The multilinear extension and the symmetric coalition Banzhaf value

J.M. Alonso-Meijide [a,*], M.G. Fiestras-Janeiro [b] and F. Carreras [c]

[a] Departamento de Estatística e I. O., Universidade de Santiago de Compostela, Facultade de Ciencias, Alfonso X s/n, Campus de Lugo, E-27002 Lugo, España.

[b] Departamento de Estatística e I. O., Universidade de Vigo, Facultade de Ciencias Económicas, E-36200 Vigo, España.

[c] Departamento de Matemática Aplicada II, Universidad Politécnica de Cataluña, ETSEIT, P.O.Box 577, E-08220 Terrassa, España.

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Abstract

Alonso-Meijide and Fiestras-Janeiro (2002) proposed a modification of the Banzhaf value for games where a coalition structure is given. In this paper we present a method to compute this value by means of the multilinear extension of the game. A real-world numerical example illustrates the application procedure.

Keywords: coalition structure, Banzhaf value, coalition value, multilinear extension

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1 Introduction

In the context of TU games, two of the most important solution concepts are the Shapley value φ (Shapley, 1953) and the Banzhaf value β, introduced by Owen (1975) as a dummy-independent but non-normalized extension to all cooperative games of the original Banzhaf-Coleman power index (Banzhaf, 1965; Coleman, 1971), which was restricted to simple games. One of the main difficulties with these values is that computation generally requires the sum of a very large number of terms. Owen (1972) defined the multilinear extension of a game, that has been proven to be useful for such a computation, as both the Banzhaf value

*Corresponding author. E-mail address: meijide@lugo.usc.es (Alonso-Meijide)

and the Shapley value of any game can be easily obtained from its multilinear extension.

Indeed, the Shapley value of a game can be calculated by integrating the partial derivatives of the multilinear extension of the game along the main diagonal $x_1 = x_2 = \cdots = x_n$ of the cube $[0,1]^N$ (Owen, 1972). In turn, the derivatives of that multilinear extension, evaluated at point $(1/2, 1/2, \ldots, 1/2)$, give the Banzhaf value of the game (Owen, 1975).

In the context of TU games endowed with a coalition structure, Owen (1977) defined and axiomatically characterized a natural extension of the Shapley value, the so-called Owen coalition value Φ. With a similar procedure, Owen (1981) suggested an extension of the Banzhaf value to this context, and we will refer to this new value as the Banzhaf-Owen coalition value Ψ. This second extension has been axiomatically characterized, only recently, by Albizuri (2001) and independently by Amer et al. (2002). The multilinear extension technique has been also applied to compute the Owen coalition value (Owen and Winter, 1992) as well as the Banzhaf-Owen coalition value (Carreras and Magaña, 1994).

Alonso-Meijide and Fiestras-Janeiro (2002) introduced and axiomatically characterized another solution concept for games with a coalition structure: the symmetric coalition Banzhaf value π. As was mentioned in their paper, only one property distinguishes the Owen coalition value Φ from the symmetric coalition Banzhaf value π: efficiency is satisfied in the former case whereas total power holds in the latter. This relationship cannot be achieved between the Owen coalition value Φ and the Banzhaf-Owen coalition value Ψ, since the quotient game and symmetry properties for unions are not satisfied by the latter (for details, see Amer et al., 2002).

In this paper, we state how to use the multilinear extension to computing the symmetric coalition Banzhaf value π and even its counterpart: a coalition value μ introduced by Amer et al. (2002). In Section 2 we recall some basic definitions. In Section 3, we introduce the procedure to calculate the symmetric coalition Banzhaf value by means of the multilinear extension and we apply this method to a real-world example. Finally, we include in Section 4 some additional remarks and discuss other possibilities for evaluating the effects of the coalition formation.

