WHAT IS THE "x" WHICH OCCURS IN "$\sin x$"? Being an Essay Towards a Conceptual Foundations of Mathematics[*†]

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TO EUDOXUS
c408-c355 BC
The Savior of Ancient Mathematics, The Founder of Modern Mathematics

> [...] .In contrast, the present exposition, in my opinion, is marred by a gross error, not against logic but against common sense, which is more serious.
> Henri Leon Lebesgue
> [1875-1941]

As Lebesgue implies, *measure* is the most fundamental concept of *science*. It is not accidental that the principal branch of mathematics is that which deals with measure: recently Analysis, before it Geometry. The transition of the study of measure from geometry to analysis is a long process which is yet to be investigated from both the logical and the *conceptual* points of view. Trigonometry, thanks to Eudoxus' theory of ratio and proportion, was developed within geometry. Now it involves analysis too. To analytically deal with the trigonometric functions, several mathematicians answered Hardy's key question "The whole difficulty lies in the question, *What is the x which occurs in* $\cos x$ *and* $\sin x$?".

Building on the works of Eudoxus and Lebesgue, it is shown that (to the best of my knowledge) *all* of these answers are logically wrong or conceptually unsound.

In contrast, based on Eudoxus' theory, measure *in general* and measuring angles in particular are investigated, and an answer to this question, which is both logically correct and conceptually sound, is presented.

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To comprehend the philosophical and historical background of the subject, briefly several problems are considered which need further investigation and several questions are raised which are yet to be answered.

In an appendix a correction of the proof of proposition XII.2 of Euclid's Elements, which practically defines π, is given.

Introduction.

"*Our teaching does not yet make full use of that historic event, which is perhaps the most important event in the history of science, namely, the invention of the decimal system of numeration.*" says Lebesgue [1966, p. 18]. As far as the computational aspect is concerned, this event is conceivably the most important. The *digital* electronic computers revolution enhances its importance even more.

Concerning the conceptual aspect, perhaps the most important event in the history of *science* is Eudoxus' invention of his thoery of ratio and proportion [see II.1 below] because it is the basis of the study of *measure* [see I.5 and II.1, 2 below], and "There is no more fundamental subject than this." [Lebesgue [1966], pp. 10-11].

It is not accidental that the principal branch of mathematics is that which deals with measure: recently Analysis, before it Geometry which dominated for more than two millennia [cf. Descartes [1954] p. 43]. The transition of the study of measure from geometry to analysis is a long historical process which is yet to be investigated, not only from the logical point of view, but mainly from the *conceptual* point of view.

Descartes [1954, pp. 2,5] bases his analytic geometry on ratios of straight line segments [cf. Bos [1997] pp. 38, 45, 55 and II.2 below]. And as Hobson [1957, p. 20] noticed, Newton defines (real) numbers as ratios of magnitudes of the same kind [cf. II.2 below]; nevertheless Newoton does not strictly observe this definition.

The Archimedean principle [cf. I.5 below] implies that the Eudoxean ratios have no infinitesimals [see II.2 below] while it is well known that Newton allows them.

For Descartes, there is no problem. In Euclidean geometry all field operations may be defined and performed on an extension of the ratios of (equivalence classes of) the straight line segments; precisely, the set of these ratios is the set of positive elements of an Archimedean ordered field [see II.2 below, also cf. Descartes [1954] p. 2]. To generalize to all ratios of magnitudes is a completely different story [see II.2 below]. This is probably one reason why the Greeks did not regard ratios as numbers or even magnitudes [cf. Stein [1990] p. 181 and II.2 below].

Concerning the process which made it possible in the seventeenth century to regard ratios as numbers [see above], it is reasonable to conjecture that the decimal system [see above] has played a crucial role. Approximating ratios by finite decimals suggests a way to define addition, subtraction and multiplication, and makes it easy to perform them [cf. Al-Khwarizmi [1939] pp. 55, 56 and II.2 below]. For division one further approximation may be needed. Substantiating this conjecture will give a good example of a conceptual development which is induced by computational considerations.

It took mathematicians over two centuries to get rid of the infinitesimals, in an endeavor which was culminated by the *Arithmetization of Analysis*. Dedekind's approach to this arithmetization resembles Eudoxus' theory of ratio and proportion [see II.1, 2 below] to the extent that made Heath [Euclid [1956] vol. 2, p. 124] assert that "Certain it is that there is an exact correspondence, almost coincidence, between Euclid's [Eudoxus'] definition of equal ratios and the modern theory of irrationals due to Dedekind.". In the light of the above discussion this view can hardly be accepted.

Though Stein is fully aware that the Greeks did not regard ratios as numbers or even magnitudes [cf. Stein [1990] p. 181] he wonders [cf. Stein [1990] pp. 186, 203] why did not they develop the seventeenth century mathematics, and says [Stein [1990] p. 203] "[...] if we had only been blessed by another Archimedes or two in antiquity, the mathematics of the seventeenth century might have begun -and on firmer and clearer foundations than when it did in fact begin- more than a millennium and a half earlier." This view, in fact, underestimates the developments that took place during this long period of time [see above and II.2 below].

Instead, the natural questions to be raised are:

(1) Why did not the Greeks develop analytic geometry? [See above, and II.2 below].

(2) Why did not analysis begin, in the seventeenth century, on firmer and clearer foundations than it did in fact begin? Why did not the mathematicians of that era exploit the potentiality of Eudoxus' thoery of ratio and proportion and his principle of exhaustion to develop the real numbers (henceforth, $\mathbb{R}$), the theory of limits and analysis in general, without recourse to infinitesimals? [See II.1, 2 below]

Unfortunately, the failure to exploit Eudoxus' work extends to our era too. This is not accidental, but is considered a prerequisite for rigor, which is believed to be incapable of being achieved apart from a complete arithmetical theory of irrational numbers [see above, also cf. the concluding remarks below].

The necessity for the "extension of the domain of Number arises not only on account of the inadequacy of rational numbers for application to ideally exact measurement, but also, as will be explained later in detail, because the theory of limits, which is an essential element in Analysis, is incapable of any rigorous formulation apart from a *complete arithmetical* [emphasis added] theory of irrational numbers.

Before the recent establishment of the theory of irrational numbers, no completely adequate theory of Magnitude was in existence. This is not surprising, if we recognize the fact that the language requisite for a complete description of relations of magnitudes must be provided by a *developed Arithmetic* [emphasis added]." Says Hobson [1957, pp. 20-1, also cf. II.1, 2 below].

This erroneous attitude is probably an over reaction to another erroneous tradition which prevailed before arithmetization of analysis [see above and I.7, below], and is best exemplified by the following striking quotation from Euler

[2000, p. 51] "Since we are going to show that an infinitely small quantity is really zero, we must first meet the objection of why we do not always use the same symbol 0 for infinitely small quantities, rather than some special ones. [...] . Every one knows that [...] $n.0 = 0$, so that $n : 1 = 0 : 0$. [...] . Since between zeros any ratio is possible, in order to indicate this diversity we use different notations on *purpose* [emphasis added], especially when a geometric ratio between two zeros is being investigated. [...] . For this reason, in these calculations, unless we use different symbols to represent these [same] quantities, we will fall into the greatest confusion with no way to extricate ourselves." [cf. I.7 below]. Is there any confusion greater than this?

In a similar context Lebesgue imputed such collective behavior of mathematicians to hypnosis [cf. Lebesgue [1966] p. 97 and I.6 below]. Are mathematicians really hypnotized? Or they need deeper psychoanalysis?

Trigonometry, thanks to Eudoxus' theory of ratio and proportion, was developed since long ago within geometry. Now it involves both geometry and analysis, the vehicle between them is the *measure* of angles [see II.5 below].

The title of this article is a variation on the key observation, concerning the trigonometric functions, which was raised by Hardy [1967, p. 316] over a century ago: "The whole difficulty lies in the question, *what is the x which occurs in* $\cos x$ *and* $\sin x$?".

During the past hundred years or so several mathematicians (Hardy included) explicitly or implicitly answered this key question in order to analytically deal with the trigonometric functions.

Building on the deep conceptual insights of Eudoxus and Lebesgue, it is shown that (to the best of my knowledge) all of these answers are logically wrong or -which is even worse - conceptually unsound.

In contrast, based mainly on the work of Eudoxus, the problem of *measure in general* and that of measuring angles in particular are investigated, and via defining two sine functions, one geometric and the other analytic, an answer to this key question, which is both logically correct and conceptually sound, is presented. The answer extends to all trigonometric functions.

To understand the philosophical and hystorical background of the subject, to comprehend its dimensions, to fathom its different aspects, and to perceive the evolution of mathematics as a process of erecting an edifice not piling a heap, briefly several problems are considered (passim) which need to be further investigated and several questions are raised (passim) which are yet to be answered.

The structure of the rest of the article is as follows:

I. BACKGROUND

I.1. Question and Answers. As was mentioned in the introduction, the title of this article is a variation on the key observation, concerning trigonometric functions, which was raised by Hardy [1967, p. 316] over a century ago: "The whole difficulty lies in the question, *what is the x which occurs in* $\cos x$ *and* $\sin x$?". "To answer this question," continues Hardy [1967, pp. 316-7] "we must define the measure of an angle, and we are now in a position to do so. The most natural definition would be this: suppose that AP is an arc of a circle whose centre is O and whose radius is unity, so that $OA = OP = 1$. Then x, the measure of the angle $[AOP]$, is *the length of the arc AP*. This is, in substance, the definition adopted in the text-books, in the accounts which they give of the theory of 'circular measure'. It has however, for our present purpose, a fatal defect; for we have not proved that the arc of a curve, even of a circle, possesses a length. The notion of the length of a curve is capable of precise mathematical analysis just as much as that of an area; but the analysis, although of the same general character as that of the preceding sections, is decidedly more difficult, and it is impossible that we should give any general treatment of the subject here.

We must therefore found our definition on the notion not of length but of *area*. We define the measure of the angle AOP as *twice the area of the sector AOP of the unit circle*."

Applying the area approach to measuring angles, Hardy answers his question via defining [Hardy [1967] p. 317] the inverse tangent function by:

$$\tan^{-1} x = \int_0^x dt/(1+t^2)$$

from this [Hardy [1967], pp. 434-8] the well known properties of the trigonometric functions were analytically deduced.

