Journal of Applied Non-Classical Logics
Vol. 00, No. 00, Month 201X, 1–40

RESEARCH ARTICLE

Logical limits of abstract argumentation frameworks

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(*v1.0 released September 2012*)

Dung's *argumentation framework* takes as input two abstract entities: a set of *arguments* and a *binary relation* encoding attacks between these arguments. It returns acceptable sets of arguments, called *extensions*, wrt a given *semantics*. While the abstract nature of this setting is seen as a great advantage, it induces a big gap with the application that it is used to. This raises some questions about the compatibility of the setting with a logical formalism (i.e., whether it is possible to instantiate it properly from a *logical* knowledge base), and about the significance of the various semantics in the application context.

In this paper we tackle the above questions. We first propose to fill in the previous gap by extending Dung's framework. The idea is to consider all the ingredients involved in an argumentation process. We start with the notion of an abstract monotonic logic which consists of a language (defining the formulas) and a consequence operator. We show how to build, in a systematic way, arguments from a knowledge base formalized in such a logic. We then recall some basic postulates that any instantiation should satisfy. We study how to choose an attack relation so that the instantiation satisfies the postulates. We show that symmetric attack relations are generally not suitable. However, we identify at least one 'appropriate' attack relation. Next, we investigate under stable, semi-stable, preferred, grounded and ideal semantics the outputs of logic-based instantiations that satisfy the postulates. For each semantics, we delimit the number of extensions an argumentation system may have, characterize the extensions in terms of subsets of the knowledge base, and finally characterize the set of conclusions that are drawn from the knowledge base. The study reveals that stable, semi-stable and preferred semantics either lead to counter-intuitive results or provide no added value w.r.t. naive semantics. Besides, naive semantics either leads to arbitrary results or generalizes the coherence-based approach initially developed by Rescher and Manor in 1970. Ideal and grounded semantics either coincide and generalize the free consequence relation developed by Benferhat, Dubois and Prade in 1997, or return arbitrary results. Consequently, Dung's framework seems problematic when applied over deductive logical formalisms.

Keywords: Abstract argumentation frameworks, Logic, Postulates.

1. Introduction

Argumentation has become an Artificial Intelligence keyword for the last twenty years, especially for handling inconsistency in knowledge bases (e.g., (6; 11; 31)), making decisions (e.g., (8; 14)), modeling different types of dialogues between agents like persuasion (e.g., (7; 34)), negotiation (e.g., (29; 32)) and inquiry (e.g., (12; 26)), and for learning concepts (e.g., (9; 25)).

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One of the most abstract argumentation formalisms in existing literature was proposed by Dung in (21). It consists of a set of *arguments* and a *binary relation* encoding *attacks* between these arguments. *Semantics* are used for defining acceptable sets of arguments, called *extensions*. Since its original formulation, Dung's framework has become very popular because it seriously abstracts away from the application to which it can be used. Indeed, the structure and the origin of arguments and attacks are left unspecified. While this can be seen as a great advantage of the framework, two important questions are raised regarding its interplay with logic, namely when it is applied for reasoning about *inconsistent information*:

(1) Is the framework *compatible* with a logical formalism? To put it differently, is it possible to instantiate "properly" the framework from a *logical* knowledge base?
(2) Are the semantics *significant* when the framework is instantiated from a logical knowledge base? Are they really different? What are the counterparts of the extensions (under each semantics) in the knowledge base? Are those counterparts meaningful? What are the plausible inferences under these semantics?

In this paper we answer the above questions. For that purpose, we first propose to fill in the gap between the abstract framework and the logical knowledge base from which it is specified. The idea is to consider all the ingredients involved in an argumentation problem. We start with the notion of an *abstract monotonic logic* as defined by Tarski in (33). According to Tarski, a monotonic logic is a set of *formulas* and a *consequence operator* that satisfies some axioms. It is worth mentioning that almost *all* well-known monotonic logics such as propositional logic, modal logic, first order logic, fuzzy logic and probabilistic logic are special cases of Tarski's notion of abstract logic. Consequently, any result that holds in the general case of a Tarskian logic obviously holds under all these particular logics. We then show how to build, in a systematic way, arguments from a knowledge base formalized in such a logic.

A 'good' instantiation of Dung's framework is one that satisfies some basic *rationality postulates*. In (17), three postulates were proposed, and it was shown that not any instantiation is acceptable since some may lead to counter-intuitive results. Examples are the instantiations proposed in (24; 28). The three postulates are tailored for rule-based formalisms, i.e., logical languages that distinguish between strict rules and defeasible ones. In (1; 2), those postulates were generalized to any Tarskian logic. Moreover, three new and intuitive ones were proposed. We study under which conditions they are satisfied or violated. The satisfaction/violation of a postulate by an instantiation depends mainly on the properties of the attack relation. We show that this relation should be based on the inconsistency of the knowledge base. We show also that symmetric relations cannot be adopted when the knowledge base contains a ternary or n-ary ($n > 2$) minimal conflict. We do establish the existence of appropriate attack relations. To sum up, there are interesting cases that cannot be captured by Dung's framework. Nevertheless, the framework can still be properly instantiated.

In the second part of the paper, we investigate the *underpinnings* of the main semantics: *stable*, *semi-stable*, *preferred*, *grounded* and *ideal*. For that purpose, we consider only the logic-based instantiations that satisfy the postulates since the remaining ones are not good. For each semantics, we delimit the number of extensions an instantiation may have, characterize its extensions in terms of subsets of the knowledge base over which the instantiation is built, and fully characterize its set of plausible inferences that may be drawn from the knowledge base. The results show that unlike the abstract framework, its good instantiations have a finite number of extensions under stable, semi-stable and preferred semantics. This is particularly the case when the knowledge base is finite. We show that the set of all formulas used in the supports of the arguments of a stable extension is a maximal (for set inclusion) consistent subset of the knowledge base. However, not every maximal consistent subset of the knowledge base has necessarily a corresponding stable extension. This leads to arbitrary plausible inferences. In case there is a full correspondence, stable semantics does not play any role since the same result is already ensured by the naive semantics. This means that stable semantics either returns arbitrary results or provide no added value. Besides, naive semantics either leads to arbitrary results or generalizes the coherence-based approach initially developed by Rescher and Manor in (30). The situation is worse for preferred semantics. The set of formulas that are used in the supports of the arguments of a preferred extension is a consistent (but not necessarily maximal for set inclusion) subset of the knowledge base. Thus, arbitrary inferences may be drawn from the knowledge base. Semi-stable extensions are shown to always coincide with stable ones. Thus, semi-stable semantics has no added value w.r.t. stable semantics. Regarding the ideal and grounded semantics; there are two cases as well: i) both semantics coincide, i.e., the ideal extension of any argumentation framework satisfying the postulates coincides with the grounded extension which itself coincides with the set of arguments that are built from the free part of a knowledge base, i.e., using the subset of formulas which are not involved in the inconsistency of the knowledge base. Consequently, under these semantics, the set of plausible conclusions is the so-called *free consequences* in the coherence-based approach for reasoning about inconsistent information (10). ii) both semantics return arbitrary conclusions.

The overall study reveals that Dung's framework can be properly instantiated, i.e., there are instantiations that satisfy some basic rationality postulates. However, stable, semi-stable, preferred, ideal and grounded semantics are not suitable. Thus, Dung's framework is problematic when applied over a logical formalism, specifically a deductive one.

The paper is structured as follows: Section 2 recalls the abstract argumentation framework of Dung. Section 3 details our instantiation of Dung's framework. Section 4 defines rationality postulates that such instantiation should satisfy, and investigates when those postulates are satisfied/violated. Section 5 analyzes the different acceptability semantics introduced by Dung. Section 6 compares our contribution with existing works. Finally, Section 7 concludes the paper with some remarks and perspectives.

2. Dung's abstract argumentation framework

In (21), an argumentation framework consists of a set of arguments and a binary relation expressing attacks among the arguments.

Definition 1 (Argumentation framework) *An* argumentation framework *is a pair* $(\mathcal{A}, \mathcal{R})$ *where* $\mathcal{A}$ *is a set of arguments and* $\mathcal{R} \subseteq \mathcal{A} \times \mathcal{A}$ *is an attack relation.*

A pair $(a, b) \in \mathcal{R}$ *means that* a *attacks* b*. A set* $\mathcal{E} \subseteq \mathcal{A}$ *attacks an argument* b *iff* $\exists a \in \mathcal{E}$ *such that* $(a, b) \in \mathcal{R}$*.*

Notation: We sometimes use the infix notation $a\mathcal{R}b$ to denote $(a, b) \in \mathcal{R}$.

An argumentation framework $(\mathcal{A}, \mathcal{R})$ is a *graph* whose nodes are the arguments of $\mathcal{A}$ and its edges are the attacks in $\mathcal{R}$. The arguments are evaluated using a semantics. In (21), different acceptability semantics were proposed. Some of them were refined, for instance in (16; 20). The basic idea behind them is the following: for a rational agent, an argument is acceptable if he can defend this argument against all attacks upon it. All the arguments jointly acceptable for a rational agent will be gathered in a so-called extension. An extension must satisfy a consistency requirement and must defend all its elements.

Definition 2 (Conflict-freeness, Defence) *Let* $(\mathcal{A}, \mathcal{R})$ *be an argumentation framework and* $\mathcal{E} \subseteq \mathcal{A}$*.*

- $\mathcal{E}$ *is* conflict-free *iff* $\nexists a, b \in \mathcal{E}$ *such that* $(a, b) \in \mathcal{R}$*.*
- $\mathcal{E}$ defends *an argument* a *iff* $\forall b \in \mathcal{A}$*, if* $(b, a) \in \mathcal{R}$*, then* $\exists c \in \mathcal{E}$ *such that* $(c, b) \in \mathcal{R}$*.*

The following definition recalls the main semantics that were proposed by Dung (21) as well as their refinements (16; 20). It is worth noticing that the fundamental semantics features admissible extensions. The other semantics are based on it.

Definition 3 (Acceptability semantics) *Let* $\mathcal{T} = (\mathcal{A}, \mathcal{R})$ *be an argumentation framework, and* $\mathcal{E} \subseteq \mathcal{A}$ *be a conflict-free set.*

- $\mathcal{E}$ *is a* naive *extension iff it is a maximal (w.r.t. set* $\subseteq$*) conflict-free set.*
- $\mathcal{E}$ *is an* admissible *set iff it defends all its elements.*
- $\mathcal{E}$ *is a* complete *extension iff it is an admissible set that contains any argument it defends.*
- $\mathcal{E}$ *is a* preferred *extension iff it is a maximal (w.r.t. set* $\subseteq$*) admissible set.*
- $\mathcal{E}$ *is a* stable *extension iff it attacks any argument in* $\mathcal{A} \setminus \mathcal{E}$*.*
- $\mathcal{E}$ *is a* semi-stable extension *iff it is a complete extension and the union of the set* $\mathcal{E}$ *and the set of all arguments attacked by* $\mathcal{E}$ *is maximal (w.r.t.* $\subseteq$*).*
- $\mathcal{E}$ *is a* grounded extension *iff it is a minimal (w.r.t. set* $\subseteq$*) complete extension.*
- $\mathcal{E}$ *is an* ideal extension *iff it is a maximal (w.r.t. set* $\subseteq$*) admissible set contained in every preferred extension.*

Notations: $\text{Ext}_x(\mathcal{T})$ denotes the set of all extensions of $\mathcal{T}$ under semantics x where $x \in \{n, p, s, ss\}$ and n (respectively p, s, ss) stands for naive (respectively preferred, stable and semi-stable). When we do not need to refer to a particular semantics, we write $\text{Ext}(\mathcal{T})$ for short. Since grounded and ideal extensions are unique for any argumentation framework $\mathcal{T}$, they will be denoted respectively by $\text{GE}(\mathcal{T})$ and $\text{IE}(\mathcal{T})$.

It is worth recalling that stable extensions are naive (respectively preferred) extensions but the converses are not always true. Moreover, an argumentation framework has *at least* one preferred extension while it may have no stable extensions. When stable extensions exist, they coincide with the semi-stable ones (i.e., if $|\text{Ext}_s(\mathcal{T})| > 0$, then $\text{Ext}_s(\mathcal{T}) = \text{Ext}_{ss}(\mathcal{T})$ for any argumentation framework $\mathcal{T}$).

Example 1 *Let us consider the argumentation framework* $\mathcal{T} = (\mathcal{A}, \mathcal{R})$ *such that:*

- $\mathcal{A} = \{a, b, c, d, e, f, g\}$
- $\mathcal{R} = \{(c, b), (b, e), (e, c), (d, c), (a, d), (d, a), (a, f), (f, g)\}$

This framework has five naive extensions:

- $\mathcal{E}_1 = \{a, c, g\}$,
- $\mathcal{E}_2 = \{d, e, f\}$,
- $\mathcal{E}_3 = \{b, d, f\}$,
- $\mathcal{E}_4 = \{a, e, g\}$, *and*
- $\mathcal{E}_5 = \{a, b, g\}$.

It has one stable/semi-stable extension $\mathcal{E}_3$ *and two preferred extensions:* $\mathcal{E}_3$ *and* $\mathcal{E}_6 = \{a, g\}$. *Both the grounded and the ideal extensions are empty, i.e.,* $\mathrm{GE}(\mathcal{T}) = \mathrm{IE}(\mathcal{T}) = \emptyset$.

An argumentation framework may be *infinite*, i.e., its set of arguments may be infinite. Consequently, it may have an infinite number of extensions (under a given semantics).

3. Logic-based instantiations of Dung's framework

Argumentation is an alternative approach for reasoning with inconsistent information. It follows three main steps: i) constructing *arguments* and counterarguments from a logical knowledge base, ii) defining the *status* of each argument, and iii) specifying the *conclusions* to be drawn from the base. In what follows, we instantiate Dung's framework by defining all these items. We start with an *abstract logic* as defined by Alfred Tarski (33) from which the notions of argument and attacks between arguments are defined.

3.1 *Tarski's abstract consequence operators*

Alfred Tarski (33) defines a logic as a pair $(\mathcal{L}, \mathsf{CN})$ where the members of $\mathcal{L}$ are called *well-formed formulas*, and CN is a *consequence operator*. No constraints are defined on the logical language $\mathcal{L}$. Thus, no particular connectors are required. However, the consequence operator CN is a function from $2^{\mathcal{L}}$ to $2^{\mathcal{L}}$ that should satisfy the following axioms:

(1) $X \subseteq \mathsf{CN}(X)$ (Expansion)
(2) $\mathsf{CN}(\mathsf{CN}(X)) = \mathsf{CN}(X)$ (Idempotence)
(3) $\mathsf{CN}(X) = \bigcup_{Y \subseteq_f X} \mathsf{CN}(Y)$ (Compactness)
(4) $\mathsf{CN}(\{x\}) = \mathcal{L}$ for some $x \in \mathcal{L}$ (Absurdity)
(5) $\mathsf{CN}(\emptyset) \neq \mathcal{L}$ (Coherence)

Notation: $Y \subseteq_f X$ means that Y is a finite subset of X.

Intuitively, $\mathsf{CN}(X)$ returns the set of formulas that are logical consequences of X according to the logic in question. In (33), it was shown that CN is a closure operator, that is, CN enjoys properties such as:

Property 1 *Let* $X, X', X'' \subseteq \mathcal{L}$.

(1) $X \subseteq X' \Rightarrow \mathsf{CN}(X) \subseteq \mathsf{CN}(X')$.
(2) $\mathsf{CN}(X) \cup \mathsf{CN}(X') \subseteq \mathsf{CN}(X \cup X')$.
(3) $\mathsf{CN}(X) = \mathsf{CN}(X') \Rightarrow \mathsf{CN}(X \cup X'') = \mathsf{CN}(X' \cup X'')$.
(4) $\mathsf{CN}(X \cap X') \subseteq \mathsf{CN}(X) \cap \mathsf{CN}(X')$.

Almost all well-known monotonic logics (classical logics, intuitionistic logics, modal logics, etc.) can be viewed as special cases of Tarski's notion of an abstract logic. AI introduced non-monotonic logics, which do not satisfy monotonicity (13).

Once $(\mathcal{L}, \mathsf{CN})$ is fixed, a notion of *consistency* arises as follows:

Definition 4 (Consistency) *Let $X \subseteq \mathcal{L}$. X is* consistent *w.r.t. the logic $(\mathcal{L}, \mathsf{CN})$ iff $\mathsf{CN}(X) \neq \mathcal{L}$. It is* inconsistent *otherwise.*

In simple English, this says that X is consistent iff its set of consequences is not the set of all formulas. The coherence requirement (absent from Tarski's original proposal but added here to avoid considering trivial systems) forces the empty set $\emptyset$ to always be consistent - this makes sense for any reasonable logic.

One can show that if a set X is consistent, then its closure under CN is also consistent and any proper subset of X is consistent.

Property 2 *Let $X \subseteq \mathcal{L}$.*

(1) If X is consistent, then $\mathsf{CN}(X)$ is consistent as well.
(2) $\forall X' \subseteq X$, if X is consistent, then X' is consistent.
(3) $\forall X' \subseteq X$, if X' is inconsistent, then X is inconsistent.

If a set $X \subseteq \mathcal{L}$ of formulas is inconsistent, this means that it contains *minimal conflicts.*

Definition 5 (Minimal conflict) *A set $C \subseteq \mathcal{L}$ is a* minimal conflict *iff:*

- *C is inconsistent*
- *$\forall x \in C$, $C \setminus \{x\}$ is consistent*

Notations: Let $X \subseteq \mathcal{L}$. $\mathcal{C}_X$ denotes the set of all minimal conflicts C such that $C \subseteq X$. $\mathrm{Max}(X)$ is the set of all maximal (for set inclusion) consistent subsets of X, $\mathrm{Free}(X) = \bigcap_{\mathcal{S}_i \in \mathrm{Max}(X)} \mathcal{S}_i$, and $\mathrm{Inc}(X) = X \setminus \mathrm{Free}(X)$.

The following properties are useful for proving our results.

Property 3 *For all $X \subseteq \Sigma \subseteq \mathcal{L}$,*

- *if X is consistent then $\mathcal{C}_X = \emptyset$.*
- *if X is consistent then $X \subseteq \mathcal{S}$ for some $\mathcal{S} \in \mathrm{Max}(\Sigma)$.*
- *if X is inconsistent then there exists at least one minimal conflict C such that $C \subseteq X$.*

The next property is true in case the underlying logic is *adjunctive*. Let us first define this new concept.

Definition 6 (Adjunctiveness) *A logic $(\mathcal{L}, \mathsf{CN})$ is* adjunctive *iff for all x and y in $\mathcal{L}$, there exists $z \in \mathcal{L}$ such that $\mathsf{CN}(\{z\}) = \mathsf{CN}(\{x, y\})$.*

Intuitively, an adjunctive logic infers, from the union of two formulas $\{x, y\}$, some formula(s) that can be inferred neither from x alone nor from y alone (except, of course, when y ensues from x or vice-versa). In fact, most well-known logics are adjunctive.[1] A logic which is not adjunctive could for instance fail to deny $x \vee y$ from the premises $\{\neg x, \neg y\}$.

Property 4 *Let $(\mathcal{L}, \mathsf{CN})$ be* adjunctive, *$C \subseteq \mathcal{L}$ be a minimal conflict. For all $X \subset C$, if $X \neq \emptyset$, then:*

[1]Some fragments of well-known logics fail to be adjunctive, e.g., the pure implicational fragment of classical logic as it is negationless, disjunctionless, and, of course, conjunctionless.

(1) $\exists x \in \mathcal{L}$ *such that* $\mathsf{CN}(\{x\}) = \mathsf{CN}(X)$.
(2) $\exists x_1 \in \mathsf{CN}(X)$ *and* $\exists x_2 \in \mathsf{CN}(C \setminus X)$ *such that the set* $\{x_1, x_2\}$ *is inconsistent.*

3.2 *Tarskian logic-based instantiations*

Let $(\mathcal{L}, \mathsf{CN})$ be a fixed abstract logic. From now on, we will consider a *knowledge base* Σ which is a subset of the logical language $\mathcal{L}$ (in symbols, $\Sigma \subseteq \mathcal{L}$). This base may be infinite, however, with no loss of generality and for the sake of simplicity, it is assumed to be free of tautologies:

Assumption 1 *Let* Σ *be a knowledge base. For all* $x \in \Sigma$, $x \notin \mathsf{CN}(\emptyset)$.