2 Preliminaries

A finite transferable utility cooperative game (in short, a TU game) is defined by a finite set of players N, where $|N| = n$, and a real valued function v defined on the subsets of N and such that $v(\emptyset) = 0$. The multilinear extension of game v is given by:

$$f(x_1, x_2, \ldots, x_n) = \sum_{S \subseteq N} \prod_{i \in S} x_i \prod_{j \notin S} (1 - x_j)\, v(S).$$

Let us consider a finite set $N = \{1, 2, \ldots, n\}$. We will denote by $P(N)$ the set of all partitions of N. An element $P \in P(N)$ is called a coalition structure or a system of unions on N. Two trivial systems of unions are given by $P^n = \{\{1\},\{2\},\ldots,\{n\}\}$ and $P^N = \{N\}$. A TU game with a coalition structure is a triple (N,v,P), where (N,v) is a TU game and $P \in P(N)$.

Given a game with a coalition structure (N,v,P), the symmetric coalition Banzhaf value of a player $i \in N$ is given by

$$\pi_i(N,v,P) = \sum_{R \subseteq M\setminus\{k\}} \sum_{T \subseteq P_k\setminus\{i\}} \frac{1}{2^{m-1}} \frac{t!(p_k-t-1)!}{p_k!}[v(Q \cup T \cup \{i\}) - v(Q \cup T)],$$

where $M = \{1,2,\ldots,m\}$, $P = \{P_1,P_2,\ldots,P_m\}$, $Q = \cup_{r\in R} P_r$, and $P_k \in P$ is the union such that $i \in P_k$. For an axiomatic introduction of this value and additional properties, see Alonso-Meijide and Fiestras-Janeiro (2002).

3 The computation procedure

In this section, we will describe the procedure by which the symmetric coalition Banzhaf value can be obtained from the multilinear extension of the game.

Theorem 1 *Given a TU game with a coalition structure (N,v,P), where $P = \{P_1,P_2,\ldots,P_m\}$, the following steps lead to the symmetric coalition Banzhaf value of a player $i \in P_k$.*

1. Obtain the multilinear extension $f(x_1,x_2,\ldots,x_n)$ of game (N,v).

2. For any $l \neq k$ and any $h \in P_l$, replace the variable x_h with y_l. This yields a new function of x_j, where $j \in P_k$, and y_l where $l \in M\setminus\{k\}$.

3. In the previous function, reduce to 1 all higher exponents, i.e., replace with y_l each y_l^a such that $a > 1$. This gives a new multilinear function that we denote as $g((x_j)_{j\in P_k},(y_l)_{l\in M\setminus\{k\}})$.

4. In the function obtained in step 3, substitute each y_l by $1/2$. This yields a new function $\alpha_k((x_j)_{j\in P_k})$ defined by

$$\alpha_k((x_j)_{j\in P_k}) = g((x_j)_{j\in P_k},(1/2)_{l\in M\setminus\{k\}}).$$

5. Finally, the symmetric coalition Banzhaf value of player $i \in P_k$ is given by

$$\pi_i(N,v,P) = \int_0^1 \frac{\partial \alpha_k}{\partial x_i}(t,t,\ldots,t)dt\,.$$

Proof.

Let (N,v,P) be a game with a priori unions and let $i \in P_k$.

Steps 1-3 have already been used by Owen and Winter (1992) and Carreras and Magaña (1994) in dealing with the Owen value and the Banzhaf-Owen value, respectively. It will be useful to recall here their argument. By second and third steps, we get a multilinear function where those terms corresponding to coalitions S such that $S \cap P_l \neq \emptyset$ and $(N \backslash S) \cap P_l \neq \emptyset$ for some $l \in M \backslash \{k\}$ are null. In step 2, the terms corresponding to these coalitions include expressions of the form $k y_l^{a_1}(1-y_l)^{a_2}$, with $a_1, a_2 \in \mathbb{N}$, and in step 3 these terms turn on $k(y_l - y_l)$ thus getting zero.