Morrey [1962, pp. 214-8] also pursues the same area approach to measuring angles, and defines:

$$\cos^{-1} x = x\sqrt{1-x^2} + 2\int_x^1 \sqrt{1-t^2}\, dt$$

unlike that of Hardy, Morrey's analytic treatment recurs partially to geometry [Morrey [1962], pp. 46, 57-8, 218, 220-5].

Besides, Hardy [1967, pp. 433-4] mentions four more approaches to answer his question:

(i) Power series.

(ii) Infinite products.

(iii) Complex analysis. This approach was pursued, from two different points of view, in Davis [2003] where arc length was made use of (but in an avoidable way) and in Bartle and Tulcea [1968] where an ad hoc definition of measuring angles is introduced and made use of.

(iv) Arc-length approach to measuring angles. This approach was pursued from different points of view, in Anton et al. [2009], Gearhart and Shultz [1990], Gillman [1991], Hughes-Hallett et al. [2005], Protter and Morrey [1963], Richman [1993], Rose [1991], and Ungar [1986].

Inspired by the arc-length approach, the treatment in Eberlein [1966] is based on the theory of complex valued functions on real numbers.

Besides, some books on the foundations of geometry, presupposing real numbers, take an axiomatic approach to define the measure of angles [Moise 1964, p. 74; Borsuk and Szmielew 1960, p. 172; and Greenberg 1980, p. 98].

We shall come back to this in I.3, 6 and II.1, 4 below.

I.2. Towards a Careful Analysis. The foundations of the theory of trigonometric functions is not as simple as it is generally supposed. It still needs to be properly and carefully analyzed [cf. Hardy [1967] p. 432].

Proper analysis must deal not only with the logical aspects, but also with the *conceptual* aspects (which include Lebesgue's "common sense"). We shall come back to this in I.4 below.

I.3. Logical Integrity. Most common approaches to the theory of trigonometric functions are based on the length of a circular arc or the area of a circular sector.

From the logical point of view, these approaches involve two problems, each potentially entails circularity:

(i) The trigonometric functions are defined in terms of arc length or sector area, while these entities are, possibly, defined or evaluated in terms of the trigonometric functions [see I.6 below].

(ii) The proof of the celebrated limit, $\lim_{x \to 0} (\sin x)/x = 1$, possibly, depends on properties of arc length or sector area which, in their turn, depend on properties of the trigonometric functions which are based on this limit [cf. Richman [1993], Rose [1991], and Ungar [1986]].

Whether there is actual circularity, no circularity or the situation is obscure, can be seen only through the detailed analysis of the treatment.

Though the two aforementioned problems are solved [Hardy [1967], Morrey [1962]] or avoided via other approaches [Davis [2003], Eberlein [1966]], they still persist, without even being pointed to, in modern calculus books [cf., Anton et al. [2009], Hughes-Hallett et al. [2005], Protter and Morrey [1963]].

I.4. Arbitrary and Accidental? Is mathematics arbitrary and its effectiveness in natural sciences accidental?

That mathematics is arbitrary is implied by Hardy, not only when he arbitrarily defined the measure of an angle [see I.1 above] but also when he said that real mathematics "must be justified as art if it can be justified at all" [Hardy [1969] p.139, also cf. Griffiths [2000] p. 4].

That the effectiveness of mathematics is accidental is implied by E. P. Wigner when, under the title "The Unreasonable Effectiveness of Mathematics in the Natural Sciences" he wrote "The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve.", as was noticed by Kac and Ulam [1971, p. 166]. Kac and Ulam agree, saying that this effectiveness "remains perhaps a philosophical mystery" [Kac and Ulam [1971] p. 12] and that it "may be philosophically puzzling" [Kac and Ulam [1971] p. 191]. Yet, they give [Kac and Ulam [1971] p. 191] the following clue for solving this puzzle "One reason for it is the necessary condition that measurements, and thus much of the discussion in physics and astronomy, can be reduced to operations with numbers."

Kac and Ulam are, in a sense, following Lebesgue in emphasizing the importance of measure of quantities, of which he says [Lebesgue [1966] pp. 10-11] "There is no more fundamental subject than this. Measure is the starting point of all mathematical applications [...] it is usually supposed that geometry originated in the measure of [angles, lengths,] areas and volumes. Furthermore, measure *provides* [emphasis added] us with numbers, the very subject of analysis."

Unlike Wigner, Kac and Ulam, Lebesgue is *not* puzzled. For him "[...] those whom we have to thank for such [practically effective] abstract considerations have been able to think in abstractions and at the same time to perform useful work precisely because they had a particularly acute sense of reality. It is this sense of reality that we must strive to waken in the young [and, a fortiori, the grownups]." [Lebesgue [1966] p. 11].

Concerning the role of conceptual considerations in developing abstract mathematics and its foundations, Marquis [2013] may be consulted.

As a metamathematician, Feferman distinguishes between *structural* and *foundational* axioms of mathematics (or parts thereof) holding that both sorts of axioms are *not arbitrary* [cf. Feferman [1999] p. 100 and Feferman et al. [2000] pp. 403, 417]. Besides, he calls for a "philosophy grounded in intersubjective *human* conceptions [...] to explain the apparent *objectivity* [emphasis added] of mathematics." [Feferman [1999] p. 110].

Though on different philosophical grounds, Feferman was anticipated by Gödel [1964b, p. 264] who asserts that Mahlo "axioms show clearly, not only that the axiomatic system of set theory as used today is incomplete, but also that it can be supplemented *without arbitrariness* [emphasis added] by new axioms which only unfold the content of the concept of set explained above."

Concerning the rigorous unfolding of informal concepts Lavers [2009] may be consulted. And concerning the relationship between intuitive mathematical concepts and the corresponding formal systems Baldwin [2013] may be consulted.

Bertrand Russell went even further than Gödel and Feferman when he once said "Logic [hence, mathematics] is concerned with the real world just as truly as zoology, though with its more abstract and general features," as was brought to my attention by Gödel [1964a, pp. 212-3]. Lukasiewicz [1998, pp. 205-7] provides logico-philosophical grounds for this view.

Lebesgue [1966, p. 101] distinguishes between "terminology" which is arbitrary, or free, to use his terminology, and "definitions" which are not, or "At the very least, some definitions, those that are meant to make practical concepts more precise, are not free.". Frege was more explicit when he said [Frege [1968] pp. $107^e - 8^e$] "[E]ven the mathematician cannot create things at will, any more than the geographer can; he too can only discover what is there and give it a name.".

Moschovakis [1995, p. 753] seconds, asserting that for a positive [integrable] function $f : [a, b] \to \mathbb{R}$ we do not mean the equation:

$$\int_a^b f(x)\,dx = \text{the area under } f \text{ and above the } x\text{-axis} \qquad (*)$$

"as a definition of area, for, if we did, we *could not* [emphasis added] use it to compute real areas; and we cannot prove it conclusively[1] without a separate, precise definition of "area"- which we might give, of course, but whose connection with actual "physical" or "geometrical" area we would then need to justify.". We shall come back to this and to footnote "1" in II.1 below.

For further discussion concerning the relationship of mathematics to reality, Amer [1980] and Maddy [2008] may be consulted. At any rate, this relationship should be a subject not only of philosophical reflection, but also of factual and historical scrutiny.

For a general view of the problem of providing foundations Sher [2013] may be consulted.

I.5. Direct Measure. The direct sense of the measure of a magnitude is the ratio of this magnitude to a standard (measuring) unit of the same kind. So, in this sense, measure is a special ratio.

The founder of the theory of ratio and proportion is Eudoxus. He developed this theory to deal with all kinds of magnitudes, not only those of geometry, and with the incommensurables as well as the commensurables [cf. Euclid [1956] vol. 2, p. 112], making use only of whole numbers. Dealing with the incommensurables practically produces irrational numbers, e.g. π. We shall come back to this in II.1 and II.2 below.

Unlike Eudoxus, Lebesgue starts with measure, then generalizes to ratios. Having at his disposal the decimal system of numeration (which he highly esteems, see the introduction above) Lebesgue [1966, pp. 19-21] defines the measure of straight line segments as follows.

Let a straight line segment u be the standard measuring unit, and let b be a generic straight line segment. Lay off u on b several times in the obvious way to determine the unique natural number n_o such that $n_o u \leq b < (n_o + 1)u$. This process may be performed, and the uniqueness of n_o proved, in Euclidean geometry. Its termination is guaranteed by the Archimedean principle (which Archimedes himself attributes to Eudoxus [Heath [1963] p. 193 and Euclid [1956] vol. 3, pp. 15-6]).

Assume $n_o u < b$. By well known Euclidean methods divide u into 10 equal (i.e. congruent) parts and determine the unique natural number n_1 (≤ 9) such that $(n_o + n_1 10^{-1})u \leq b < (n_o + (n_1 + 1)10^{-1})u$. Iterate this procedure whenever possible. If the iteration terminates at stage m, the length of b in terms of u is, by definition, $\displaystyle\sum_{i=0}^{m} n_i 10^{-i}$. Otherwise the perpetual procedure will give rise to an infinite decimal, which is *itself* (by *definition*), at the same time, both a *real number* and the *length* of b in terms of u. In this sense direct measures, here lengths of straight line segments, *produce* real numbers, while making use only of natural numbers.

Lebesgue [1966, pp. 35-8] generalizes lengths to ratios via replacing the unit u by a generic segment c, the denominator of the ratio.

The decimal system may be generalized to what may be called "positional system" by replacing 10 by any natural number $k \geq 2$.

For angles, the value of k which facilitates geometric constructions most is 2. Replacing 10 by 2 and repeating everything else almost verbatim, measures of angles (in terms of a standard unit angle) and ratios among them may be discussed, giving rise to similar results as above [cf. Lebesgue [1966] p.38 and II.1 below].

Not only angles and straight line segments are measured in this way, but also time intervals, masses, ... [cf. Lebesgue [1966] p. 39].

Theoretical measure is not quite the same as measure in everyday life, or even in scientific laboratories. As the former is meant to make the latter precise, the former is an abstraction and idealization of the latter, and in this sense it is not arbitrary, but it is essentially dictated by our understanding of the actual facts of the real world [cf. I.4 above].

For comparison of the above treatment with that of Eudoxus see II.1, 2 below.

I.6. Indirect Measure. What makes direct measure, and ratios in general, possible is that the two terms of the ratio are capable, when multiplied, of exceeding one another [see II.1 below].