The first parameter of an argumentation framework is the set of arguments. In Dung's framework, an argument is an abstract entity. In what follows, it is built from a knowledge base Σ. It gives a reason for believing a conclusion. Formally, an argument satisfies three main requirements: i) the reason is a subset of the knowledge base, thus restricting the origin of the arguments, ii) the reason should be consistent, thus avoiding absurd reasons, and iii) the reason is minimal. The latter means that only relevant information w.r.t. the conclusion are considered.

Definition 7 (Argument) *Let* Σ *be a knowledge base. An* argument *is a pair* (X, x) *such that:*

(1) $X \subseteq \Sigma$ *and* $x \in \mathcal{L}$
(2) X *is consistent*
(3) $x \in \mathsf{CN}(X)$
(4) $\nexists X' \subset X$ *such that* $x \in \mathsf{CN}(X')$

An argument (X, x) *is a* sub-argument *of another argument* (X', x') *iff* $X \subseteq X'$.

Let us introduce some notations that will be used throughout the paper.

Notations: Supp and Conc are two functions that return respectively the *support* X and the *conclusion* x of an argument (X, x). Sub is a function that returns all the sub-arguments of a given argument. For $X \subseteq \mathcal{L}$, $\mathrm{Arg}(X)$ denotes the set of all arguments that can be built from X by means of Definition 7. For a set $\mathcal{E}$ of arguments, $\mathrm{Concs}(\mathcal{E}) = \{\mathrm{Conc}(a) \mid a \in \mathcal{E}\}$ and $\mathrm{Base}(\mathcal{E}) = \bigcup_{a \in \mathcal{E}} \mathrm{Supp}(a)$.

The following property shows that the conclusion of any argument is consistent.

Property 5 *For all* $(X, x) \in \mathrm{Arg}(\Sigma)$*, the set* $\{x\}$ *is consistent.*

Due to Assumption 1 ($x \notin \mathsf{CN}(\emptyset)$ for all $x \in \Sigma$), it can also be shown that each consistent formula in Σ gives birth to an argument:

Property 6 *Let* Σ *be a knowledge base such that for all* $x \in \Sigma$, $x \notin \mathsf{CN}(\emptyset)$*. For all* $x \in \Sigma$ *such that the set* $\{x\}$ *is consistent,* $(\{x\}, x) \in \mathrm{Arg}(\Sigma)$.

Since CN is monotonic, constructing arguments is a monotonic process: Additional knowledge never makes the set of arguments to shrink but only gives rise to extra arguments that may interact with the existing ones.

Property 7 $\mathrm{Arg}(\Sigma) \subseteq \mathrm{Arg}(\Sigma')$ *whenever* $\Sigma \subseteq \Sigma' \subseteq \mathcal{L}$.

We show now that any proper subset of a minimal conflict is the support of at least one argument. This is in particular true in case of adjunctive logics. This result is of utmost importance as regards encoding the attack relation.

Proposition 1 *Let* $(\mathcal{L}, \mathsf{CN})$ *be adjunctive and* Σ *be a knowledge base. For all non-empty proper subset* X *of some minimal conflict* $C \in \mathcal{C}_\Sigma$*, there exists* $a \in \mathrm{Arg}(\Sigma)$ *such that* $\mathrm{Supp}(a) = X$*.*

Proposition 1 is fundamental because it says that if statements from Σ contradict others then it is always possible to define an argument exhibiting the conflict.

In the sequel, we use the term *system* instead of framework in order to distinguish the framework of Dung from its logical instantiations which are defined as follows.

Definition 8 (Argumentation system) *Let* $(\mathcal{L}, \mathsf{CN})$ *be a given Tarskian logic and* $\Sigma \subseteq \mathcal{L}$ *be a knowledge base. An* argumentation system *over* Σ *is a pair* $\mathcal{T} = (\mathrm{Arg}(\Sigma), \mathcal{R})$ *such that* $\mathcal{R} \subseteq \mathrm{Arg}(\Sigma) \times \mathrm{Arg}(\Sigma)$ *(intuitively, it is an* attack *relation).*

In the previous definition, the attack relation is left unspecified. However, in Section 4 we show that it should enjoy some properties otherwise the system may return counter-intuitive results.

The arguments of a system $\mathcal{T} = (\mathrm{Arg}(\Sigma), \mathcal{R})$ are evaluated using one of the semantics given in Definition 3. Recall that the structure of arguments is not taken into account in those semantics. The extensions are used in order to define the conclusions that may be drawn from Σ according to the system $\mathcal{T}$. The idea is to conclude x if it is the conclusion of an argument in every extension of the system.

Definition 9 (Output) *Let* $\mathcal{T} = (\mathrm{Arg}(\Sigma), \mathcal{R})$ *be an argumentation system over a knowledge base* Σ *and* $\mathrm{Ext}(\mathcal{T})$ *its set of extensions under a given semantics. For* $x \in \mathcal{L}$*,* x *is a* conclusion *of* $\mathcal{T}$ *iff* $\forall \mathcal{E}_i \in \mathrm{Ext}(\mathcal{T})$*,* $\exists a \in \mathcal{E}_i$ *such that* $\mathrm{Conc}(a) = x$*. We write* $\mathrm{Output}(\mathcal{T})$ *to denote the set of all conclusions of* $\mathcal{T}$*.*

It follows immediately from the definition that the set of conclusions exactly consists of the formulas which happen to be, in each extension, the conclusion of an argument of the extension.

Property 8 *Let* $\mathcal{T} = (\mathrm{Arg}(\Sigma), \mathcal{R})$ *be an argumentation system over a knowledge base* Σ *and* $\mathrm{Ext}(\mathcal{T})$ *its set of extensions under a given semantics. It holds that*

$$\mathrm{Output}(\mathcal{T}) = \bigcap_{\mathcal{E}_i \in \mathrm{Ext}(\mathcal{T})} \mathrm{Concs}(\mathcal{E}_i).$$

Finally, it is obvious that the outputs of an argumentation system are consequences of the corresponding knowledge base under the consequence operator CN.

Property 9 *Let* $\mathcal{T} = (\mathrm{Arg}(\Sigma), \mathcal{R})$ *be an argumentation system over a knowledge base* Σ*.* $\mathrm{Output}(\mathcal{T}) \subseteq \mathsf{CN}(\Sigma)$*.*

4. On the quality of logic-based argumentation systems

We have shown so far how to build an argumentation system from a logical knowledge base. The system is still incomplete since the attack relation is not specified. As already mentioned, Dung is silent on how to proceed in order to obtain a reasonable $\mathcal{R}$ in practice. It happens that it is in fact a delicate step. We will show next that the choice of this relation is crucial for the "soundness" of the system.

Soundness is determined by some *rationality postulates* that any system should satisfy.

The first work on rationality postulates in argumentation was done by Caminada and Amgoud (17). The authors focused *only* on *rule-based systems* (i.e., systems that distinguish between strict and defeasible rules in their underlying logical language). They proposed the following postulates that such systems should satisfy:

- *Closure:* The idea is that if a system concludes x and there is a strict rule[1] $x \to y$, then the system should also conclude y.
- *Direct consistency:* the set of conclusions of arguments of each extension should be consistent.
- *Indirect consistency:* the closure of the set of conclusions of arguments of each extension should be consistent.

As obvious as they may appear, these postulates are violated by most rule-based systems (like (5; 24; 27; 28)). Besides, they are tailored for rule-based logics. Their counterparts for any Tarskian logic were defined in (1; 2). Moreover, three new postulates were proposed in (1). In what follows, we recall all the postulates that are necessary for our study.

4.1 *Rationality postulates for logic-based argumentation systems*

The first rationality postulate that an argumentation system should satisfy concerns the closure of its output. The basic idea is that the conclusions of a formalism should be "complete". There should be no case that a user performs on her own some extra reasoning to derive statements that the formalism apparently "forgot" to entail. In (17), closure is defined for rule-based argumentation systems. In what follows, we extend this postulate to systems that are grounded on any Tarskian logic. The idea is to define closure using the consequence operator CN.

Postulate 1 (Closure under CN) *Let $\mathcal{T} = (\mathrm{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ. For all $\mathcal{E} \in \mathrm{Ext}(\mathcal{T})$, $\mathrm{Concs}(\mathcal{E}) = \mathsf{CN}(\mathrm{Concs}(\mathcal{E}))$. We say that $\mathcal{T}$ is closed under CN.*

In (17), closure is imposed both on the extensions of a system and on its output set. The next result shows that the closure of the output set does not deserve to be a separate postulate since it follows immediately from the closure of extensions.

Proposition 2 *Let $\mathcal{T} = (\mathrm{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ. If $\mathcal{T}$ is closed under CN, then $\mathrm{Output}(\mathcal{T}) = \mathsf{CN}(\mathrm{Output}(\mathcal{T}))$.*

The second rationality postulate concerns *sub-arguments*. An argument may have one or several sub-arguments, reflecting the different premises on which it is based. Thus, the acceptance of an argument should imply also the acceptance of all its sub-parts.

Postulate 2 (Closure under sub-arguments) *Let $\mathcal{T} = (\mathrm{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ. For all $\mathcal{E} \in \mathrm{Ext}(\mathcal{T})$, if $a \in \mathcal{E}$, then $\mathrm{Sub}(a) \subseteq \mathcal{E}$. We say that $\mathcal{T}$ is closed under sub-arguments.*

These two postulates have a great impact on the extensions of an argumentation system as shown by the following result.

[1] A strict rule $x \to y$ means that if x holds, then y holds with no exception whatsoever.

Proposition 3 *Let $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base. If $\mathcal{T}$ is closed under sub-arguments and under* CN*, then for all $\mathcal{E} \in \text{Ext}(\mathcal{T})$, $\text{Concs}(\mathcal{E}) = \mathsf{CN}(\text{Base}(\mathcal{E}))$.*

The third rationality postulate concerns the *consistency* of the results. It ensures that the set of conclusions supported by each extension is consistent. The following postulate generalizes the *direct* consistency postulate which was proposed for rule-based argumentation systems in (17).

Postulate 3 (Consistency) *Let $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ. For all $\mathcal{E} \in \text{Ext}(\mathcal{T})$, $\text{Concs}(\mathcal{E})$ is consistent. We say that $\mathcal{T}$ satisfies consistency.*

As for closure, in (17) a postulate imposing the consistency of the output is defined. We show next that such a postulate is not necessary. Indeed, an argumentation system that satisfies Postulate 3 has necessarily a consistent output.

Proposition 4 *If an argumentation system $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ satisfies consistency, then the set $\text{Output}(\mathcal{T})$ is consistent.*

We show next that argumentation systems that satisfy both consistency and closure under sub-arguments enjoy a strong version of consistency. Indeed, the set of formulae used in arguments of each extension is consistent. It is worth mentioning that this result is *very general* as it holds under any semantics, any attack relation and any Tarskian logic.

Proposition 5 *Let $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that for all $x \in \Sigma$, $x \notin \mathsf{CN}(\emptyset)$. If $\mathcal{T}$ satisfies consistency and is closed under sub-arguments, then for all $\mathcal{E} \in \text{Ext}(\mathcal{T})$, $\text{Base}(\mathcal{E})$ is consistent.*

Since the free formulas of a knowledge base (i.e., the ones that are not involved in any minimal conflict) are the "hard" part in the base, it is natural that any argument that is built only from this part should be in every extension of an argumentation system built over the knowledge base.

Postulate 4 (Free Precedence) *Let $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ. For all $\mathcal{E} \in \text{Ext}(\mathcal{T})$, $\text{Arg}(\text{Free}(\Sigma)) \subseteq \mathcal{E}$. We say that $\mathcal{T}$ satisfies* free precedence.

We show next that the free formulas are drawn by any argumentation system satisfying Postulate 4.

Proposition 6 *Let $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ. If $\mathcal{T}$ satisfies free precedence, then $\text{Free}(\Sigma) \subseteq \text{Output}(\mathcal{T})$ (under any of the reviewed semantics).*

The last postulate says that if the support and the conclusion of an argument are part of the conclusions of a given extension, then the argument should belong to the extension. Informally: If each step in the argument is good enough to be in the extension, then so is the argument itself.

Postulate 5 (Exhaustiveness) *An argumentation system $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ over a knowledge base Σ satisfies* exhaustiveness *iff for all $\mathcal{E} \in \text{Ext}(\mathcal{T})$, for all $(X, x) \in \text{Arg}(\Sigma)$, if $X \cup \{x\} \subseteq \text{Concs}(\mathcal{E})$, then $(X, x) \in \mathcal{E}$.*

The following result shows that when this postulate is satisfied, then extensions are closed in terms of arguments.

Proposition 7 *If an argumentation system $\mathcal{T}$ is closed under both* CN *and sub-arguments and satisfies the exhaustiveness postulate, then* $\forall \mathcal{E} \in \text{Ext}(\mathcal{T})$, $\mathcal{E} = \text{Arg}(\text{Base}(\mathcal{E}))$ *(under any of the reviewed semantics).*

The five postulates are generally independent. However, in case of naive and stable semantics, closure under the consequence operator CN is induced from closure under sub-arguments and consistency. This is in particular the case when the attack relation is based on inconsistency.

Definition 10 (Conflict-dependent) *An attack relation $\mathcal{R}$ is* conflict-dependent *iff for all* $a, b \in \text{Arg}(\Sigma)$, *if* $a\mathcal{R}b$ *then* $\text{Supp}(a) \cup \text{Supp}(b)$ *is inconsistent.*

The above definition says that $\mathcal{R}$ should show no attack from a to b unless Σ provides evidence (according to CN) that the supports of a and b conflict with each other. That is, being conflict-dependent ensures that, when passing from Σ to $(\text{Arg}(\Sigma), \mathcal{R})$, no conflict is "invented" in $\mathcal{R}$. Note that all the attack relations that are used in existing structured argumentation systems are conflict-dependent (see (23) for a summary of existing relations).

Proposition 8 *Let* $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ *be an argumentation system over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent. If $\mathcal{T}$ satisfies consistency and is closed under sub-arguments (under naive and stable semantics), then it is closed under* CN *(under naive and stable semantics).*

It was shown in (1) that the five postulates are *compatible*, i.e., they can be satisfied all together by an argumentation system. This is in particular witnessed by the argumentation system studied in (18). This system is grounded on propositional logic (a Tarskian logic) and uses the *assumption attack* relation defined in (22). According to this relation, an argument attacks another if its conclusion is the negation of an element of the support of the second argument. This relation was generalized to any Tarskian logic in (3) as follows:

Definition 11 (Assumption attack relation) *Let* $(\mathcal{L}, \mathsf{CN})$ *be a Tarskian logic. An argument* (X, x) *attacks another argument* (X', x') *iff* $\exists y \in X'$ *such that the set* $\{x, y\}$ *is inconsistent. This relation will be denoted by* $\mathcal{R}_{as}$.

It was shown in (1) that any argumentation system $(\text{Arg}(\Sigma), \mathcal{R}_{as})$ satisfies the five postulates. This is certainly a positive result as it shows that Dung's abstract framework can be correctly instantiated with logical formalisms. However, in the next section we show that there is a broad class of natural instantiations that are not possible since they violate the postulates.

4.2 *On the violation of consistency postulate*

In sub-section 3.2 we provided a clear definition of an argument and how it is built from a knowledge base Σ. However, there still is no indication on how the attack relation $\mathcal{R}$ is chosen and how it is related to Σ. Moreover, in (17) it was shown that there are some instantiations of Dung's framework that violate the consistency postulate. This means that the choice of the attack relation has a direct impact on the postulates. This means also that conflict-freeness is not sufficient to ensure consistency. Thus, an attack relation should enjoy some basic properties. The first one concerns its *origin*. We show that an attack relation should be based on inconsistency, thus conflict-dependent.

When the attack relation is conflict-dependent, then it is empty when the knowledge base is consistent.

Proposition 9 *Let* $(\text{Arg}(\Sigma), \mathcal{R})$ *be such that* $\mathcal{R}$ *is conflict-dependent. If* Σ *is consistent, then* $\mathcal{R} = \emptyset$.

It follows that when the attack relation is conflict-dependent, if a set of arguments is such that its corresponding base (set-theoretic union of supports) is consistent then it is a conflict-free set:

Proposition 10 *Let* $(\text{Arg}(\Sigma), \mathcal{R})$ *be such that* $\mathcal{R}$ *is conflict-dependent.* $\forall \mathcal{E} \subseteq \text{Arg}(\Sigma)$*, if* $\text{Base}(\mathcal{E})$ *is consistent, then* $\mathcal{E}$ *is conflict-free.*

It is also worth pointing out that an attack relation which is conflict-dependent exhibits no self-attacks.

Proposition 11 *Let* $(\text{Arg}(\Sigma), \mathcal{R})$ *be such that* $\mathcal{R}$ *is conflict-dependent. For all* $a \in \text{Arg}(\Sigma)$, $(a, a) \notin \mathcal{R}$.

Let us now consider the following example of an argumentation system that is built from a propositional knowledge base and that uses the symmetric attack relation known as *rebut* (22). According to this relation, *an argument a attacks another argument b iff* $\text{Conc}(a) \equiv \neg\text{Conc}(b)$ *(in case* $(\mathcal{L}, \mathsf{CN})$ *is propositional logic). This relation will be denoted by* $\mathcal{R}_{re}$.

Example 2 *Let* $(\mathcal{L}, \mathsf{CN})$ *be propositional logic and* $\Sigma = \{x, y, x \to \neg y\}$*. Let us consider the following set of arguments:*

- $a_1 = (\{x\}, x)$
- $a_2 = (\{y\}, y)$
- $a_3 = (\{x \to \neg y\}, x \to \neg y)$
- $a_4 = (\{x, x \to \neg y\}, \neg y)$
- $a_5 = (\{y, x \to \neg y\}, \neg x)$
- $a_6 = (\{x, y\}, x \wedge y)$

The rebut relation is as follows: $\{(a_1, a_5), (a_5, a_1), (a_2, a_4), (a_4, a_2), (a_3, a_6), (a_6, a_3)\}$*. The set* $\{a_1, a_2, a_3\}$ *(as a finite representation (4) for all its "mates", i.e., the arguments* $(\{x\}, \ldots)$ *and* $(\{y\}, \ldots)$ *and* $(\{x \to \neg y\}, \ldots)$*) is an admissible extension of the system* $(\text{Arg}(\Sigma), \mathcal{R}_{re})$*. However, the set* $\{\text{Conc}(a_1), \text{Conc}(a_2), \text{Conc}(a_3)\}$ *is inconsistent. Similarly, the set* $\{a_4, a_5, a_6\}$ *is (a finite representation of) another admissible extension whose set of conclusions is inconsistent.*

This example shows that an admissible set of arguments may fail to have a consistent set of conclusions. The problem encountered with the rebut relation is due to the fact that it is binary, in compliance with Dung's definitions imposing the attack relation to be binary. Thus, the ternary conflict between a_1, a_2 and a_3 is not captured. Particularly, symmetric attack relations are crippled by non-binary minimal conflicts. Indeed, we show that when the attack relation is symmetric, Postulate 3 is violated.

Proposition 12 *Let* $(\mathcal{L}, \mathsf{CN})$ *be adjunctive and* Σ *be a knowledge base such that* $\exists C \in \mathcal{C}_\Sigma$ *and* $|C| > 2$*. If* $\mathcal{R}$ *is conflict-dependent and symmetric, then the argumentation system* $(\text{Arg}(\Sigma), \mathcal{R})$ *violates consistency.*

This result shows a broad class of attack relations that cannot be used in argumentation: the symmetric ones. Relations like rebut or a union of rebut and any other conflict-dependent attack relation would lead to the violation of consistency, namely when there exist n-ary ($n > 2$) minimal conflicts in the knowledge base.

Consequently, the symmetric systems studied in (19) cannot be adopted in a concrete application.

5. The outcomes of logic-based argumentation systems

The aim of this section is to investigate the underpinnings of the different acceptability semantics introduced in (16; 20; 21) and to check whether they make sense in a concrete application. Recall that those semantics are defined without considering neither the internal structure nor the origin of arguments and attacks. In this section, we fully characterize for the first time both the extensions and the output set of any Tarskian logic-based argumentation system under naive, stable, semi-stable, preferred, grounded and ideal semantics. For that purpose, we consider only systems that enjoy the rationality postulates introduced in the previous section (as the other systems are regarded as ill-fated instantiations of Dung's framework).

5.1 *Naive semantics*

In this section, we characterize the outputs of an argumentation system under naive semantics. We show that the naive extensions of *any* argumentation system that satisfies consistency and closure under sub-arguments *always* return maximal (for set inclusion) consistent subsets of Σ.

