JUMP DEGREES OF TORSION-FREE ABELIAN GROUPS

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ABSTRACT. We show, for each computable ordinal α and degree $\mathbf{a} > \mathbf{0}^{(\alpha)}$, the existence of a torsion-free abelian group with proper α^{th} jump degree $\mathbf{a}$.

1. INTRODUCTION

An important aim of effective algebra is concerned with determining how close to computable an algebraic structure is. For an algebraic structure $\mathcal{A}$, this property is reflected in the *degree spectum of* $\mathcal{A}$:

$$\mathrm{DegSpec}(\mathcal{A}) := \{\mathbf{d} : \mathcal{A} \text{ is } \mathbf{d}\text{-computable}\}.$$

We recall an infinite algebraic structure $\mathcal{A}$ is $\mathbf{d}$-*computable* if its universe can be identified with the natural numbers ω in such a way that the atomic diagram of $\mathcal{A}$ becomes $\mathbf{d}$-computable.

In many cases, the degree spectrum of $\mathcal{A}$ has no least element (for example, see [18]). As a result, there has been a line of study into the jump degrees of structures.

Definition 1.1. If $\mathcal{A}$ is a countable structure, α is a computable ordinal, and $\mathbf{a} \geq \mathbf{0}^{(\alpha)}$ is a degree, then $\mathcal{A}$ has α^{th} *jump degree* $\mathbf{a}$ if the set

$$\{\mathbf{d}^{(\alpha)} : \mathbf{d} \in \mathrm{DegSpec}(\mathcal{A})\}$$

has $\mathbf{a}$ as its least element. In this case, the structure $\mathcal{A}$ is said to have α^{th} *jump degree.*

A structure $\mathcal{A}$ has *proper* α^{th} *jump degree* $\mathbf{a}$ if $\mathcal{A}$ has α^{th} jump degree $\mathbf{a}$ but not β^{th} jump degree for any $\beta < \alpha$. In this case, the structure $\mathcal{A}$ is said to have *proper* α^{th} *jump degree.*

For a computable ordinal α, it is well-known that an arbitrary structure may not have α^{th} jump degree (for example, see [6]). The existence or nonexistence of a structure with proper α^{th} jump degree $\mathbf{a}$ for $\mathbf{a} \geq \mathbf{0}^{(\alpha)}$ depends heavily on the class of algebraic structures considered. Within the context of linear orders, if an order type has a degree, it must be $\mathbf{0}$; if an order type has first jump degree, it must be $\mathbf{0}'$; and yet for each computable ordinal $\alpha \geq 2$ and degree $\mathbf{a} \geq \mathbf{0}^{(\alpha)}$, there is a linear order having proper α^{th} jump degree $\mathbf{a}$ (see [6], culminating results in [1], [13], [15] and [18]). Within the context of Boolean algebras, if a Boolean algebra has n^{th} jump degree (for any $n \in \omega$), it must be $\mathbf{0}^{(n)}$; yet for each $\mathbf{a} \geq \mathbf{0}^{(\omega)}$, there is a Boolean algebra with proper ω^{th} jump degree $\mathbf{a}$ (see [14]).

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The subject of this paper is the existence of jump degrees of torsion-free abelian groups. For $\alpha \in \{0,1,2\}$, it is known that every possible proper α^{th} jump degree is realized.

Theorem 1.2 (Downey [5]; Downey and Jockusch [5])**.** *For every degree* $\mathbf{a} \geq \mathbf{0}$*, there is a (rank one) torsion-free abelian group having degree* $\mathbf{a}$*.*

For every degree $\mathbf{b} \geq \mathbf{0}'$*, there is a (rank one) torsion-free abelian group having proper first jump degree* $\mathbf{b}$*.*

Indeed, every finite rank torsion-free abelian group (see Definition 2.2) has first jump degree as a consequence of a difficult computability-theoretic theorem of Coles, Downey, and Slaman [3]. (See Melnikov [17] for a complete discussion.)