Let c be a circular arc, and u be a straight line segment. Within Euclidean geometry, none of c and u is capable of exceeding the other, no matter how (finitely) many times it is multiplied. So, accepting the Euclidean notion of multiplicity [see II.1 below] the measure (here, the "length") of c in terms of u can but be indirect. This concept should be carefully analyzed if the arc-length approach to measuring angles is to be made precise.

Indirect measure is probably at least as ancient as the XII dynasty of the ancient Egyptian middle kingdom (c1900 B.C.). It is believed that it was shown, via an example (which may be generalized) and approximate calculations, that the area of the surface of a hemisphere is (exactly) twice as much as that of the corresponding greater circle [Vafea [1998] pp. 31-3].

The first known treatment of indirect measure after the incommensurability crisis is that of Archimedes. In his *Measurement of a circle*, Archimedes shows that (i) "the area of a circle is equal to that of a right angled triangle having for perpendicular the radius of the circle and for base its circumference." and (ii) "the ratio of the circumference of any circle to its diameter is $< 3\frac{1}{7}$ but $> 3\frac{10}{71}$." [Heath [1963] p. 305].

Within the framework of Euclidean geometry [Euclid 1956], each of the statements (i) and (ii) does not make sense, simply because the circumference of a circle is not straight. It is surprising that Heath [1963] did not raise this point, while Al-Khwarizmi [1939, pp. 55-6] and Descartes [1954, p. 91] practically raised it.

Stein [1990, p. 180] sheds more light on this issue: "In phys: VII iv $248a_{18}$-b_7 he [Aristotle] contends that a circular arc cannot be greater or smaller than a [straight] line segment, on the grounds that if it could be greater or smaller, it could also be equal.". For other views see [Stein [1990] footnote 48, pp. 208-9].

Under the urge of practical applications, and guided by physical insight and geometric intuition, the work of Archimedes was incorporated into mainstream mathematics some way or the other.

In modern mathematics this is done through developing classical analysis and Euclidean geometry (with continuity). It is well known that these theories can be formalized in Zermelo-Fraenkel set theory with choice. However, almost all of the scientifically applicable portions of them may be formalized in systems which are as weak as Peano Arithmetic [cf. Feferman [1999] p. 109].

Concerning rectifiable curves, the common practice, partially following Archimedes, is to adopt a treatment which concentrates only on inscribed polygons. Even an advanced book like [Buck [1965] p. 321] does this. Lebesgue [1966, p. 97] describes this as being "hypnotized by the word "inscribed"".

The theory of rectifiable curves thus treated, is susceptible to a paradox (similar to Schwarz paradox concerning the area of curved surfaces) which entails that $\pi = 2$ [Lebesgue [1966] p. 98].

Inspired by his physical insight, Lebesgue [1966, pp. 98-106, 117-9] provides for plane curves two alternatives which are free from this paradox. The first [Lebesgue [1966] pp. 103-5] depends on measuring angles, which makes the arc-length definition of the measure of an angle logically circular. The second [Lebesgue [1966] pp. 117-9] depends on measuring areas, which makes the arc-length definition of measuring angles less direct, hence conceptually more questionable, than the area definition. We shall come back to this in II.1, 4 below.

Accepting the traditional arc-length definition of measuring angles, and the traditional definition of arc-length, it is worthwhile to note that the celebrated limit, $\lim_{\theta \to 0} (\sin\theta)/\theta = 1$, or rather $\lim_{\theta \to 0} 2\theta/(2\sin\theta) = 1$, is a special case of the evident:

$$\lim_{h \to 0} \int_x^{x+h} \sqrt{1 + f'^2(u)}du / \sqrt{h^2 + (f(x+h) - f(x))^2} = 1,$$

where f is continuously differentiable [cf., Hardy [1967] p. 270].

I.7. Erroneous but Fecund Mathematics. The aforementioned work of Archimedes and the current treatment of the trigonometric functions are examples of (logically or conceptually) erroneous but fecund mathematics.

The most widely spread example of erroneous but fecund mathematics is the pre-Cauchy calculus. Look at the following striking passage of Euler [2000, pp. 51-2] "If we accept the notation used in the analysis of the infinite, then dx indicates a quantity that is infinitely small, so that both $dx = 0$ and $adx = 0$, where a is any finite quantity. Despite this, the geometric ratio $adx : dx$ is finite, namely $a : 1$. For this reason the two infinitely small quantities dx and adx, both being *equal* [emphasis added] to 0, *cannot* [emphasis added] be confused [*considered equal*] when we consider their ratio. In a similar way, we will deal with infinitely small quantities dx and dy. *Although these are both equal to* 0, *still their ratio is not that of equals* [emphasis added]. Indeed the whole force of differential calculus is concerned with the investigation of the ratios of any two infinitely small quantities of this kind. The application of these ratios at first sight might seem to be minimal. Nevertheless, it turns out to be very great, which becomes clearer with each passing day."

Correct mathematics may be sterile. The first attempt to algebraize logic, due to Leibniz [cf. Kneale and Kneale [1966], pp. 320-45] is a good example. The second attempt, due to Boole [1948], is well known to be erroneous but fecund.

Erroneous but fecund mathematics helps to develop mathematics in two ways: (i) It incorrectly arrives to "correct" results, then (ii) It forces mathematicians to endeavor to establish these results on more solid foundations.

The subject needs to be further investigated through concerted endeavors of mathematicians, philosophers of mathematics and historians of mathematics.

II. FROM ANTIQUITY TO MODERNITY

II.1. Eudoxus Theory of Ratio and Proportion. That the theory of ratio and proportion expounded in Book V of Euclid's Elements [Euclid [1956] vol. 2, pp. 112-86] is due to Eudoxus, is beyond reasonable doubt [Heath [1963] p. 190]. Nevertheless, the actual arrangement and sequence of book V is not attributed to Eudoxus, but to Euclid [Heath [1963] p. 224].

It should be emphasized that it was understood from the very beginning that Eudoxus' theory of ratio and proportion (henceforth ETRP) is the foundation not only of geometry, arithmetic and music (harmonics) but also of *all* mathematical sciences [Euclid [1956] vol. 2, p. 112]. At Eudoxus' time these sciences used to include, in addition, mechanics, astronomy and optics [cf. Amer [1980] p. 569].

Moreover, it is to be noted that Euclid distinguishes between two sets of indemonstrable principles: postulates [Euclid [1956] vol. 1, pp. 154-5] and common notions [Euclid [1956] vol. 1, p. 155]. Postulates are peculiar to geometry, in contrast, common notions, which are instrumental in ETRP, are common to all demonstrative sciences [cf. Euclid [1956] vol.1, p. 221].

The key definitions of ETRP are definitions V.3 of Euclid - V.7 of Euclid (this is an abbreviation of "definitions 3-7 of book V of Euclid [1956]"; in the sequel similar abbreviations will be made use of in similar situations, without further notice).

DEFINITION V.3 of Euclid [1956, vol. 2, p. 114]. A **ratio** is a sort of relation in respect to size between two magnitudes of the same kind.

DEFINITION V.4 of Euclid [1956, vol. 2, p. 114]. Magnitudes are said to **have a ratio** to one another which are capable, when multiplied, of exceeding one another.

Magnitudes do not have to be geometric. They may be parameters of physical objects: masses, electric charges, time intervals, ...

If definition V.3 of Euclid is understood to the effect that being of the same kind is a necessary and sufficient condition for two magnitudes to have a ratio; definition V.4 of Euclid may be considered as a definition of "being of the same kind", or simply, as a definition of "Kind". Though Heath says that this view (which is adopted here) is accepted by De Morgan, he sees the matter differently [Euclid [1956] vol. 2, p. 120].

In definition V.4 of Euclid "multiplied" means multiplied by a positive integer: to multiply a magnitude by 1, is to leave it as it is, to multiply it by $n+1$ is to add it to itself n times [cf. Euclid [1956] vol. 2, pp. 138-9, also see below]. This may be made precise via recursion.

The term "added" occurs in the formulation of common notion 2 [Euclid

[1956] vol. 1, p. 155] which is made use of from the very beginning, e.g., in Pythagoras theorem [Euclid [1956] vol. 1, pp. 349-50]. As a matter of fact "addition" means essentially-disjoint (i.e., only boundary points may be in common) union [cf. Euclid [1956] vol. 1, pp. 349-50 and I.5 above, also see below].

To appreciate the subtleness of definition V.4 of Euclid, the following may be easily seen in the light of the above discussion. There is a ratio between a side of a square and its diameter and there is a ratio between a circle and the square on its radius. Also there is a ratio between any two circular arcs of any two congruent circles. In contrast, at least within Euclidean geometry, there is no ratio between any circular arc and any straight line segment [cf. I.6 above]. Also arcs of non-congruent circles do not have ratios to one another, nor do elliptic arcs have a ratio to one another, even if they belong to congruent ellipses (except in some obvious special cases). Whenever there is a ratio, it is direct, no rectification or integration is involved.

Calling it a "species of quantity" Stein [1990, p. 167] formalized Eudoxus' notion of a kind of magnitude (henceforth a kind) [cf. Avigad et al. [2009] p. 722]. Slightly modifying Stein's formalization and requiring the Archimedean principle to hold, an ordered quadruple $\mathbf{K} =< K, +, <, \dot{-} >$ is said to be a kind if it satisfies:

(i) $< K, + >$ is a commutative semigroup.

(ii) $x < y \leftrightarrow \exists z(y = x + z)$. Hence $<$ is transitive.

(iii) $<$ is irreflexive and trichotomous.

(iv) $\dot{-}\ (x, y, z) \leftrightarrow x = y + z$. In this case it may be written that $x \dot{-} y = z$, for uniqueness see below.

(v) $<$ is Archimedean.

This entails that $\mathbf{K}$ is an Archimedean strictly linearly ordered cancellative commutative semigroup with partial subtraction. K is said to be the universe of $\mathbf{K}$, and is denoted also by "$|\mathbf{K}|$". $\mathbf{K}$ is said to be a quasi-kind if it satisfies (i) - (iv). A quasi-kind $\mathbf{K}$ is Archimedean (hence a kind) iff every pair of elements of $|\mathbf{K}|$ have a ratio to one another. If $x, y(\epsilon|\mathbf{K}|)$ do not have a ratio to one another, then every multiple of x, say, is less than y. In this case x is said to be infinitesimal relative to y and y is infinite relative to x.