Theorem 1 *Let $\mathcal{T} = (\mathrm{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent. If $\mathcal{T}$ satisfies consistency and is closed under sub-arguments (under naive semantics), then:*

- *For all $\mathcal{E} \in \mathrm{Ext}_n(\mathcal{T})$, $\mathrm{Base}(\mathcal{E}) \in \mathrm{Max}(\Sigma)$.*
- *For all $\mathcal{E} \in \mathrm{Ext}_n(\mathcal{T})$, $\mathcal{E} = \mathrm{Arg}(\mathrm{Base}(\mathcal{E}))$.*
- *For all $\mathcal{E}_i, \mathcal{E}_j \in \mathrm{Ext}_n(\mathcal{T})$, if $\mathrm{Base}(\mathcal{E}_i) = \mathrm{Base}(\mathcal{E}_j)$ then $\mathcal{E}_i = \mathcal{E}_j$.*

The next theorem confirms that *any* maximal consistent subset of Σ defines a naive extension of an argumentation system which satisfies consistency and closure under sub-arguments. This is the case when the logic $(\mathcal{L}, \mathsf{CN})$ is adjunctive.

Theorem 2 *Let $(\mathcal{L}, \mathsf{CN})$ be adjunctive. Let $\mathcal{T} = (\mathrm{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent. If $\mathcal{T}$ satisfies consistency and is closed under sub-arguments (under naive semantics), then:*

- *For all $\mathcal{S} \in \mathrm{Max}(\Sigma)$, $\mathrm{Arg}(\mathcal{S}) \in \mathrm{Ext}_n(\mathcal{T})$.*
- *For all $\mathcal{S}_i, \mathcal{S}_j \in \mathrm{Max}(\Sigma)$, if $\mathrm{Arg}(\mathcal{S}_i) = \mathrm{Arg}(\mathcal{S}_j)$ then $\mathcal{S}_i = \mathcal{S}_j$.*
- *For all $\mathcal{S} \in \mathrm{Max}(\Sigma)$, $\mathcal{S} = \mathrm{Base}(\mathrm{Arg}(\mathcal{S}))$.*

It follows that any argumentation system that satisfies the two postulates 2 and 3 enjoys a full correspondence between the maximal consistent subsets of Σ and the naive extensions of the system.

Corollary 1 *Let $(\mathcal{L}, \mathsf{CN})$ be adjunctive. Let $\mathcal{T} = (\mathrm{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent. $\mathcal{T}$ satisfies consistency and is closed under sub-arguments (under naive semantics) iff there is a bijection between the naive extensions of $\mathcal{T}$ and the elements of $\mathrm{Max}(\Sigma)$.*

A direct consequence of the previous result is that the number of naive extensions of an argumentation system is less or equal to the number of maximal consistent subbases of the knowledge base over which the system is built. Thus, if the knowledge base is finite, then the system has a finite number of naive extensions.

Corollary 2 *Let $\mathcal{T} = (\mathrm{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies consistency and is closed under sub-arguments (under naive semantics).*

- $|\mathrm{Ext}_n(\mathcal{T})| \leq |\mathrm{Max}(\Sigma)|$
- *If Σ is finite, then $\mathcal{T}$ has a finite number of naive extensions.*

The following result characterizes the case where an argumentation system has an empty naive extension. It shows that the knowledge base contains only inconsistent formulae.

Corollary 3 *Let $\mathcal{T} = (\mathrm{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies consistency and is closed under sub-arguments (under naive semantics). If $\mathrm{Ext}_n(\mathcal{T}) = \{\emptyset\}$, then for all $x \in \Sigma$, $\mathsf{CN}(\{x\})$ is inconsistent.*

Let us now characterize the set of inferences that may be drawn from a knowledge base Σ by any argumentation system under naive semantics. It coincides with the set of inferences that are drawn from *some* maximal consistent subsets of Σ.

Theorem 3 *Let $\mathcal{T} = (\mathrm{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent, $\mathcal{T}$ satisfies consistency and is closed under sub-arguments (under naive semantics). $\mathrm{Output}(\mathcal{T}) = \bigcap \mathsf{CN}(\mathcal{S}_i)$ where $\mathcal{S}_i$ ranges over $\{\mathcal{S}_i \in \mathrm{Max}(\Sigma) \mid \exists \mathcal{E}_i \in \mathrm{Ext}_n(\mathcal{T})$ and $\mathcal{S}_i = \mathrm{Base}(\mathcal{E}_i)\}$.*

When the number of naive extensions of an argumentation system is less than the number of maximal consistent subsets of the knowledge base over which the system is built, the system returns arbitrary conclusions.

Example 3 *Assume that $(\mathcal{L}, \mathsf{CN})$ is non adjunctive, $\Sigma = \{x, \neg x \wedge y\}$ and assume that this base has two maximal consistent subsets:*

- $\mathcal{S}_1 = \{x\}$
- $\mathcal{S}_2 = \{\neg x \wedge y\}$

According to Theorem 1, any argumentation system $\mathcal{T} = (\mathrm{Arg}(\Sigma), \mathcal{R})$ satisfying the postulates and whose attack relation $\mathcal{R}$ is conflict-dependent will have one or two naive extensions: $\mathcal{E}_1 = \mathrm{Arg}(\mathcal{S}_1)$ and $\mathcal{E}_2 = \mathrm{Arg}(\mathcal{S}_2)$. Assume that $\mathrm{Ext}_n(\mathcal{T}) = \{\mathcal{E}_1\}$. It follows that $x \in \mathrm{Output}(\mathcal{T})$ and $\neg x \notin \mathrm{Output}(\mathcal{T})$. If $\mathrm{Ext}_n(\mathcal{T}) = \{\mathcal{E}_2\}$, $x \notin \mathrm{Output}(\mathcal{T})$ and $\neg x \in \mathrm{Output}(\mathcal{T})$. Both results are arbitrary.

In case of adjunctive logics, the output of an argumentation system is the set of conclusions that follow from *all* the maximal consistent subsets of Σ.

Corollary 4 *Let $(\mathcal{L}, \mathsf{CN})$ be adjunctive. Let $\mathcal{T} = (\mathrm{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent, $\mathcal{T}$ satisfies consistency and is closed under sub-arguments (under naive semantics).*

$$\mathrm{Output}(\mathcal{T}) = \bigcap_{\mathcal{S}_i \in \mathrm{Max}(\Sigma)} \mathsf{CN}(\mathcal{S}_i).$$

Example 3 (Cont): Assume now that $(\mathcal{L}, \mathsf{CN})$ is propositional logic (which is adjunctive). The base $\Sigma = \{x, \neg x \wedge y\}$ has two maximal consistent subsets:

- $\mathcal{S}_1 = \{x\}$
- $\mathcal{S}_2 = \{\neg x \wedge y\}$

According to Corollary 1, any argumentation system $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ satisfying the postulates and whose attack relation $\mathcal{R}$ is conflict-dependent will have exactly two naive extensions: $\text{Arg}(\mathcal{S}_1)$ and $\text{Arg}(\mathcal{S}_2)$. Moreover, $\text{Output}(\mathcal{T}) = \mathsf{CN}(\mathcal{S}_1) \cap \mathsf{CN}(\mathcal{S}_2)$.

In short, under naive semantics, *any* 'good' instantiation of Dung's abstract framework returns exactly the formulas that are drawn (with CN) by all the maximal consistent subsets of the knowledge base Σ. So whatever the attack relation that is chosen, the result will be the same. It is worth recalling that the output set contains exactly the so-called *universal conclusions* in the *coherence-based approach* developed by Rescher and Manor in (30) for reasoning from inconsistent propositional bases. Indeed, in their approach Rescher and Manor takes as input a (possibly inconsistent) propositional knowledge base, then compute all its maximal (for set inclusion) consistent subsets. The universal conclusions to be drawn from the base are the formulae that follow logically from all those subsets. Thus, argumentation systems generalize (under naive semantics) this approach to any *adjunctive Tarskian logic*. As a consequence, the argumentation approach is syntax-dependent, and may thus lead to undesirable results as discussed in the following example.

Example 3 (Cont): Assume again that $(\mathcal{L}, \mathsf{CN})$ is propositional logic. Thus, $\text{Output}(\mathcal{T}) = \mathsf{CN}(\mathcal{S}_1) \cap \mathsf{CN}(\mathcal{S}_2)$. Note that $y \notin \text{Output}(\mathcal{T})$.

Assume that x stands for 'Sunny day' and y for 'It is cloudy'. The fact that $y \notin \text{Output}(\mathcal{T})$ seems reasonable. Assume now that y stands for 'The temperature is 18 degrees'. In this case, y should be inferred from Σ according to the idea that it is not part of the conflict.

Should $\neg x \wedge y$ instead be written as two formulas, namely $\neg x$ and y, then y is out of the conflict and is inferred.

Figure 1 summarizes the main results under naive semantics.

$(\mathcal{L}, \mathsf{CN})$	Output
Non-adjunctive	Arbitrary conclusions may be drawn from Σ
Adjunctive	Generalized universal conclusions of Rescher and Manor

Figure 1. Reasoning under naive semantics

5.2 *Stable - Semi-stable semantics*

We show that the stable extensions of *any* argumentation system satisfying consistency and closure under sub-arguments return maximal consistent subsets of Σ. This means that if one instantiates Dung's framework and does not get maximal consistent subsets with stable extensions, then the instantiation certainly violates one or both of the two key postulates: consistency and closure under sub-arguments.

Theorem 4 *Let $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent. If $\mathcal{T}$ satisfies consistency and closure under sub-arguments (under stable semantics) and $\text{Ext}_s(\mathcal{T}) \neq \emptyset$, then:*

- *For all $\mathcal{E} \in \text{Ext}_s(\mathcal{T})$, $\text{Base}(\mathcal{E}) \in \text{Max}(\Sigma)$.*

- *For all* $\mathcal{E} \in \text{Ext}_s(\mathcal{T})$, $\mathcal{E} = \text{Arg}(\text{Base}(\mathcal{E}))$.
- *For all* $\mathcal{E}_i, \mathcal{E}_j \in \text{Ext}_s(\mathcal{T})$, *if* $\text{Base}(\mathcal{E}_i) = \text{Base}(\mathcal{E}_j)$ *then* $\mathcal{E}_i = \mathcal{E}_j$.

This result characterizes the stable extensions of a large class of argumentation systems, namely the ones that are built using (adjunctive and non-adjunctive) Tarskian logics. However, it does not guarantee that each maximal consistent subset of Σ has a corresponding stable extension in an argumentation system $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$. To put it differently, it does not guarantee a bijection from the set $\text{Ext}_s(\mathcal{T})$ to the set $\text{Max}(\Sigma)$. The bijection (thus the equality $|\text{Ext}_s(\mathcal{T})| = |\text{Max}(\Sigma)|$) depends broadly on the attack relation that is chosen.

Let $\Re_s$ be the set of *all* attack relations that ensure the postulates under stable semantics:

$$\Re_s = \bigcup_{\Sigma \subseteq \mathcal{L}} \{\mathcal{R} \subseteq \text{Arg}(\Sigma) \times \text{Arg}(\Sigma) \mid \mathcal{R} \text{ is conflict-dependent and } (\text{Arg}(\Sigma), \mathcal{R}) \text{ satisfies Postulates 1, 2, 3, 4 and 5 under stable semantics}\}.$$

This set contains three *disjoint* subsets of attack relations: $\Re_s = \Re_{s_1} \cup \Re_{s_2} \cup \Re_{s_3}$:

- $\Re_{s_1}$: the relations which lead to $|\text{Ext}_s(\mathcal{T})| = 0$.
- $\Re_{s_2}$: the relations which ensure $0 < |\text{Ext}_s(\mathcal{T})| < |\text{Max}(\Sigma)|$.
- $\Re_{s_3}$: the relations which ensure $|\text{Ext}_s(\mathcal{T})| = |\text{Max}(\Sigma)|$.

Let us analyze separately each category of attack relations. The following result shows that the set $\Re_{s_1}$ is empty, meaning that there is no attack relation which prevents the existence of stable extensions. In other words, any argumentation system satisfying the rationality postulates has at least one stable extension. It is worth recalling that in the general case, Dung has shown that stable semantics does not guarantee the existence of extensions. This was considered as a weakness of this semantics.

Theorem 5 *It holds that* $\Re_{s1} = \emptyset$.

What about the attack relations of category $\Re_{s_2}$? Systems that use these relations choose a proper subset of the maximal consistent subsets of Σ and make inferences from them. Their output sets are as follows:

Theorem 6 *Let* $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ *be an argumentation system over a knowledge base* Σ *such that* $\mathcal{R} \in \Re_{s_2}$. $\text{Output}(\mathcal{T}) = \bigcap \mathsf{CN}(\mathcal{S}_i)$ *where* $\mathcal{S}_i$ *ranges over* $\{\mathcal{S}_i \in \text{Max}(\Sigma) \mid \exists \mathcal{E}_i \in \text{Ext}_s(\mathcal{T})$ *and* $\mathcal{S}_i = \text{Base}(\mathcal{E}_i)\}$.

These attack relations lead to an unjustified discrimination between maximal consistent subsets of a knowledge base. Unfortunately, this is fatal for the argumentation systems which use them as they may return arbitrary results. Note that the situation is similar to the one encountered under naive semantics when the logic is non adjunctive (see Example 3).

Attack relations of category $\Re_{s3}$ induce a one-to-one correspondence between the stable extensions of an argumentation system and the maximal consistent subsets of the knowledge base over which it is built.

Theorem 7 *Let* $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ *be an argumentation system over a knowledge base* Σ *such that* $\mathcal{R} \in \Re_{s3}$. *For all* $\mathcal{S} \in \text{Max}(\Sigma)$, $\text{Arg}(\mathcal{S}) \in \text{Ext}_s(\mathcal{T})$.

The stable extensions of any argumentation system using an attack relation of category $\Re_{s3}$ coincide with the naive extensions. They even coincide with the preferred extensions of the system meaning that this latter is *coherent* (21).

Theorem 8 *Let $\mathcal{T} = (\mathrm{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R} \in \Re_{s3}$. The equality $\mathrm{Ext}_n(\mathcal{T}) = \mathrm{Ext}_s(\mathcal{T})$ holds. If $\mathcal{T}$ satisfies the postulates under preferred semantics, then $\mathrm{Ext}_s(\mathcal{T}) = \mathrm{Ext}_p(\mathcal{T})$.*

An argumentation system using an attack relation of this category leads exactly to the same result under naive semantics. It returns the universal conclusions (of the coherence-based approach) under any monotonic logic and not only propositional logic as in (30). Finally, it is worth mentioning that the set $\Re_{s3}$ is not empty. Indeed, the *assumption attack relation* ($\mathcal{R}_{as}$) recalled in Definition 11 is one of its elements. In (18), it was shown that there is a full correspondence between the stable extensions of an argumentation system (defined over propositional logic) and the maximal consistent subsets of the propositional knowledge base over which it is built. This result was generalized to any Tarskian logic in (3). Consequently, any argumentation system using $\mathcal{R}_{as}$ is coherent.

Corollary 5 *For all $\Sigma \subseteq \mathcal{L}$, the argumentation system $\mathcal{T} = (\mathrm{Arg}(\Sigma), \mathcal{R}_{as})$ is coherent.*

From the previous results, it follows that any argumentation system satisfying the four postulates has stable extensions. Moreover, it is possible to delimit their maximum number.

Corollary 6 *Let $\mathcal{T} = (\mathrm{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a base Σ such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies consistency, closure under sub-arguments and free precedence (under stable semantics). It holds that*

$$0 < |\mathrm{Ext}_s(\mathcal{T})| \leq |\mathrm{Max}(\Sigma)|.$$

It follows that when the knowledge base is finite, the number of stable extensions is finite as well.

Corollary 7 *If Σ is finite, then the set $\mathrm{Ext}_s(\mathcal{T})$ is finite, whenever $\mathcal{T} = (\mathrm{Arg}(\Sigma), \mathcal{R})$ satisfies consistency and closure under sub-arguments (under stable semantics).*

To sum up, there are two possible categories of attack relations that lead to the satisfaction of the rationality postulates: $\Re_{s_2}$ and $\Re_{s_3}$. Relations of $\Re_{s_2}$ should be avoided as they lead to arbitrary results. Relations of category $\Re_{s_3}$ lead to "correct" results, but argumentation systems based on them return exactly the same results under naive semantics. This means that stable semantics does not play any particular role in the logic-based argumentation systems studied in the paper. Thus, stable semantics either leads to undesirable results or offers no added value w.r.t. naive semantics. Figure 2 summarizes the different situations that may be encountered under this semantics.

$\mathcal{R} \in \Re_{s_1}$	Impossible
$\mathcal{R} \in \Re_{s_2}$	Arbitrary conclusions are drawn from Σ
$\mathcal{R} \in \Re_{s_3}$	$\mathrm{Ext}_n(\mathcal{T}) = \mathrm{Ext}_s(\mathcal{T}) = \mathrm{Ext}_{ss}(\mathcal{T}) = \mathrm{Ext}_p(\mathcal{T})$

Figure 2. Reasoning under stable semantics

From the definitions of the two categories $\Re_{s2}$ and $\Re_{s3}$, stable extensions exist. Besides, it was shown in (16) that when this is the case, semi-stable extensions coincide with the stable ones.

Corollary 8 *Let $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a base Σ such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies consistency, closure under sub-arguments and free precedence (under stable semantics). The equality $\text{Ext}_s(\mathcal{T}) = \text{Ext}_{ss}(\mathcal{T})$ holds.*

Thus, in practice semi-stable semantics does not offer an added value w.r.t. stable semantics which is itself problematic.

5.3 *Preferred semantics*

Preferred semantics was mainly proposed in (21) as an alternative to stable semantics since the latter does not guarantee (for abstract frameworks) the existence of extensions. In this section, we study the outcomes of logic-based argumentation systems under preferred semantics and check whether it has an added value w.r.t. stable semantics in the context of handling inconsistency in knowledge bases.

We have previously shown in Proposition 5 that the extensions (under any admissibility-based semantics) of an argumentation system satisfying the postulates are made up of consistent subsets of the knowledge base over which the system is defined. Thus, the subset $\text{Base}(\mathcal{E})$ computed from any preferred extension $\mathcal{E}$ is a subset of maximal consistent subbase of the knowledge base at hand.

Theorem 9 *Let $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ be an argumentation system such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies consistency and closure under sub-arguments (under preferred semantics). For all $\mathcal{E} \in \text{Ext}_p(\mathcal{T})$, there exists $\mathcal{S} \in \text{Max}(\Sigma)$ such that $\text{Base}(\mathcal{E}) \subseteq \mathcal{S}$.*

Unlike stable extensions, the subsets of a knowledge base that are computed from preferred extensions are not necessarily maximal (for set inclusion). This is due to the existence of *undecided* arguments under preferred semantics. In (15) another way of defining Dung's semantics was provided. It consists of labeling the nodes of the graph corresponding to the argumentation system with three possibles values: {in, out, undec}. An argument is labeled *in* iff all its attackers are labeled *out*, it is labeled *out* iff one of its attackers is labeled *in*. Finally, it is labeled *undec* iff it is not possible to assign neither in nor out. When the subset of Σ which is computed from a preferred extension is not maximal then some formulae of Σ appear only in the support of undecided arguments.

This does not mean that a preferred extension can never return a maximal consistent subset. Remember that stable extensions exist, thus, there is at least one preferred extension whose base is maximal for set inclusion.

Corollary 9 *Let $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ be an argumentation system such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies consistency, closure under sub-arguments and free precedence (under preferred semantics). There exists $\mathcal{E} \in \text{Ext}_p(\mathcal{T})$ such that $\text{Base}(\mathcal{E}) \in \text{Max}(\Sigma)$.*

The following result shows that the subsets computed from the preferred extensions of an argumentation system are pairwise different.

Theorem 10 *Let $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ be an argumentation system such that $\mathcal{R}$ is* *conflict-dependent and $\mathcal{T}$ satisfies consistency and closure under sub-arguments (under preferred semantics). For all $\mathcal{E}_i, \mathcal{E}_j \in \mathrm{Ext}_p(\mathcal{T})$, if $\mathrm{Base}(\mathcal{E}_i) \subseteq \mathrm{Base}(\mathcal{E}_j)$ then $\mathcal{E}_i = \mathcal{E}_j$.*

We show that every maximal consistent subset of a knowledge base is captured by at most one preferred extension.