In contrast, not every infinite rank torsion-free abelian group has first jump degree as a consequence of the following theorem of Melnikov. Recall that a nonzero degree $\mathbf{a}$ is *low* if $\mathbf{a}' = \mathbf{0}'$ and *nonlow* otherwise.

Theorem 1.3 (Melnikov [17])**.** *There is a torsion-free abelian group* $\mathcal{G}$ *such that DegSpec*$(\mathcal{G}) = \{\mathbf{a} : \mathbf{a}$ *is nonlow*$\}$*. Consequently, there is a torsion-free abelian group with proper second jump degree* $\mathbf{0}''$*.*

As a consequence of work by Ash, Jockusch, and Knight (see [1]), two observations of Melnikov (see Theorem 3 and Proposition 10 of [17]) have implications about proper second jump degrees and proper third jump degrees.

Theorem 1.4. *For every degree* $\mathbf{a} > \mathbf{0}''$*, there is a torsion-free abelian group having proper second jump degree* $\mathbf{a}$*. For every degree* $\mathbf{b} > \mathbf{0}'''$*, there is a torsion-free abelian group having proper third jump degree* $\mathbf{b}$*.*

The main purpose of this paper is to generalize Theorem 1.4 to an arbitrary computable ordinal α.

Theorem 1.5. *For every computable ordinal* α *and degree* $\mathbf{a} > \mathbf{0}^{(\alpha)}$*, there is a torsion-free abelian group having proper* α^{th} *jump degree* $\mathbf{a}$*.*

Unfortunately, our techniques do not allow us to produce examples with $\mathbf{a} = \mathbf{0}^{(\alpha)}$.

Question 1.6. For each computable ordinal $\alpha \geq 3$, is there a torsion-free abelian group having proper α^{th} jump degree $\mathbf{0}^{(\alpha)}$?

Fixing α, we prove Theorem 1.5 by coding sets $S \subseteq \omega$ into groups $\mathcal{G}_S^\alpha$ in such a way that $\mathcal{G}_S^\alpha$ is X-computable if and only if $S \in \Sigma_\alpha^0(X)$. The coding method is based on techniques in Fuchs (see Section XIII, Chapter 88 and Chapter 89, of [11]) and Hjorth (see [12]). In particular, given torsion-free abelian groups $\mathcal{A}$ and $\mathcal{B}$ of a certain type and elements $a \in \mathcal{A}$ and $b \in \mathcal{B}$, Fuchs adds elements of the form $p^{-n}(a+b)$ for $n \in \omega$ to $\mathcal{A} \oplus \mathcal{B}$ to build an indecomposable group containing $\mathcal{A} \oplus \mathcal{B}$. Hjorth uses this idea to encode certain labeled graphs into countable torsion-free abelian groups. Hjorth's ideas were later used by Downey and Montalbán to show the isomorphism problem for torsion-free abelian groups is analytic complete (see [7]) and further studied by Fokina, Knight, Maher, Melnikov, and Quinn (see [8]). We refer the reader to these papers for additional background, though we do not assume knowledge of them in our presentation. Our notation and terminology will follow Fuchs (see [10] and [11]) rather than Hjorth.

Section 2 discusses background, notation, and conventions, though we also refer the reader to [10] and [11] for classical background on torsion-free abelian groups and to [2] for background on effective algebra. Section 3 describes the encoding of sets $S \subseteq \omega$ into groups $\mathcal{G}_S^\alpha$ (this encoding depends on α). Theorem 1.5 is demonstrated in Section 4.

2. Background, Notation, and Conventions

In this section, we review basic terminology and results relevant to torsion-free abelian groups. We also introduce some classical notation and adopt some conventions that will simplify the exposition.

Definition 2.1. An abelian group $\mathcal{G} = (G : +, 0)$ is *torsion-free* if

$$(\forall x \in G)(\forall n \in \omega)\,[x \neq 0 \wedge n \neq 0 \implies nx \neq 0]\,,$$

where nx denotes $\underbrace{x + \cdots + x}_{n \text{ times}}$.[1]

Definition 2.2. If $\mathcal{G}$ is a torsion-free abelian group, a (finite or infinite) set of nonzero elements $\{g_i\}_{i\in I} \subset G$ is *linearly independent* if $\alpha_1 g_{i_1} + \cdots + \alpha_k g_{i_k} = 0$ has no solution with $\alpha_i \in \mathbb{Z}$ for each i, $\{i_1, \ldots, i_k\} \subseteq I$, and $\alpha_i \neq 0$ for some i.