A simple example of a kind is $\mathbb{N}^*$ (the non-zero natural numbers) with the usual addition ($<$, $\dot{-}$ are definable). Nevertheless Euclid [1956] did not consider it so. Instead of the general definition of proportion (definition V.6 of Euclid [1956, vol. 2, p. 114]) $\mathbf{N}^*$ was given a special one (definition VII.20 of Euclid [1956, vol. 2, p. 278]) which is equivalent, in this case, to the general definition but easier to apply.

Neither the cardinals with cardinal addition, nor the ordinals with ordinal addition, nor the positive reals with the usual multiplication can be expanded to become a kind.

However, keeping the intuitive notion of the number (of the elements) of a set in mind and guided by Euclid's common notions (henceforth cns, and cn for the singular) [Euclid [1956] vol. 1, p. 155], a set theoretic kind which is isomorphic to the one given above may be obtained as follows. In accordance with cn 4 "Things which coincide with [congruent to, bijective to, ...] one another are equal [equivalent] to one another." equivalence classes of sets under bijection are considered. This is the only conceptually sound choice as long as applying numbers to sets is the objective [cf. Gödel [1964b] pp. 258-9]. And in accordance with cn 5 "The whole is gneater than the part." only sets which are not bijective to any of its respective proper subsets are taken. Addition is readily definable via disjoint union.

In the forthcoming applications it is needed to deal with different kinds: straight line segments (henceforth slss, and sls for the singular), arcs of congruent circles, angles, polygons and, to generalize, plane regions bounded by slss and circular arcs.

These kinds are explications of corresponding implicit kinds in Euclid [1956]. In all cases the explications as well as the corresponding implicit notions are not arbitrary inasmuch as they are generalizing the corresponding practical concepts (ratios, hence direct measures) and making them precise [cf. Lebesgue [1966] pp. 45, 65, 101 and Borceux [2014] pp. vii, 45, 305]. This abstract formalization which is grounded on practical dealing with reality [cf. Lebesgue [1966] p. 11, see I.4 above] probably provides the justification Moschovakis [1995, p. 753, see I.4 above] is seeking to connect mathematics with the *real* and the *actual*, which is an essential issue regarding conceptual soundness.

To see how this would go, consider footnote "1" of Moschovakis [1995, p. 753, see I.4 above]. It reads, in part, as follows "We can almost prove (*), by making some natural assumptions about "area" (that we know it for rectangles and that it is an additive set function when defined), and then showing that there is a unique way to assign area to "nice" sets subject to these asumptions.".

The problem is that the assumption that we know the area of rectangles needs justification. And, which is more basic: why is it taken for granted that the area function is real valued?

In his first exposition, Lebesgue [1966, pp. 42-5] deals with rectangles as a subset of a set of regions which includes circles too. The area of rectangles and that of the other regions are treated on an equal footing, making use of *limits*. Besides, Lebesgue [1966, pp. 51-9] presents other expositions dealing with the area of polygons only. In each of them the well known formula for the area of a triangle is *preassumed*. Likewise, Moise [1964, p. 168] begins building his area function for polygons by taking this well known formula as a *definition* of the area of a triangle. Earlier he [Moise [1964] pp. 154-5] adopts the well known formula for the area of a rectangle as one of the *postulates* to be satisfied by an area real valued function on polygons which is supposed to be given. Then Moise [1964, pp. 165-7] proves this postulate from its special case of a square of unit side, yet making use of nontrivial properties of the reals. One can hardly say that any of these alternatives would help Moschovakis.

On the contrary the Eudoxean approach proceeds as follows. Practically speaking, the measure of the area of a polygon P in terms of a (unit square) tile T is obtained as the result of comparison of P with T. In laying off T on P, or vice versa, any of them may be reoriented, broken and regrouped, or ... [cf. Lebesgue [1966] pp. 19, 42]. So, in fact what are dealt with are equivalences of T and P. The equivalence relation is inspired by cns $1-4$ as shown in (iv) below [cf. Borceux [2014] p. 61].

Lebesgue [1966, p. 65] says "one must have the concept of area before calculating [measuring] areas". This is achieved via kind (iv) which, regarding its definition, embodies the concept of area of polygons. This what makes Borceux [2014, p. 61] say "In a sense, the Greek approach [to area] is more "intrinsic" than ours, because it does not depend on the choice of a unit to perform the measure, and of course, different choices of unit yield different measures of the same area.". An aspect of the ingenuity of the Eudoxean approach is that there is a universal definition of (direct) measure once a kind is defined: it is the ratio to a specified (unit) element. Based on this, the ratio form of the well known formula for the area of a rectangle (hence that of a triangle) may be deduced (see II.2 below) without recourse to real numbers, limits, nor exhaustion (the definition of kind (iv) makes use only of Euclid [1956]'s book I). Though extremely deep, the Eudoxean approach is very natural and intuitively quite acceptable.

Moreover, let F be the set of all positively Riemann integrable non-negative real valued functions on closed bounded real intervals, and let $F^{\daleth}$ be the set of the regions under their graphs. Along the lines of (v) below, $F^{\daleth}$ may be shown to be the universe of a kind $\mathbf{F}^{\daleth}$. With regard to $\mathbf{F}^{\daleth}$ it may be shown that the integral of any of these functions is the positive real number corresponding (see II.2 below) to the ratio of the corresponding region to the unit square. This would provide Moschovakis with the needed justification. Furthermore this shows why the area function may be real valued.

In view of the *current*, not only the historic, strong, probably matchless, conceptual explanatory power of the Eudoxean approach, the following thesis may be proposed: for each of the kinds considered below, the corresponding Edoxean direct measure (as here explicated) is conceptually sound; any other measure on the same kind or a subset thereof is to be assessed with reference to it. Those who may have reservations about adopting this thesis are invited to accept it as a working hypothesis. In the extreme case of complete rejection, this article may still be of significance as an attempt to precisely explicate a notion which had prevailed for more than two millennia, and to discuss its role in the evolution and applications of mathematics. At any rate, further discussion towards deeper analysis is always welcome.

Following are the kinds to be dealt with:

(i) For slss, K is the set of all congruence classes of slss. The definitions of $+, <$ and $\dot{-}$ are obvious. The Archimedean principle is assumed. Notice that for the Archimedean principle to hold in each of cases (ii) - (v) discussed below, it is sufficient that it holds for the slss.

(ii) For arcs of congruent circles, K is the set of their congruence classes. For the definition of $+$, multiples of complete circumferences also should be allowed.

(iii) The treatment of angles parallels that of arcs of congruent circles.

(iv) The case of polygons is more involved. Hilbert [1950, p. 37] distinguishes between two equivalence relations on polygons: of *equal area* and of *equal content*, which will be denoted by "ρ_0" and "ρ_1" respectivily [cf. Euclid [1956] vol. 1, p. 328].

For a pair of polygons $P, P', P\rho_0 P'$ iff they are piecewise congruent; while $P\rho_1 P'$ iff there are two polygons P_1 and P_1' such that:

(a) P and P_1 are essentially-disjoint. Same for P' and P_1'.

(b) $P_1\rho_0 P_1'$ and $(P \cup P_1)\rho_0(P' \cup P_1')$.

The definitions of ρ_0 and $\rho_1$are guided by Euclid's common notions [see above]. The definition of ρ_0 is guided by cn 4 and cn 2 "If equals are added to equals, the wholes are equal". The definition of ρ_1 is guided, in addition, by cn 3 "If equals are subtracted from equals, the remainders are equal". Obviously $\rho_0 \subseteq \rho_1$.

If K is taken to be the ρ_0-equivalence classes, the definition of $+$, hence of $<$ and $\dot{-}$, will present no problem, but the trichotomy of $<$ will be problematic. Replacing ρ_0 by ρ_1 will solve the problem. For, from proposition I.45 of Euclid and its proof [Euclid [1956] vol. 1, pp. 345-6] one may easily deduce:

COROLLARY (E). For every sls l and every polygon P, there is a rectangle $R(l, P)$ which is ρ_1-related to P and which has a side congruent to l.

The uniqueness (up to congruence) of $R(l, P)$ is guaranteed by cn 5 (cf. the proof of proposition 39 of Euclid [1956, vol. 1, p. 336]). From this the trichotomy of $<$ readily follows by the corresponding property of kind (i). So K may be taken to be the ρ_1-equivalence classes.

Why not to go beyond ρ_1? To be sure, guided by Euclid's cns, a monotonic increasing sequence $< \rho_n >_{n\in\mathbb{N}}$ of equivalence relations may be recursively defined as follows. ρ_0 is already defined. For every $n \in \mathbb{N}$, ρ_{n+1} is to be obtained from ρ_n in a way similar to that by which ρ_1 is obtained from ρ_0 above. The union of this sequence, σ say, also is an equivalence relation.

As a matter of fact, the proof of proposition I.45 of Euclid [1956, vol. 1, pp. 345-6] did not go beyond ρ_1. Moreover $\sigma = \rho_1$. Following is a proof based on cns 2-5 and the terminology thereof.

By cns 4, 2, for every pair of polygons P, P', if $P\rho_0 P'$, then P and P' are equal. Let $n \in \mathbb{N}$ and assume that for every pair of polygons P, P', if $P\rho_n P'$, then P and P' are equal. Then by cn 3, for every pair of polygons P, P', if $P\rho_{n+1} P'$, then P, P' are equal.

By mathematical induction and the definition of σ, it follows that for every pair of polygons P, P', if $P\sigma P'$, then P and P', are equal.

Let P, P', be two polygons such that $P\sigma P'$, and let l be a sls. By corollary(E), $R(l,P)\rho_1 P\sigma P'\rho_1\ R(l,P')$, hence $R(l,P)\sigma R(l,P')$, consequently $R(l,P)$ and $R(l,P')$ are equal. So, by cn 5 none of them may be a part of the other. From this it follows that they are congruent, then $R(l,p)\rho_0 R(l,P')$, a fortiori $R(l,P)\rho_1 R(l,P')$. Summing up:

$P\rho_1 R(l,P)\rho_1 R(l,P')\rho_1 P'$. By the transitivity of ρ_1, $P\rho_1 P'$, which completes the proof.