Theorem 11 *Let $\mathcal{T} = (\mathrm{Arg}(\Sigma), \mathcal{R})$ be an argumentation system such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies consistency and closure under sub-arguments (under preferred semantics). Let $\mathcal{S} \in \mathrm{Max}(\Sigma)$. For all $\mathcal{E}_i, \mathcal{E}_j \in \mathrm{Ext}_p(\mathcal{T})$, if $\mathrm{Base}(\mathcal{E}_i) \subseteq \mathcal{S}$ and $\mathrm{Base}(\mathcal{E}_j) \subseteq \mathcal{S}$, then $\mathcal{E}_i = \mathcal{E}_j$.*

The previous result allows us to delimit the maximum number of preferred extensions a system may have. Like stable semantics, it is the number of maximal (for set inclusion) consistent subsets of the knowledge base at hand.

Theorem 12 *Let $\mathcal{T} = (\mathrm{Arg}(\Sigma), \mathcal{R})$ be an argumentation system such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies consistency and closure under sub-arguments (under preferred semantics). It holds that*

$$1 \leq |\mathrm{Ext}_p(\mathcal{T})| \leq |\mathrm{Max}(\Sigma)|.$$

When a knowledge base is finite, each argumentation system enjoying the rationality postulates has a finite number of preferred extensions.

Corollary 10 *If a knowledge base Σ is finite, then for all $\mathcal{T} = (\mathrm{Arg}(\Sigma), \mathcal{R})$ such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies consistency and closure under sub-arguments (under preferred semantics), $\mathrm{Ext}_p(\mathcal{T})$ is finite.*

Let us characterize the inferences that are drawn from a knowledge base Σ by an argumentation system $\mathcal{T}$ satisfying the rationality postulates under preferred semantics. Let $\Re_p$ be the set of *all* attack relations that ensure the postulates under preferred semantics:

$$\Re_p = \bigcup_{\Sigma \subseteq \mathcal{L}} \{\mathcal{R} \subseteq \mathrm{Arg}(\Sigma) \times \mathrm{Arg}(\Sigma) \mid \mathcal{R} \text{ is conflict-dependent and } (\mathrm{Arg}(\Sigma), \mathcal{R}) \text{ satisfies Postulates 1, 2, 3, 4 and 5 under preferred semantics}\}.$$

In his seminal paper (21), Dung has shown that the stable extensions of an argumentation system are also preferred extensions of the system. Consequently, the set $\Re_p$ is a subset of $\Re_s$.

Property 10 *It holds that $\Re_p \subseteq \Re_s$.*

The set $\Re_p$ contains thus three *disjoint* subsets of attack relations: $\Re_p = \Re_{p_1} \cup \Re_{p_2} \cup \Re_{p_3}$:

- $\Re_{p_1}$: the relations which are in $\Re_p \cap \Re_{s_1}$.
- $\Re_{p_2}$: the relations which are in $\Re_p \cap \Re_{s_2}$.
- $\Re_{p_3}$: the relations which are in $\Re_p \cap \Re_{s_3}$.

Let us analyze each category of attack relations separately. The first set is empty (i.e., $\Re_{p_1} = \emptyset$) since we have shown previously that there is no attack relation which prevents an argumentation system from having stable extensions ($\Re_{s_1} = \emptyset$).

Attack relations of category $\Re_{p_3}$ lead to coherent argumentation systems (their stable extensions coincide with the preferred extensions) as shown in Theorem

8. Moreover, the preferred extensions coincide with the naive ones meaning that preferred semantics does not provide an added value w.r.t. naive semantics.

Theorem 13 *Let* $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ *be an argumentation system. If* $\mathcal{R} \in \Re_{p_3}$ *then:*

- *For all* $\mathcal{S} \in \text{Max}(\Sigma)$*,* $\text{Arg}(\mathcal{S}) \in \text{Ext}_p(\mathcal{T})$*.*
- $|\text{Ext}_p(\mathcal{T})| = |\text{Max}(\Sigma)|$

The output of an argumentation system is in this case the same as under naive semantics, i.e., the universal conclusions given in Corollary 4.

Corollary 11 *Let* $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ *be an argumentation system such that* $\mathcal{R} \in \Re_{p_3}$*.*

$$\text{Output}(\mathcal{T}) = \bigcap_{\mathcal{S}_i \in \text{Max}(\Sigma)} \mathsf{CN}(\mathcal{S}_i)$$

Let us now analyze attack relations of the second category $\Re_{p_2}$. Remember that in this case stable semantics chooses only some maximal consistent subsets of the knowledge base at hand. Four situations may be encountered:

(1) The stable extensions and the preferred extensions of an argumentation system coincide. Thus, preferred semantics has no added value w.r.t. stable semantics. Moreover, it leads to arbitrary results as discussed in the previous subsection (i.e., when $\mathcal{R} \in \Re_{s_2}$).
(2) The preferred extensions consider additional but *not all* maximal consistent subsets (other than the ones chosen by stable semantics). This case is similar to the previous one and the argumentation system returns arbitrary results.
(3) The preferred extensions return *all* the maximal consistent subsets of the knowledge base. This means that stable semantics chooses *some* maximal consistent subsets and preferred semantics considers the remaining ones. This case collapses with the case of attack relations of category $\Re_{p_3}$. We have seen that the result of preferred semantics is already ensured by naive and stable semantics in this case.
(4) Some of the preferred extensions provide *non-maximal* consistent subsets of the knowledge base. In this case, the result of the argumentation system is arbitrary.

To sum up, attack relations of category $\Re_{p_2}$ may lead either to *arbitrary* results or to results which can be provided by naive semantics. The results are characterized in the following theorem.

Theorem 14 *Let* $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ *be an argumentation system over a knowledge base* Σ *such that* $\mathcal{R} \in \Re_{p_2}$*.* $\text{Output}(\mathcal{T}) = \bigcap \mathsf{CN}(\mathcal{S}_i)$ *where* $\mathcal{S}_i$ *ranges over* $\{\mathcal{S}_i \in \text{Cons}(\Sigma) \mid \exists \mathcal{E}_i \in \text{Ext}_p(\mathcal{T})$ *and* $\mathcal{S}_i = \text{Base}(\mathcal{E}_i)\}$ *and* $\text{Cons}(\Sigma) = \{\mathcal{S} \mid \mathcal{S} \subseteq \Sigma,\ \mathcal{S}$ *is consistent and* $\text{Free}(\Sigma) \subseteq \mathcal{S}\}$*.*

The results of this section show that reasoning under preferred semantics is not recommended since it leads either to arbitrary results or to the results got under naive semantics. Figure 3 summarizes the different situations that may be encountered under this semantics.

$\mathcal{R} \in \Re_{p_1}$	Impossible
$\mathcal{R} \in \Re_{p_2}$	Arbitrary conclusions are drawn from Σ
$\mathcal{R} \in \Re_{p_3}$	$\mathrm{Ext}_n(\mathcal{T}) = \mathrm{Ext}_s(\mathcal{T}) = \mathrm{Ext}_{ss}(\mathcal{T}) = \mathrm{Ext}_p(\mathcal{T})$

Figure 3. Reasoning under preferred semantics

5.4 *Grounded - Ideal semantics*

This section analyzes the outputs of argumentation systems under existing skeptical semantics, namely grounded and ideal. Grounded semantics was proposed by Dung in (21). It ensures a unique extension for every argumentation system, and is based on a skeptical principle. It starts by non-attacked arguments to which are added arguments they defend. This reinstatement process is repeated until a fixpoint is reached. Argumentation systems that do not have non-attacked arguments have empty grounded extensions. In (20), this semantics was extended to the so-called ideal semantics. The new semantics returns a unique extension which is an admissible set of arguments contained by every preferred extension of an argumentation system. The following properties were shown in (20).

Property 11 *(20) Let $\mathcal{T}$ be an argumentation system.*

- *$\mathcal{T}$ admits a unique ideal extension.*
- $\mathrm{GE}(\mathcal{T}) \subseteq \mathrm{IE}(\mathcal{T}) \subseteq \bigcap_{\mathcal{E}_i \in \mathrm{Ext}_p(\mathcal{T})} \mathcal{E}_i$.
- *If $\bigcap_{\mathcal{E}_i \in \mathrm{Ext}_p(\mathcal{T})} \mathcal{E}_i$ is admissible, then $\mathrm{IE}(\mathcal{T}) = \bigcap_{\mathcal{E}_i \in \mathrm{Ext}_p(\mathcal{T})} \mathcal{E}_i$.*

The following example, borrowed from (20), shows some differences between ideal and grounded semantics.

Example 4 *Let us consider the argumentation framework $\mathcal{T} = (\mathcal{A}, \mathcal{R})$ where*

- $\mathcal{A} = \{a, b, c, d\}$
- $\mathcal{R} = \{(a, a), (a, b), (b, a), (c, d), (d, c)\}$

It can be checked that:

- $\mathrm{GE}(\mathcal{T}) = \emptyset$,
- $\mathrm{Ext}_p(\mathcal{T}) = \{\{b, c\}, \{b, d\}\}$, *and*
- $\mathrm{IE}(\mathcal{T}) = \{b\}$.

Before analyzing the argumentation systems' outputs under ideal and grounded semantics, we provide a result of great importance. It shows that the set of arguments built from $\mathrm{Free}(\Sigma)$ is an admissible extension of any argumentation system whose attack relation is conflict-dependent. Thus, this is true even for systems that do not satisfy the free precedence postulate.

Theorem 15 *Let $\mathcal{T} = (\mathrm{Arg}(\Sigma), \mathcal{R})$ be such that $\mathcal{R}$ is conflict-dependent.*

- *For all $a \in \mathrm{Arg}(\mathrm{Free}(\Sigma))$, a neither attacks nor is attacked by another argument in $\mathrm{Arg}(\Sigma)$.*
- *$\mathrm{Arg}(\mathrm{Free}(\Sigma))$ is an admissible extension of $\mathcal{T}$.*

Since ideal semantics is based on preferred semantics, we analyze the two cases that may be encountered with the latter. We start by the case of an argumentation system that uses an attack relation of category $\Re_{p_3}$. We show that the ideal extension of such system coincides with the intersection of all its preferred extensions. Moreover, it is exactly the set of arguments built from the free part of the knowledge base.

Theorem 16 *Let* $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ *be an argumentation system such that* $\mathcal{R} \in \Re_{p_3}$.

$$\text{IE}(\mathcal{T}) = \bigcap_{\mathcal{E}_i \in \text{Ext}_p(\mathcal{T})} \mathcal{E}_i = \text{Arg}(\text{Free}(\Sigma)).$$

Since arguments of $\text{Arg}(\text{Free}(\Sigma))$ are not attacked by any argument, then they belong to the grounded extension of the argumentation system. Consequently, grounded and ideal extensions coincide.

Corollary 12 *Let* $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ *be an argumentation system such that* $\mathcal{R} \in \Re_{p_3}$. $\text{IE}(\mathcal{T}) = \text{GE}(\mathcal{T}) = \text{Arg}(\text{Free}(\Sigma))$.

From the previous results, it is possible to characterize the set of conclusions drawn from a knowledge base using grounded and ideal semantics. It is the set of all formulae that follow using the consequence operator CN from $\text{Free}(\Sigma)$.

Theorem 17 *Let* $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ *be an argumentation system such that* $\mathcal{R} \in \Re_{p_3}$. *The output of* $\mathcal{T}$ *under grounded/ideal semantics is:*

$$\text{Output}(\mathcal{T}) = \mathsf{CN}(\text{Free}(\Sigma)).$$

It is worth pointing out that in this case, argumentation systems generalize the *free consequences* proposed by Benferhat, Dubois and Prade for reasoning about inconsistent propositional knowledge bases (10). Indeed, argumentation systems consider not only propositional logic but also any other Tarskian logic.

Recall that the assumption attack relation leads to coherent argumentation systems, thus their ideal and grounded semantics coincide.

Corollary 13 *Let* $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R}_{as})$. *For any* $\Sigma \subseteq \mathcal{L}$, $\text{Output}(\mathcal{T}) = \mathsf{CN}(\text{Free}(\Sigma))$.

Let us now consider the case where the attack relation of an argumentation system is of category $\Re_{p_2}$. Here again, since the attack relation is conflict-dependent, the set of arguments $\text{Arg}(\text{Free}(\Sigma))$ is contained by both ideal and grounded extensions.

Corollary 14 *Let* $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ *be an argumentation system such that* $\mathcal{R} \in \Re_{p_2}$. *The inclusions* $\text{Arg}(\text{Free}(\Sigma)) \subseteq \text{GE}(\mathcal{T}) \subseteq \text{IE}(\mathcal{T}) \subseteq \mathcal{S}$ *hold for some* $\mathcal{S} \in \text{Max}(\Sigma)$.

In case the above inclusions are strict, i.e., $\text{Arg}(\text{Free}(\Sigma)) \subset \text{GE}(\mathcal{T})$ (respectively $\text{Arg}(\text{Free}(\Sigma)) \subset \text{IE}(\mathcal{T})$), we show that the argumentation system $\mathcal{T}$ returns arbitrary conclusions.

Theorem 18 *Let* $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ *be an argumentation system such that* $\mathcal{R}$ *is conflict-dependent and* $\mathcal{T}$ *satisfies the five postulates under grounded (respectively ideal) semantics. If* $\text{Arg}(\text{Free}(\Sigma)) \subset \text{GE}(\mathcal{T})$ *(respectively* $\text{Arg}(\text{Free}(\Sigma)) \subset \text{IE}(\mathcal{T})$*) then there exists* $C \in \mathcal{C}_\Sigma$ *such that there exist* $x, x' \in C$ *and* $x \in \text{Output}(\mathcal{T})$ *and* $x' \notin \text{Output}(\mathcal{T})$.

To sum up, ideal and grounded semantics either coincide and return as output the set of all formulas that follow from the safe part of a knowledge base, i.e., $\mathsf{CN}(\text{Free}(\Sigma))$, or may both return arbitrary results. Figure 4 summarizes the different situations encountered under these two semantics.

$\mathcal{R} \in \Re_{p_2}$	Arbitrary conclusions are drawn from Σ
$\mathcal{R} \in \Re_{p_3}$	$\text{Output}(\mathcal{T}) = \mathsf{CN}(\text{Free}(\Sigma))$

Figure 4. Reasoning under grounded and ideal semantics

6. Related work

This paper investigated the compatibility of Dung's argumentation framework with logical formalisms. For that purpose, it showed how to instantiate the framework from a logical knowledge base, namely how to define in a systematic way the arguments. Then, it proposed some basic postulates that the logical instantiations should satisfy, and studied under which conditions the postulates may be violated. Next, it investigated the outputs of such instantiations under various semantics.

There are some works in the literature which are somehow related to our. In (17), rationality postulates were proposed for instantiations that use a particular language (it distinguishes between strict rules and defeasible rules). In our work, we extended some of those postulates to Tarskian logics and proposed three new ones. In (17), the authors investigated when the ASPIC system satisfies the postulates. In (23), the authors focused on some argumentation systems that are defined using propositional logic. They studied when those systems satisfy some of the rationality postulates presented in this paper. Our work is more general since we considered a larger class of logics, and we did not focus on particular attack relations. Our results holds for any attack relation that is conflict-dependent. Note that all the attack relations that were studied in (23) are conflict-dependent.

The second part of our paper on the outputs of argumentation systems under various semantics is novel. There is almost no work on the topic except the one by Cayrol in (18). Cayrol studied the underpinnings of stable semantics for one particular argumentation system: the one that is grounded on propositional logic (a particular case of Tarski's logics) and uses the "assumption attack" relation. She showed that there is a one-to-one correspondence between the stable extensions of the system and the maximal consistent subsets of the knowledge base over which the system is built. In our paper, this result is generalized to any Tarskian logic and any attack relation. Moreover, we have shown that this result is already ensured by naive semantics. Thus, in the case of the system studied by Cayrol, stable semantics is useless. Finally, we have shown that this particular system is coherent, i.e., its stable extensions coincide with its preferred ones. Our work is more general since it presented a *complete* view of the outputs of argumentation systems not only under stable semantics but also under various other semantics.

7. Conclusion

The paper investigated Dung's argumentation framework. It started by pointing out its main limits, namely the gap between the abstract framework and the application to which it may be used, the lack of rationality postulates that would describe the kind of results expected from the framework, the lack of methodology for defining arguments and the attacks between them, and finally some overlook of the underpinnings of the different acceptability semantics as well as the basic concepts of the framework like defense and conflict-freeness. The paper gave an answer to each of these issues.

The paper extended Dung's argumentation framework by taking into account the logic from which arguments are built. The new framework is general since it is grounded on any abstract logic in Tarski's sense. Thus, a wide variety of logics can be used even those that have not been considered yet in argumentation, like temporal logic, modal logic, etc. The extension has two main advantages: First, it enforces the framework to avoid unsound conclusions. Second, it relates the different notions of Dung's approach, like the attack relation and conflict-freeness, to the knowledge base at hand.

In (17), three rationality postulates were defined for rule-based argumentation systems. The paper generalized the two postulates on direct consistency and closure to any argumentation framework built over a monotonic logic, and proposed three new postulates on sub-arguments, free precedence and exhaustiveness. It then showed that indirect consistency is always satisfied if direct consistency is ensured. Moreover, it showed that if consistency (respectively closure) is satisfied by the different extensions, then it is also satisfied by the output of the framework.

The paper then presented a formal methodology for defining arguments from a knowledge base, and for eliciting an appropriate attack relation. By appropriate, we mean an attack relation that satisfies the postulates. It showed that an attack relation should be grounded on the minimal conflicts that occur in the knowledge base at hand. An important result shows that when ternary or more minimal conflicts occur in the knowledge base, then symmetric attack relations should be avoided since they lead to the violation of direct consistency.

Using well-behaved attack relations, the paper analyzed the different acceptability semantics introduced in (21) in terms of the subsets that are returned by each extension. The results of the analysis are very surprising and, unfortunately, disappointing. In fact, they show to what extent the rationality of Dung's approach is at stake. Moreover, it behaves in a completely arbitrary way. The first important result shows that maximal conflict-free sets of arguments are sufficient in order to derive reasonable conclusions from a knowledge base. Indeed, there is a one-to-one correspondence between maximal consistent subsets of a knowledge base and maximal conflict-free sets of arguments. This means that the different acceptability semantics defined in the literature are not necessary, and the notion of defense is useless. It is also shown that under naive semantics, argumentation systems generalize the coherence-based approach of Rescher and Manor (30) to any Tarskian logic. This is particularly the case for adjunctive logics.

Remember that stable extensions are maximal conflict-free sets of arguments. Does this mean that stable semantics is appropriate? The answer is unfortunately no. Indeed, stable extensions amount either to an arbitrary pick of *some* maximal consistent subsets of the knowledge base, or to consider all of them. In the first case, they lead to arbitrary inferences whereas in the second case they lead to the result already ensured by naive semantics. The case of preferred semantics is even worse and the semantics should be avoided. The corresponding extensions represent *some* consistent subsets but not necessarily maximal ones. Thus, they lead to arbitrary inferences. There are, however, two good news: The first one is that the number of (stable/preferred/naive) extensions is finite as soon as the knowledge base is finite. The second one is that stable extensions always exist meaning that semi-stable semantics proposed in (16) is useless since it does not provide an added value wrt stable semantics. Reasoning under ideal and grounded semantics may also be problematic. There are two possible situations: i) the situation where the two extensions (grounded and ideal) coincide with the set of arguments built from the free part of a knowledge base, ii) the situation where both semantics lead to arbitrary results.

One of the main reasons for this defective behavior of Dung's approach is the attack relation. Indeed, we have shown that in order to ensure reasonable results, this relation should capture the minimal conflicts that occur in the knowledge base at hand. A minimal conflict is an inconsistent set of formulas. It is well-known that the notion of inconsistency is not oriented. Thus, it *should* be captured by a symmetric attack relation. However, we have also shown that, due to the binary character of the attack relation, if it is symmetric, then consistency often fails. Which is when n-ary (other than binary) minimal conflicts occur in the knowledge base. The second reason is the definition of semantics without taking into account the application to which the system is applied, and thus without considering the structure of arguments.

Acknowledgment

The authors are very grateful to the anonymous referee for her/his valuable suggestions for improving the presentation of this work.

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Appendix

Property 1 *Let $X, X', X'' \subseteq \mathcal{L}$.*

(1) $X \subseteq X' \Rightarrow \mathsf{CN}(X) \subseteq \mathsf{CN}(X')$.
(2) $\mathsf{CN}(X) \cup \mathsf{CN}(X') \subseteq \mathsf{CN}(X \cup X')$.
(3) $\mathsf{CN}(X) = \mathsf{CN}(X') \Rightarrow \mathsf{CN}(X \cup X'') = \mathsf{CN}(X' \cup X'')$.
(4) $\mathsf{CN}(X \cap X') \subseteq \mathsf{CN}(X) \cap \mathsf{CN}(X')$.

Proof Let $X, X', X'' \subseteq \mathcal{L}$.

