

Tensorial Relativistic Quantum Mechanics in (1 + 1) Dimensions and Boundary Conditions

Vidal Alonso,¹ Salvatore De Vincenzo,¹ and Luigi Mondino¹

Received August 20, 1997; revised October 28, 1998

The tensorial relativistic quantum mechanics in (1 + 1) dimensions is considered. Its kinematical and dynamical features are reviewed as well as the problem of finding the Dirac spinor for given finite multivectors. For stationary states, the dynamical tensorial equations, equivalent to the Dirac equation, are solved for a free particle, for a particle inside a box, and for a particle in a step potential.

1. INTRODUCTION

From the beginning of the quantum theory it has been accepted that the spinors are essential in the quantum domain. However, a physical system can be described by giving simultaneously the observables and the state in the form of tensorial densities, that is, probability densities, currents and fields etc. that are sesquilinearly defined in terms of the relativistic wave function and/or its derivatives. Physicists like Pauli, Gordon, Belinfante, Proca,⁽¹⁾ among others, studied this type of quantities by their utility in the interpretation of the relativistic quantum theory. The idea of formulating the relativistic quantum mechanics without using spinors, in the form of a hydrodynamic of tensorial densities, which satisfy dynamical equations equivalent to the Dirac equation, was considered by Costa de Beauregard and Takabayasi.⁽²⁾ The inversion of the bilinear relations, which permits to express the Dirac spinor in terms of multivectors has been considered.⁽³⁾ The dynamical aspects of the tensorial theory show the existence of an equivalence theorem between the Dirac equation and Maxwell-like equations;

¹ Escuela de Física. Facultad de Ciencias, Universidad Central de Venezuela. A.P. 47145. Caracas 1041-A, Venezuela. E-mail: valonso@tierra.ciens.ucv.ve; svincenz@tierra.ciens.ucv.ve.

thus, the quantum theory can be regarded as a generalized electromagnetism.⁽⁴⁾ Despite of this, solving problems by using the tensorial theory in $3 + 1$ dimensions is troublesome. The reason is the non-linear character of the resultant dynamical equations, and the complexity of the probabilistic fluid arising from the spin. The tensorial theory in $1 + 1$ dimensions is more simple and is linear; but as far as we know, it has not been considered in the scientific literature.⁽⁵⁾

We shall study the tensorial theory in $1 + 1$ dimensions obtaining a dynamical equation for the probability density and we will solve this equation for some standard problems. In particular, we shall consider a particle inside a one-dimensional box.⁽⁶⁾ That is, the problem of the various boundary conditions that may be imposed to a relativistic "free" particle inside a one-dimensional box. By the way, one might be interested in studying relativistic equations with covariant boundary conditions. However, without loss of generality, the Lorentz covariance of a dynamical equation can be used so as to choose the privileged frame in which the basic properties of the physical system present themselves in the simplest form. For a particle in a box, the convenient privileged frame is the one in which the space-time Lorentz transformations are frozen and the box is at rest in a determined space region. In this paper, we use the formally covariant tensorial theory in this privileged frame, what will allow us to understand the physics behind some of the spinorial boundary conditions that make self-adjoint the "free" hamiltonian of the Dirac particle in the box.

A detailed spinorial study of the boundary conditions, i.e., self-adjoint extensions for a relativistic particle inside a box, as well as their non-relativistic limits, has been considered by two of us (V. A. and S. De V.).⁽⁷⁾

An analogous problem in $1 + 1$ dimensions but in the frame of quantum field theory was considered by Schwinger,⁽⁸⁾ and has been the subject of several investigations.⁽⁹⁾

The localization problem⁽¹⁰⁾ needs no longer to be regarded as such in the tensorial theory. This is because the intrinsic extended nature of the Dirac particle becomes an essential feature, so that various concepts associated to the point particle lose their meaning. Of course, this is especially evident in $3 + 1$ dimensions, where the tensorial theory appears as a very rich hydrodynamic theory which includes vortical currents.

In Secs. 2 and 3, we review the kinematical and dynamical structures of the tensorial theory, as well as the problem of finding the Dirac spinor in terms of finite multivectors. In Sec. 4 we particularize the obtained results for stationary states. Finally, in Secs. 5 and 6 we study the free particle, the problem of a particle inside a box, and that of a particle in a step potential; here we consider the Klein paradox,^(11, 12) recovering the results reported by Greiner.

2. KINEMATICAL STRUCTURE OF THE TENSORIAL THEORY

Let $\Psi = \Psi(x, t)$ be a Dirac-like spinor which does not necessarily satisfy the Dirac equation. In 1 + 1 dimensions Ψ is a two components spinor and represents the quantum state; but a pure state can also be described by an observable density matrix, such as the Clifford number

$$C = 2 \in \Psi \otimes \bar{\Psi} = 2 \in \Psi \bar{\Psi} \tag{1}$$

where $\in = -e/|e|$, that is, for electrons $\in = 1$. The Dirac adjoint is $\bar{\Psi} = \psi^\dagger \gamma^0$ where Ψ^\dagger is the hermitian conjugate of Ψ and γ^0 is the first one of the two gamma matrices $\gamma^\mu = (\gamma^0, \gamma^1) = (\beta, \beta\alpha)$, which satisfy the Clifford relation: $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} = 2 \text{diag}(1, -1)$. It can be proved that the charge-conjugation anticommutes with the Dirac adjoint operation: $(\bar{\Psi})^c = -(\Psi^c)$, where $\Psi^c = \alpha \Psi^*$ is the charge conjugate of Ψ , with Ψ^* the complex conjugate of Ψ and $\in^c = -\in$.