Then, the only coalitions S for which the corresponding term of the initial multilinear extension does not vanish after steps 2 and 3 are those of the form

$$S = Q \cup R,$$

where $R \subseteq P_k$, $Q = \bigcup_{l \in L} P_l$, and $L \subseteq M \backslash \{k\}$. The function arising after step 3 is

$$g((x_j)_{j \in P_k}, (y_l)_{l \in M \backslash \{k\}}) =$$

$$\sum_{R \subseteq P_k} \sum_{L \subseteq M \backslash \{k\}} \prod_{j \in R} x_j \prod_{j \in P_k \backslash R} (1-x_j) \prod_{l \in L} y_l \prod_{l \notin L \cup \{k\}} (1-y_l)\, v(Q \cup R).$$

Substituting each y_l by $1/2$ (step 4) gives

$$\alpha_k((x_j)_{j \in P_k}) = \sum_{R \subseteq P_k} \sum_{L \subseteq M \backslash \{k\}} \prod_{j \in R} x_j \prod_{j \in P_k \backslash R} (1-x_j) \frac{1}{2^{m-1}}\, v(Q \cup R).$$

By differentiating function $\alpha_k((x_j)_{j \in P_k})$ with respect to x_i

$$\frac{\partial \alpha_k}{\partial x_i}((x_j)_{j \in P_k}) =$$

$$\sum_{R \subseteq P_k \backslash \{i\}} \sum_{L \subseteq M \backslash \{k\}} \prod_{j \in R} x_j \prod_{j \in P_k \backslash (R \cup \{i\})} (1-x_j) \frac{1}{2^{m-1}} [v(Q \cup R \cup \{i\}) - v(Q \cup R)].$$

Finally, by step 5,

$$\int_0^1 \frac{\partial \alpha_k}{\partial x_i}(t, t, \ldots, t) dt =$$

$$\int_0^1 \sum_{R \subseteq P_k \backslash \{i\}} \sum_{L \subseteq M \backslash \{k\}} \prod_{j \in R} t \prod_{j \in P_k \backslash (R \cup \{i\})} (1-t) \frac{1}{2^{m-1}} [v(Q \cup R \cup \{i\}) - v(Q \cup R)] dt =$$

$$\sum_{R \subseteq P_k \backslash \{i\}} \sum_{L \subseteq M \backslash \{k\}} \frac{1}{2^{m-1}} [v(Q \cup R \cup \{i\}) - v(Q \cup R)] \int_0^1 t^r (1-t)^{p_k - r - 1} dt =$$

$$\sum_{R \subseteq P_k \setminus \{i\}} \sum_{L \subseteq M \setminus \{k\}} \frac{1}{2^{m-1}} \frac{(r!(p_k - r - 1)!}{p_k!} [v(Q \cup R \cup \{i\}) - v(Q \cup R)] = \pi_i(N, v, P). \square$$

As is readily seen, the symmetric coalition Banzhaf value π reflects the result of a bargaining procedure by which: (a) the a priori unions receive in the quotient game they play, i.e. in (M, v_P) where $M = \{1, 2, \dots, m\}$ and $v_P(K) = v(\bigcup_{k \in K} P_k)$ for every $K \subseteq M$, the payoff given by the Banzhaf value β; and (b) within each union P_k, the original players share this payoff $\beta_k(M, v_P)$ among themselves by using the Shapley value φ.

This is a mixed procedure, already suggested by Carreras and Magaña (1997) in a more general set up, that may make sense in specific contexts. After all, unions are in general of a different nature from original, single players, and the quotient game v_P may well possess features not found in the initial game v. The question is not therefore "why will the unions follow, as entities, a different way from players' one?" but "why not?" Thus, it is only natural to go further and to consider the possibility to interchange criteria and allow unions to use the Shapley value and players within unions to adopt the Banzhaf value: indeed, this is what is reflected by value μ, a "counterpart" of π introduced by Amer et al. (2002) and defined for player $i \in P_k$ by

$$\mu_i(N, v, P) = \sum_{R \subseteq M \setminus \{k\}} \sum_{T \subseteq P_k \setminus \{i\}} \frac{1}{2^{p_k - 1}} \frac{r!(m - r - 1)!}{m!} [v(Q \cup T \cup \{i\}) - v(Q \cup T)],$$

where $M = \{1, 2, \dots, m\}$, $P = \{P_1, P_2, \dots, P_m\}$, $Q = \cup_{r \in R} P_r$, and $P_k \in P$ is the union such that $i \in P_k$. (A full generalization of the mixed bargaining procedure idea will be dealt with in Remark 6 below.) Minor changes in the steps of Theorem 1 give rise to a computation method for μ.