A *basis* for $\mathcal{G}$ is a maximal linearly independent set. The *rank* of $\mathcal{G}$ is the cardinality of a basis.

The groups $\mathcal{G}$ constructed for Theorem 1.5 will have countably infinite rank. The key coding mechanism will be the existence or nonexistence of elements divisible by arbitrarily high powers of a prime.

Definition 2.3. If $\mathcal{G}$ is a torsion-free abelian group and $x \in G$, we write $p^\infty \mid x$ and say p *infinitely divides* x if $(\forall k \in \omega)(\exists y \in G)\left[x = p^k y\right]$, i.e., if p divides x arbitrarily many times. We write $p^\infty \nmid x$ and say p *finitely divides* x otherwise.

Definition 2.4. A subgroup $\mathcal{G}$ of an abelian group $\mathcal{H}$ is *pure* if for every $n \in \mathbb{Z}$ and every $b \in G$:

$$\text{If } \mathcal{H} \models n \mid b, \text{ then } \mathcal{G} \models n \mid b.$$

In other words, if an integer n divides an element $b \in G$ within $\mathcal{H}$, then n divides b within $\mathcal{G}$.

We note that if $\mathcal{G}$ is torsion-free, then $x = p^k y$ can have at most one solution y for any nonnegative integer k.

Remark 2.5. Within any presentation of $\mathcal{G}$, the set $\{x \in G : p^\infty \mid x\}$ of elements infinitely divisible by p is $\Pi_2^c(\mathcal{G})$. Indeed, this set is a subgroup of $\mathcal{G}$ under the group operation (which we use without further mention).

Definition 2.6. A torsion-free abelian group is *divisible* if it is the additive group of a $\mathbb{Q}$-vector space (or equivalently, if $x = ny$ has a solution $y \in G$ for every $x \in G$ and $n \geq 1$).

If $\mathcal{G}$ is a torsion-free abelian group, its *divisible closure* (denoted $D(\mathcal{G})$) is the smallest (under inclusion) divisible torsion-free abelian group containing $\mathcal{G}$.

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[1] We use such abbreviations freely throughout the paper.

Thus, the countable divisible torsion-free abelian groups are the groups $\mathbb{Q}^n$ (for $n \in \omega$) and $\mathbb{Q}^\omega$, and the divisible closure of $\mathbb{Z}$ is $\mathbb{Q}$. Classically, the divisible closure $D(\mathcal{G})$ exists, is unique, and contains $\mathcal{G}$ as a subgroup. In terms of effective algebra, Smith (see [20]) proved that every computable torsion-free abelian group has a computable divisible closure and that there is a uniform procedure for passing from $\mathcal{G}$ to $D(\mathcal{G})$.[2] However, in general the divisible closure is not effectively unique (i.e., unique up to computable isomorphism) and the canonical image of $\mathcal{G}$ in $D(\mathcal{G})$ is computably enumerable but not necessarily computable (see [9] and [21] for a complete discussion of these issues). Therefore, when we consider a particular copy $\mathcal{G}$ of a torsion-free abelian group, we use $D(\mathcal{G})$ to denote the canonical divisible closure as in [20]. Thus, we have a uniform way to pass from any given copy of $\mathcal{G}$ to a copy of $D(\mathcal{G})$.

In our construction, we will use a more limited notion of closure under divisibility by certain primes.

Definition 2.7. If $p \in \omega$ is prime and $\mathcal{G}$ is a torsion-free abelian group, define the *p-closure of* $\mathcal{G}$ (denoted $[\mathcal{G}]_p$) to be the smallest subgroup $\mathcal{H}$ of $D(\mathcal{G})$ containing $\mathcal{G}$ having the property $(\forall g \in G)\, [p^\infty \mid g]$.