The Clifford number C is a 2×2 spinor complex matrix, and is tensorially a scalar. The Dirac adjoint of C is defined as $\bar{C} = \gamma^0 C^\dagger \gamma^0$. Note that $\bar{C} \neq C^\dagger \neq C$, but $\bar{\bar{C}} = C$. The Clifford algebra shows that C can be written in a unique way as a linear combination of four matrices basis: $\Gamma^A = \{1, \gamma^\mu, -i\gamma^5\}$, where $\gamma^5 = \alpha$ verifies $\gamma^5 \gamma^\mu + \gamma^\mu \gamma^5 = 0$, which constitute the Clifford basis; that is

$$C = \sum_{A=1}^4 \lambda_A \Gamma^A = S1 + V_\mu \gamma^\mu - i\omega \gamma^5 \tag{2}$$

The components $\lambda_A = \{S, V_\mu, \omega\}$ of C are real because $\bar{C} = C$, and can be obtained by using the scalar product of matrices: $\lambda_A = (\Gamma^A, C) = \frac{1}{2} \text{Tr}[(\Gamma^A)^\dagger C]$, where Tr means trace. So,

$$S = \bar{\Psi} \Psi \tag{3}$$

$$V^\mu = \bar{\Psi} \gamma^\mu \Psi \tag{4}$$

$$\omega = i \bar{\Psi} \gamma^5 \Psi \tag{5}$$

which are the basic finite multivectors. We shall refer to V^μ as to the two-vector probability current, to S and V^0 as to the scalar and the probability densities respectively, V^1 is the spatial component of the probability current density, and ω is the pseudo-scalar density.⁽⁷⁾ The symbol + denotes the dual operation, so ω is a pseudo-tensor.

It is convenient to write explicitly (3)–(5) as: $S = \Psi^\dagger \beta \Psi$, $V^0 = \Psi^\dagger \Psi$, $V^1 = \Psi^\dagger \alpha \Psi$, and $\omega = i \Psi^\dagger \beta \alpha \Psi$. Any of the 2×2 Pauli matrices can be used as the Dirac matrices α and β . In the standard or Dirac representation in

1 + 1 dimensions, $\alpha = \sigma_x$ and $\beta = \sigma_z$. In the so-called Weyl representation, $\alpha = \sigma_z$ and $\beta = \sigma_x$.

From (1), we get

$$\left(\frac{C}{2S}\right)^2 = \frac{C}{2S} \tag{6}$$

moreover, the Dirac adjoint of $C/2S$ verifies $\overline{(C/2S)} = C/2S$. Then, $C/2S$ is a projection operator with eigenvalues 0, 1, and therefore it represents a pure state.

Substituting (2) in (6) and using the identities $\gamma^\mu \gamma^\nu = g^{\mu\nu} - \gamma^5 D^{\mu\nu}$ and $\gamma^\mu \gamma^5 = D^{\mu\nu} \gamma_\nu$, where $D_{\mu\nu} = -D_{\nu\mu}$ and $D_{01} = 1$, with $D_{\mu\nu}$ the permutation pseudo-tensor, we obtain as a consequence of the purity of the quantum state

$$S^2 + \omega^2 = V^\mu V_\mu \tag{7}$$

This relation implies that only the three finite multivectors (3)–(5) are independents.

Introducing the covariant derivative of $\Psi : D_\mu = \partial_\mu + (ieA_\mu/\hbar c)$, with $\partial_\mu = ((1/c) \partial_t, \partial_x)$, and letting $\lambda = \hbar/2mc$, we define the set of Clifford numbers

$$C_\mu = 2i\lambda(D_\mu \Psi \otimes \bar{\Psi} - \Psi \otimes \overline{D_\mu \Psi}) \tag{8}$$

which tensorially constitute a covariant vector and may also be written as

$$C_\mu = I_\mu 1 + T_{\nu\mu} \gamma^\nu - i\hbar_\mu \gamma^5 \tag{9}$$

The differential multivectors defined by (9) are obtained by using the scalar product between matrices. That is,

$$I_\mu = i\lambda(\bar{\Psi} D_\mu \Psi - \overline{D_\mu \Psi} \Psi) = i\lambda(\bar{\Psi} \partial_\mu \Psi - \overline{\partial_\mu \Psi} \Psi) - \frac{eA_\mu S}{mc^2} \tag{10}$$

$$T_{\mu\nu} = i\lambda(\bar{\Psi} \gamma_\mu D_\nu \Psi - \overline{D_\nu \Psi} \gamma_\mu \Psi) = i\lambda(\bar{\Psi} \gamma_\mu \partial_\nu \Psi - \overline{\partial_\nu \Psi} \gamma_\mu \Psi) - \frac{eA_\nu V_\mu}{mc^2} \tag{11}$$

$$\hbar_\mu = i\lambda(i\bar{\Psi} \gamma^5 D_\mu \Psi - \overline{iD_\mu \Psi} \gamma^5 \Psi) = i\lambda(i\bar{\Psi} \gamma^5 \partial_\mu \Psi - \overline{i\partial_\mu \Psi} \gamma^5 \Psi) - \frac{eA_\mu \omega}{mc^2} \tag{12}$$

where $e = -|e|$ is the electron charge and m is its mass. The electromagnetic potential is $A^\mu = (\varphi, A)$. Obviously, $A = 0$ in 1 + 1 dimensions.

The differential multivectors I_μ and h_μ^+ are called the convective and the pseudo-current respectively, and $T_{\mu\nu}$ the probability tensor.

Using (1) and (8), it may be verified that

$$CC_\mu = 4i\lambda[\bar{\Psi}D_\mu\Psi(\Psi\otimes\bar{\Psi}) - \bar{\Psi}\Psi(\Psi\otimes\overline{D_\mu\Psi})] \quad (13)$$

and

$$C_\mu C = 4i\lambda[\bar{\Psi}\Psi(D_\mu\Psi\otimes\bar{\Psi}) - \overline{D_\mu\Psi}\Psi(\Psi\otimes\bar{\Psi})] \quad (14)$$

Then,

$$CC_\mu + C_\mu C = 2(SC_\mu + I_\mu C) \quad (15)$$

$$CC_\mu - C_\mu C = 2i\lambda(\partial_\mu S C - S \partial_\mu C) \quad (16)$$

Substituting (2) and (9) in (15) and (16), we obtain from (15) the following relation between multivectors:

$$SI^\mu + \overset{+}{\omega}h^\mu = V_\nu T^{\nu\mu} \quad (17)$$

and from (16),

$$\overset{+}{D}_{\mu\nu}(V^\nu\overset{+}{h}_\rho - \overset{+}{\omega}T^\nu{}_\rho) = \lambda(V_\mu\partial_\rho S - S\partial_\rho V_\mu) \quad (18)$$

$$\overset{+}{D}_{\mu\nu}V^\mu T^\nu{}_\rho = \lambda(S\partial_\rho\overset{+}{\omega} - \overset{+}{\omega}\partial_\rho S) \quad (19)$$