Making use of the Archimedean principle (for slss), Lebesgue [1966, p. 58] strengthened corollary (E) showing that ρ_1 may be replaced by ρ_0. So, by a simplification of the above argument, $\sigma = \rho_0$, and K may be taken to be the ρ_0-equivalence classes.

It is worth noting that no recourse to the real numbers is needed here. For other views see Hilbert [1950, pp. 37-45] and Lebesgue [1966, pp. 42-62]. We shall come back to this at the end of this section.

(v) Regions bounded by slss and circular arcs: the set C of all of these regions may be defined as follows. First, recursively define $< A_n >_{n\in\mathbb{N}}$ by:

A_o= the set of all triangles $\cup$ the set of all circles.

$A_{n+1} = A_n \cup \{X$: there are $Y, Z \in A_n$ such that the set theoretic difference $Y-Z$ has a non-empty interior and X is the (topological) closure of $Y-Z\}$.

Put:

$$B = \bigcup_{n\in\mathbb{N}} A_n$$

$C =$ the set of all finite non-empty essentially-disjoint unions of elements of B.

Every element of B is regular closed (i.e. is the closure of its interior), hence perfect (i.e. equals the set of all its limit points). Same applies to every element of C. The intersection of two elements of B may not belong to B, but if the interior of this intersection is non-empty, then its closure belongs to B.

Let η be the binary relation defined on C by: $X\eta Y$ iff for every inner polygon P^i_X of X and every outer polygon P^o_Y of Y:

$$P^i_X/\rho_1 \leq P^o_Y/\rho_1$$

where $<$ is the ordering relation defined in (iv) above.

η is reflexive and, by the principle of exhaustion (which is due to Eudoxus and is equivalent to the Archimedean principle [Euclid [1956] vol. 3, pp. 14-6, 365-8, 374-7 and vol. 1, p. 234]) for C, it is transitive. I.e. η is a pre-ordering, hence it defines an equivalence relation on C, to be denoted by "ζ", by: $X\zeta Y$ iff $X\eta Y$ and $Y\eta X$. K may be taken as the set of all ζ-equivalence classes.

Notice that the Archimedean principles for C and slss are equivalent. Assuming any of them, ρ_1 may be replaced by ρ_0 and the mapping $P/\rho_0 \longmapsto P/\zeta$, where P is a polygon, is an embedding of the kind defined in (iv) into the kind here defined.

For each kind direct measure [see I.5 above] may be defined. The direct measure of an element of any of the kinds (i)-(v) may be considered to be at the same time the direct measure of each element of this element. Following what may be called the Euclidean tradition, we may confuse equivalence classes with elements thereof; the intention will be clear from the context.

The advantage of Eudoxus' treatment over that of Lebesgue is that it allows direct measures for more kinds, e.g. these defined in (iv) and (v) above [cf. Lebesgue [1966] ch. III, pp. 42-67]. Eudoxus' treatment practically produces π [cf. Euclid [1956] proposition XII.2, vol. 3, p. 371 and the appendix below], while Lebesgue's treatment pre-assumes π [cf. Lebesgue [1966] pp. 62-3].

Going back to Hilbert, it is worth noting that Hilbert [1950, p. 37] takes into consideration not only ρ_0, but also ρ_1 "to establish Euclid's [Eudoxus'] theory of areas by means of the axioms already mentioned; that is to say, *for the plane geometry, and that is independent of the axiom of Archimedes*.". Nevertheless, this promise is not completely fulfilled. Hilbert [1950] does not establish a theory of areas for kind (v), it does [pp. 37-45] only for kind (iv) taking the well-known formula for the area of a triangle as a definition after proving that it does not depend on the choice of the base, without any further justification.

Likewise, without assuming the axiom of Archimedes, Hilbert [1950, pp. 23-36] establishes a theory of proportion; it deals only with kind (i), in particular, it does not deal with kind (iii): angles.

It is noteworthy that Tarski [1959, p. 21] adopted this theory of proportion. Unlike that of Hilbert, Tarski's system of geometry is one sorted which makes it better suited for metamathematical investigations.

As a matter of fact, in addition to being indirectly based on $\mathbb{R}$, Hilbert [1950, pp. 23-45]'s theories of propertion and areas are obtained from the corresponding theories of Eudoxus by some sort of reverse technology (without mentioning Eudoxus). Computationally, the corresponding theories are equivalent (assuming the axiom of Archimedes); conceptually, they are not. In particular, from the measure-theoretic viewpoint, those of Hilbert are entirely conceptually unsound as they disregard the concept of direct measure. Moreover, unlike Euclid [1956], Hilbert [1950] and Tarski [1959] do not touch the problem of measuring angles.

The serious defect of all current approaches to trigonometric functions [see I.1 above] is that they ignore altogether that there is a direct measure for angles [see II.4, 5 below]. Hardy and Morrey [cf. I.1 above] ignore, moreover, that there is a direct measure for the plane regions they are concerned with. The axiomatic and the purely analytic approaches to defining the measure of angles or the trigonometric functions do not solve the problem, they just transfer it [cf. the quotation from Moschovakis in I.4 above].

II.2. Ratios and Real Numbers. Define the function L from the union $\mathfrak{K}$ of all sets of the form $|\mathbf{K}| \times |\mathbf{K}|$, where $\mathbf{K}$ is a quasi-kind, to $\wp(Q^+)$, where Q^+ is the set of all positive rationals, by:

$$L(x,y) = \{\frac{m}{n} \in Q^+ : my \leq nx\}$$

This function is "well defined" as we may, and shall, assume that the universes of different quasi-kinds are pairwise disjoint; also the choice of m, n is irrelevant as long as the fraction is the same.

The function L defines an equivalence relation $=_L$ on $\mathfrak{K}$ by: $< w, x >=_L < y, z >$ iff $L(w,x) = L(y,z)$. This equivalence relation includes the equivalence relation, to be denoted by "$=_E$", practically defined by Eudoxus and formulated by Euclid in:

DEFINITION V.5 of Euclid [1956, vol. 2, p. 114]. Magnitudes are said to **be in the same ratio** [or to be proportional], the first to the second and the third to the fourth, when, if any [positive integral] equimultiples whatever be taken of the first and the third, and any [positive integral] equimutiples whatever of the second and the fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order.

That is, for $< x_1, x_2 >, < x_3, x_4 > \in \mathfrak{K}, < x_1, x_2 >=_E < x_3, x_4 >$ iff for every pair of positive integers m, n, $mx_1 >$ (respectively $=$ or $<$) nx_2 iff $mx_3 >$ (respectively $=$ or $<$) nx_4.

$\mathfrak{K}$ may have elements $< x, y >$ such that x, y do not have a ratio to one another. The set of all other elements of $\mathfrak{K}$ is denoted by "$\mathfrak{R}$": That is

$$\mathfrak{R} = \{< x, y > \in \mathfrak{K} : \phi \neq L(x,y) \neq Q^+\}.$$

The following proposition [cf. Stein [1990] pp. 170-2] presents equivalences to requiring that all quasi-kinds are kinds.

PROPOSITION. The following are equivalent:

(i) $=_E = =_L$.

(ii) No quasi-kind has infinitesimals.

(iii) For every x, y, z, if $< x, z >=_E < y, z, >$ then $x = y$.

(iv) All quasi-kinds are Archimedean.

(v) All quasi-kinds are kinds.

(vi) $\mathfrak{R} = \mathfrak{K}$.

(vii) For every $< x, y > \in \mathfrak{K}$, $\{L(x,y), (Q^+ - L(x,y))\}$ is a partition of Q^+.

(viii) The function induced by L on $\mathfrak{K} \,/ =_E$ is injective.

PROOF. It is easy to see that: (a) (i) and (viii) are equivalent, (b) (ii), (iv), (v), (vi) and (vii) are equivalent, (c) $\urcorner$(ii) $\rightarrow \urcorner$(i), and (d) $\urcorner$(ii) $\rightarrow$ $\urcorner$(iii).

Also, it is not hard to see that $\urcorner$(i) $\rightarrow \urcorner$(ii) [cf. Stein [1990] p. 170]. Finally, (iv) $\rightarrow$ (iii) is essentially the first part of proposition V.9 of Euclid [1956, vol. 2, p. 153].

Define the equivalence relations $\approx_E$ and $\approx_L$ on $\mathfrak{R}$ by:

$$\approx_E \;=\; =_E \cap (\mathfrak{R} \times \mathfrak{R}) \text{and} \approx_L \;=\; =_L \cap (\mathfrak{R} \times \mathfrak{R})$$

Then for every $< x, y > \in \mathfrak{R}$, $< x, y > / \approx_E \; = \; < x, y > / =_E \; \subseteq \; < x, y > / =_L \; = \; < x, y > / \approx_L \; \subseteq \mathfrak{R}$.

Following the modern practice, the ratio of x to y, for $< x, y > \in \mathfrak{R}$, is defined to be $< x, y > / \approx_E$. Thus, the set of all ratios is $\mathfrak{R} / \approx_E$.

Presumably, Frege was planning to develop the real numbers along the above lines before he was stopped by the discovery of Russell's paradox [cf. Frege [1968] pp. $73^e - 80^e$, $114^e - 9^e$ and Simons [2011] p. 19]. To honour Eudoxus and Frege, $\mathfrak{R} / \approx_E$ may be called the Eudoxus/Frege (positive) real numbers. However, to simplify our naming system the usage of "real numbers" will be restricted in what follows to denote the usual (Dedekind) real numbers only.

For $< x, y > \in \mathfrak{K}$, $L(x,y)$ is an initial segment of Q^+, i.e., for every $q, q' \in Q^+$, $q \in L(x,y)$ if $q' \in L(x,y)$ and $q \leq q'$. Hence, the ordered pair $< L(x,y), (Q^+ - L(x,y)) >$ is a Dedekind cut of Q^+ iff $< x, y > \in \mathfrak{R}$.