More generally, if P is a set of prime numbers and $\mathcal{G}$ is a torsion-free abelian group, define the *P-closure of* $\mathcal{G}$ (denoted $[\mathcal{G}]_P$) to be the smallest subgroup $\mathcal{H}$ of $D(\mathcal{G})$ containing $\mathcal{G}$ having the property $(\forall g \in G)(\forall p \in P)\, [p^\infty \mid g]$. We often write $[\mathcal{G}]_{p_0,p_1}$ for $[\mathcal{G}]_P$ with $P = \{p_0, p_1\}$, $[\mathcal{G}]_{P,q}$ for $[\mathcal{G}]_{P\cup\{q\}}$, and so on.

If $\mathcal{G}$ is any torsion-free abelian group and P is any set of prime numbers, we say that $\mathcal{G}$ is *P-closed* if $\mathcal{G} \cong [\mathcal{G}]_P$.

The following lemma says that if $\mathcal{G}$ is P-closed, then the result of closing $\mathcal{G}$ under additional primes will still be P-closed. In particular, we can view the prime closure $[\mathcal{G}]_P$ as the result of closing $\mathcal{G}$ under each of the individual primes in P in any order.

Lemma 2.8. *If $\mathcal{G}$ is a torsion-free abelian group, P is a set of primes, and q is a prime not in P, then $[[\mathcal{G}]_P]_q \cong [\mathcal{G}]_{P,q}$.*

Proof. Since $[[\mathcal{G}]_P]_q$ is clearly a subgroup of $[\mathcal{G}]_{P,q}$ and since every element of $[[\mathcal{G}]_P]_q$ is infinitely divisible by q, it suffices to show that each element of $[[\mathcal{G}]_P]_q$ is infinitely divisible by each prime $p \in P$. Fix $p \in P$ and $g \in [[\mathcal{G}]_P]_q$. We need to find $h \in [[\mathcal{G}]_P]_q$ such that $ph = g$. By the definition of $[[\mathcal{G}]_P]_q$, there is a $k \geq 0$ such that $q^k g \in [\mathcal{G}]_P$; let $\widehat{h}$ be this element. Let $\widehat{g} \in [\mathcal{G}]_P$ be such that $p\widehat{g} = \widehat{h}$ and let $h \in [[\mathcal{G}]_P]_q$ be such that $q^k h = \widehat{g}$. Then

$$q^k(ph) = p(q^k h) = p\widehat{g} = \widehat{h} = q^k g.$$

Since $\mathcal{G}$ is torsion-free, the equality $q^k(ph) = q^k g$ implies $ph = g$ as required. □

By an obvious variation of the construction in [20], there is an effective way to pass from $\mathcal{G}$ to a copy of $[\mathcal{G}]_P$ which is uniform in both $\mathcal{G}$ and P. As above, the closure operation sending $\mathcal{G}$ to $[\mathcal{G}]_P$ is not necessarily effectively unique so we fix this uniform procedure to define a particular copy of $[\mathcal{G}]_P$ given a particular copy of $\mathcal{G}$.

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[2] One forms $D(\mathcal{G})$ from pairs $\langle g, n\rangle$ with $g \in \mathcal{G}$ and $n \geq 1$ modulo the computable equivalence relation $\langle g, n\rangle \sim \langle h, m\rangle$ if and only if $mg = nh$.

Convention 2.9. We will write statements such as $([\mathbb{Z}]_{\rho_1,P}\backslash[\mathbb{Z}]_P)\cap[\mathbb{Z}]_{\rho_2,\rho_3,P}=\emptyset$. Such statements are intended to apply within a fixed (one-dimensional) copy of $\mathbb{Q}$, where $\mathbb{Z}\leqq\mathbb{Q}$ is fixed as well. In particular, the indicated prime closures of $\mathbb{Z}$ should all be seen as being taken within a fixed copy of $\mathbb{Q}$.