Using (15), (16) and (6), we also obtain

$$i\frac{\lambda}{2}(C\partial_\mu C - \partial_\mu C C) = 2(I_\mu C - SC_\mu) \quad (20)$$

Following the same procedure adopted for deriving (17)–(19), two new relations may be obtained from (20)

$$\lambda\overset{+}{D}_{\mu\nu}V^\mu\partial_\rho V^\nu = \overset{+}{\omega}I_\rho - S\overset{+}{h}_\rho \quad (21)$$

$$\overset{+}{\lambda}D_{\mu\nu}(V^\nu\partial_\rho\overset{+}{\omega} - \overset{+}{\omega}\partial_\rho V^\nu) = ST_{\mu\rho} - I_\rho V^\mu \quad (22)$$

The expression (21) is particularly interesting because it relates the three two-vector currents which are present in the 1 + 1 tensorial theory.

We emphasize that (17)–(19), as well as (21) and (22), have been systematically derived without involving the Dirac equation, so they answer to non-dynamical features.

3. DYNAMICAL STRUCTURE OF THE TENSORIAL THEORY

3.1. Dynamical Equations

Let $O_D = i\gamma^\mu D_\mu - (mc/\hbar)$ be the Dirac operator. If Ψ satisfies the Dirac equation

$$O_D \Psi = 0 \quad (23)$$

Ψ becomes a Dirac spinor.

Let us define the Clifford number

$$C_D = O_D \Psi \otimes \bar{\Psi} \quad (24)$$

In view of (23), each complex multivector belonging to C_D is null, that is

$$\bar{\Psi} \Gamma^A O_D \Psi = 0 \quad (25)$$

Thus, the real and imaginary parts of (25) are zero. Making full use of the definitions of the multivectors and of the relations between the gamma matrices given above, in addition to: $\gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0$; $\gamma^{5\dagger} = \gamma^5$ and $D^{\mu\nu} D_{\alpha\beta} = -\delta^\mu_\alpha \delta^\nu_\beta + \delta^\mu_\beta \delta^\nu_\alpha$, we obtain the dynamical equations implied by the Dirac equation. These may be conveniently grouped in three pairs:

$$\lambda D^{\mu\nu} \partial_\nu \omega = I^\mu - V^\mu \quad (26)$$

$$\lambda D^{\nu\mu} \partial_\nu S = h^\mu \quad (27)$$

$$T^\mu_\mu = S \quad (28)$$

$$D^{\mu\nu} T_{\mu\nu} = 0 \quad (29)$$

$$\partial_\mu V^\mu = 0 \quad (30)$$

$$\lambda D^{\mu\nu} \partial_\mu V_\nu = \dot{\omega} \quad (31)$$

These equations are the fundamental dynamical relations implied by the Dirac equation.

Note that, from (26), (27) and (30), the currents I^μ and h^μ verify

$$\partial_\mu I^\mu = 0 \quad (32)$$

$$\partial_\mu h^\mu = 0 \quad (33)$$

Hence, these currents, as well V^μ according to (30), are conserved, inasmuch as each one of them satisfies a continuity equation.

The above three pairs of dynamical equations (26)–(31) look like a rather large set of relations to be satisfied. However, the first pair, which we call the Maxwell-like equations, may be considered as the fundamental one. We will see that the other two pairs may be derived from these Maxwell-like equations and the non-dynamical relations (7), (17) and (18), (19) being implied by (7) and (18).

3.2. Mathematical Equivalence Between the Maxwell-like Equations and the Dirac Equation

Multiplying (26) with S and (27) with $\overset{+}{\omega}$, we have

$$\lambda \overset{+}{D}{}^{\mu\nu} (S \partial_\nu \overset{+}{\omega} - \overset{+}{\omega} \partial_\nu S) = SI^\mu + \overset{+}{\omega} \overset{+}{h}{}^\mu - SV^\mu \quad (34)$$

Using (17) and (19), and the product $\overset{+}{D}{}^{\mu\nu} \overset{+}{D}{}_{\alpha\beta}$ given earlier, this expression takes the form

$$V^\mu (T^\nu{}_\nu - S) = V_\nu (T^{\mu\nu} - T^{\nu\mu}) \quad (35)$$

Contracting (35) with V_μ ,

$$V_\mu V^\mu (T^\nu{}_\nu - S) = 0 \quad (36)$$

From (7) $V_\mu V^\mu$ is not zero, thus we get (28).

Returning to (35) and using the last result, we have $T^{\mu\nu} = T^{\nu\mu}$, and contracting it with $\overset{+}{D}{}^{\mu\nu}$ we obtain (29).

From (18) we can write

$$\overset{+}{D}{}_{\mu\nu} (V^\nu \overset{+}{h}{}^\mu - \overset{+}{\omega} T^{\nu\mu}) = \lambda (V^\mu \partial_\mu S - S \partial^\mu V_\mu) \quad (37)$$

using (27) and (29) and the identity $\overset{+}{D}{}_{\mu\nu} \overset{+}{D}{}^{\mu\rho} = -\delta_\nu{}^\rho$, we get (30).

Contracting (18) with $\overset{+}{D}{}^{\rho\mu}$ we find

$$V^\rho \overset{+}{h}{}_\rho - \overset{+}{\omega} T^\rho{}_\rho = \lambda \overset{+}{D}{}^{\rho\mu} (V_\mu \partial_\rho S - S \partial_\rho V_\mu) \quad (38)$$

Taking into account (28) and (27) in this equation, we finally get (31).