Let $\mathfrak{C}$ be the set of all Dedekind cuts of Q^+, define the equivalence relation $\sim$ on $\mathfrak{C}$ by:

$$< A, B > \;\; \sim \;\; < A', B' > \text{ iff } \overline{\overline{A \oplus A'}} \leq 1$$

where $\oplus$ is the set theoretic symmetric difference, and $\overline{\overline{X}}$ is the cardinal number of X. Then $\mathfrak{C} \,/ \sim$ is the set of all (Dedekind) positive real numbers, which is denoted also by "$\mathbb{R}^+$". $\mathfrak{R} \,/ \approx_E$ may be mapped into $\mathbb{R}^+$ via:

$$\begin{aligned} Re \quad &: \quad \mathfrak{R} \,/ \approx_E \;\; \rightarrow \; \mathbb{R}^+ \\ Re(\;\; < \;\; x, y > / \approx_E) \;\; &= \;\; < L(x,y), (Q^+ - L(x,y)) > / \sim \end{aligned}$$

Q^+ is a subset of the range of Re (the abuse of notation is easily resolvable), and Re is injective iff $\mathfrak{R} = \mathfrak{K}$. In this case (which is here adopted), $\approx_E \, = \, =_E \, = \, =_L \, = \, \approx_L$. So $\mathfrak{K} \,/ =_E$ ($= \mathfrak{K} \,/ =_L$) will be the set of all ratios, and two distinct ratios will correspond to two distinct positive reals.

Moreover, if two magnitudes have a ratio in the sense of Lebesgue [cf. I.5 above] they will have a ratio in the sense of Eudoxus, and both ratios correspond to the same (positive) real number.

The usual strict linear order on $\mathbb{R}^+$ defines, via the injection Re, a strict linear order $<_L$ on the set $\mathfrak{K} \,/ =_E$ of all ratios. Explicitly:

$$< x, y > / =_E \quad <_L \quad < x', y' > / =_E \text{ iff } L(x, y) \subsetneq L(x', y').$$

This ordering relation is the same as the ordering relation, to be denoted by $<_E$, which is obtained in an obvious way from the relation defined by Eudoxus and formulated by Euclid in:

DEFINITION. V.7 of Euclid [1956, vol. 2, p. 114]. When, of equimultiples, the multiple of the first magnitude exceeds the multiple of the second, but the multiple of the third does not exceed the multiple of the fourth, then the first is said to **have a greater ratio** to the second than the third has to the fourth.

Taking $<_E$ into consideration, Re is in fact an order embedding of $\mathfrak{K} \,/ =_E$ into $\mathbb{R}^+$, and the restriction of the inverse of Re to Q^+ is an order dense embedding of Q^+ into $\mathfrak{K} \,/ =_E$.

Euclid [1956] has all needed ingredients to define two partial binary operations on $\mathfrak{K} \,/ =_E$. The first [cf. Euclid [1956] vol. 2, p. 187 and Stein [1990] pp. 177, 181-3] to be called addition and to be denoted by "+", is defined as follows: For $< a, b >, < c, d > \in \mathfrak{K}$, $<< a, b > / =_E, < c, d > / =_E>$ belongs to the domain of $+$ if there are slss u, v and w such that $< a, b > =_E < u, w >$ and $< c, d > =_E < v, w >$. In this case,

$$< a, b > / =_E + < c, d > \;\; / \;\; =_E \;\; = \;\; < u + v, w > / =_E$$

This justifies the name "addition" and the symbol "+", though there is some abuse of notation. That $+$ is well defined is guaranteed by proposition V.24 of Euclid [1956, vol. 2, p. 183]. It is not hard to see that $+$ is commutative in the strong sense (i.e. if one side exists so does the other and they are equal) and that $<_E$ is compatible with $+$.

The second [cf. Euclid [1956] vol. 2, p. 187 and Stein [1990] pp. 183, 185] to be called "multiplication" and to be denoted by "$\times$" (not to be confused with the Cartesian product, the intention will be clear from the context), is defined as follows: For $< a, b >, < c, d > \in \mathfrak{K}, << a, b > / =_E, < c, d > / =_E>$ belongs to the domain of $\times$ if there are u, v, w all of the same kind such that:

$$\begin{array}{lcl} & < \;\; a, b > \;\; =_E \;\; < u, w > \text{ and } < c, d > \;\; =_E \;\; < w, v >, \\ \text{or,} & < \;\; a, b > \;\; =_E \;\; < w, v > \text{ and } < c, d > \;\; =_E \;\; < u, w > . \end{array}$$

In both cases, $< a, b > / =_E \times < c, d > / =_E = < u, v > / =_E$.

That $\times$ is well defined is guaranteed by propositions V.22, 23 of Euclid [1956, vol. 2, pp. 179, 181]. It is not hard to see that $\times$ is commutative in the strong sense and that $<_E$ is compatible with $\times$.

This definition makes the ratio of a rectangle of sides a, b to a square of side u (unit square) equal to $(< a, u > / =_E \times < b, u > / =_E>)$ [cf., Euclid [1956] vol. 2, prop. VI.23, p. 247], which justifies the choice of the name and the symbol.

Taking addition and multiplication into consideration, Re will be an order embedding partial monomorphism, and the restriction of its inverse to Q^+ will be a monomorphism.

Contrary to what most mathematicians believe, Descartes did *not* establish his analytic geometry on the real numbers (they were not developed yet), but on ratios of slss [Descartes [1954] pp. 2,5; also cf. Bos [1997] pp. 38, 45, 55]. As a matter of fact, if $\mathfrak{L}_E$ is the set of all ratios defined by slss, i.e.:

$$\mathfrak{L}_E = \{< a, b > / =_E \ \in \mathfrak{K}/ =_E\text{: there are two slss } c, d \text{ such that}$$

$$< a, b > \ =_E \ < c, d >\}$$

then each of $+, \times$ will be defined for every pair of elements of $\mathfrak{L}_E$, so that the restrictions of $+, \times$ to $\mathfrak{L}_E \times \mathfrak{L}_E$ will be total operations on $\mathfrak{L}_E$ which make it the set of positive elements of an Archimedean ordered field; the ordering is an extension of the restrietion of $<_L$to $\mathfrak{L}_E$. The restriction of Re to $\mathfrak{L}_E$ will be a monomorphism whose range includes Q^+.

The crucial property needed for the proof is the existence of the fourth proportional [cf. Euclid [1956] vol. 2, Heath's commentary, p. 187; prop. VI. 12, p. 215 and Stein [1990] pp. 174-7]. To specify, the fourth proportional is needed to bridge a gap in the proof of proposition V.18 of Euclid [1956, vol.2, p.170]. This proposition is made use of in proving proposition V.24 of Euclid [1956, vol. 2, pp. 183-4], which is made use of to prove that $+$ is well defined [see above]. Also, the fourth proportional is made use of in proving that each of $+, \times$ is defined for each pair of elements of $\mathfrak{L}_E$. Proposition VI.12 of Euclid provides a method to construct a fourth proportional for any three given slss [Euclid [1956] vol. 2, pp. 215-6].

Moreover, the square root function on $\mathfrak{L}_E$ is geometrically definable [cf. Euclid [1956] prop. II.14, vol. 1, p. 409 and prop. VI.13, vol. 2, p. 216; and Stein [1990] p. 183]. This plays an important role in Descartes' analytic geometry [Descartes [1954] pp. 2,5; also cf. Bos [1997] pp. 38, 55].

This is the origin of the modern development of real numbers. As Hobson [1957, p. 20] testifies. "In later times [after Euclid], the idea was current that, to the ratio of any two magnitudes of the same kind, there corresponds a definite number; and in fact Newton in his *Arithmetica Universalis* expressly defines a number as a ratio of any two quantities [of the same kind]." To a considerable extent, even after the development of the arithmetical theories of irrational numbers, "a number has been regarded as the ratio of a segment of a straight line to a unit segment, and the conception of irrational number as the ratio of incommensurable segments has been accepted as a sufficient basis for the use of such numbers in Analysis." adds Hobson [1957, p. 20].

It seems that this tradition is what made Heath say [Euclid [1956] vol. 2, p. 124] "Certain it is that there is an exact correspondence, almost coincidence, between Euclid's [Eudoxus'] definition of equal ratios and the modern theory of irrationals due to Dedekind.". Understanding this in the strong sense would entail that the mapping Re is onto, which holds only if the (geometric) axiom of continuity is accepted or inferred. But [cf. Stein [1990] p. 178] this axiom is not necessary for Euclidean geometry.

Indeed, the Greeks did not even regard ratios as kinds of magnitude or, to use Stein's terminology, quantities [cf. Stein [1990] p. 181]. However, Stein advocates that Eudoxus' theory of proportion made them very close to develop $\mathbb{R}^+$ and wonders [Stein [1990] p. 186] "[W]hy the Greeks failed to exploit this potentiality of a theory they actually possessed.". And adds [Stein [1990] p. 203] "[...] in my own opinion, [...] if we had only been blessed by another Archimedes or two in antiquity, the mathematics of the seventeenth century might have begun -and on firmer and clearer foundations than when it did in fact begin- more than a millennium and a half earlier.".

These are only speculations. The development from ratios to real numbers is a long historical process which should be dealt with on the basis of solid factual historical grounds, which is yet to be done.

In particular the following developments need to be given careful attention:

(1) The development of the decimal system of numeration, which "[I]s perhaps the most important event in the history of science", says Lebesgue [1966, p. 18].

Through imprecise but fecund mathematical practice, this system made addition and multiplication of ratios (or rather their finite decimal approximations) possible, whether these ratios come from the same kind or not, and whether the result of the operation corresponds to a ratio or not.

(2) The development of (verbal, then symbolic) algebra which -among other things- was instrumental in both the conceptual and computational development of analytic geometry [cf. Bos [1997] pp. 38, 43, 45, 49, 55].

(3) The development of analytic geometry, which -among other things- set the stage for the development of the differential and integral calculus.

It is to be noted that all the geometric ingredients Descartes made use of in developing his analytic geometry, were available to the Greeks since Eudoxus [see above]. So a natural question to be asked is why did not they develop it then? A preliminary reply is: Algebra was lacking.

On the other hand, it is also natural to ask: Having the Eudoxean legacy, exhaustion included, why did not analysis begin, in the seventeenth century, on firmer and clearer foundations than it did in fact begin? [see the introduction above].

II.3. Angles. Euclid's definition of a (rectilineal) angle is given [Euclid [1956] vol. 1, p. 153] in two steps:

DEFINITION I.8 of Euclid [1956, vol. 1, p. 153]. **A plane angle** is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.

DEFINITION I.9 of Euclid [1956, vol. 1, p. 153]. And when the lines containing the angle are straight, the angle is called **rectilineal**.