(1) Assume that $X \subseteq X'$. According to the compactness axiom, it holds that $\mathsf{CN}(X) = \bigcup_{Y \subseteq_f X} \mathsf{CN}(Y)$. $\bigcup_{Y \subseteq_f X} \mathsf{CN}(Y) \subseteq \bigcup_{Y \subseteq_f X'} \mathsf{CN}(Y) = \mathsf{CN}(X')$ since $Y \subseteq_f X \subset X'$.
(2) $X \subseteq X \cup X'$, thus by monotonicity, $\mathsf{CN}(X) \subseteq \mathsf{CN}(X \cup X')$ (a). Similarly, $X' \subseteq X \cup X'$ thus $\mathsf{CN}(X') \subseteq \mathsf{CN}(X \cup X')$ (b). From (a) and (b), $\mathsf{CN}(X) \cup \mathsf{CN}(X') \subseteq \mathsf{CN}(X \cup X')$.
(3) Assume that $\mathsf{CN}(X) = \mathsf{CN}(X')$. According to the expansion axiom, the following inclusions hold: $X' \subseteq \mathsf{CN}(X')$ and $X'' \subseteq \mathsf{CN}(X'')$. Thus, $X' \cup X'' \subseteq \mathsf{CN}(X') \cup \mathsf{CN}(X'') = \mathsf{CN}(X) \cup \mathsf{CN}(X'')$ (a). Moreover, $X \subseteq X \cup X''$ thus $\mathsf{CN}(X) \subseteq \mathsf{CN}(X \cup X'')$ (a'). Similarly, $X'' \subseteq X \cup X''$ thus $\mathsf{CN}(X'') \subseteq \mathsf{CN}(X \cup X'')$ (b'). From (a') and (b'), $\mathsf{CN}(X) \cup \mathsf{CN}(X'') \subseteq \mathsf{CN}(X \cup X'')$. From (a), it follows that $X' \cup X'' \subseteq \mathsf{CN}(X \cup X'')$. By monotonicity, $\mathsf{CN}(X' \cup X'') \subseteq \mathsf{CN}(\mathsf{CN}(X \cup X''))$. Finally, by the idempotence axiom, the inclusion $\mathsf{CN}(X' \cup X'') \subseteq \mathsf{CN}(X \cup X'')$ holds. To show that $\mathsf{CN}(X \cup X'') \subseteq \mathsf{CN}(X' \cup X'')$, the same reasoning is applied by starting with X instead of X'.
(4) $X \cap X' \subseteq X$ thus $\mathsf{CN}(X \cap X') \subseteq \mathsf{CN}(X)$. Similarly, $X \cap X' \subseteq X'$ thus $\mathsf{CN}(X \cap X') \subseteq \mathsf{CN}(X')$. Consequently, $\mathsf{CN}(X \cap X') \subseteq \mathsf{CN}(X) \cap \mathsf{CN}(X')$.

□

Property 2 *Let $X \subseteq \mathcal{L}$.*

(1) If X is consistent, then $\mathsf{CN}(X)$ is consistent as well.
(2) $\forall X' \subseteq X$, if X is consistent, then X' is consistent.
(3) $\forall X' \subseteq X$, if X' is inconsistent, then X is inconsistent.

Proof Let $X \subseteq \mathcal{L}$. Assume that X is consistent, then $\mathsf{CN}(X) \neq \mathcal{L}$ (1)

(1) Let us now assume that $\mathsf{CN}(X)$ is inconsistent. This means that $\mathsf{CN}(\mathsf{CN}(X)) = \mathcal{L}$. However, according to the idempotence axiom, $\mathsf{CN}(\mathsf{CN}(X)) = \mathsf{CN}(X)$. Thus, $\mathsf{CN}(X) = \mathcal{L}$, this contradicts (1).
(2) Assume that $\exists X' \subseteq X$ such that X' is inconsistent. This means that

$\mathsf{CN}(X') = \mathcal{L}$. However, since $X' \subseteq X$ then $\mathsf{CN}(X') \subseteq \mathsf{CN}(X)$ (according to the monotonicity axiom). Thus, $\mathcal{L} \subseteq \mathsf{CN}(X)$. Since $\mathsf{CN}(X) \subseteq \mathcal{L}$, then $\mathsf{CN}(X) = \mathcal{L}$. Thus, X is inconsistent. Contradiction.

(3) Let $X' \subseteq X$. Assume that X' is inconsistent. Since $X' \subseteq X$ thus $\mathsf{CN}(X') \subseteq \mathsf{CN}(X)$ (by monotonicity axiom). Since X' is inconsistent, $\mathsf{CN}(X') = \mathcal{L}$. Consequently, $\mathsf{CN}(X) = \mathcal{L}$ which means that X is inconsistent.

□

Property 3 *For all $X \subseteq \Sigma \subseteq \mathcal{L}$,*

- *if X is consistent then $\mathcal{C}_X = \emptyset$.*
- *if X is consistent then $X \subseteq \mathcal{S}$ for some $\mathcal{S} \in \mathrm{Max}(\Sigma)$.*
- *if X is inconsistent then there exists at least one minimal conflict C such that $C \subseteq X$.*

Proof The first and third items are obvious. Let us now prove the second item. Let us construct a maximal consistent subset of Σ that contains X. Consider an enumeration $s_1, s_2, \ldots$ of Σ. Define a series $S_0, S_1, S_2, \ldots$ of subsets of Σ as follows: S_0 is X, and $S_{n+1} = S_n \cup \{s_{n+1}\}$ if $S_n \cup \{s_{n+1}\}$ is consistent, otherwise $S_{n+1} = S_n$. By construction, S_n is consistent for all $n \geq 0$. Let $\mathcal{S} = \bigcup_{n \geq 0} S_n$. Trivially, $X \subseteq \mathcal{S}$. We now show $\mathcal{S} \in \mathrm{Max}(\Sigma)$. Assume first that $\mathcal{S}$ is inconsistent, i.e., $\mathsf{CN}(\mathcal{S}) = \mathcal{L}$ according to Definition 4. Tarski's absurdity axiom yields that $x \in \mathsf{CN}(\mathcal{S})$ for some x satisfying $\mathsf{CN}(\{x\}) = \mathcal{L}$. Since $S_n \subseteq S_{n+1}$, Tarski's compactness axiom means that $x \in \mathsf{CN}(\mathcal{S})$ iff $x \in \mathsf{CN}(S_k)$ for some k. However, $x \in \mathsf{CN}(S_k)$ means that $x \in S_k$ is inconsistent, a contradiction. Hence, we have shown that $\mathcal{S}$ is consistent. By construction, it is a subset of Σ. There remains to show that it is a maximal consistent subset of Σ. Let y be in $\Sigma \setminus \mathcal{S}$. Assume $\mathcal{S} \cup \{y\}$ is consistent. Since $y \in \Sigma$, y is some s_h in the above enumeration. Should $\mathcal{S} \cup \{y\}$ be consistent, so would be $S_h \cup \{y\}$ in view of Tarski's compactness axiom (monotony direction). However, $S_h \cup \{y\}$ consistent would mean that $y \in S_{h+1}$ hence $y \in \mathcal{S}$, a contradiction. □

Property 4 *Let $(\mathcal{L}, \mathsf{CN})$ be* adjunctive*, $C \subseteq \mathcal{L}$ be a minimal conflict. For all $X \subset C$, if $X \neq \emptyset$, then:*

(1) $\exists x \in \mathcal{L}$ such that $\mathsf{CN}(\{x\}) = \mathsf{CN}(X)$.
(2) $\exists x_1 \in \mathsf{CN}(X)$ and $\exists x_2 \in \mathsf{CN}(C \setminus X)$ such that the set $\{x_1, x_2\}$ is inconsistent.

Proof Let C be a minimal conflict. Consider $X \subset C$ such that $X \neq \emptyset$.

We prove the first item of the property by induction, after we first take care to show that X is finite. By Tarski's requirements, there exists $x_0 \in \mathcal{L}$ s.t. $\mathsf{CN}(\{x_0\}) = \mathcal{L}$. Since C is a conflict, $\mathsf{CN}(C) = \mathsf{CN}(\{x_0\})$. As a consequence, $x_0 \in \mathsf{CN}(C)$. However, $\mathsf{CN}(C) = \bigcup_{C' \subseteq_f C} \mathsf{CN}(C')$ by Tarski's requirements. Thus, $x_0 \in \mathsf{CN}(C)$ means that there exists $C' \subseteq_f C$ s.t. $x_0 \in \mathsf{CN}(C')$. This says that C' is a conflict. Since C is a minimal conflict, $C = C'$ and it follows that C is finite. Of course, so is X: Let us write $X = \{x_1, \ldots, x_n\}$. *Base step:* $n = 1$. Taking x to be x_1 is enough. *Induction step:* Assume the lemma is true up to rank $n-1$. As CN is a closure operator, $\mathsf{CN}(\{x_1, \ldots, x_n\}) = \mathsf{CN}(\mathsf{CN}(\{x_1, \ldots, x_{n-1}\}) \cup \{x_n\})$. The induction hypothesis entails $\exists x \in \mathcal{L}$ s.t. $\mathsf{CN}(\mathsf{CN}(\{x_1, \ldots, x_{n-1}\}) \cup \{x_n\}) = \mathsf{CN}(\mathsf{CN}(\{x\}) \cup \{x_n\})$. Then, $\mathsf{CN}(\{x_1, \ldots, x_n\}) = \mathsf{CN}(\{x, x_n\})$. Hence, there exists $y \in \mathcal{L}$ s.t. $\mathsf{CN}(\{x, x_n\}) = \mathsf{CN}(\{y\})$ because $(\mathcal{L}, \mathsf{CN})$ is adjunctive. Since $\mathsf{CN}(\{x_1, \ldots, x_n\}) = \mathsf{CN}(\{x, x_n\})$ was just proved, it follows that $\mathsf{CN}(\{y\}) = \mathsf{CN}(\{x_1, \ldots, x_n\})$.

Take $X_1 = X$ and $X_2 = C \setminus X_1$. Since X is a non-empty proper subset of C, so are both X_1 and X_2. Then, the first bullet of this property can be applied to X_1 and X_2. As a result, $\exists x_1 \in \mathcal{L}$ s.t. $\mathsf{CN}(\{x_1\}) = \mathsf{CN}(X_1)$ and $\exists x_2 \in \mathcal{L}$ s.t. $\mathsf{CN}(\{x_2\}) = \mathsf{CN}(X_2)$. The expansion axiom gives $\{x_1\} \subseteq \mathsf{CN}(\{x_1\})$ and $\{x_2\} \subseteq \mathsf{CN}(\{x_2\})$. Thus, $x_1 \in \mathsf{CN}(X_1)$ and $x_2 \in \mathsf{CN}(X_2)$. Using the expansion axiom again, $X_1 \subseteq \mathsf{CN}(X_1)$ and $X_2 \subseteq \mathsf{CN}(X_2)$. Thus, $X_1 \cup X_2 \subseteq \mathsf{CN}(X_1) \cup \mathsf{CN}(X_2) = \mathsf{CN}(\{x_1\}) \cup \mathsf{CN}(\{x_2\})$. It follows that $C \subseteq \mathsf{CN}(\{x_1\}) \cup \mathsf{CN}(\{x_2\})$. Using Property 1, $\mathsf{CN}(\{x_1\}) \cup \mathsf{CN}(\{x_2\}) \subseteq \mathsf{CN}(\{x_1, x_2\})$, thus $C \subseteq \mathsf{CN}(\{x_1, x_2\})$. Since C is inconsistent, Property 2 gives that $\mathsf{CN}(\{x_1, x_2\})$ is inconsistent as well. By the definition of inconsistency, it follows that $\mathsf{CN}(\mathsf{CN}(\{x_1, x_2\})) = \mathcal{L}$. Applying the idempotence axiom, $\mathsf{CN}(\{x_1, x_2\}) = \mathcal{L}$, thus the set $\{x_1, x_2\}$ is inconsistent. □

Property 5 *For all* $(X, x) \in \mathrm{Arg}(\Sigma)$*, the set* $\{x\}$ *is consistent.*

Proof Let $(X, x) \in \mathrm{Arg}(\Sigma)$. From Definition 7, the support X is consistent. Thus, according to Property 2, $\mathsf{CN}(X)$ is consistent as well. Since $x \in \mathsf{CN}(X)$, then $\{x\} \subseteq \mathsf{CN}(X)$ thus $\{x\}$ is consistent. □

Property 6 *Let* Σ *be a knowledge base such that for all* $x \in \Sigma$, $x \notin \mathsf{CN}(\emptyset)$*. For all* $x \in \Sigma$ *such that the set* $\{x\}$ *is consistent,* $(\{x\}, x) \in \mathrm{Arg}(\Sigma)$*.*

Proof Let $x \in \Sigma$ be such that the set $\{x\}$ is consistent. Since $x \notin \mathsf{CN}(\emptyset)$, then $\{x\}$ is a minimal set such that $x \in \mathsf{CN}(\{x\})$. It follows that $(\{x\}, x) \in \mathrm{Arg}(\Sigma)$. □

Property 7 $\mathrm{Arg}(\Sigma) \subseteq \mathrm{Arg}(\Sigma')$ *whenever* $\Sigma \subseteq \Sigma' \subseteq \mathcal{L}$*.*

Proof Let $\Sigma \subseteq \Sigma' \subseteq \mathcal{L}$. Assume that $(X, x) \in \mathrm{Arg}(\Sigma)$ and $(X, x) \notin \mathrm{Arg}(\Sigma')$. Since $(X, x) \notin \mathrm{Arg}(\Sigma')$, then there are four possible cases:

(1) X is not a subset of Σ'. This is impossible since $(X, x) \in \mathrm{Arg}(\Sigma)$, thus $X \subseteq \Sigma \subseteq \Sigma'$.
(2) X is inconsistent, this is impossible since $(X, x) \in \mathrm{Arg}(\Sigma)$.
(3) $x \notin \mathsf{CN}(X)$. This is impossible since $(X, x) \in \mathrm{Arg}(\Sigma)$ hence $x \in \mathsf{CN}(X)$.
(4) X is not minimal. This means that $\exists X' \subset X$ that satisfies conditions 1-3 of Def. 7. This is impossible since $(X, x) \in \mathrm{Arg}(\Sigma)$.

□

Property 8 *Let* $\mathcal{T} = (\mathrm{Arg}(\Sigma), \mathcal{R})$ *be an argumentation system over a knowledge base* Σ *and* $\mathrm{Ext}(\mathcal{T})$ *its set of extensions under a given semantics. It holds that* $\mathrm{Output}(\mathcal{T}) = \bigcap_{\mathcal{E}_i \in \mathrm{Ext}(\mathcal{T})} \mathrm{Concs}(\mathcal{E}_i)$*.*

Proof Let $\mathcal{T} = (\mathrm{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a base Σ.

(1) Let $x \in \mathrm{Output}(\mathcal{T})$. Thus, for all $\mathcal{E}_i \in \mathrm{Ext}(\mathcal{T})$, $\exists a_i \in \mathcal{E}_i$ such that $\mathrm{Conc}(a_i) = x$. It follows that $x \in \mathrm{Concs}(\mathcal{E}_i)$, $\forall \mathcal{E}_i$ and hence $x \in \cap \mathrm{Concs}(\mathcal{E}_i)$.

(2) Assume that $x \in \cap \mathrm{Concs}(\mathcal{E}_i)$. Thus, $\forall \mathcal{E}_i \in \mathrm{Ext}(\mathcal{T})$, $\exists a_i \in \mathcal{E}_i$ such thatConc$(a_i) = x$. Consequently, $x \in \mathrm{Output}(\mathcal{T})$.

□

Property 9 *Let* $\mathcal{T} = (\mathrm{Arg}(\Sigma), \mathcal{R})$ *be an argumentation system over a knowledge base* Σ. $\mathrm{Output}(\mathcal{T}) \subseteq \mathsf{CN}(\Sigma)$.

Proof Let $\mathcal{T} = (\mathrm{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ. Let $x \in \mathrm{Output}(\mathcal{T})$. Thus, $\exists a \in \mathrm{Arg}(\Sigma)$ such that $\mathrm{Conc}(a) = x$. Besides, $x \in$

$\mathsf{CN}(\text{Supp}(a))$ and $\text{Supp}(a) \subseteq \Sigma$. By monotonicity of CN, $\mathsf{CN}(\text{Supp}(a)) \subseteq \mathsf{CN}(\Sigma)$. Thus, $x \in \mathsf{CN}(\Sigma)$. □

Property 10 *It holds that $\Re_p \subseteq \Re_s$.*

Proof Let $\mathcal{R} \in \Re_p$. Thus, for all $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$, $\mathcal{T}$ satisfies all the postulates, i.e., for all $\mathcal{E} \in \text{Ext}_p(\mathcal{T})$,

- $\text{Concs}(E)$ is consistent.
- $\text{Concs}(E) = \mathsf{CN}(\text{Concs}(E))$
- For all $a \in \mathcal{E}$, $\text{Sub}(a) \subseteq \mathcal{E}$
- $\text{Arg}(\text{Free}(\Sigma)) \subseteq \mathcal{E}$
- for all $(X, x) \in \text{Arg}(\Sigma)$, if $X \cup \{x\} \subseteq \text{Concs}(\mathcal{E})$, then $(X, x) \in \mathcal{E}$

Since $\text{Ext}_s(\mathcal{T}) \subseteq \text{Ext}_p(\mathcal{T})$, thus the previous properties are satisfies by all the stable extensions of $\mathcal{T}$. Consequently, $\mathcal{R} \in \Re_s$. □

Proposition 1 *Let $(\mathcal{L}, \mathsf{CN})$ be adjunctive and Σ be a knowledge base. For all non-empty proper subset X of some minimal conflict $C \in \mathcal{C}_\Sigma$, there exists $a \in \text{Arg}(\Sigma)$ such that $\text{Supp}(a) = X$.*

Proof Let $C \in \mathcal{C}_\Sigma$ and $X \subseteq C$ such that X is non-empty. Assume $\nexists a \in \text{Arg}(\Sigma)$ such that $\text{Supp}(a) = X$. I.e., there exists no x such that X is a minimal consistent set satisfying $x \in \mathsf{CN}(X)$. So, for all $x \in \mathcal{L}$, if $x \in \mathsf{CN}(X)$ then $\exists Y \subset X$ such that $x \in \mathsf{CN}(Y)$. In short, for all $x \in \mathsf{CN}(X)$, there exists $Y \subset X$ such that $x \in \mathsf{CN}(Y)$. Property 4 says that there exists $z \in \mathcal{L}$ such that $\mathsf{CN}(\{z\}) = \mathsf{CN}(X)$. However, $z \notin \mathsf{CN}(Y)$ for all $Y \subset X$ otherwise C would fail to be minimal (because $Y \cup (C \setminus X) \subset C$ while $z \in \mathsf{CN}(Y)$ implies $\mathsf{CN}(C) = \mathsf{CN}(X \cup (C \setminus X)) = \mathsf{CN}(\{z\} \cup (C \setminus X)) \subseteq \mathsf{CN}(Y \cup (C \setminus X)))$. A contradiction arises. □

Proposition 2 *Let $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ. If $\mathcal{T}$ is closed under CN, then $\text{Output}(\mathcal{T}) = \mathsf{CN}(\text{Output}(\mathcal{T}))$.*

Proof Let $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ, and $\text{Ext}(\mathcal{T})$ be its set of extensions under a given semantics. Assume that $\mathcal{T}$ satisfies closure under CN. From Expansion axiom, it follows that $\text{Output}(\mathcal{T}) \subseteq \mathsf{CN}(\text{Output}(\mathcal{T}))$.