So, the dynamical information of the Dirac equation is contained in only two vectorial equations. Indeed, if the Dirac equation is verified, it implies three pairs of dynamical equations which with help of the non-dynamical or algebraic relations may be reduced to a single pair. On the other hand, if the three pairs of dynamical relations are verified, then (25) is satisfied for every Γ^A . The spinor $\bar{\Psi}$ is in general not null, thus the Dirac

equation is satisfied. In this way, the equivalence between Maxwell-like equations and the Dirac equation is demonstrated.

Let us assume that we know the necessary multivectors and that we want to obtain their corresponding spinor. In the next section we will show how to obtain the spinor for a given set of finite multivectors.

3.3. Spinors from Multivectors

Let us first consider the following general spinor in the Dirac representation:

$$\Psi(x, t) = \begin{pmatrix} \sqrt{a} e^{i(\varepsilon + \Omega)} \\ \sqrt{b} e^{i(\varepsilon - \Omega)} \end{pmatrix} \quad (39)$$

From (3)–(5) one obtains: $S = a - b$, $V^0 = a + b$, $V^1 = 2\sqrt{ab} \cos(2\Omega)$, and $\overset{+}{\omega} = 2\sqrt{ab} \sin(2\Omega)$. From which one gets

$$a = \frac{1}{2}(V^0 + S) \quad (40)$$

$$b = \frac{1}{2}(V^0 - S) \quad (41)$$

$$\cos(2\Omega) = V^1 / [(V^1)^2 + (\overset{+}{\omega})^2]^{1/2} \quad (42)$$

In order to obtain the overall phase ε up to an integration constant, one may use any of the Maxwell-like equations. It is convenient to write them only in terms of finite multivectors, the external electromagnetic potential A_μ , and the gradient of ε . For this, let us write the currents I^μ and h^μ in terms of the spinor (39):

$$I_\mu = -2\lambda(S \partial_\mu \varepsilon + V^0 \partial_\mu \Omega) - \frac{eA_\mu S}{mc^2} \quad (43)$$

$$\overset{+}{h}_\mu = -2\lambda \overset{+}{\omega} \left(\partial_\mu \varepsilon + \frac{V^0}{S} \partial_\mu \Omega \right) - \frac{eA_\mu \overset{+}{\omega}}{mc^2} - \frac{\lambda}{S} \overset{+}{D}^{\alpha\rho} V_\alpha \partial_\mu V_\rho \quad (44)$$

where we have used (21) in order to express $\overset{+}{h}_\mu$ in term of I_μ .

Substituting (43) in (26) and (44) in (27), one gets

$$\lambda \overset{+}{D}_{\mu\nu} \partial^\nu \overset{+}{\omega} = -2\lambda(S \partial_\mu \varepsilon + V^0 \partial_\mu \Omega) - \frac{eA_\mu S}{mc^2} - V_\mu \quad (45)$$

$$\lambda S \overset{+}{D}_{\mu\nu} \partial^\nu S = \lambda \overset{+}{D}^{\alpha\rho} V_\alpha \partial_\mu V_\rho + 2\lambda \overset{+}{\omega} (S \partial_\mu \varepsilon + V^0 \partial_\mu \Omega) + \frac{eA_\mu S \overset{+}{\omega}}{mc^2} \quad (46)$$

In this way, with (45) or (46), both implied by the Dirac equation, one obtains $\partial_\mu \varepsilon$ from the finite multivectors and the external electromagnetic potential. This quantity is observable. Integrating it, one can calculate the overall phase ε up to an integration constant.

4. STATIONARY STATES

In the case of a stationary state the overall phase may be written as: $\varepsilon(x, t) = \varepsilon = -(E/\hbar)t + f(x)$, moreover $\partial_t \Omega = 0$ and $\partial_t A_\mu = 0$. So, from the general spinor in the Dirac representation (39), one obtains $\Psi(x, t) = \psi(x) e^{-i(E/\hbar)t}$, where $\psi(x) = (\sqrt{a} e^{i(f+\Omega)}, \sqrt{b} e^{i(f-\Omega)})$. The finite multivectors may be written as

$$S = \psi^\dagger \beta \psi = \phi^* \phi - \chi^* \chi \quad (47)$$

$$V^0 = \psi^\dagger \psi = \phi^* \phi + \chi^* \chi \quad (48)$$

$$V^1 = \psi^\dagger \alpha \psi = \phi^* \chi + \chi^* \phi \quad (49)$$

$$\overset{+}{\omega} = i\psi^\dagger \beta \alpha \psi = i(\phi^* \chi - \chi^* \phi) \quad (50)$$

where $\phi = \sqrt{a} e^{i(f+\Omega)}$ and $\chi = \sqrt{b} e^{i(f-\Omega)}$ are respectively the spatial parts of the so-called large and small components of the spinor Ψ in the Dirac representation.

Denoting hereafter with primes the differentiation with respect to x and choosing the axial gauge $A^1 = 0$, and $eA_0 \equiv U$, the components of the differential multivectors (10)–(12) are

$$I^0 = \frac{E-U}{mc^2} S \quad (51)$$

$$I^1 = i\lambda(\psi^{\dagger'} \beta \psi - \psi^\dagger \beta \psi') \quad (52)$$

$$T^{00} = \frac{E-U}{mc^2} V^0 \quad (53)$$

$$T^{01} = i\lambda(\psi^{\dagger'} \psi - \psi^\dagger \psi') \quad (54)$$

$$T^{10} = \frac{E-U}{mc^2} V^1 \quad (55)$$

$$T^{11} = i\lambda(\psi^{\dagger'} \alpha \psi - \psi^\dagger \alpha \psi') \quad (56)$$

$$\overset{+}{h}^0 = \frac{E-U}{mc^2} \overset{+}{\omega} \quad (57)$$

$$\overset{+}{h}^1 = i\lambda(i\psi^{\dagger'} \beta \alpha \psi - i\psi^\dagger \beta \alpha \psi') \quad (58)$$

From the Maxwell-like equations the following relations are obtained:

$$-\lambda \overset{+}{\omega}' = I^0 - V^0 \quad (59)$$

$$0 = I^1 - V^1 \quad (60)$$

$$\lambda S' = \overset{+}{h}^0 \quad (61)$$

$$0 = \overset{+}{h}^1 \quad (62)$$

The remainder dynamical equations implied by the Maxwell-like equations are

$$T^{00} - T^{11} = S \quad (63)$$

$$T^{01} - T^{10} = 0 \quad (64)$$

$$(V^1)' = 0 \quad (65)$$

$$\lambda(V^0)' = \overset{+}{\omega} \quad (66)$$

We emphasize that the pseudo-scalar $\overset{+}{\omega}$ is nothing but a gradient of the probability density V^0 . Taking into account the set of non-dynamical relations and the definitions of the differential multivectors components, it is possible to find a set of equations that only involve finite multivectors and that can be solved for a given external electromagnetic potential. Certainly, boundary conditions on the multivectors must also be considered.