Modern writers try to avoid the obscurity of the term "inclination" by defining the rectilineal angle (henceforth, angle) as the configuration formed by two straight lines [rays] having the end point in common [Hardy [1967] p. 316], or as the locus consisting of two rays which have a common end point and lie along different lines [Morrey [1962] p. 203].

These definitions share the following two characteristics:

(i) They make use of infinite objects: (full) rays, which is alien to Greek geometry.

(ii) Different angles, according to these definitions, are equal according Euclidean geometry.

Noting that in Euclidean geometry "equal" is sometimes used where "congruent" or "equivalent" should have been used, it is reasonable to retain the second characteristic. To avoid the first one, the following definition is proposed:

An angle is (the equivalence class generated by) an ordered triple of three pairwise distinct (or non-collinear, to exclude the zero and the straight angles) points. Two such triples $< a, b, c >$ and $< a', b', c' >$ are said to be equivalent if $b = b'$ and

(1) one of a, a' lies between b and the other one, same for c, c'; or

(2) one of a, c' lies between b and the other one, same for c, a'.

II.4. Measure of Angles. As was mentioned before [I.5 and II.1 above], to measure angles, an angle (henceforth called unit angle) is to be fixed, and the measure of any angle is to be its ratio to the unit angle [cf. Lebesgue [1966] pp. 38-40]. This section and the next one link this measure to the measures currently given in the literature, and show how to move from geometry to analysis.

PROPOSITION VI.33 of Euclid [1956, vol. 2, pp. 273-6]. In equal circles angles have the same ratio as the circumferences [arcs] on which they stand, whether they stand at the centers or at the circumferences.

There is a (bridgeable) gap in the proof of proposition III.26 of Euclid [1956, vol. 2, pp. 56-7] on which the proof of the above proposition is based. As a matter of fact, had the proof of proposition III.26 of Euclid been completed as expected, it would have proved:

PROPOSITION III.26$'$. In equal circles equal angles have equal (central) sectors, whether the angles stand at the centers or at the circumferences.

Instead of the actually proven proposition, which is:

PROPOSITION III.26 of Euclid [1956, vol. 2, p. 56]. In equal circles equal angles stand on equal circumferences [arcs], whether they stand at the centers or at the circumferences.

Based on proposition III.26$'$, the proof of proposition VI.33 of Euclid may be modified to yield:

PROPOSITION VI.33$'$. In equal circles angles have the same ratio as their (central) sectors, whether they stand at the centers or at the circumferences.

Accordingly, in measuring angles, ratios between angles may be replaced by ratios between the corresponding arcs or sectors, after fixing a (radius of a) circle.

To pursue discussing the measure of angles, in the next section and the rest of this section, Euclidean geometry will be augmented by the (geometric) axiom of continuity [see I.6 and II.2 above].

This second order axiom guarantees that all gaps are filled. It may be replaced by first order axioms [cf., I.6 above] which will guarantee that the required, but not necessarily all, gaps are filled. The resulting theory will be weaker.

(i) Let C be a circle of radius r, and let c be an arc of C. Consider the set of all (open) polygons inscribed in c. To each of these polygons there corresponds a sls, the sum of its edges. The set of all of these slss is non-empty and is bounded above, so it has a supremum sls. This defines a function b which assigns to each circular arc the corresponding supremum sls.

It is easy to see that b is finitely additive, hence for every pair c_1, c_2 of arcs of C, $< c_1, c_2 > =_E < b(c_1), b(c_2) >$. On the other hand finite additivity follows from this equality, by proposition V.24 of Euclid [1956, vol. 2, p. 183].

Let a be an angle, and let u be a chosen (unit) angle. Regarding a and u as central angles, let c_a and c_u be, respectively, the corresponding C arcs, then by proposition VI.33 of Euclid [1956, vol. 2, p. 273]:

$$< a, u > \quad =_E \quad < c_a, c_u > \quad =_E \quad < b(c_a), b(c_u) > .$$

Further, let C' be a circle of radius r', and let c'_a and c'_u correspond, respectively, to c_a and c_u. By similarity of triangles and the properties of the suprema, $< r, b(c_u) > =_E < r', b(c'_u) >$, hence $< r, b(c_u) > / =_E = < r', b(c'_u) > / =_E$. By continuity and the Archimedean principle there is a unique angle (call it "radian" and denote it by "d") which makes this common ratio equal to $< r, r > / =_E$. Consequently:

$$\begin{aligned} < \ & a, d > / =_E \ = \ < b(c_a), b(c_d) > / =_E \ = \\ < \ & b(c_a), r > / =_E \times \ < r, b(c_d) > / =_E \\ = \ & < b(c_a), r > / =_E \times \ < r, r > / =_E \ = \ < b(c_a), r) > / =_E \\ = \ & < b(c'_a), r' > / =_E \ . \end{aligned}$$

(ii) Enrich the vocabulary of (i) by letting s_a and s_u (s'_a and s'_u) be, respectively, the sectors corresponding to a and u in C (C'). Also, let q and q' be, respectively, the squares on r and r'.

By proposition VI.33' [see above]:

$$< a, u > / =_E \ = \ < s_a, s_u > / =_E$$

A corollary of this and proposition XII.2 of Euclid [1956, vol. 3, p. 371 and the appendix of this article] is:

$$\begin{aligned} & < \ s_u, s'_u > / =_E \ = \ < q, q' > / =_E \ , \\ \text{hence} \ & < \ q, s_u > / =_E \ = \ < q', s'_u > / =_E \ . \end{aligned}$$

In the proof, proposition V.16 of Euclid [1956, vol. 2, p. 164] is made use of.

By continuity and the Archimedean principle there is a unique angle (denote it by "e") which makes this common ratio equal to $< q, q > / =_E$. Consequently:

$$\begin{aligned} < \ & a, e > / =_E \ = \ < s_a, s_e > / =_E \ = \ < s_a, q > / =_E \times \ < q, s_e > / =_E \\ = \ & < s_a, q > / =_E \times \ < q, q > / =_E \ = \ < s_a, q > / =_E \\ = \ & < s'_a, q' > / =_E \ . \end{aligned}$$

(iii) So, the (direct) measure of a in terms of the unit angle d (e), to be denoted by $m(a)$ ($\mu(a)$), is given by:

$$m(a) \ = \ < a, d > / =_E \ = \ < b(c_a), r > / =_E \ (\mu(a) \ = \ < a, e > / =_E \ = \ < \ s_a, q > / =_E).$$

Notice that the right hand side does not depend on r (q), it depends only on a. This defines two functions m and μ from angles to ratios.

In modern terminology it may be written:

$$a = m^*(a)d = \mu^*(a)e$$

where for every ratio $\rho \in \mathfrak{R}/ =_E, \rho^*$ is $Re(\rho)$, and for every function g taking values in $\mathfrak{R}/ =_E$, g^* is the composition $Reog$ [see II.1, 2 above]. Notice that these equations involve no multiplications; a,d and e are not numbers.

(iv) To find out the relationship between m, d and, respectively, μ, e, let $p_i(p_o)$ be the sls formed by adding the edges of an inscribed (a circumscribed) polygon in (around) c_a, and let $Rl(x, y)$ be the rectangle whose sides are the slss x and y:

(1) $p_i < p_o$.

(2) $2s_a < Rl(r, p_o)$. By exhaustion, the r.h.s. may be made arbitrarily close to the l.h.s.

(3) $Rl(r, p_i) < 2s_a$. The proof is by contradiction making use of (1) and (2). Again by exhaustion, the l.h.s. may be made arbitrarily close to the r.h.s.

(4) $Rl(r, b(c_a)) \; \zeta \; 2s_a$. Recall that ζ is defined in part (v) of II.1 above.

(5) For positive integers m, n and a ratio $< x, y > / =_E$, $\frac{m}{n} < x, y > / =_E$ is defined to be $< mx, ny > / =_E$.

The choice of the representative $< x, y >$ is immaterial. Obvious properties of this multiplication, in particular $\frac{m}{n} < x, y > / =_E = \frac{m'}{n'} < x, y > / =_E$ whenever $\frac{m}{n} = \frac{m'}{n'}$, are readily provable.

(6) $2\mu(a) = < 2s_a, q > / =_E = < Rl(r, b(c_a)), q > / =_E \quad =< b(c_a), r > / =_E = m(a)$.

The equality before the last follows from proposition VI.1 of Euclid [1956, vol. 2, p. 191].

(7) With the obvious definition, $m = 2\mu$.

(8) $< a, e > / =_E = \mu(a) = \frac{1}{2}m(a) = \frac{1}{2} < a, d > / =_E =< a, 2d > / =_E$.

So, by proposition V.9 of Euclid [1956, vol. 2, p. 153], $e = 2d$ [cf. the proposition in II.2 above].

This is the reason behind the factor 2 in [Hardy [1967] p. 317] and [Morrey [1962] p. 214]. There the computation is correct, while the treatment is conceptually completely inappropriate [cf. II.1 above, II.5 below, and Lebesgue [1966] pp. 63-4].

(v) The above discussion raises the question: Why did not Euclid prove propositions along the lines of III.26′ and VI.33′, though he has developed all the necessary infrastructure for proving them?

This question becomes more pressing when it is remembered that in Euclid [1956] π is (almost) defined as the (constant) ratio of a circle to the square on its radius. This is an immediate consequence of:

PROPOSITION XII.2 of Euclid [1956, vol. 3, pp. 371-8]. Circles are to one another as the squares on their diameters.
There is a bridgeable gap in Euclid's proof [see the appendix of this article].

II.5. From Geometry to Analysis. It is known since ages that by similarity of triangles, there is a function, to be denoted by "Sin", from acute angles to ratios, defined by:

$$\text{Sin } \theta =< \text{opposite side, hypotenuse} > / =_E$$

Restricting the measure function m to acute angles (call the new function also "m", it will be clear from the context which function is intended). Each of the two functions Sin^* and m^* from acute angles to reals is injective, hence invertible. Composing each of them with the inverse of the other gives two functions from reals to reals, which are the inverses of each other. They are denoted by "f" and "f^{-1}" as shown in figure 1, where A is the set of all acute

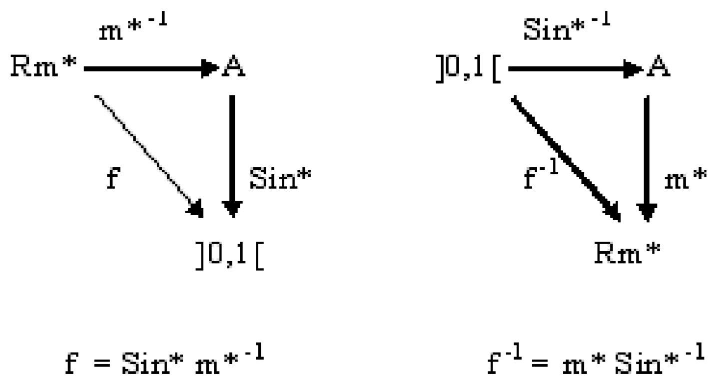

Figure 1:

angles and Rm* is the range of the (restricted) function m*.

