes the classical one

$$v^{(0)}(x,t) = \frac{1}{m}\nabla S^{(0)}(x,t) + O(\hbar^2). \tag{10}$$

and the quantum potential

$$U = -\frac{\hbar^2}{m}\frac{\nabla^2 R^{(0)}}{R^{(0)}} \tag{11}$$

goes zero because $R^{(0)}$ doesn't depend on $\hbar$. In other words, in the limit $\hbar \to 0$, we have convergence of modified Hamilton-Jacobi equation to classical Hamilton-Jacobi equation.

4 Slowly Varying Potentials

Another special case of the limit $\epsilon \to 0$ is given by the situation in which there is a slowly varying external potential,

$$V_L = V\left(\frac{x}{L}\right),$$

[1] We are considering here times shorter than the "first caustic time", i.e. the time at which the function $p(x,t) = \nabla S^0(x,t)$ becomes multivalued. This in not a restriction, as explained in [1] and [3].

with scale of variation L very large. Given that $\epsilon = \lambda/L$, the slowly varying potential limit is a special limit corresponding to keep λ fixed (that is, the initial wave function) and letting $L \to +\infty$. This limit is equivalent to a long time limit. In fact, if the potential is slowly varying ($L \to +\infty$), to see its effect the particle has to travel for a time of order

$$T = \frac{mL\lambda}{\hbar},$$

and for $L \to \infty$ also $T \to \infty$.

Since time and space rescaling are of the same order, it is useful to rescale space and time as

$$x \to \frac{x}{\epsilon}, \quad t \to \frac{t}{\epsilon}. \tag{12}$$

Under these rescaling, the initial conditions of Bohmian dynamics become

$$\psi_0^\epsilon(x) = \frac{1}{\epsilon^{1/2}}\psi_0(\frac{x}{\epsilon}), \quad X_0^\epsilon = \frac{X_0}{\epsilon} \tag{13}$$

and Bohmian equations become the usual equations of motion for $\psi^\epsilon(x,t)$ with $\hbar$ substituted by $\hbar\epsilon$

$$\begin{aligned} i\hbar\epsilon\frac{\partial\psi^\epsilon(x,t)}{\partial t} &= -\frac{\hbar^2\epsilon^2}{2m}\nabla^2\psi^\epsilon(x,t) + V(x)\psi^\epsilon(x,t), & (14)\\ \frac{dX_t^\epsilon}{dt} &= v^\epsilon(X_t^\epsilon,t) = \frac{\hbar\epsilon}{m}\mathrm{Im}\left[\frac{\nabla\psi^\epsilon(X_t^\epsilon,t)}{\psi^\epsilon(X_t^\epsilon,t)}\right]. & (15)\end{aligned}$$

The solution $\psi^\epsilon(x,t)$ of equation (14) can be expressed in terms of the rescaled propagator $K^\epsilon(x,t;x_0,0)$, in which $\hbar$ is replaced by $\hbar\epsilon$, and the Fourier transform of the initial wave function $\hat{\psi^\epsilon}_0(k)$,

$$\psi^\epsilon(x,t) = \frac{1}{(2\pi\epsilon)^{d/2}}\int\int K^\epsilon(x,t;x_0,0)\mathrm{e}^{i\frac{x_0\cdot k}{\epsilon}}\hat{\psi}_0(k)d^dx_0d^dk, \tag{16}$$

where d is the dimension of the space. In general, the asymptotic form of the propagator in the limit $\epsilon \to 0$ is [9]

$$K^\epsilon(x,t;x_0,0) = \frac{1}{(2\pi i\hbar\epsilon)^{d/2}}\sqrt{C(x,x_0;t)}\mathrm{e}^{\frac{i}{\hbar\epsilon}S^0(x,t;x_0,0)}[1+\hbar\epsilon z], \tag{17}$$

where $z = z(t,x_0,x,\epsilon)$ and $||z||_{L^2(\mathbf{R}^\mathbf{d})} \leq c$ where c is a constant [2]. S^0 is the classical action

$$S^0(x,t) = \int_0^\tau L(x(\tau),\dot{x}(\tau),\tau)d\tau, \qquad x(0) = x_0, \quad x(t) = x, \tag{18}$$

[2]We are again assuming times shorter than the "first caustic time". See footnote (1).

and

$$C(x,x_0;t) = \det\left[-\nabla^2_{x,x_0} S^0(x,t;x_0,0)\right]. \tag{19}$$

In order to find the asymptotic $\epsilon \to 0$ of $\psi^\epsilon(x,t)$, we apply the method of stationary phase: the main contribution to $\psi^\epsilon(x,t)$ comes from the x_0 and the k which make stationary the phase $\phi(x_0,k) = \frac{1}{\hbar}[S^0(x,t;x_0,0) + x_0 \cdot \hbar k]$. They are

$$x_0 = 0 \quad \text{and} \quad k_0(x,t) = -\frac{1}{\hbar}\, \nabla_{x_0} S^0(x,t;x_0,0)\Big|_{x_0=0}. \tag{20}$$