From (51) and (59), one has

$$-\lambda \overset{+}{\omega}' = \left(\frac{E - U}{mc^2} \right) S - V^0 \quad (67)$$

using (57) and (61),

$$\lambda S' = \left(\frac{E - U}{mc^2} \right) \overset{+}{\omega} \quad (68)$$

In addition to these relations, it is also convenient to write here (65), (66) and (7):

$$(V^1)' = 0, \quad \lambda(V^0)' = \overset{+}{\omega}, \quad S^2 + \overset{+}{\omega}^2 = (V^0)^2 - (V^1)^2$$

This set of equations involves only finite multivectors. The differential multivectors $\{I^\mu, T^{\mu\nu}, \overset{+}{h}^\mu\}$ can be obtained from (51), (60), (53), (63), (55), (64), (57) and (62).

Returning to the problem of calculating the spinor starting from multivectors for stationary states, we obtain from the spatial component of (45) a differential equation that permits us to get the overall phase $f(x)$ up to an integration constant, with Ω obtained from (42),

$$2\lambda(Sf' + V^0\Omega') = V^1 \tag{69}$$

The time component of (45) is precisely (67).

Given a potential energy $U(x)$, the Maxwell-like equations yield a linear differential equation for the probability density. By using (66)–(68) we obtain

$$(V^0)''' + g(V^0)'' + (2k)^2(V^0)' - \frac{g}{\lambda^2}V^0 = 0 \tag{70}$$

where $g(x) = U'(x)/(E - U(x))$ and $k^2 = ((E - U)^2 - (mc^2)^2)/\hbar^2c^2$. In the next section we solve this equation for $g(x) = 0$, in this case a harmonic type equation is obtained.

5. FREE PARTICLE

Let us now solve the Maxwell-like equations for a free particle. Letting $U = 0$ in (70), and then $k^2 = (E^2 - (mc^2)^2)/\hbar^2c^2$, we obtain the fundamental equation for the probability density $V^0(x)$,

$$(V^0)''' + (2k)^2(V^0)' = 0 \tag{71}$$

The general solution of (71) and (65) may be written as:

$$V^0(x) = \frac{mc^2}{\hbar ck} [A + B \sin(2kx) - C \cos(2kx)] > 0 \tag{72}$$

$$V^1(x) = D \tag{73}$$

where A, B, C and D are arbitrary constants. For the other two finite multivectors the general solution is given by:

$$S(x) = \frac{E}{mc^2} V^0(x) - \frac{\hbar ck}{E} A \tag{74}$$

$${}^+ \omega(x) = B \cos(2kx) + C \sin(2kx) \tag{75}$$

Using the constraint (7) one obtains:

$$-\frac{(mc^2)^2}{E^2}A^2 + B^2 + C^2 + D^2 = 0 \quad (76)$$

The simplest solution of (72) is a constant probability density:

$$V^0(x) = \frac{mc^2}{hck}A > 0 \quad (77)$$

Thus, (74) and (75) can be written as

$$S(x) = \frac{(mc^2)^2}{hckE}A \quad (78)$$

$${}^+\omega(x) = 0 \quad (79)$$

The probability current is given by

$$V^1(x) = \pm \frac{mc^2}{E}A \quad (80)$$

where the constraint (76) has been used. Obviously, A is a normalization constant.

For a free particle the differential multivectors can be easily obtained in terms of finite multivectors by using: $I^0 = ES/mc^2$, $I^1 = V^1$, $h^0 = E^+\omega/mc^2$, $h^1 = 0$, $T^{00} = EV^0/mc^2$, $T^{11} = T^{00} - S$, and $T^{01} = T^{10} = EV^1/mc^2$.

The Dirac spinor (39), that is $\Psi = \left(\frac{\sqrt{a}e^{i(f+a)}}{\sqrt{b}e^{i(f-a)}} \right) e^{-i(E/\hbar)t}$, may be obtained by using (40)–(42) and (69), from which: $a = (mc^2/2hckE)(E + mc^2)A$, $b = (mc^2/2hckE)(E - mc^2)A$, $\cos(2\Omega) = \pm 1$ and $f' = k$.

Substituting in the spinor Ψ the last expressions for a and b , and choosing $\Omega = 0$ or $\Omega = \pi/2$ and $f = kx$ up to a constant, one finds

$$\Psi \propto \sqrt{\frac{mc^2}{hck}}A \begin{pmatrix} 1 \\ \pm \frac{hck}{E + mc^2} \end{pmatrix} e^{\pm ikx} e^{-i(E/\hbar)t} \quad (81)$$

This spinor is the well known solution to the Dirac equation for a free particle.

6. “FREE” PARTICLE INSIDE A BOX

Let us now consider a relativistic “free” particle confined inside a one-dimensional box with fixed walls at $x = 0$ and $x = L$. In order to obtain the four arbitrary constants and the energy eigenvalues, instead of considering a confinement potential at the walls of the box, we impose adequate boundary conditions upon the solutions (72)–(75).

Using (73), the constraint (7) becomes

$$(V^0)^2(x) - S^2(x) - \overset{+}{\omega}^2(x) = D^2 \tag{82}$$

For a particle confined inside a box, we put $V^1(0) = V^1(L) = 0$, then $D = 0$ everywhere and

$$S^2(x) + \overset{+}{\omega}^2(x) = (V^0)^2(x) \tag{83}$$

So, $V^0(x)$ cannot vanish unless $S(x) = \overset{+}{\omega}(x) = 0$, but this yields the trivial solution.