Assume now that $x \in \mathsf{CN}(\text{Output}(\mathcal{T}))$. Thus, $\exists x_1, \ldots, x_n \in \text{Output}(\mathcal{T})$ such that$x \in \mathsf{CN}(\{x_1, \ldots, x_n\})$. From Property 8, $x_1, \ldots, x_n \in \cap\text{Concs}(\mathcal{E}_i)$ where $\mathcal{E}_i \in \text{Ext}(\mathcal{T})$. From Property 1, it holds that $\mathsf{CN}(\{x_1, \ldots, x_n\}) \subseteq \mathsf{CN}(\cap\text{Concs}(\mathcal{E}_i))$. Again, from Property 1, $x \in \mathsf{CN}(\text{Concs}(\mathcal{E}_1)) \cap \ldots \cap \mathsf{CN}(\text{Concs}(\mathcal{E}_n))$. Since $\mathcal{T}$ satisfies closure under CN, then for each $\mathcal{E}_i$ it holds that $\mathsf{CN}(\text{Concs}(\mathcal{E}_i)) = \text{Concs}(\mathcal{E}_i)$. Thus, $x \in \text{Concs}(\mathcal{E}_1) \cap \ldots \cap \text{Concs}(\mathcal{E}_n)$. From Property 8, it holds that $x \in \text{Output}(\mathcal{T})$. □

Proposition 3 *Let $\mathcal{T} = (\text{Args}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base. If $\mathcal{T}$ is closed under sub-arguments and under CN, then for all $\mathcal{E} \in \text{Ext}(\mathcal{T})$, $\text{Concs}(\mathcal{E}) = \mathsf{CN}(\text{Base}(\mathcal{E}))$.*

Proof Assume that $\mathcal{T} = (\text{Args}(\Sigma), \mathcal{R})$ is closed under sub-arguments and under CN. From Property 4 in (1), since $\mathcal{T}$ is closed under sub-arguments, then it follows that $\text{Base}(\mathcal{E}) \subseteq \text{Concs}(\mathcal{E})$. By monotonicity of CN, we get $\mathsf{CN}(\text{Base}(\mathcal{E})) \subseteq \mathsf{CN}(\text{Concs}(\mathcal{E}))$. Since $\mathcal{T}$ is closed under CN, then $\mathsf{CN}(\text{Base}(\mathcal{E})) \subseteq \text{Concs}(\mathcal{E})$. Besides, by definition of $\text{Concs}(\mathcal{E})$, $\text{Concs}(\mathcal{E}) \subseteq \bigcup \mathsf{CN}(\text{Supp}(a_i))$ with $a_i \in \mathcal{E}$. From Property 1, it follows that $\text{Concs}(\mathcal{E}) \subseteq \mathsf{CN}(\bigcup \text{Supp}(a_i))$, thus $\text{Concs}(\mathcal{E}) \subseteq \mathsf{CN}(\text{Base}(\mathcal{E}))$. □

Proposition 4 *If an argumentation system $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ satisfies consistency,* *then the set* Output($\mathcal{T}$) *is consistent.*

Proof Let $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ be an argumentation system built over a knowledge base Σ. Assume that $\mathcal{T}$ satisfies consistency. Thus, $\forall \mathcal{E}_i \in \text{Ext}(\mathcal{T})$, $\text{Concs}(\mathcal{E}_i)$ is consistent. Let $\mathcal{E}$ be a given extension in the set $\text{Ext}(\mathcal{T})$. Since $\cap\text{Concs}(\mathcal{E}_i) \subseteq \text{Concs}(\mathcal{E})$, then $\cap\text{Concs}(\mathcal{E}_i)$ is consistent as well. Besides, from Property 8, $\text{Output}(\mathcal{T}) = \cap\text{Concs}(\mathcal{E}_i)$. It follows that $\text{Output}(\mathcal{T})$ is consistent. □

Proposition 5 *Let* $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ *be an argumentation system over a knowledge base* Σ *such that for all* $x \in \Sigma$, $x \notin \mathsf{CN}(\emptyset)$. *If* $\mathcal{T}$ *satisfies consistency and is closed under sub-arguments, then for all* $\mathcal{E} \in \text{Ext}(\mathcal{T})$, $\text{Base}(\mathcal{E})$ *is consistent.*

Proof Let $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ. Assume that $\mathcal{T}$ satisfies consistency and closure under sub-arguments. From closure under sub-arguments, it follows that for all $\mathcal{E} \in \text{Ext}(\mathcal{T})$, $\text{Base}(\mathcal{E}) \subseteq \text{Concs}(\mathcal{E})$ (from Property 4 in (1)). Since $\mathcal{T}$ satisfies consistency, the set $\text{Concs}(\mathcal{E})$ is consistent. From Property 2, it follows that $\text{Base}(\mathcal{E})$ is consistent. □

Proposition 6 *Let* $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ *be an argumentation system over a knowledge base* Σ. *If* $\mathcal{T}$ *satisfies free precedence, then* $\text{Free}(\Sigma) \subseteq \text{Output}(\mathcal{T})$ *(under any of the reviewed semantics).*

Proof Let $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ, and let $\text{Ext}(\mathcal{T})$ be its set of extensions under any of the reviewed semantics. Assume that $\mathcal{T}$ satisfies Postulate 4. Thus, for all $\mathcal{E} \in \text{Ext}(\mathcal{T})$, $\text{Arg}(\text{Free}(\Sigma)) \subseteq \mathcal{E}$. Assume that $\text{Free}(\Sigma) \neq \emptyset$. Thus, for all $x \in \text{Free}(\Sigma)$, $(\{x\}, x) \in \text{Arg}(\text{Free}(\Sigma))$ (indeed, $x \notin \mathsf{CN}(\emptyset)$) and thus, $(\{x\}, x) \in \mathcal{E}$ (for all $\mathcal{E} \in \text{Ext}(\mathcal{T})$). Consequently, $\text{Free}(\Sigma) \subseteq \text{Concs}(\mathcal{E})$ (for all $\mathcal{E} \in \text{Ext}$). So, $\text{Free}(\Sigma) \subseteq \text{Output}(\mathcal{T})$. □

Proposition 7 *If an argumentation system* $\mathcal{T}$ *is closed under both* CN *and sub-arguments and satisfies the exhaustiveness postulate, then* $\forall \mathcal{E} \in \text{Ext}(\mathcal{T})$, $\mathcal{E} = \text{Arg}(\text{Base}(\mathcal{E}))$ *(under any of the reviewed semantics).*

Proof Let $\mathcal{T}$ be an argumentation system that satisfies exhaustiveness and that is closed under both CN and sub-arguments. Let $\text{Ext}(\mathcal{T})$ be its extensions under any of the reviewed semantics. Let $\mathcal{E} \in \text{Ext}(\mathcal{T})$. From the definition of Arg and Base, it follows that $\mathcal{E} \subseteq \text{Arg}(\text{Base}(\mathcal{E}))$.

Assume now $a \in \text{Arg}(\text{Base}(\mathcal{E}))$ and let $a = (X, x)$. Thus, $X \subseteq \text{Base}(\mathcal{E})$. By monotonicity of CN, $\mathsf{CN}(X) \subseteq \mathsf{CN}(\text{Base}(\mathcal{E}))$. From Proposition 3, since $\mathcal{T}$ is closed under both CN and sub-arguments, then $\text{Concs}(\mathcal{E}) = \mathsf{CN}(\text{Base}(\mathcal{E}))$. Thus, $\mathsf{CN}(X) \subseteq \text{Concs}(\mathcal{E})$. Besides, $X \subseteq \mathsf{CN}(X)$ (from Expansion Axiom of CN) and $x \in \mathsf{CN}(X)$ (from the definition of an argument), thus, $X \cup \{x\} \subseteq \text{Concs}(\mathcal{E})$. By exhaustiveness of $\mathcal{T}$, $a \in \mathcal{E}$. □

Proposition 8 *Let* $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ *be an argumentation system over a knowledge base* Σ *such that* $\mathcal{R}$ *is conflict-dependent. If* $\mathcal{T}$ *satisfies consistency and is closed under sub-arguments (under naive and stable semantics), then it is closed under* CN *(under naive and stable semantics).*

Proof Let $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent. Assume that $\mathcal{T}$ is closed under sub-arguments and satisfies consistency. Assume also that $\mathcal{T}$ violates closure under CN. Thus, $\exists \mathcal{E} \in \text{Ext}_n(\mathcal{T})$ such that $\text{Concs}(\mathcal{E}) \neq \mathsf{CN}(\text{Concs}(\mathcal{E}))$. This means that $\exists x \in \mathsf{CN}(\text{Concs}(\mathcal{E}))$ and $x \notin \text{Concs}(\mathcal{E})$. Besides, $\mathsf{CN}(\text{Concs}(\mathcal{E})) \subseteq \mathsf{CN}(\text{Base}(\mathcal{E}))$. Thus, $x \in \mathsf{CN}(\text{Base}(\mathcal{E}))$. Since CN verifies compactness, then $\exists X \subseteq \text{Base}(\mathcal{E})$ such that X is finite and $x \in$

$\mathsf{CN}(X)$. Moreover, from Proposition 5, $\text{Base}(\mathcal{E})$ is consistent. Then, X is consistent as well (from Property 2). Consequently, the pair (X, x) is an argument. Besides, since $x \notin \text{Concs}(\mathcal{E})$ then $(X, x) \notin \mathcal{E}$. This means that $\exists a \in \mathcal{E}$ such that $a\mathcal{R}(X, x)$ or $(X, x)\mathcal{R}a$. Finally, since $\mathcal{R}$ is conflict-dependent, then $\text{Supp}(a) \cup X$ is inconsistent and consequently $\text{Base}(\mathcal{E})$ is inconsistent. This contradicts the assumption.

Since $\text{Ext}_s(\mathcal{T}) \subseteq \text{Ext}_n(\mathcal{T})$, then since $\mathcal{T}$ satisfies closure under CN under naive semantics, then it also satisfies the postulate under stable semantics. □

Proposition 9 *Let* $(\text{Arg}(\Sigma), \mathcal{R})$ *be such that* $\mathcal{R}$ *is conflict-dependent. If* Σ *is consistent, then* $\mathcal{R} = \emptyset$.

Proof Let $(\text{Arg}(\Sigma), \mathcal{R})$ be s.t. $\mathcal{R}$ is conflict-dependent. Assume that Σ is consistent. Then, for all $a, b \in \text{Arg}(\Sigma)$, $\text{Supp}(a) \cup \text{Supp}(b)$ is consistent. Since $\mathcal{R}$ is conflict-dependent, then $(a, b) \notin \mathcal{R}$. Thus, $\mathcal{R} = \emptyset$. □

Proposition 10 *Let* $(\text{Arg}(\Sigma), \mathcal{R})$ *be such that* $\mathcal{R}$ *is conflict-dependent.* $\forall \mathcal{E} \subseteq \text{Arg}(\Sigma)$, *if* $\text{Base}(\mathcal{E})$ *is consistent, then* $\mathcal{E}$ *is conflict-free.*

Proof Let $\mathcal{E} \subseteq \text{Arg}(\Sigma)$. Since $\text{Base}(\mathcal{E})$ is consistent, then so is $\text{Supp}(a) \cup \text{Supp}(b)$ for all a and b in $\mathcal{E}$ (according to Property 2). Hence, there exist no minimal conflict $C \subseteq \text{Supp}(a) \cup \text{Supp}(b)$. By the definition of $\mathcal{R}$ being conflict-dependent, $(a, b) \notin \mathcal{R}$ ensues. □

Proposition 11 *Let* $(\text{Arg}(\Sigma), \mathcal{R})$ *be such that* $\mathcal{R}$ *is conflict-dependent. For all* $a \in \text{Arg}(\Sigma)$, $(a, a) \notin \mathcal{R}$.

Proof Assume that $\mathcal{R}$ is conflict-dependent and $a \in \text{Arg}(\Sigma)$ such that$(a, a) \in \mathcal{R}$. Since $\mathcal{R}$ is conflict-dependent, then $\exists C \in \mathcal{C}_\Sigma$ such that$C \subseteq \text{Supp}(a)$. This means that $\text{Supp}(a)$ is inconsistent. This contradicts the fact that a is an argument. □

Proposition 12 *Let* $(\mathcal{L}, \mathsf{CN})$ *be adjunctive and* Σ *be a knowledge base such that* $\exists C \in \mathcal{C}_\Sigma$ *and* $|C| > 2$. *If* $\mathcal{R}$ *is conflict-dependent and symmetric, then the argumentation system* $(\text{Arg}(\Sigma), \mathcal{R})$ *violates consistency.*

Proof Let C be a minimal conflict in a knowledge base Σ. Consider a partition $\{X_1, X_2, X_3\}$ of C. Due to Proposition 1, there exist a_1, a_2, a_3 in $\text{Arg}(\Sigma)$ such that $\mathsf{CN}(\text{Supp}(a_i)) = \mathsf{CN}(X_i)$ for $i = 1..3$. For $\{a_1, a_2, a_3\}$ not to be an admissible extension, either it is not conflict-free or it fails to defend its elements. Assume that $\{a_1, a_2, a_3\}$ is not conflict-free. I.e., $a_i\mathcal{R}a_j$ for some i and j in $\{1, 2, 3\}$. Since $\mathcal{R}$ is conflict-dependent, there exists $C' \in \mathcal{C}_\Sigma$ such that $C' \subseteq \text{Supp}(a_i) \cup \text{Supp}(a_j)$. Hence, $\mathsf{CN}(\text{Supp}(a_i) \cup \text{Supp}(a_j)) = \mathcal{L}$. However, $\mathsf{CN}(\text{Supp}(a_i) \cup \text{Supp}(a_j)) = \mathsf{CN}(\mathsf{CN}(\text{Supp}(a_i)) \cup \mathsf{CN}(\text{Supp}(a_j))) = \mathsf{CN}(\mathsf{CN}(X_i) \cup \mathsf{CN}(X_j)) = \mathsf{CN}(X_i \cup X_j)$, meaning that $X_i \cup X_j$ is an inconsistent proper subset of C, contradicting $C \in \mathcal{C}_\Sigma$. Otherwise, assume that $\{a_1, a_2, a_3\}$ fails to defend its elements. It obviously cannot be the case because $\mathcal{R}$ is symmetric: If $a\mathcal{R}a_i$ then $a_i\mathcal{R}a$. □

Theorem 1 *Let* $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ *be an argumentation system over a knowledge base* Σ *such that* $\mathcal{R}$ *is conflict-dependent. If* $\mathcal{T}$ *satisfies consistency and is closed under sub-arguments (under naive semantics), then:*

- *For all* $\mathcal{E} \in \text{Ext}_n(\mathcal{T})$, $\text{Base}(\mathcal{E}) \in \text{Max}(\Sigma)$.
- *For all* $\mathcal{E} \in \text{Ext}_n(\mathcal{T})$, $\mathcal{E} = \text{Arg}(\text{Base}(\mathcal{E}))$.
- *For all* $\mathcal{E}_i, \mathcal{E}_j \in \text{Ext}_n(\mathcal{T})$, *if* $\text{Base}(\mathcal{E}_i) = \text{Base}(\mathcal{E}_j)$ *then* $\mathcal{E}_i = \mathcal{E}_j$.

Proof Let $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent. Assume that $\mathcal{T}$ satisfies consistency and is closed under sub-arguments. Let $\mathcal{E} \in \text{Ext}_n(\mathcal{T})$. From Proposition 5, Base($\mathcal{E}$) is consistent.

Assume now that Base($\mathcal{E}$) is not maximal (for set inclusion) consistent. Thus, $\exists x \in \Sigma \setminus \text{Base}(\mathcal{E})$ such that $\text{Base}(\mathcal{E}) \cup \{x\}$ is consistent. This means that $\{x\}$ is consistent. Thus, $\exists a \in \mathcal{A}$ such that $\text{Supp}(a) = \{x\}$. Since $x \notin \text{Base}(\mathcal{E})$, then $a \notin \mathcal{E}$. Since $\mathcal{E}$ is a naive extension, then $\exists b \in \mathcal{E}$ such that $a\mathcal{R}b$ or $b\mathcal{R}a$. Since $\mathcal{R}$ is conflict-dependent, then $\text{Supp}(a) \cup \text{Supp}(b)$ is inconsistent. But, $\text{Supp}(b) \subseteq \text{Base}(\mathcal{E})$, this would mean that $\text{Base}(\mathcal{E}) \cup \{x\}$ is inconsistent. Contradiction.

Let $\mathcal{E} \in \text{Ext}_n(\mathcal{T})$. It is obvious that $\mathcal{E} \subseteq \text{Arg}(\text{Base}(\mathcal{E}))$ since the construction of arguments is monotonic. Let $a \in \text{Arg}(\text{Base}(\mathcal{E}))$. Thus, $\text{Supp}(a) \subseteq \text{Base}(\mathcal{E})$. Assume that $a \notin \mathcal{E}$, then $\exists b \in \mathcal{E}$ such that $a\mathcal{R}b$ or $b\mathcal{R}a$. Since $\mathcal{R}$ is conflict-dependent, then $\text{Supp}(a) \cup \text{Supp}(b)$ is inconsistent. Besides, $\text{Supp}(a) \cup \text{Supp}(b) \subseteq \text{Base}(\mathcal{E})$. This means that Base($\mathcal{E}$) is inconsistent. Contradiction.

Let now $\mathcal{E}_i, \mathcal{E}_j \in \text{Ext}_n(\mathcal{T})$. Assume that $\text{Base}(\mathcal{E}_i) = \text{Base}(\mathcal{E}_j)$. Then, $\text{Arg}(\text{Base}(\mathcal{E}_i)) = \text{Arg}(\text{Base}(\mathcal{E}_j))$. Besides, from the previous bullet, $\mathcal{E}_i = \text{Arg}(\text{Base}(\mathcal{E}_i))$ and $\mathcal{E}_j = \text{Arg}(\text{Base}(\mathcal{E}_j))$. Consequently, $\mathcal{E}_i = \mathcal{E}_j$. □

Theorem 2 *Let* $(\mathcal{L}, \mathsf{CN})$ *be adjunctive. Let* $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ *be an argumentation system over a knowledge base* Σ *such that* $\mathcal{R}$ *is conflict-dependent. If* $\mathcal{T}$ *satisfies consistency and is closed under sub-arguments (under naive semantics), then:*

- *For all* $\mathcal{S} \in \text{Max}(\Sigma)$, $\text{Arg}(\mathcal{S}) \in \text{Ext}_n(\mathcal{T})$.
- *For all* $\mathcal{S}_i, \mathcal{S}_j \in \text{Max}(\Sigma)$, *if* $\text{Arg}(\mathcal{S}_i) = \text{Arg}(\mathcal{S}_j)$ *then* $\mathcal{S}_i = \mathcal{S}_j$.
- *For all* $\mathcal{S} \in \text{Max}(\Sigma)$, $\mathcal{S} = \text{Base}(\text{Arg}(\mathcal{S}))$.

Proof Let $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that$\mathcal{R}$ is conflict-dependent. Assume that $\mathcal{T}$ satisfies Postulates 2 and 3.

Let $\mathcal{S} \in \text{Max}(\Sigma)$, and assume that $\text{Arg}(\mathcal{S}) \notin \text{Ext}_n(\mathcal{T})$. Since $\mathcal{R}$ is conflict-dependent and $\mathcal{S}$ is consistent, then it follows from Proposition 10 that Arg($\mathcal{S}$) is conflict-free. Thus, Arg($\mathcal{S}$) is not maximal for set inclusion. So, $\exists a \in \mathcal{A}$ such that $\text{Arg}(\mathcal{S}) \cup \{a\}$ is conflict-free. There are two possibilities:

(1) $\mathcal{S} \cup \text{Supp}(a)$ is consistent. But since $\mathcal{S} \in \text{Max}(\Sigma)$, then $\text{Supp}(a) \subseteq \mathcal{S}$, and this would mean that $a \in \text{Arg}(\mathcal{S})$.

(2) $\mathcal{S} \cup \text{Supp}(a)$ is inconsistent. Thus, $\exists C \in \mathcal{C}_\Sigma$ such that $C \subseteq \mathcal{S} \cup \text{Supp}(a)$. Let $X_1 = C \cap \mathcal{S}$ and $X_2 = C \cap \text{Supp}(a)$ with $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$ (since $\mathcal{S}$ and Supp(a) are consistent). From Property 4, $\exists x_1 \in \mathsf{CN}(X_1)$ and $\exists x_2 \in \mathsf{CN}(X_2)$ such that the set $\{x_1, x_2\}$ is inconsistent. Note that (X_1, x_1) and (X_2, x_2) are arguments. Moreover, $(X_1, x_1) \in \text{Arg}(\mathcal{S})$ and $(X_2, x_2) \in \text{Sub}(a)$. Besides, since $\text{Arg}(\mathcal{S}) \cup \{a\}$ is conflict-free, then $\exists \mathcal{E} \in \text{Ext}(\mathcal{T})$ such that $\text{Arg}(\mathcal{S}) \cup \{a\} \subseteq \mathcal{E}$. Thus, $(X_1, x_1) \in \mathcal{E}$. Since $\mathcal{T}$ is closed under sub-arguments then $(X_2, x_2) \in \mathcal{E}$. Thus, $\{x_1, x_2\} \subseteq \text{Concs}(\mathcal{E})$. From Property 2, it follows that Concs($\mathcal{E}$) is inconsistent. This contradicts the fact that $\mathcal{T}$ satisfies consistency.