Theorem 2 *Given a TU game with a coalition structure (N, v, P), where $P = \{P_1, P_2, \dots, P_m\}$, the following steps lead to the expression of $\mu_i(N, v, P)$ for every $i \in P_k$.*

1. Obtain the multilinear extension $f(x_1, x_2, \dots, x_n)$ of game (N, v).

2. For any $l \neq k$ and any $h \in P_l$, replace the variable x_h with y_l. This yields a new function of x_j, where $j \in P_k$, and y_l where $l \in M \setminus \{k\}$.

3. In the previous function, reduce to 1 all higher exponents, i.e., replace with y_l each y_l^a such that $a > 1$. This gives a new multilinear function that we denote as $g((x_j)_{j \in P_k}, (y_l)_{l \in M \setminus \{k\}})$.

4. In the function obtained in step 3, substitute each y_l by r and integrate with respect to r to get a new function $\alpha_k((x_j)_{j \in P_k})$ given by

$$\alpha_k((x_j)_{j \in P_k}) = \int_0^1 g((x_j)_{j \in P_k}, (r)_{l \in M \setminus \{k\}}) dr.$$

Table 1: Seat distribution in the Catalonia Parliament, 1995-1999

i party	number of seats w_i
1: Convergència i Unió (CiU)	60
2: Partit dels Socialistes de Catalunya (PSC)	34
3: Partido Popular (PP)	17
4: Esquerra Republicana de Catalunya (ERC)	13
5: Iniciativa per Catalunya-Verds (IC-V)	11

5. *Finally, differentiate with respect to* x_i *and evaluate at point* $(1/2, 1/2, \ldots, 1/2)$*:*

$$\mu_i(N, v, P) = \frac{\partial \alpha_k}{\partial x_i}(1/2, 1/2, \ldots, 1/2).$$

Proof.
Steps 1-3 are, of course, the same as in Theorem 1. The remaining of the proof is, *mutatis mutandis*, analogous to that of Theorem 1. □

Both the Shapley and Banzhaf values, as well as their initial extensions to games with a coalition structure (Owen, 1977; Owen, 1981), have been often used as measures of power, by applying them to simple games and looking at proportions rather than absolute payoffs (see Laruelle and Valenciano, 2002). In order to illustrate the procedures stated in Theorems 1 and 2, let us present and analyze a real-world numerical example of this kind of games. We will compare the payoffs given by the symmetric coalition Banzhaf value π, its counterpart μ, and the Owen and Banzhaf-Owen values Φ and Ψ, which will be computed using the analogues of our method provided, respectively, by Owen and Winter (1992) and Carreras and Magaña (1994).

Example 3 *(The Catalonia Parliament, 1995-1999). We will consider here the structure of the Catalonia Parliament, a typical western Europe parliamentary body, during Legislature 1995-1999. Five parties elected members to this Spanish regional house and shared the 135 seats as indicated in Table 1.*

The straight majority rule requires here 68 votes to pass a bill. Because of voting discipline, in this scenario the players are the parties rather than the elected representatives. Then $N = \{1, 2, 3, 4, 5\}$ *and a simple game* v *is defined by setting* $v(S) = 1$ *iff* $\sum_{i \in S} w_i \geq 68$ *and* $v(S) = 0$ *otherwise. The set of minimal winning coalitions of this game is*

$$W^m = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 3, 4, 5\}\}.$$

The multilinear extension of game v *is therefore*

$$f(x_1, x_2, x_3, x_4, x_5) =$$
$$\sum_{i=2}^{4} x_1 x_i - \sum_{i,j=2}^{4} x_1 x_i x_j + \sum_{i,j,k=2}^{4} x_1 x_i x_j x_k + x_2 x_3 x_4 x_5 - 2x_1 x_2 x_3 x_4 x_5.$$