Using (83) at the boundaries of the box

$$\begin{aligned} S^2(0) + \overset{+}{\omega}^2(0) &= (V^0)^2(0) \\ S^2(L) + \overset{+}{\omega}^2(L) &= (V^0)^2(L) \end{aligned} \tag{84}$$

In order to satisfy this set of relations, one may write, for $0 \leq \theta, \xi < 2\pi$,

$$S(0) = -\cos \theta V^0(0), \quad \overset{+}{\omega}(0) = -\sin \theta V^0(0) \tag{85}$$

$$S(L) = \cos \xi V^0(L), \quad \overset{+}{\omega}(L) = -\sin \xi V^0(L) \tag{86}$$

where the parameters θ, ξ label the subfamilies of boundary conditions. It can be shown that this two-parameters family of boundary conditions is the most general one for a particle confined in a box, and that these conditions define the domain of the Dirac Hamiltonian for a “free” particle inside a box. We will only consider those boundary conditions that are symmetrical under space inversions. The fundamental equation (71) is invariant under space inversions if

$$V^0(x) = V^0(L - x) \tag{87}$$

Then, by using (66) and (74), we obtain

$$\overset{+}{\omega}(x) = -\overset{+}{\omega}(L - x) \tag{88}$$

$$S(x) = S(L - x) \tag{89}$$

In this case, we have only a one-parameter family

$$S(0) = \cos \xi V^0(0), \quad \overset{+}{\omega}(0) = \sin \xi V^0(0) \quad (90)$$

$$S(L) = \cos \xi V^0(L), \quad \overset{+}{\omega}(L) = -\sin \xi V^0(L) \quad (91)$$

Among the infinite boundary conditions parametrized by ξ we choose the simplest ones: $\xi = 0, \pi$ and $\xi = \pi/2, 3\pi/2$. In the following list we specify the tensorial boundary conditions for the local observables (TBC), their corresponding spinorial boundary conditions (SBC), and the energy eigenvalue equations (EEE). For the first case $\xi = 0, \pi$, we obtain:

$$\overset{+}{\omega}(0) = \overset{+}{\omega}(L) = 0 \quad (92)$$

$$(a) \text{ TBC: } S(0) = -V^0(0), \quad S(L) = -V^0(L) \quad (93)$$

$$\text{SBC: } \phi(0) = \phi(L) = 0$$

$$\text{EEE: } \cos(2kL) = 1$$

$$(b) \text{ TBC: } S(0) = V^0(0), \quad S(L) = V^0(L) \quad (94)$$

$$\text{SBC: } \chi(0) = \chi(L) = 0$$

$$\text{EEE: } \cos(2kL) = 1$$

For the second case $\xi = \pi/2, 3\pi/2$, we obtain:

$$S(0) = S(L) = 0 \quad (95)$$

$$(c) \text{ TBC: } \overset{+}{\omega}(0) = -V^0(0), \quad \overset{+}{\omega}(L) = V^0(L) \quad (96)$$

$$\text{SBC: } \chi(L) = -i\phi(L), \quad \chi(0) = i\phi(0)$$

$$\text{EEE: } \tan(kL) - \frac{\hbar ck}{mc^2} = 0$$

$$(d) \text{ TBC: } \overset{+}{\omega}(0) = V^0(0), \quad \overset{+}{\omega}(L) = -V^0(L) \quad (97)$$

$$\text{SBC: } \chi(L) = i\phi(L), \quad \chi(0) = -i\phi(0)$$

$$\text{EEE: } \tan(kL) + \frac{\hbar ck}{mc^2} = 0$$

Then, using the constraint (7) the multivectors for the first case are:

$$V^0(x) = A \frac{mc^2}{\hbar ck} \left[\pm 1 - \frac{mc^2}{E} \cos(2kx) \right] \quad (98)$$

$$V^1(x) = 0 \quad (99)$$

$$S(x) = \frac{E}{mc^2} V^0(x) - \frac{\hbar ck}{E} A \quad (100)$$

$${}^+ \omega(x) = A \frac{mc^2}{E} \sin(2kx) \quad (101)$$

The upper sign corresponds to the boundary condition (a) and the lower one to the boundary condition (b). Note that in order that $V^0(x)$ be positive the lower sign in (98) must be used only for electrons with negative energy.

The multivectors for the second case are

$$V^0(x) = A \frac{(mc^2)^2}{E^2} \left[\frac{E^2}{\hbar ck mc^2} \mp \sin(2kx) - \frac{mc^2}{\hbar ck} \cos(2kx) \right] \quad (102)$$

$$V^1(x) = 0 \quad (103)$$

$$S(x) = \frac{E}{mc^2} V^0(x) - \frac{\hbar ck}{E} A \quad (104)$$

$${}^+ \omega(x) = A \frac{(mc^2)^2}{E^2} \left[\sin(2kx) \mp \frac{\hbar ck}{mc^2} \cos(2kx) \right] \quad (105)$$

Where the upper signs corresponds to the boundary condition (c) and the lower one to the boundary condition (d).

Knowing the finite multivectors, the Dirac spinors may be obtained using the relations (40)–(42) and (69). For the first case of boundary conditions the following spinors are obtained for (a) and (b)

$$\Psi \propto \sqrt{A} \begin{pmatrix} \sin(kx) \\ \frac{-\hbar ck}{E + mc^2} \cos(kx) \end{pmatrix} e^{-i(E/\hbar)t} \quad (106)$$

$$\Psi \propto \sqrt{A} \begin{pmatrix} \cos(kx) \\ \frac{\hbar ck}{E + mc^2} \sin(kx) \end{pmatrix} e^{-i(E/\hbar)t} \quad (107)$$

where the upper and the lower sign respectively has been used in order to obtain the spinors (106) and (107) respectively.