Let now $\mathcal{S}_i, \mathcal{S}_j \in \text{Max}(\Sigma)$ be such that $\text{Arg}(\mathcal{S}_i) = \text{Arg}(\mathcal{S}_j)$. Assume that $\mathcal{S}_i \neq \mathcal{S}_j$, thus $\exists x \in \mathcal{S}_i$ and $x \notin \mathcal{S}_j$. Besides, $\mathcal{S}_i$ is consistent, then so is the set $\{x\}$. Consequently, $\exists a \in \mathcal{A}$ such that $\text{Supp}(a) = \{x\}$. It follows also that $a \in \text{Arg}(\mathcal{S}_i)$ and thus $a \in \text{Arg}(\mathcal{S}_j)$. By definition of an argument, $\text{Supp}(a) \subseteq \mathcal{S}_j$. Contradiction.

Let $\mathcal{S} \in \text{Max}(\Sigma)$. Since $\mathcal{S}$ is consistent, then $\forall x \in \mathcal{S}$, it holds that the set $\{x\}$ is consistent as well (from Property 2). Then, $(\{x\}, x)$ is an argument in Arg($\mathcal{S}$) (from Property 6). Thus, $\mathcal{S} \subseteq \text{Base}(\text{Arg}(\mathcal{S}))$. Conversely, let $x \in \text{Base}(\text{Arg}(\mathcal{S}))$. By definition of the function Base, $\exists a \in \text{Arg}(\mathcal{S})$ such that $x \in \text{Supp}(a)$. Besides, by definition of an argument, $\text{Supp}(a) \subseteq \mathcal{S}$. Consequently, $x \in \mathcal{S}$. □

Theorem 3 *Let $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent, $\mathcal{T}$ satisfies consistency and is closed under sub-arguments (under naive semantics).* $\text{Output}(\mathcal{T}) = \bigcap \mathsf{CN}(\mathcal{S}_i)$ *where $\mathcal{S}_i$ ranges over $\{\mathcal{S}_i \in \text{Max}(\Sigma) \mid \exists \mathcal{E}_i \in \text{Ext}_n(\mathcal{T})$ and $\mathcal{S}_i = \text{Base}(\mathcal{E}_i)\}$.*

Proof Let $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent. Assume that $\mathcal{T}$ satisfies consistency and is closed under sub-arguments. From Proposition 8, $\mathcal{T}$ enjoys closure under CN (under naive semantics). From Proposition 3, for all $\mathcal{E} \in \text{Ext}_n(\mathcal{T})$, $\text{Concs}(\mathcal{E}) = \mathsf{CN}(\text{Base}(\mathcal{E}))$. Besides, from Theorem 1, for all $\mathcal{E}_i \in \text{Ext}_n(\mathcal{T})$, $\exists! \mathcal{S}_i \in \text{Max}(\Sigma)$ such that $\text{Base}(\mathcal{E}_i) = \mathcal{S}_i$. Thus, $\text{Concs}(\mathcal{E}_i) = \mathsf{CN}(\mathcal{S}_i)$. By definition, $\text{Output}(\mathcal{T}) = \bigcap \text{Concs}(\mathcal{E}_i)$, thus $\text{Output}(\mathcal{T}) = \bigcap \mathsf{CN}(\mathcal{S}_i)$. □

Theorem 4 *Let $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent. If $\mathcal{T}$ satisfies consistency and closure under sub-arguments (under stable semantics) and $\text{Ext}_s(\mathcal{T}) \neq \emptyset$, then:*

- *For all $\mathcal{E} \in \text{Ext}_s(\mathcal{T})$, $\text{Base}(\mathcal{E}) \in \text{Max}(\Sigma)$.*
- *For all $\mathcal{E} \in \text{Ext}_s(\mathcal{T})$, $\mathcal{E} = \text{Arg}(\text{Base}(\mathcal{E}))$.*
- *For all $\mathcal{E}_i, \mathcal{E}_j \in \text{Ext}_s(\mathcal{T})$, if $\text{Base}(\mathcal{E}_i) = \text{Base}(\mathcal{E}_j)$ then $\mathcal{E}_i = \mathcal{E}_j$.*

Proof Let $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent. Let $\mathcal{E} \in \text{Ext}_s(\mathcal{T})$. Since $\mathcal{T}$ satisfies Postulates 1, 2 and 3, then $\text{Base}(\mathcal{E})$ is consistent (from Proposition 5). Assume now that $\text{Base}(\mathcal{E})$ is not maximal for set inclusion. Thus, $\exists x \in \Sigma \setminus \text{Base}(\mathcal{E})$ such that $\text{Base}(\mathcal{E}) \cup \{x\}$ is consistent. This means that $\{x\}$ is consistent. Thus, from Property 6, $\exists a \in \text{Arg}(\Sigma)$ such that $\text{Supp}(a) = \{x\}$. Since $x \notin \text{Base}(\mathcal{E})$, then $a \notin \mathcal{E}$. Since $\mathcal{E}$ is a stable extension, then $\exists b \in \mathcal{E}$ such that $b\mathcal{R}a$. Since $\mathcal{R}$ is conflict-dependent, then $\text{Supp}(a) \cup \text{Supp}(b)$ is inconsistent. But, $\text{Supp}(b) \subseteq \text{Base}(\mathcal{E})$, this would mean that $\text{Base}(\mathcal{E}) \cup \{x\}$ is inconsistent. Contradiction.

Let $\mathcal{E} \in \text{Ext}_s(\mathcal{T})$. It is obvious that $\mathcal{E} \subseteq \text{Arg}(\text{Base}(\mathcal{E}))$ since the construction of arguments is monotonic. Let $a \in \text{Arg}(\text{Base}(\mathcal{E}))$. Thus, $\text{Supp}(a) \subseteq \text{Base}(\mathcal{E})$. Assume that $a \notin \mathcal{E}$, then $\exists b \in \mathcal{E}$ such that $b\mathcal{R}a$. Since $\mathcal{R}$ is conflict-dependent, then $\text{Supp}(a) \cup \text{Supp}(b)$ is inconsistent. Besides, $\text{Supp}(a) \cup \text{Supp}(b) \subseteq \text{Base}(\mathcal{E})$. This means that $\text{Base}(\mathcal{E})$ is inconsistent. Contradiction.

Let now $\mathcal{E}_i, \mathcal{E}_j \in \text{Ext}_s(\mathcal{T})$. Assume that $\text{Base}(\mathcal{E}_i) = \text{Base}(\mathcal{E}_j)$. Then, $\text{Arg}(\text{Base}(\mathcal{E}_i)) = \text{Arg}(\text{Base}(\mathcal{E}_j))$. Besides, from bullet 2 of this proof, $\mathcal{E}_i = \text{Arg}(\text{Base}(\mathcal{E}_i))$ and $\mathcal{E}_j = \text{Arg}(\text{Base}(\mathcal{E}_j))$. Consequently, $\mathcal{E}_i = \mathcal{E}_j$. □

Theorem 5 *It holds that $\Re_{s1} = \emptyset$.*

Proof Let $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R} \in \Re_{s1}$. Thus, $\text{Output}(\mathcal{T}) = \emptyset$. Assume that $\text{Free}(\Sigma) \neq \emptyset$. Thus, $\mathcal{T}$ violates free precedence postulate. This contradicts the fact that $\mathcal{R} \in \Re_{s1}$. □

Theorem 6 *Let $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R} \in \Re_{s_2}$.* $\text{Output}(\mathcal{T}) = \bigcap \mathsf{CN}(\mathcal{S}_i)$ *where $\mathcal{S}_i$ ranges over $\{\mathcal{S}_i \in \text{Max}(\Sigma) \mid \exists \mathcal{E}_i \in \text{Ext}_s(\mathcal{T})$ and $\mathcal{S}_i = \text{Base}(\mathcal{E}_i)\}$.*

Proof Let $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R} \in \Re_{s2}$. From Proposition 3, for all $\mathcal{E}_i \in \text{Ext}_s(\mathcal{T})$, $\text{Concs}(\mathcal{E}_i) = \mathsf{CN}(\text{Base}(\mathcal{E}_i))$. Thus, $\text{Output}(\mathcal{T}) = \bigcap \mathsf{CN}(\text{Base}(\mathcal{E}_i))$ with $\mathcal{E}_i \in \text{Ext}_s(\mathcal{T})$. From Theorem 4, for all $\mathcal{E}_i \in \text{Ext}_s(\mathcal{T})$, $\exists! \mathcal{S}_i \in \text{Max}(\Sigma)$ such that $\text{Base}(\mathcal{E}_i) = \mathcal{S}_i$. Thus, $\text{Output}(\mathcal{T}) = \bigcap \mathsf{CN}(\mathcal{S}_i)$. □

Theorem 7 *Let $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R} \in \Re_{s3}$. For all $\mathcal{S} \in \text{Max}(\Sigma)$, $\text{Arg}(\mathcal{S}) \in \text{Ext}_s(\mathcal{T})$.*

Proof Let $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R} \in \Re_{s3}$. Let $\mathcal{S} \in \text{Max}(\Sigma)$. Since $|\text{Ext}_s(\mathcal{T})(\mathcal{T})| = |\text{Max}(\Sigma)|$, then from Theorem 4, $\exists \mathcal{E} \in \text{Ext}_s(\mathcal{T})$ such that $\text{Base}(\mathcal{E}) = \mathcal{S}$. Besides, from the same theorem, $\mathcal{E} = \text{Arg}(\text{Base}(\mathcal{E}))$, thus $\mathcal{E} = \text{Arg}(\mathcal{S})$. Consequently, $\text{Arg}(\mathcal{S}) \in \text{Ext}_s(\mathcal{T})$. □

Theorem 8 *Let $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R} \in \Re_{s3}$. The equality $\text{Ext}_n(\mathcal{T}) = \text{Ext}_s(\mathcal{T})$ holds. If $\mathcal{T}$ satisfies the postulates under preferred semantics, then $\text{Ext}_s(\mathcal{T}) = \text{Ext}_p(\mathcal{T})$.*

Proof Let $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R} \in \Re_{s3}$. From Theorem 7, for all $\mathcal{S} \in \text{Max}(\Sigma)$, $\text{Arg}(\mathcal{S}) \in \text{Ext}_s(\mathcal{T})$. From Theorem 1, $|\text{Ext}_n(\mathcal{T})| \leq |\text{Max}(\Sigma)|$. Thus, $\text{Ext}_n(\mathcal{T}) = \text{Ext}_s(\mathcal{T})$. Assume now that $\mathcal{T}$ satisfies the postulates under preferred semantics, then from Theorem 10, $|\text{Ext}_p(\mathcal{T})| = |\text{Max}(\Sigma)|$ and $\text{Ext}_s(\mathcal{T}) = \text{Ext}_p(\mathcal{T})$. □

Theorem 9 *Let $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ be an argumentation system such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies consistency and closure under sub-arguments (under preferred semantics). For all $\mathcal{E} \in \text{Ext}_p(\mathcal{T})$, there exists $\mathcal{S} \in \text{Max}(\Sigma)$ such that $\text{Base}(\mathcal{E}) \subseteq \mathcal{S}$.*

Proof Let $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ be an argumentation system such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies consistency and closure under sub-arguments. Let $\mathcal{E} \in \text{Ext}_p(\mathcal{T})$. Due to consistency and closure under sub-arguments, $\text{Base}(\mathcal{E})$ is consistent (cf. Proposition 5). From Property 3, there exists $\mathcal{S} \in \text{Max}(\Sigma)$ such that $\text{Base}(\mathcal{E}) \subseteq \mathcal{S}$. □

Theorem 10 *Let $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ be an argumentation system such that$\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies consistency and closure under sub-arguments (under preferred semantics). For all $\mathcal{E}_i, \mathcal{E}_j \in \text{Ext}_p(\mathcal{T})$, if $\text{Base}(\mathcal{E}_i) \subseteq \text{Base}(\mathcal{E}_j)$ then $\mathcal{E}_i = \mathcal{E}_j$.*

Proof Let $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ be an argumentation system such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies consistency and closure under sub-arguments. Assume that $\mathcal{E}_i, \mathcal{E}_j \in \text{Ext}_p(\mathcal{T})$ and $\text{Base}(\mathcal{E}_i) \subseteq \text{Base}(\mathcal{E}_j)$.

We first show that $\forall x \in \{i, j\}$, $\text{Arg}(\text{Base}(\mathcal{E}_x))$ is conflict-free. Assume that $\text{Arg}(\text{Base}(\mathcal{E}_x))$ is not conflict-free. Thus, $\exists a, b \in \text{Arg}(\text{Base}(\mathcal{E}_x))$ such that $a\mathcal{R}b$ or $b\mathcal{R}a$. Since $\mathcal{R}$ is conflict-dependent, then $\text{Supp}(a) \cup \text{Supp}(b)$ is inconsistent. Besides, $\text{Supp}(a) \cup \text{Supp}(b) \subseteq \text{Base}(\mathcal{E}_x)$. Thus, $\text{Base}(\mathcal{E}_x)$ is inconsistent. This contradicts Proposition 5.

Assume that $\mathcal{E}_i \setminus \mathcal{E}_j \neq \emptyset$. Let $\mathcal{E} = \mathcal{E}_j \cup (\mathcal{E}_i \setminus \mathcal{E}_j)$. $\mathcal{E} \subseteq \text{Arg}(\text{Base}(\mathcal{E}_j))$. Thus, $\mathcal{E}$ is conflict-free (since $\text{Arg}(\text{Base}(\mathcal{E}_j))$ is conflict-free). Moreover, $\mathcal{E}$ defends any element in $\mathcal{E}_j$ (since $\mathcal{E}_j \in \text{Ext}_p(\mathcal{T})$) and any element in $\mathcal{E}_i \setminus \mathcal{E}_j$ (since $\mathcal{E}_i \in \text{Ext}_p(\mathcal{T})$). Thus, $\mathcal{E}$ is an admissible set. This contradicts the fact that $\mathcal{E}_j \in \text{Ext}_p(\mathcal{T})$. The same reasoning holds for the case $\mathcal{E}_j \setminus \mathcal{E}_i \neq \emptyset$ and having $\mathcal{E} = \mathcal{E}_i \cup \mathcal{E}_j \setminus \mathcal{E}_i$. □

Theorem 11 *Let $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ be an argumentation system such that $\mathcal{R}$ is* *conflict-dependent and $\mathcal{T}$ satisfies consistency and closure under sub-arguments (under preferred semantics). Let $\mathcal{S} \in \mathrm{Max}(\Sigma)$. For all $\mathcal{E}_i, \mathcal{E}_j \in \mathrm{Ext}_p(\mathcal{T})$, if $\mathrm{Base}(\mathcal{E}_i) \subseteq \mathcal{S}$ and $\mathrm{Base}(\mathcal{E}_j) \subseteq \mathcal{S}$, then $\mathcal{E}_i = \mathcal{E}_j$.*

Proof Let $\mathcal{T} = (\mathrm{Arg}(\Sigma), \mathcal{R})$ be an argumentation system such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies consistency and closure under sub-arguments (under preferred semantics). Assume there exist two distinct preferred extensions $\mathcal{E}_1$ and $\mathcal{E}_2$ such that for some $\mathcal{S} \in \mathrm{Max}(\Sigma)$, both $\mathrm{Base}(\mathcal{E}_1) \subseteq \mathcal{S}$ and $\mathrm{Base}(\mathcal{E}_2) \subseteq \mathcal{S}$ hold. Then, $\mathrm{Base}(\mathcal{E}_1) \cup \mathrm{Base}(\mathcal{E}_2) \subseteq \mathcal{S}$ and $\mathrm{Base}(\mathcal{E}_1) \cup \mathrm{Base}(\mathcal{E}_2)$ is consistent. Since $\mathcal{R}$ is conflict-dependent, $a\mathcal{R}b$ would demand $\mathrm{Supp}(a) \cup \mathrm{Supp}(b)$ to be inconsistent. Therefore, $a\mathcal{R}b$ is impossible for a and b both in $\mathcal{E}_1 \cup \mathcal{E}_2$. That is, $\mathcal{E}_1 \cup \mathcal{E}_2$ is conflict-free. Since $\mathcal{E}_1$ is an extension, it defends all arguments in $\mathcal{E}_1$. Since $\mathcal{E}_2$ is an extension, it defends all arguments in $\mathcal{E}_2$. Hence, $\mathcal{E}_1 \cup \mathcal{E}_2$ defends all arguments in $\mathcal{E}_1 \cup \mathcal{E}_2$. That is, $\mathcal{E}_1 \cup \mathcal{E}_2$ is an admissible set, and there exists a preferred extension $\mathcal{E}_3$ that contains it (possibly improperly). According to Theorem 10, $\mathrm{Base}(\mathcal{E}_1) \subseteq \mathrm{Base}(\mathcal{E}_1 \cup \mathcal{E}_2) = \mathrm{Base}(\mathcal{E}_3)$ yield $\mathcal{E}_1 = \mathcal{E}_3$, and, similarly, $\mathrm{Base}(\mathcal{E}_2) \subseteq \mathrm{Base}(\mathcal{E}_1 \cup \mathcal{E}_2) = \mathrm{Base}(\mathcal{E}_3)$ yield $\mathcal{E}_2 = \mathcal{E}_3$. Therefore, $\mathcal{E}_1 = \mathcal{E}_2$, contradicting the assumption that $\mathcal{E}_1$ and $\mathcal{E}_2$ are distinct. □

Theorem 12 *Let $\mathcal{T} = (\mathrm{Arg}(\Sigma), \mathcal{R})$ be an argumentation system such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies consistency and closure under sub-arguments (under preferred semantics). It holds that $1 \leq |\mathrm{Ext}_p(\mathcal{T})| \leq |\mathrm{Max}(\Sigma)|$.*

Proof Let $\mathcal{T} = (\mathrm{Arg}(\Sigma), \mathcal{R})$ be an argumentation system such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies consistency and closure under sub-arguments (under preferred semantics). From Theorem 9, for all $\mathcal{E} \in \mathrm{Ext}_p(\mathcal{T})$, $\exists \mathcal{S} \in \mathrm{Max}(\Sigma)$ such that $\mathrm{Base}(\mathcal{E}) \subseteq \mathcal{S}$. From Theorem 11, there cannot exist two distinct preferred extensions $\mathcal{E}_i$ and $\mathcal{E}_j$ such that for some $\mathcal{S} \in \mathrm{Max}(\Sigma)$, both $\mathrm{Base}(\mathcal{E}_i) \subseteq \mathcal{S}$ and $\mathrm{Base}(\mathcal{E}_j) \subseteq \mathcal{S}$. Thus, every $\mathcal{S} \in \mathrm{Max}(\Sigma)$ is captured by at most one preferred extension of $\mathcal{T}$. It follows that $1 \leq |\mathrm{Ext}_p(\mathcal{T})| \leq |\mathrm{Max}(\Sigma)|$. □

Theorem 13 *Let $\mathcal{T} = (\mathrm{Arg}(\Sigma), \mathcal{R})$ be an argumentation system. If $\mathcal{R} \in \Re_{p_3}$ then:*

- *for all $\mathcal{S} \in \mathrm{Max}(\Sigma)$, $\mathrm{Arg}(\mathcal{S}) \in \mathrm{Ext}_p(\mathcal{T})$.*
- $|\mathrm{Ext}_p(\mathcal{T})| = |\mathrm{Max}(\Sigma)|$

Proof Let $\mathcal{T} = (\mathrm{Arg}(\Sigma), \mathcal{R})$ be an argumentation system such that $\mathcal{R} \in \Re_{p_3}$. From Theorem 7, for all $\mathcal{S} \in \mathrm{Max}(\Sigma)$, $\mathrm{Arg}(\mathcal{S}) \in \mathrm{Ext}_s(\mathcal{T})$. Thus, $\mathrm{Arg}(\mathcal{S}) \in \mathrm{Ext}_p(\mathcal{T})$.