Often, we will write elements of the form $\frac{x+y}{p}$ as $\frac{x}{p}+\frac{y}{p}$ even though $\frac{x}{p}$ and $\frac{y}{p}$ may not exist within the group. We justify this by passing to the divisible closure of the group and considering the canonical image of the group within its divisible closure. Thus, $\frac{x+y}{p}=\frac{x}{p}+\frac{y}{p}$ in $D(\mathcal{G})$ even though $\frac{x}{p}$ and $\frac{y}{p}$ may not be in the image of $\mathcal{G}$.

Definition 2.10. A *rooted torsion-free abelian group* $\mathcal{G}$ is a torsion-free abelian group with a distinguished element (termed the *root* of $\mathcal{G}$).

We use rooted torsion-free abelian groups to help build our groups inductively. When we consider isomorphisms, we always consider group isomorphisms with no assumption that roots are preserved. That is, the root is only used as a tool in the inductive definitions and is not a formal part of the algebraic structure.

Definition 2.11. Let $\mathcal{G}$ be a torsion-free abelian group and $\{d_i\}_{i\in I}\subseteq D(\mathcal{G})$ be a subset of its divisible closure. We define the *extension of* $\mathcal{G}$ *by* $\{d_i\}_{i\in I}$, denoted

$$\langle\mathcal{G};\, d_i : i\in I\rangle\,,$$

to be the smallest subgroup of $D(\mathcal{G})$ containing $\mathcal{G}$ and d_i for $i\in I$.

Note that if $\mathcal{G}$ is computable and $\{d_i\}_{i\in I}$ is a computable set of elements of $D(\mathcal{G})$ (indeed, computably enumerable suffices), then the subgroup $\langle\mathcal{G}; d_i : i\in I\rangle$ is computably enumerable in $D(\mathcal{G})$. Since there is a uniform procedure to produce a computable copy of any computably enumerable subgroup of $D(\mathcal{G})$ (by letting n denote the n-th element enumerated into the subgroup and defining the group operations accordingly) we have a uniform procedure to pass from $\mathcal{G}$ to $\langle\mathcal{G}; d_i : i\in I\rangle$.

We continue by introducing some (important) conventions that will be used throughout the paper without further mention.

Convention 2.12. If β is any nonzero ordinal, when we write $\beta=\delta+i$ or $\beta=\delta+2\ell+i$ for some $i\in\omega$, we require δ to be either zero or a limit ordinal (allowing zero only if $\beta<\omega$) and ℓ to be a nonnegative integer.

If i is even, we say the ordinal β is *even*; if i is odd, we say the ordinal β is *odd*.

When at limit ordinals, it will be necessary to approximate the ordinal effectively from below. We therefore fix a computable ordinal λ and increasing cofinal sequences for ordinals less than λ.

Definition 2.13. Fix a computable ordinal λ.

Fix a computable function $f:\lambda\times\omega\to\lambda$ such that $f(\alpha+1,n)=\alpha$ for all successor ordinals $\alpha+1\in\lambda$ and $n\in\omega$, and such that $\{f(\alpha,n)\}_{n\in\omega}$ is a sequence of increasing odd ordinals (greater than one) with $\alpha=\cup_{n\in\omega}f(\alpha,n)$ for all limit ordinals $\alpha\in\lambda$.

We denote $f(\alpha,n)$ by $f_\alpha(n)$.

3. The Group $\mathcal{G}_S^\alpha$ (For Successor Ordinals α)

Fixing a computable successor ordinal α below λ, the group $\mathcal{G}_S^\alpha$ will be a direct sum of rooted torsion-free abelian groups $\mathcal{G}_S^\alpha(n)$ coding whether n is or is not in S. It will be useful to have a plethora of disjoint sets of primes. We therefore partition the prime numbers into uniformly computable sets $P = \{p_\beta\}_{\beta \in \alpha+1}$, $Q = \{q_\beta\}_{\beta\in\alpha+1}$, $U = \{u_{\beta,k}\}_{\beta\in\alpha+1,k\in\omega}$, $V = \{v_{\beta,k}\}_{\beta\in\alpha+1,k\in\omega}$, $D = \{d_n\}_{n\in\omega}$, and $E = \{e_n\}_{n\in\omega}$.