For the second case of boundary conditions $\xi = \pi/2, 3\pi/2$, using the upper sign of (102), (105), we obtain the spinor

$$\Psi \propto \sqrt{A} \begin{pmatrix} \sin\left(kx - \frac{\zeta}{2}\right) \\ \frac{-ihck}{E + mc^2} \cos\left(kx - \frac{\zeta}{2}\right) \end{pmatrix} e^{-i(E/\hbar)t} \quad (108)$$

where $\tan(\zeta) = hck/mc^2$. Using the lower sign, we obtain

$$\Psi \propto \sqrt{A} \begin{pmatrix} \cos\left(kx - \frac{\delta}{2}\right) \\ \frac{ihck}{E + mc^2} \sin\left(kx - \frac{\delta}{2}\right) \end{pmatrix} e^{-i(E/\hbar)t} \quad (109)$$

where $\tan(\delta) = -hck/mc^2$.

7. PARTICLE IN A STEP POTENTIAL

Let us consider the scattering problem of an electron with energy E and momentum hck_1 at the potential step

$$U(x) = \begin{cases} 0, & x \leq 0 \\ U, & x > 0 \end{cases} \quad (110)$$

The electrons of positive energy are incoming from the left side ($x < 0$).

In the region $x < 0$, with $hck_1 = \sqrt{E^2 - (mc^2)^2}$, the solutions (72)–(75) are

$$\begin{aligned} V^0(x) &= \frac{mc^2}{hck_1} [A_1 + B_1 \sin(2k_1x) - C_1 \cos(2k_1x)] \\ V^1(x) &= D_1 \\ S(x) &= \frac{E}{mc^2} V^0(x) - hck_1 A_1 \\ \omega^+(x) &= B_1 \cos(2k_1x) + C_1 \sin(2k_1x) \end{aligned} \quad (111)$$

Likewise, in the region $x > 0$, with $(\hbar ck_2)^2 = (E - U)^2 - (mc^2)^2$ we get

$$\begin{aligned}
 V^0(x) &= \frac{mc^2}{\hbar ck_2} A_2 \\
 V^1(x) &= D_2 \\
 S(x) &= \frac{(mc^2)^2}{\hbar ck_2(E - U)} A_2 \\
 \overset{+}{\omega}(x) &= 0
 \end{aligned} \tag{112}$$

where the sign of k_2 will be choose later according to the physical boundary conditions.

Using the continuity of the multivectors at $x = 0$,

$$\begin{aligned}
 V^0(-0) &= V^0(+0) \\
 V^1(-0) &= V^1(+0) \\
 S(-0) &= S(+0) \\
 \overset{+}{\omega}(-0) &= \overset{+}{\omega}(+0)
 \end{aligned} \tag{113}$$

we obtain for $x \leq 0$,

$$\begin{aligned}
 V^0(x) &= \frac{2E |A|^2}{E + mc^2} \left[1 + R + 2 \sqrt{R} \frac{mc^2}{E} \cos(2k_1 x) \right] \\
 V^1(x) &= 2 \sqrt{\frac{E - mc^2}{E + mc^2}} (1 - R) |A|^2 \\
 S(x) &= \frac{2mc^2}{E + mc^2} |A|^2 \left[1 + R + 2 \sqrt{R} \frac{E}{mc^2} \cos(2k_1 x) \right] \\
 \overset{+}{\omega}(x) &= -4 \sqrt{\frac{E - mc^2}{E + mc^2}} \sqrt{R} |A|^2 \sin(2k_1 x)
 \end{aligned} \tag{114}$$

and for $x \geq 0$,

$$\begin{aligned}
 V^0(x) &= \frac{2(E - U)}{\hbar ck_2} \sqrt{\frac{E - mc^2}{E + mc^2}} T |A|^2 \\
 V^1(x) &= 2 \sqrt{\frac{E - mc^2}{E + mc^2}} T |A|^2 \\
 S(x) &= \frac{2mc^2}{\hbar ck_2} \sqrt{\frac{E - mc^2}{E + mc^2}} T |A|^2 \\
 \overset{+}{\omega}(x) &= 0
 \end{aligned} \tag{115}$$

where $|A|^2 = (mc^2/2(E-U)) \sqrt{(E+mc^2)/(E-mc^2)} 1/(1-R) A_2$; $R = ((1-\mu)/(1+\mu))^2$ and $T = 4\mu/(1+\mu)^2$ being respectively the reflection and transmission coefficients which satisfy

$$R + T = 1 \quad (116)$$

with $\mu \equiv \eta_2/\eta_1 = (k_2/k_1)((E+mc^2)/(E-U+mc^2))$. The corresponding spinor solutions are

$$\Psi = A \begin{pmatrix} e^{ik_1x} + \sqrt{T} R e^{-ik_1x} \\ \eta_1 (e^{ik_1x} + \sqrt{T} R e^{-ik_1x}) \end{pmatrix} e^{-i(E/\hbar)t} \quad (117)$$

for $x \leq 0$, and

$$\Psi = A \sqrt{T} \sqrt{\frac{\eta_1}{\eta_2}} \begin{pmatrix} 1 \\ \eta_2 \end{pmatrix} e^{ik_2x} e^{-i(E/\hbar)t} \quad (118)$$

in the region $x \geq 0$, with $\eta_1 = \sqrt{(E-mc^2)/(E+mc^2)}$ and $\eta_2 = \sqrt{(E-U-mc^2)/(E-U+mc^2)}$.

If $(k_2)^2 < 0$ and $x > 0$, we have an exponentially damped solution in (118), that is, a classically forbidden region. In this case, $mc^2 > E - U > -mc^2$.