As a matter of fact, f is essentially the explication of the sine function studied in analysis. To see this (with the same notation as above):

(i) The multiplicative inverse of a ratio $< x, y > / =_E$ is defined to be $< y, x > / =_E$. Obvious properties of this inverse, such as $Re(< v, w > / =_E \times < w, x > / =_E) = Re(< v, w > / =_E)/Re(< x, w > / =_E)$, are readily provable.

(ii) Choose a (unit) sls $\widehat{u}$. Define the function l on the set of all slss by $l(x) = Re(< x, \widehat{u} > / =_E)$. $l(x)$ is said to be the length of x in terms of $\widehat{u}$.

(iii) For $a \in A$, $m^*(a) = Re(< b(c_a), r > / =_E) = l(b(c_a))/l(r)$.

(iv) $l(b(c_a)) = l(r) \displaystyle\int_0^{\mathrm{Sin}^*\, a} dt/\sqrt{1-t^2}$

(v) $m^*(a) = \displaystyle\int_0^{\mathrm{Sin}^*\, a} dt/\sqrt{1-t^2}$.

(vi) For $x \in]0,1[$, $f^{-1}(x) = m^*(\mathrm{Sin}^{*-1}(x)) = \int_0^x dt/\sqrt{1-t^2}$.

(vii) Put $\pi = Re(< C, q > / =_E)$ [cf. Euclid [1956] , vol. 3, p. 371, prop. XII.2].

(viii) Let ρ be a right angle [Euclid [1956] , vol. 1, p. 153, def. I.10 and vol. 1, p. 154, postulate I.4]. Then:

$$m^*(\rho) = 2\mu^*(\rho) = 2Re(< s_\rho, q > / =_E) = 2\frac{\pi}{4} = \frac{\pi}{2}.$$

(ix) $\frac{\pi}{2} = m^*(\rho) = 2m^*(\frac{\rho}{2}) = 2\int_0^{\mathrm{Sin}^* \frac{\rho}{2}} dt/\sqrt{1-t^2} = 2\int_0^{\frac{1}{\sqrt{2}}} dt/\sqrt{1-t^2} = \lim_{x \to 1} \int_0^x dt/\sqrt{1-t^2}$.

(x) $Rm^* =]0, \frac{\pi}{2}[$

(xi) Let ξ be the zero angle, put:

$\overline{A} = A \cup \{\xi, \rho\}, \qquad \overline{m} = m^* \cup \{< \xi, 0 >, < \rho, \frac{\pi}{2} >\}$

$\overline{\mathrm{Sin}} = \mathrm{Sin}^* \cup \{< \xi, 0 >, < \rho, 1 >\}$

$\overline{f} = f \cup \{< 0, 0 >, < \frac{\pi}{2}, 1 >\}$.

(xii) The sine function studied in analysis, denoted by "sin", may be explicated by the obvious extension of $\overline{f}$. Notice that the notion of arc length was never needed.

(xiii) The analytic cosine function may be similarly obtained from the geometric cosine function. Also, it may be directly obtained from the analytic sine function in the usual way.

That the two approaches give the same function may be proved making use of: the relationship between Sin and sin, the Pythagorean theorem, corollary (E) [see above], the properties of proportions and ordered fields, and analytic and geometric square root extraction.

(xiv) The other trigonometric functions may be similarly dealt with.

Based on the above results, the proofs need only basic properties of proportions and fields.

Distinguishing between the two functions Sin and sin, and defining them as above, answer Hardy's question raised in the title of this article in a conceptually sound and logically correct manner.

III. CONCLUDING REMARKS

Retracing Elementary Mathematics [Henkin et al. 1962] is the outcome of three Summer Institutes for Teachers of Mathematics, organized by the National

Science Foundation, and taught and directed by the authors [Henkin et al. [1962] p. vii]. It is supposed to initiate a new beginning, and is proposed to be "Used as a college text [...] for a one-year course of the type commonly given under a title such as "Foundations of Mathematics," or "*Fundamental Concepts of Mathematics*." [emphasis added]" [Henkin et al. [1962] p. viii]. Moreover, the *detailed* and *explicit* treatment of the material makes the book "accessible to almost all undergraduate students of mathematics, and even to the attentive layman with an interest in this subject." [Henkin et al. [1962] p. viii].

Nevertheless the book deals principally with Arithmetization of $\mathbb{R}$ [cf. the introduction above] from "the modern deductive point of view" [Henkin et al. [1962] p. viii] considering the Fundamental Concepts of Elementary Mathematics such as the length of a sls and the measure of an angle already known, with no word concerning retracing them [cf. Henkin et al. [1962] pp. 316, 328, 332].

This is compatible with the current practice in mathematics. It is well known that treatises on the structure of $\mathbb{R}$ go along the lines of Henkin et al [1962] with varying degrees of formality; treatments of Euclidean geometry are based on $\mathbb{R}$ or on Hilbert [1950]'s theory of proportion, or variations thereof [cf. Borceux [2014, pp. vii-ix, 305, 351,352], Hartshorne [2000, pp. 2,3,168], Meyer [2006, pp. 23,29], Sossinsky [2012, pp. 1,2], Szmielew [1983, pp. 72-84, 109, 180, 181], Tarski and Givant [1999, p. 211] and Venema [2006, pp. ix, 398-410]]; dealing with measure (even of slss) is well known to presuppose $\mathbb{R}$; ... . The result is that-to the best of my knowledge-current mathematics almost entirely ignores Eudoxs' theory of proportion, the few exceptions which deal with it treat it as if it were merely a piece of antique. This being so, though this theory is the origin of $\mathbb{R}$, hence of modern mathematics; moreover it is indispensable for: (i) developing the theory of direct measure, (ii) comprehending $\mathbb{R}$, not just making use of it, (iii) understanding the reletionship of mathematics to reality and to the other sciences, and (iv) asking the correct questions to understand the evolution of mathematics through history.

The issue is not only confined to mathematics, but it extends also to *science* in general [cf., I.5, II.1 above]. Consider mechanics, for example. Let "g", "m" and "x" denote, respectively, one gram, a mass and a real number. In the equation:

$$m = xg$$

what is x if it is not the real number corresponding to the Eudoxean ratio $< m, g > / =_E$? [cf. II.2 above]. Notice that the equation involves no multiplications, g is not a real number.

Indeed, theoretical science is essentially the search for relevant relationships among Eudoxean ratios.

When will Eudoxus' work be given all the attention and glorification due a unique event in the life of humanity?

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APPENDIX

The gap in each of the proofs of proposition XII.2 of Euclid [1956, vol. 3, pp. 371-8; cf., II.4 above] attributed to Eudoxus and Legendre, is the unjustified assumption of the existence of the fourth proportional. This error is recently reiterated in the proofs of a version of this proposition [Borceux [2014] pp. 35-7, 100-2]. Concerning this existence, the arguments attributed to De Morgan [see Euclid [1956] vol. 2, p. 171] and Simson [see Euclid [1956] vol. 3, p. 375] do not solve the problem. Nor would proposition VI.12 of Euclid [1956] vol. 2, p. 215] help, for it proves the existence of the fourth proportional only for straight-line segments.

Following is a proof of proposition XII.2 of Euclid which does not make use of this problematic assumption. It is so simple that it may replace the proof given by Euclid in his book.

Let c_1 and c_2 be circles, and let s_1 and s_2 be the squares on their respective diameters.

To prove that $< c_1, c_2 >=_E< s_1, s_2 >$, assume the contrary. Then by definition V.5 of Euclid [1956, vol. 2, p. 114], there are positive integers n_1 and n_2 such that:

(i) $n_1c_2 > n_2c_1$ and $n_1s_2 \leq n_2s_1$,

(ii) $n_2c_1 > n_1c_2$ and $n_2s_1 \leq n_1s_2$,

(iii) $n_1c_2 = n_2c_1$ and $n_1s_2 < n_2s_1$,

or

(iv) $n_2c_1 = n_1c_2$ and $n_2s_1 < n_1s_2$.

Since (ii) and (iv) may, respectively, be obtained from (i) and (iii) by interchanging the indices 1 and 2, it suffices to deal with (i) and (iii).

Let p_1 and p_2 be similar polygons inscribed in c_1 and c_2 respectively. By proposition XII.1 of Euclid [1956, vol. 3, p. 369]:

$$< p_1, p_2 >=_E< s_1, s_2 > \tag{*}$$

Assume (i), then by (*) and definition V.5 of Euclid,

$$n_1p_2 \leq n_2p_1 < n_2c_1 < n_1c_2$$

hence,

$$(n_1c_2 \dot{-} n_2c_1) < (n_1c_2 \dot{-} n_1p_2),$$

which entails a contradiction. For, in view of exhaustion, the right-hand side may be made less than the left-hand side, by choosing an appropriate p_2.

To deal with the other case, let p'_1 and p'_2 be similar polygons inscribed in c_1 and c_2 respectively. As above,

$$< p'_1, p'_2 >=_E< s_1, s_2 >$$

so by (*)

$$< p'_1, p'_2 >=_E< p_1, p_2 >$$

Moreover, let p'_1, p'_2 properly include p_1, p_2 respectively, then by propositions V.16, 17 of Euclid [1956, vol. 2, pp. 164-6],

$$< (p'_1 \dot{-} p_1), (p'_2 \dot{-} p_2) >=_E< p_1, p_2 >$$

Finally, assume (iii), then,

$$n_2 p_1 \dot{-} n_1 p_2 < n_2 p'_1 \dot{-} n_1 p'_2 < n_1 c_2 \dot{-} n_1 p'_2,$$

which entails a contradiction.

This completes the proof.

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