So we have

$$\psi^{(0)}(x,t) = \sqrt{C(x,0;t)}\left(\frac{i}{\hbar}\right)^{d/2} \hat{\psi}_0(k_0(x,t))\mathrm{e}^{\frac{i}{\hbar\epsilon}S^0(x,t;0,0)} + O(\epsilon). \tag{21}$$

We can rewrite this as

$$\psi^{(0)}(x,t) = R^0(x,t)\mathrm{e}^{\frac{i}{\hbar\epsilon}S^0(x,t)} + O(\epsilon^2), \tag{22}$$

where

$$R^0(x,t) = \sqrt{C(x,0;t)}\left(\frac{i}{\hbar}\right)^{d/2} \hat{\psi}_0(k_0(x,t)). \tag{23}$$

In the limit $\epsilon \to 0$, the velocity field becomes

$$v^{(0)}(x,t) = \frac{1}{m}\, \nabla_x S^0(x,t;x_0,0)\Big|_{x_0=0} + O(\epsilon^2). \tag{24}$$

Therefore, also in this case, we have convergence of modified Hamilton-Jacobi equation to the classical Hamilton-Jacobi equation (and thus vanishing of the quantum potential, as before) although for different initial conditions. In fact, in the case of the family of quasi classical wave functions ($\hbar \to 0$ limit), the limiting position is distributed according to the classical probability distribution $\rho(x,t) = |R^0(x)|^2$ and the limiting velocity is distributed according to

$$\rho(v,t) = \int \delta\left(v - \frac{\nabla S^0(x,t)}{m}\right)\left|R^0(x)\right|^2 dx. \tag{25}$$

On the other hand, in the case of slowly varying potential, the probability distribution of the limiting position is

$$\rho(x,t) = \frac{C(x,0;t)}{\hbar^d}|\hat{\psi}_0(k_0(x,t))|^2, \tag{26}$$

where $k_0(x,t)$ is defined by equation (20). This is the probability distribution transported along the classical flow for the initial position is $X_0 = 0$ and initial velocity distributed according to

$$\rho(v,0) = \left(\frac{m}{\hbar}\right)^d \left|\hat{\psi}_0\left(\frac{mv}{\hbar}\right)\right|^2. \tag{27}$$

Note the presence of the Fourier transform of the initial wave function in the probability distribution of the initial velocity, which is somehow connected to the fact that the slowly varying potential limit is equivalent to a long time limit in which the initial transient behaviour is completely canceled out.

5 General Structure of the Classical Limit

The examples discussed in the previous sections show that there is a particular structure of the wave function (see equations (9) and (22)) that emerges when we are in the classical regime. This structure is what we call a *local plane wave*, a wave function that locally can be regarded as a plane wave having a local wave length.

A precise notion of local plane wave can be given starting from the usual notion of wave length λ (the spatial period). For simplicity, we shall analyze the problem in one dimension. Consider a wave function of the polar form, then the following relations should hold

$$R(x,t) \simeq R(x+\lambda,t), \tag{28}$$

$$S(x,t) \simeq S(x+\lambda,t) + 2\pi\hbar. \tag{29}$$

By expanding in Taylor series in λ the right hand side of equation (28), one gets

$$\left|\frac{\nabla R(x,t)}{R(x,t)}\right| \lambda(x,t) \ll 1, \tag{30}$$

$$\frac{1}{2}\left|\frac{\nabla^2 R(x,t)}{R(x,t)}\right| \lambda^2(x,t) \ll 1, \quad \ldots \tag{31}$$

Similarly for $S(x,t)$ we obtain, up to the second order terms, the definition of the local wave length $\lambda(x,t)$

$$\lambda(x,t) = \frac{\hbar}{|\nabla S(x,t)|}. \tag{32}$$

The smallness of the second order term, together with equation (32), gives the condition

$$|\nabla\lambda(x,t)| \ll 1. \tag{33}$$

It should be stressed that condition (31) directly implies that the quantum potential is smaller than the kinetic energy for a given time t, i.e.

$$\frac{\hbar^2}{2m}\left|\frac{\nabla^2 R(x,t)}{R(x,t)}\right| \ll \frac{1}{2m}\left|\nabla S(x,t)\right|^2, \tag{34}$$

which, in its turn, implies the validity of the classical Hamilton-Jacobi equation. We may then conclude that the association between the emergence of classical behavior and the formation of local plane waves is indeed the hallmark of the classical limit. This conclusion receives further supports from observing the expansive character of the Laplacian in Schrödinger's equation which tends to produce spreading of the wave function insofar the potential energy is dominated by the kinetic energy (that is, for bounded motion in a potential far from the turning points).