With $(k_2)^2 > 0$ we have an oscillatory solution and this occurs not only when $E - U > mc^2$ but also when the potential becomes so strongly repulsive that $E - U < -mc^2$. Semiclassically speaking, an electron initially confined in the region $x < 0$, can tunnel through the region $x > 0$ where behaves as it were in an attractive potential. The oscillatory solution inside a potential step, where a non-relativistic solution would decay exponentially, is called the Klein's paradox.⁽¹¹⁾ Several analysis of this problem have been considered in the literature, some of them in the context of quantum field theory.⁽¹²⁾

In the present, let us assume that $U > E + mc^2 > 2mc^2$. We will study this problem from the point of view of the one-particle interpretation of the relativistic quantum theory. In the region $x > 0$ the group velocity of the moving wave packet is given by

$$v = \frac{V^1(x)}{V^0(x)} = \frac{c^2 \hbar k_2}{E - U} \quad (119)$$

Since $E - U < 0$, this is a classically forbidden region. Clearly, v and k_2 will have opposite directions. So, we choose $k_2 > 0$, it looks as if the transmitted wave packet came in from $x = +\infty$, which contradicts the hypothesis of

the incoming wave packet from $x = -\infty$. We then have to choose $k_2 < 0$, so that $\mu > 0$ and the reflection coefficient is $R \leq 1$.

When $U \rightarrow E + mc^2$, $R \rightarrow 1$ since $\mu \rightarrow \infty$. On the other hand, with $U \rightarrow \infty$, $\mu \rightarrow \sqrt{(E + mc^2)(E - mc^2)}$ and the reflection and transmissions coefficients are

$$R \rightarrow \frac{E - \hbar ck_1}{E + \hbar ck_1}, \quad T \rightarrow \frac{2\hbar ck_1}{E + \hbar ck_1} \quad (120)$$

As also pointed out by Greiner, the unexpected largeness of the transmission coefficient, which is classically not understandable, holds for a finite large U of the order of several rest masses. For a smooth potential, which gave the same qualitative results, it was shown⁽¹³⁾ that there is a much smaller transmission coefficient when the gradient of U occurs in regions larger than the Compton wave length.

8. CONCLUSIONS

We reviewed the kinematical and dynamical aspects of the tensorial quantum theory in $1 + 1$ dimensions. We proved the mathematical equivalence between the Dirac equation and the Maxwell-like equations, showing that the dynamics of quantum mechanics is of the Maxwell type. We were able to write the Dirac spinor in terms of finite multivectors, i.e., currents and fields. By considering stationary states, we obtained a linear differential equation for the observable probability density. It is worth mentioning that the dynamical equation for the probability density has been from the beginnings an important aim of quantum mechanics.

We solved the problems of a free particle, of a particle inside a box, and of a particle in a step potential, considering the Klein paradox. In the problem of a "free" particle inside a one-dimensional box, the tensorial theory turned out to be useful in the choice among the infinite sets of boundary conditions that make self-adjoint the Hamiltonian of the system. These can be grouped in specific and simple families in terms of local observable densities. Thus, the imposition of symmetries and boundary conditions is easier, and is more physically meaningful.

We have seen that from the tensorial viewpoint, the relativistic quantum mechanics appears as a hydrodynamic-like theory, whose objects are explicitly the local mean values of the quantum observables and obey evolution equations equivalent to the Dirac equation. Thus, the tensorial formulation is by itself a quantum mechanics and is essential to the interpretation of this theory. Certainly it exhibits important features of the

extended structure of the electron; which brings a fresh insight in the localization problem and should be investigated thoroughly in $3 + 1$ dimensions. But of course, the tensorial formulation does not substitute the spinorial formulation of quantum mechanics, which as a counterpart of its implicate character is linear in all dimensions and therefore very much simpler, being clearly essential to quantum algorithm.

REFERENCES

1. W. Pauli, *General Principles of Quantum Mechanics* (Springer, Berlin, 1980); W. Gordon, *Z. Phys.* **50**, 630 (1928); A. Proca, *Ann. Phys. (Paris)* **20**, 429 (1933); F. J. Belinfante, *Physica* **6**, 887 (1939).
2. O. Costa de Beauregard, *Thèse*, Paris (1943); T. Takabayasi, *Progr. Theor. Phys. Suppl.* Number 4, Part 1 (1957).
3. L. Mondino and V. Alonso, "Kinematical structure of the Dirac matter," Universidad Central de Venezuela, unpublished (1995).
4. L. Mondino and V. Alonso, "Electromagnetic structure of the Dirac matter," Universidad Central de Venezuela, unpublished (1995).
5. S. De Vincenzo, "Dirac particle and boundary conditions." M.Sc. Thesis (in Spanish), Universidad Central de Venezuela, Caracas, Venezuela (1996).
6. V. Alonso, S. De Vincenzo, and L. Mondino, *Eur. J. Phys.* **18**, 315 (1997); R. Rajaraman and J. S. Bell, *Phys. Lett. B* **116**, 151 (1982); S. M. Roy and V. Singh, *Phys. Lett. B* **143**, 179 (1984); P. Falkensteiner and H. Grosse, *J. Math. Phys.* **28**, 850 (1987); S. M. Roy and V. Singh, *J. Phys. A Math. Gen.* **22**, L425 (1989).
7. V. Alonso and S. De Vincenzo, *J. Phys. A Math. Gen.* **30**, 8573 (1997).
8. J. Schwinger, *Phys. Rev.* **128**, 2425 (1962).
9. W. E. Thirring and J. E. Wess, *Ann. Phys.* **27**, 331 (1964); J. H. Lowenstein and J. A. Swieca, *Ann. Phys.* **68**, 172 (1971).
10. T. D. Newton and E. P. Wigner, *Rev. Mod. Phys.* **21**, 400 (1949); G. C. Hegerfeldt and S. N. M. Ruijsenaars, *Phys. Rev. D* **22**, 377 (1980); A. S. Wightman, *Rev. Mod. Phys.* **34**, 845 (1962).
11. O. Klein, *Z. Phys.* **53**, 157 (1928); P. J. M. Bongaarts and S. N. M. Ruijsenaars, *Ann. Inst. H. Poincaré* **A26**, 1 (1977); B. Thaller, *The Dirac Equation* (Springer, Berlin, 1992). W. Greiner, *Relativistic Quantum Mechanics: Wave Equations* (Springer, Berlin 1997).
12. P. J. M. Bongaarts and S. N. M. Ruijsenaars, *Ann. Phys.* **101**, 289 (1976).
13. F. Sauter, *Z. Phys.* **73**, 547 (1931).