Table 2: Power distributions in the Catalonia Parliament, 1995-1999

values	CiU	PSC	PP	ERC	IC-V	sum
weight fraction	0.4444	0.2519	0.1259	0.0963	0.0815	1.0000
$\varphi(N,v)$	0.6000	0.1000	0.1000	0.1000	0.1000	1.0000
$\beta(N,v)$	0.8750	0.1250	0.1250	0.1250	0.1250	1.3750
$\varphi(M,v_P)$	0.3333	0.3333	0.3333			1.0000
$\beta(M,v_P)$	0.5000	0.5000	0.5000			1.5000
$\Phi(N,v,P)$	0.3333	0.1111	0.3333	0.1111	0.1111	1.0000
$\Psi(N,v,P)$	0.5000	0.1250	0.5000	0.1250	0.1250	1.3750
$\pi(N,v,P)$	0.5000	0.1667	0.5000	0.1667	0.1667	1.5000
$\mu(N,v,P)$	0.3333	0.0833	0.3333	0.0833	0.0833	0.9167

Given the underlying political framework (the nationalist middle-of-the-road party CiU was losing for the first time the absolute majority and hence its dictator role held in previous legislatures, and PP, the catalan section of a right-to-center national party, got its best result and confirmed the growing of this tendency in Spain elections), a natural strategy in this situation would have been, in principle, the union of the three more or less left-wing parties, namely, PSC, ERC and IC-V. In order to discuss the effects of such a coalition, we provide in Table 2 the results of applying the Shapley and Banzhaf values φ *and* β *to games* (N,v) *and* (M,v_P)*, where* $P=\{\{1\},\{2,4,5\},\{3\}\}$ *and* $M=\{1,2,3\}$*, and our four extensions* Φ, Ψ, π *and* μ *to the game with coalition structure* (N,v,P)*. (When dealing with the quotient game, the payoffs will be given in the CIU, PSC and PP columns.)*

As a matter of illustration, let us show how to compute $\pi_2(N,v,P)$ *using Theorem 1. After applying steps 1-3 we get*

$$g(y_1,x_2,x_3,x_4,y_3)=y_1(\sum_{i=2}^{4}x_i-\sum_{i,j=2}^{4}x_ix_j+x_2x_3x_4)$$

$$+y_1(1-\sum_{i=2}^{4}x_i+\sum_{i,j=2}^{4}x_ix_j-2x_2x_3x_4)y_3+y_3x_2x_3x_4.$$

Then

$$\alpha_2(x_2x_3x_4)=\frac{1}{4}(1+\sum_{i=2}^{4}x_i-\sum_{i,j=2}^{4}x_ix_j+2x_2x_3x_4),$$

$$\frac{\partial\alpha_2}{\partial x_2}=\frac{(1-x_3-x_4+2x_3x_4)}{4}$$

and, finally,

$$\pi_2(N,v,P)=\int_0^1\frac{(1-2t+2t^2)}{4}dt=1/6.$$

Several comments are in order. (1) All values show that the union of the left-wing parties favours them in the sense of decreasing the main party's relative *power, but at the same time it enhances the strategic position of PP, a most natural opposite to the union. (2) The symmetry among the four minor parties existing in the original game is kept in the union but is broken with regard to PP. (3) Among the four extensions, only the Owen coalition value Φ and the coalition symmetric Banzhaf value π satisfy the quotient game property, i.e., they share within each union exactly the amount this union gets in the quotient game. (4) The Banzhaf-Owen value Ψ preserves the payoffs of the parties joining, but the changes concerning the outside parties CiU and PP make clear that the power proportion has been modified by the union.*

4 Final Remarks

Remark 4 *In Alonso-Meijide and Fiestras-Janeiro (2002) it is noted that the symmetric coalition Banzhaf value is an extension of the Shapley and Banzhaf values. More precisely, given a TU game (N, v),*

(i) $\pi(N, v, P^n) = \beta(N, v)$ and

(ii) $\pi(N, v, P^N) = \varphi(N, v)$.