Since $\mathcal{R} \in \Re_{p_3}$, thus $|\mathrm{Ext}_s(\mathcal{T})| = |\mathrm{Max}(\Sigma)|$. Assume now that there exists $\mathcal{E} \in \mathrm{Ext}_p(\mathcal{T})$ such that $\mathrm{Base}(\mathcal{E}) \notin \mathrm{Max}(\Sigma)$. Thus, $\exists \mathcal{S} \in \mathrm{Max}(\Sigma)$ such that $\mathrm{Base}(\mathcal{E}) \subseteq \mathcal{S}$. From the previous result, $\mathrm{Arg}(\mathcal{S}) \in \mathrm{Ext}_p(\mathcal{T})$. From Theorem 10, $\mathcal{E} = \mathrm{Arg}(\mathcal{S})$. Consequently, $|\mathrm{Ext}_p(\mathcal{T})| = |\mathrm{Max}(\Sigma)|$. □

Theorem 14 *Let $\mathcal{T} = (\mathrm{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R} \in \Re_{p_2}$. $\mathrm{Output}(\mathcal{T}) = \bigcap \mathsf{CN}(\mathcal{S}_i)$ where $\mathcal{S}_i$ ranges over $\{\mathcal{S}_i \in \mathrm{Cons}(\Sigma) \mid \exists \mathcal{E}_i \in \mathrm{Ext}_p(\mathcal{T})$ and $\mathcal{S}_i = \mathrm{Base}(\mathcal{E}_i)\}$.*

Proof Let $\mathcal{T} = (\mathrm{Arg}(\Sigma), \mathcal{R})$ be an argumentation system such that $\mathcal{R} \in \Re_{p_2}$. Let $\mathcal{E} \in \mathrm{Ext}_p(\mathcal{T})$. From Proposition 3, since $\mathcal{T}$ is closed both under CN and under sub-arguments, then $\mathrm{Concs}(\mathcal{E}) = \mathsf{CN}(\mathrm{Base}(\mathcal{E}))$. From Definition 9, $\mathrm{Output}(\mathcal{T}) = \bigcap \mathrm{Concs}(\mathcal{E}_i)$, $\mathcal{E}_i \in \mathrm{Ext}_p(\mathcal{T})$. Thus, $\mathrm{Output}(\mathcal{T}) = \bigcap \mathsf{CN}(\mathrm{Base}(\mathcal{E}_i))$, $\mathcal{E}_i \in \mathrm{Ext}_p(\mathcal{T})$. Besides, $\mathrm{Free}(\Sigma) \subseteq \mathrm{Base}(\mathcal{E})$ for all $\mathcal{E} \in \mathrm{Ext}_p(\mathcal{T})$ and from Proposition 5, $\mathrm{Base}(\mathcal{E})$ is consistent. Thus, $\mathrm{Base}(\mathcal{E}) \in \mathrm{Cons}(\Sigma)$. □

Theorem 15 *Let $\mathcal{T} = (\mathrm{Arg}(\Sigma), \mathcal{R})$ be such that $\mathcal{R}$ is conflict-dependent.*

- *For all $a \in \mathrm{Arg}(\mathrm{Free}(\Sigma))$, a neither attacks nor is attacked by another argument* *in* $\text{Arg}(\Sigma)$.

- $\text{Arg}(\text{Free}(\Sigma))$ *is an admissible extension of* $\mathcal{T}$.

Proof Let $(\text{Arg}(\Sigma), \mathcal{R})$ be such that $\mathcal{R}$ is conflict-dependent. Let $a \in \text{Arg}(\text{Free}(\Sigma))$. Assume that $\exists b \in \mathcal{A}$ s.t. $a\mathcal{R}b$ or $b\mathcal{R}a$. Since $\mathcal{R}$ is conflict-dependent, then $\exists C \in \mathcal{C}_\Sigma$ such that $C \subseteq \text{Supp}(a) \cup \text{Supp}(b)$. By definition of an argument, both $\text{Supp}(a)$ and $\text{Supp}(b)$ are consistent. Then, $C \cap \text{Supp}(a) \neq \emptyset$. This contradicts the fact that $\text{Supp}(a) \subseteq \text{Free}(\Sigma)$. Thus, $\text{Arg}(\text{Free}(\Sigma))$ is *conflict-free and can never be attacked.* Consequently, $\text{Arg}(\text{Free}(\Sigma))$ is an admissible extension of $\mathcal{T}$. □

Theorem 16 *Let* $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ *be an argumentation system such that* $\mathcal{R} \in \Re_{p_3}$.

$$\text{IE}(\mathcal{T}) = \bigcap_{\mathcal{E}_i \in \text{Ext}_p(\mathcal{T})} \mathcal{E}_i = \text{Arg}(\text{Free}(\Sigma)).$$

Proof Let $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ be an argumentation system such that $\mathcal{R} \in \Re_{p_3}$. Since $\mathcal{T}$ satisfies exhaustiveness, then from Proposition 7, for all $\mathcal{E}_i \in \text{Ext}_p(\mathcal{T})$, $\mathcal{E}_i = \text{Arg}(\text{Base}(\mathcal{E}_i))$. So,

$$\bigcap_{\mathcal{E}_i \in \text{Ext}_p(\mathcal{T})} \mathcal{E}_i = \text{Arg}(\bigcap_{\mathcal{E}_i \in \text{Ext}_p(\mathcal{T})} \text{Base}(\mathcal{E}_i)).$$

From Theorem 13, for all $\mathcal{E}_i \in \text{Ext}_p(\mathcal{T})$, $\text{Base}(\mathcal{E}_i) \in \text{Max}(\Sigma)$. Moreover, $|\text{Ext}_p(\mathcal{T})| = |\text{Max}(\Sigma)|$. Thus, $\bigcap_{\mathcal{E}_i \in \text{Ext}_p(\mathcal{T})} \text{Base}(\mathcal{E}_i) = \text{Free}(\Sigma)$. Consequently,

$$\bigcap_{\mathcal{E}_i \in \text{Ext}_p(\mathcal{T})} \mathcal{E}_i = \text{Arg}(\text{Free}(\Sigma)).$$

From Theorem 15, $\text{Arg}(\text{Free}(\Sigma))$ is an admissible set of $\mathcal{T}$. Thus, from Property 11,

$$\text{IE}(\mathcal{T}) = \bigcap_{\mathcal{E}_i \in \text{Ext}_p(\mathcal{T})} \mathcal{E}_i = \text{Arg}(\text{Free}(\Sigma)).$$

□

Theorem 17 *Let* $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ *be an argumentation system such that* $\mathcal{R}$ *is conflict-dependent and* $\mathcal{T}$ *satisfies the five postulates (under preferred semantics). If* $|\text{Ext}_p(\mathcal{T})| = |\text{Max}(\Sigma)|$, *then the output of* $\mathcal{T}$ *under grounded/ideal semantics is:* $\text{Output}(\mathcal{T}) = \mathsf{CN}(\text{Free}(\Sigma))$.

Proof Let $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ be an argumentation system such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies the five postulates (under preferred semantics). Assume that $|\text{Ext}_p(\mathcal{T})| = |\text{Max}(\Sigma)|$. Then, from Corollary 12, $\text{IE}(\mathcal{T}) = \text{GE}(\mathcal{T}) = \bigcap_{\mathcal{E}_i \in \text{Ext}_p(\mathcal{T})} \mathcal{E}_i = \text{Arg}(\text{Free}(\Sigma))$.

Let us first show that $\mathcal{T}$ is closed under CN under grounded/ideal semantics, that is

$$\text{Concs}(\bigcap_{\mathcal{E}_i \in \text{Ext}_p(\mathcal{T})} \mathcal{E}_i) = \mathsf{CN}(\text{Concs}(\bigcap_{\mathcal{E}_i \in \text{Ext}_p(\mathcal{T})} \mathcal{E}_i)).$$

Let $x \in \mathsf{CN}(\mathrm{Concs}(\bigcap_{\mathcal{E}_i \in \mathrm{Ext}_p(\mathcal{T})} \mathcal{E}_i))$. Thus, for all $\mathcal{E}_i \in \mathrm{Ext}_p(\mathcal{T})$, $x \in \mathsf{CN}(\mathrm{Concs}(\mathcal{E}_i))$. Since $\mathcal{T}$ satisfies closure under CN, then $x \in \mathrm{Concs}(\mathcal{E}_i)$. Besides, since $x \in \mathsf{CN}(\mathrm{Concs}(\bigcap_{\mathcal{E}_i \in \mathrm{Ext}_p(\mathcal{T})} \mathcal{E}_i))$, then there exists a finite set $x_1, \ldots, x_n \in \mathrm{Concs}(\bigcap_{\mathcal{E}_i \in \mathrm{Ext}_p(\mathcal{T})} \mathcal{E}_i)$ such that $x \in \mathsf{CN}(\{x_1, \ldots, x_n\})$. Moreover, $\mathsf{CN}(\{x_1, \ldots, x_n\} \subseteq \mathsf{CN}(\mathrm{Concs}(\bigcap_{\mathcal{E}_i \in \mathrm{Ext}_p(\mathcal{T})} \mathcal{E}_i))$ (by monotonicity of CN). Thus, $x \in \mathsf{CN}(\mathrm{Concs}(\bigcap_{\mathcal{E}_i \in \mathrm{Ext}_p(\mathcal{T})} \mathcal{E}_i))$.

Let us now show that $\mathcal{T}$ is closed under sub-arguments under grounded/ideal semantics. Assume that $a \in \mathrm{IE}(\mathcal{T})$. Thus, $a \in \bigcap_{\mathcal{E}_i \in \mathrm{Ext}_p(\mathcal{T})} \mathcal{E}_i$. Since $\mathcal{T}$ is closed under sub-arguments (under preferred semantics), then for all $\mathcal{E}_i \in \mathrm{Ext}_p(\mathcal{T})$, $\mathrm{Sub}(a) \subseteq \mathcal{E}_i$. Thus, $\mathrm{Sub}(a) \subseteq \bigcap_{\mathcal{E}_i \in \mathrm{Ext}_p(\mathcal{T})} \mathcal{E}_i$. So, $\mathrm{Sub}(a) \subseteq \mathrm{IE}(\mathcal{T})$.

Since $\mathcal{T}$ satisfies closure under sub-arguments and CN (under ideal/grounded semantics), then from Proposition 3, $\mathrm{Concs}(\mathrm{IE}(\mathcal{T})) = \mathrm{Concs}(\mathrm{GE}(\mathcal{T})) = \mathrm{Output}(\mathcal{T}) = \mathsf{CN}(\mathrm{Free}(\Sigma))$. □

Theorem 18 *Let $\mathcal{T} = (\mathrm{Arg}(\Sigma), \mathcal{R})$ be an argumentation system such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies the five postulates under grounded (respectively ideal) semantics. If $\mathrm{Arg}(\mathrm{Free}(\Sigma)) \subset \mathrm{GE}(\mathcal{T})$ (respectively $\mathrm{Arg}(\mathrm{Free}(\Sigma)) \subset \mathrm{IE}(\mathcal{T})$) then there exists $C \in \mathcal{C}_\Sigma$ such that there exist $x, x' \in C$ and $x \in \mathrm{Output}(\mathcal{T})$ and $x' \notin \mathrm{Output}(\mathcal{T})$.*

Proof Let $\mathcal{T} = (\mathrm{Arg}(\Sigma), \mathcal{R})$ be an argumentation system such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies the five postulates under grounded semantics. Assume that $\mathrm{Arg}(\mathrm{Free}(\Sigma)) \subset \mathrm{GE}(\mathcal{T})$. Thus, there exists an argument $a \in \mathrm{GE}(\mathcal{T})$ such that $a \notin \mathrm{Arg}(\mathrm{Free}(\Sigma))$. Thus, $\mathrm{Supp}(a) \not\subseteq \mathrm{Free}(\Sigma)$. So, there exists $x \in \mathrm{Inc}(\Sigma)$ such that $x \in \mathrm{Supp}(a)$. Then, there exists $C \in \mathcal{C}_\Sigma$ such that $x \in C$. Since $\mathcal{T}$ satisfies closure and CN and under sub-arguments, then $\mathrm{Output}(\mathcal{T}) = \mathrm{Concs}(\mathrm{GE}(\mathcal{T})) = \mathsf{CN}(\mathrm{Base}(\mathrm{GE}(\mathcal{T})))$. Since $x \in \mathrm{Supp}(a)$, then $x \in \mathrm{Base}(\mathrm{GE}(\mathcal{T}))$. From expansion axiom, $x \in \mathsf{CN}(\mathrm{Base}(\mathrm{GE}(\mathcal{T})))$. Since $\mathcal{T}$ satisfies consistency, then $C \not\subseteq \mathrm{Concs}(\mathrm{GE}(\mathcal{T}))$. Thus, there exists $x' \in C$ such that $x' \notin \mathrm{Concs}(\mathrm{GE}(\mathcal{T}))$.

The same proof holds for ideal semantics. □

Corollary 1 *Let $(\mathcal{L}, \mathsf{CN})$ be adjunctive. Let $\mathcal{T} = (\mathrm{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent. $\mathcal{T}$ satisfies consistency and is closed under sub-arguments (under naive semantics) iff there is a bijection between the naive extensions of $\mathrm{Ext}_n(\mathcal{T})$ and the elements of $\mathrm{Max}(\Sigma)$.*

Proof Let $\mathcal{T} = (\mathrm{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent. Assume that $\mathcal{T}$ satisfies Postulates 2 and 3. Then, from Theorems 1 and 2, it follows that there is a one-to-one correspondence between $\mathrm{Max}(\Sigma)$ and $\mathrm{Ext}_n(\mathcal{T})$.

Assume now that there is a one-to-one correspondence between $\mathrm{Max}(\Sigma)$ and $\mathrm{Ext}_n(\mathcal{T})$. Then, $\forall \mathcal{E} \in \mathrm{Ext}(\mathcal{T})$, $\mathrm{Base}(\mathcal{E})$ is consistent. Consequently, $\mathcal{T}$ satisfies consistency. From (1), $\mathcal{T}$ is also closed under sub-arguments. □

Corollary 2 *Let $\mathcal{T} = (\mathrm{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies consistency and is closed under sub-arguments (under naive semantics).*

- $|\mathrm{Ext}_n(\mathcal{T})| \leq |\mathrm{Max}(\Sigma)|$
- *If Σ is finite, then $\mathcal{T}$ has a finite number of naive extensions.*

Proof Let $\mathcal{T} = (\mathrm{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies consistency and is closed under sub-arguments. From Theorem 1, it follows that $|\mathrm{Ext}_n(\mathcal{T})| \leq |\mathrm{Max}(\Sigma)|$. If Σ is finite, then it has a finite number of maximal consistent subsets. Thus, the number of naive extensions is finite as well. □

Corollary 3 *Let* $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ *be an argumentation system over a knowledge base* Σ *such that* $\mathcal{R}$ *is conflict-dependent and* $\mathcal{T}$ *satisfies consistency and is closed under sub-arguments (under naive semantics). If* $\text{Ext}_n(\mathcal{T}) = \{\emptyset\}$*, then for all* $x \in \Sigma$*,* $\mathsf{CN}(\{x\})$ *is inconsistent.*

Proof Let $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies consistency and is closed under sub-arguments. Assume that $\text{Ext}_n(\mathcal{T}) = \{\emptyset\}$. Thus, $\text{Base}(\emptyset) = \emptyset$. From Theorem 1, $\emptyset \in \text{Max}(\Sigma)$. Thus, for all $x \in \Sigma$, $\mathsf{CN}(\{x\})$ is inconsistent. □

Corollary 4 *Let* $(\mathcal{L}, \mathsf{CN})$ *be adjunctive. Let* $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ *be an argumentation system over a knowledge base* Σ *such that* $\mathcal{R}$ *is conflict-dependent,* $\mathcal{T}$ *satisfies consistency and is closed under sub-arguments (under naive semantics).* $\text{Output}(\mathcal{T}) = \bigcap_{\mathcal{S}_i \in \text{Max}(\Sigma)} \mathsf{CN}(\mathcal{S}_i)$.

Proof It follows immediately from Theorem 2 and Theorem 3. □

Corollary 6 *Let* $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ *be an argumentation system over a base* Σ *such that* $\mathcal{R}$ *is conflict-dependent and* $\mathcal{T}$ *satisfies consistency, closure under sub-arguments and free precedence (under stable semantics). It holds that*

$$0 < |\text{Ext}_s(\mathcal{T})| \leq |\text{Max}(\Sigma)|.$$

Proof Let $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies consistency, closure under sub-arguments and free precedence. From Theorem 4, $|\text{Ext}_s(\mathcal{T})| \leq |\text{Max}(\Sigma)|$. From Theorem 5 $\Re_{s1} = \emptyset$, then $|\text{Ext}_s(\mathcal{T})| > 0$. □

Corollary 7 *If* Σ *is finite, then the set* $\text{Ext}_s(\mathcal{T})$ *is finite, whenever* $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ *satisfies consistency and closure under sub-arguments (under stable semantics).*

Proof It follows from Corollary 6. □

Corollary 8 *Let* $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ *be an argumentation system over a base* Σ *such that* $\mathcal{R}$ *is conflict-dependent and* $\mathcal{T}$ *satisfies consistency, closure under sub-arguments and free precedence (under stable semantics). The equality* $\text{Ext}_s(\mathcal{T}) = \text{Ext}_{ss}(\mathcal{T})$ *holds.*

Proof Let $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a base Σ such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies consistency, closure under sub-arguments and free precedence (under stable semantics). From Corollary 6, $|\text{Ext}_s(\mathcal{T})| > 0$. From (16), $\text{Ext}_s(\mathcal{T}) = \text{Ext}_{ss}(\mathcal{T})$. □

Corollary 9 *Let* $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ *be an argumentation system such that* $\mathcal{R}$ *is conflict-dependent and* $\mathcal{T}$ *satisfies consistency, closure under sub-arguments and free precedence (under preferred semantics). There exists* $\mathcal{E} \in \text{Ext}_p(\mathcal{T})$ *such that* $\text{Base}(\mathcal{E}) \in \text{Max}(\Sigma)$.

Proof Let $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ be an argumentation system such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies consistency, closure under sub-arguments and free precedence (under preferred semantics). Thus, $\mathcal{T}$ satisfies the same postulates under stable semantics since $\text{Ext}_s(\mathcal{T}) \subseteq \text{Ext}_p(\mathcal{T})$. Then, $|\text{Ext}_s(\mathcal{T})| > 0$. Thus, $\exists \mathcal{E} \in \text{Ext}_s(\mathcal{T})$

and thus $\mathcal{E} \in \text{Ext}_p(\mathcal{T})$. From Theorem 4, $\text{Base}(\mathcal{E}) \in \text{Max}(\Sigma)$. □

Corollary 10 *If a knowledge base* Σ *is finite, then for all* $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ *such that* $\mathcal{R}$ *is conflict-dependent and* $\mathcal{T}$ *satisfies consistency and closure under sub-arguments,* $\text{Ext}_p(\mathcal{T})$ *is finite.*

Proof This follows from the compactness of the knowledge base and Theorem 12. □

Corollary 11 *Let* $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ *be an argumentation system such that* $\mathcal{R} \in \Re_{p_3}$.

$$\text{Output}(\mathcal{T}) = \bigcap_{\mathcal{S}_i \in \text{Max}(\Sigma)} \mathsf{CN}(\mathcal{S}_i)$$

Proof Let $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ be an argumentation system such that $\mathcal{R} \in \Re_{p_3}$. From Theorem 13, for all $\mathcal{E} \in \text{Ext}_p(\mathcal{T})$, $\text{Base}(\mathcal{E}) \in \text{Max}(\Sigma)$. Moreover, since $\mathcal{T}$ is closed under both CN and sub-arguments, then $\text{Concs}(\mathcal{E}) = \mathsf{CN}(\text{Base}(\mathcal{E}))$. Thus, $\text{Output}(\mathcal{T}) = \bigcap_{\mathcal{S}_i \in \text{Max}(\Sigma)} \mathsf{CN}(\mathcal{S}_i)$. □

Corollary 12 *Let* $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ *be an argumentation system such that* $\mathcal{R} \in \Re_{p_3}$. $\text{IE}(\mathcal{T}) = \text{GE}(\mathcal{T}) = \text{Arg}(\text{Free}(\Sigma))$.

Proof Let $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ be an argumentation system such that $\mathcal{R} \in \Re_{p_3}$. From Theorem 16, $\text{IE}(\mathcal{T}) = \text{Arg}(\text{Free}(\Sigma))$. Moreover, from Theorem 15 $\text{Arg}(\text{Free}(\Sigma)) \subseteq \text{GE}(\mathcal{T})$ (since arguments of $\text{Arg}(\text{Free}(\Sigma))$ are not attacked). Thus, $\text{GE}(\mathcal{T}) = \text{IE}(\mathcal{T})$. □

Corollary 14 *Let* $\mathcal{T} = (\text{Arg}(\Sigma), \mathcal{R})$ *be an argumentation system such that* $\mathcal{R} \in \Re_{p_2}$. *The inclusions* $\text{Arg}(\text{Free}(\Sigma)) \subseteq \text{GE}(\mathcal{T}) \subseteq \text{IE}(\mathcal{T}) \subseteq \mathcal{S}$ *hold for some* $\mathcal{S} \in \text{Max}(\Sigma)$.

Proof Follows from Theorem 15, Property 3 and Property 11. □