Moreover, observe that to have classical limit, the quantum potential should be smaller than the kinetic energy for a sufficiently large time interval. In other words, classicality requires that the local plane wave structure should be preserved by the dynamics. There is an argument, based on Ehrenfest theorem, that allows to determine the correct notion of scale of variation of the potential L and explains the the stability of the local plane wave structure in terms of the condition $\lambda \ll L$. It is outside the scope of this paper to enter into the details of this argument (see [1] and [3]). It turns out that L is given by

$$L = \sqrt{\frac{|V'(x)|}{|V'''(x)|}}, \tag{35}$$

where V' and V''' denote respectively the first and the third derivative of the potential (for simplicity we are restricting again to the one dimensional case).

6 Some Simple Examples of ϵ

It can be useful to compute directly, in some simple special cases, what is the small adimensional parameter ϵ relevant for the classical behavior.

Note that, for quadratic potentials, from equation (35) it follows that $L = +\infty$ and thus $\epsilon = 0$.[3] Since $\epsilon = 0$, we have that the motion is classical on any scale L_o chosen by the experimenter, provided that $\lambda \ll L_o$. (This is in complete agreement with the standard understanding of the classical limit for quadratic potentials in terms of Wigner function, Feynman path integrals or Weyl quantization [12].)

A more interesting example is the case of the Coulomb potential

$$V(r) = \frac{qq'}{r}. \tag{36}$$

We find that L is

$$L \simeq r \tag{37}$$

If we consider the bound states of the hydrogen atom (with small spread in energy as it is the case, e.g., for coherent states), L becomes of the order of the Bohr radius a_0. Thus,

$$\epsilon \simeq \frac{\lambda}{a_0} \simeq \frac{1}{n}, \tag{38}$$

where n is the principal quantum number. For scattering states, $L \simeq r$, where r is simply the distance from the scattering center. This means that the scale on which the motion is classical is not fixed but it is varying.

Consider now the case of the Yukawa potential

$$V(r) = \frac{\mathrm{e}^{-\mu r}}{r}. \tag{39}$$

The scale of variation of this potential, according to definition (35), is

$$L \simeq \sqrt{\frac{(\mu r + 1)r^2}{\mu^3 r^3}} \tag{40}$$

For large distances, i.e. $r \to +\infty$ (scattering states), we have $L \simeq \frac{1}{\mu}$, the range of the potential; for small distances, i.e. $r \to 0$, we have $L \simeq r$, like in the Coulomb case.

[3] By quadratic potential we mean $V(x) = ax^2 + bx + c$ (for simplicity in one dimension). Linear and constant potentials are included as limiting cases respectively for a only and a and b going to zero.

7 Remarks and Perspectives

We would like to underline that, while in standard quantum mechanics the emergence of classicality is always connected to the permanence of a narrow wave function during the motion, what arises from the above discussion is that the crucial feature of the classical limit is the formation of very spread out wave function: the local plane wave. Only in the framework of Bohmian mechanics, given that we have also configurations and not only the wave function, we can explain the emergence of the classical behavior in a coherent way for delocalized wave functions.

Moreover, from the very notion of local plane wave it follows that the local plane wave can be thought as composed by a sum of "virtual" wave packets with a definite local wave length (for further details, see [1] and [3]). Since in Bohmian mechanics each particle has its own trajectory, the local plane wave undergoes a sort of collapse in the following sense: not all the wave function is relevant for the dynamics but there is an *effective* guiding wave packet for the particle (which is the part of the wave function in a local neighborhood of the trajectory at the time t). This fact is the key to understand the emergence of the classical limit in Bohmian mechanics for spread out wave functions, by applying one of the simplest argument used in standard quantum mechanics, i.e. the Ehrenfest theorem. While a detailed explanation of how this comes about will be given in [2] and [3], we want to stress here that this argument provides a general support for the emergence of the classical world whenever $\lambda \ll L$ and supplies with the precise notion of the scale of variation of the potential L given by equation (35).

Another delicate issue associated to the derivation of the classical limit is the following: as soon as there is a potential, caustics appear and our analysis of sections 3 and 4 for times larger than the "first caustic time" breaks down. It turns out, however, that caustics are indeed not a problem because they arise in the highly idealized model we have considered here: we have in fact neglected the term H_{int} in (6) describing the interaction between the center of mass x of the body and the relative coordinates y, as well as any perturbation due to the unavoidable interaction of the body with the external environment. These interactions produce *entanglement* between the center of mass x of the system and the other degrees of freedom y (where now y includes both the relative coordinates and the degrees of freedom of the environment). Taking into account these interactions is what nowadays people call decoherence (see, e.g., [8] and reference therein), which however is nothing but an *effective* description of all the effects that cannot be described by the external potential acting on the center of mass x. The role of the external environment is to suppress the interference produced by the presence of caustics [1].

Finally, for a complete understanding of the classical limit, there is further difficulty to consider: even for very small perturbations due to the interaction with the environment, Schrödinger's evolution for very narrow wave function of the center of mass is quickly destroyed. To solve this problem is not easy, but the key to overcome this difficulty is to observe that the emergence of classicality should be associated with the production of local plane waves in the center of mass coordinate x of the body, with weak dependence on the other degrees of freedom y (see [1] and [3]).

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