Taking this fact into account, it is easy to prove that the procedure described in Theorem 1 above can be used to compute the Banzhaf value and the Shapley value of a TU game. To compute the Banzhaf value of a game (N, v), we follow the proposed steps taking $P = P^n$, and to compute the Shapley value of this game we follow the proposed steps taking $P = P^N$.

Remark 5 *In Alonso-Meijide and Bowles (2002), new procedures based in generating functions are described to compute coalition values for the particular case of weighted majority games.*

Remark 6 *Following a suggestion of Carreras and Magaña (1997), there is a wide variety of ways in which unions and players can evaluate their strategic positions, in the quotient game the former and within each union the latter. In fact, even the starting point, that is, the evaluation of the original game, could be done by means of a semivalue, as both the Shapley and the Banzhaf value belong to this class of solution concepts. Semivalues were introduced by Weber (1979) on simple games and extended to all cooperative games by Dubet et al. (1981), and generalized later on by Weber (1988), dropping symmetry, to get the notion of probabilistic value. For details on the usefulness of semivalues in order to incorporate, into the evaluation of a game, additional information not included in the characteristic function, we refer the interested reader to Carreras and Freixas (2002).*

As to the use of the multilinear extension to compute semivalues, we refer to Giménez (2001), where it is shown that (a) binomial semivalues can be computed in a way very close to that of the Banzhaf value and (b) any other semivalue requires using a geometrical reference system of the semivalue simplex, given by any n binomial semivalues, and a linear map whose matrix depends on (the partial derivatives of the multilinear extension of) the game and the reference *system; this computation procedure applies even to the Shapley value, so that no integration step is needed.*

Then, by adopting the wider semivalue viewpoint, we can speak of homogeneous evaluation when the same semivalue is used by unions and by players within each union, as is the case of the Owen coalition value and the Banzhaf-Owen coalition value, and even that of modified semivalues introduced by Giménez (2001), which are coalition values of the original semivalue. A parallel way to compute is also given by Giménez (2001) for modified semivalues by means of a bilinear form whose matrix depends, again, on the game and the reference system. It is worthy mentioning here that in the case of modified binomial semivalues Carreras and Magaña's (1994) procedure applies as well.

However, one can also speak of heterogeneous evaluations. Players within each union might use a value different from that used by unions, as it is the case of the symmetric coalition Banzhaf value π *and also of its counterpart* μ*. But even it is also possible, as suggested by Carreras and Magaña (1997), that unions use some value in the quotient game and, then, the players of each union use a value different from that of the unions and from those used within other unions! After all, freedom is a human aspiration that we should take into account in our mathematical modelling of real life behavior and the contract for forming a union can (in fact, it should) perfectly specify the way to share profits among its members.*

Then, a formal notation will help us to better distinguish the several types of evaluation that can arise. Let σ *be the semivalue used by unions and* $\rho_1, \rho_2, \ldots, \rho_m$ *the semivalues used by each union. We denote the compound rule as*

$$\sigma\rho_1, \rho_2, \ldots, \rho_m.$$

With this notation, a first level of homogeneity is attained in case $\sigma\rho, \rho, \ldots, \rho = \sigma\rho^m$ *for a some common* ρ*. Then we have* $\pi = \varphi\beta^m$ *and* $\mu = \beta\varphi^m$*. A further homogeneity level is, finally, found in the case where* $\sigma = \rho$*, like in the classical extensions provided by Owen (1977, 1981): in these cases,* $\Phi = \varphi\varphi^m$ *and* $\Psi = \beta\beta^m$*.*

We think that these ideas open a wide research field, on both the technical side and the interpretative side.

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