More specifically, the isomorphism type of $\mathcal{G}^\alpha_S(n)$ will be either $\left[\mathcal{G}(\Sigma^0_\alpha)\right]_{d_n}$ or $\left[\mathcal{G}(\Pi^0_\alpha)\right]_{d_n}$, or $\left[\mathcal{H}(\Sigma^0_\alpha)\right]_{d_n}$ or $\left[\mathcal{H}(\Pi^0_\alpha)\right]_{d_n}$ (all described later) depending on whether α is even or odd (deciding $\mathcal{G}$ versus $\mathcal{H}$) and whether n is in S (deciding Σ versus Π). The group $\mathcal{G}^\alpha_S(n)$ will be X-computable (uniformly in n) if $S \in \Sigma^0_\alpha(X)$. Conversely, there will be an effective enumeration $\{\Upsilon_n\}_{n\in\omega}$ of computable infinitary Σ^c_α sentences such that $\mathcal{G}^\alpha_S \models \Upsilon_n$ if and only if $n \in S$.[3] Thus, the group $\mathcal{G}^\alpha_S$ will be X-computable if and only if $S \in \Sigma^0_\alpha(X)$.

The definition of the rooted torsion-free abelian groups $\mathcal{G}(\Sigma^0_\alpha)$, $\mathcal{G}(\Pi^0_\alpha)$, $\mathcal{H}(\Sigma^0_\alpha)$, and $\mathcal{H}(\Pi^0_\alpha)$ is done by recursion. Unfortunately, the recursion is not straightforward for technical reasons within the algebra (discussed in Remark 3.2). Indeed, we introduce additional rooted torsion-free abelian groups $\mathcal{G}(\Sigma^0_\alpha(m))$ for $m \in \omega$ if α is an even ordinal.

We define some of these groups pictorially in Section 3.1. The hope is these examples provide enough intuition to the reader so that the formal definition of $\mathcal{G}^\alpha_S$ (and all the auxiliary groups) is not (too) painful.

3.1. **Defining $\mathcal{G}(\Sigma^0_\beta)$, $\mathcal{G}(\Sigma^0_\beta(m))$, $\mathcal{G}(\Pi^0_\beta)$, $\mathcal{H}(\Sigma^0_\beta)$, and $\mathcal{H}(\Pi^0_\beta)$ Pictorially.** For each successor ordinal $\beta \geq 3$, we give a pictorial description of the groups $\mathcal{G}(\Sigma^0_\beta)$ (if β is odd), $\mathcal{G}(\Sigma^0_\beta(m))$ (if β is even), and $\mathcal{G}(\Pi^0_\beta)$. The recursion starts with $\mathcal{G}(\Sigma^0_2(m))$ as $\mathbb{Z}$ with root $r = p_1^m$ and $\mathcal{G}(\Pi^0_2)$ as $[\mathbb{Z}]_{p_1}$ with root $r = 1$. The recursion continues as illustrated in Figure 1 and Figure 2.

For each even ordinal $\beta = \delta + 2\ell + 2 > 2$, we give a pictorial description of the groups $\mathcal{H}(\Sigma^0_\beta)$ and $\mathcal{H}(\Pi^0_\beta)$. Though their definition relies on $\mathcal{G}(\Sigma^0_{\beta-1})$ and $\mathcal{G}(\Pi^0_{\beta-1})$ as illustrated in Figure 3, no further recursion is required.

Within these figures, the recursively defined rooted torsion-free abelian groups are denoted with triangles (the text inside specifies which recursively defined group), with the root denoted by a circle at the top. A line segment connecting two roots and with a label p denotes the sum of the roots is made infinitely divisible by p. Brackets around a recursively defined rooted group with a label p denotes the p-closure of the recursively defined rooted group is taken. A prime p next to a root r denotes r is made infinitely divisible by p.

3.2. **Defining $\mathcal{G}^\alpha_S$ Formally.** Having pictorially described some of the associated groups, we formalize the definition of $\mathcal{G}^\alpha_S$. Of course, doing so requires formalizing the definition of all the auxiliary groups.

Definition 3.1. For each ordinal β with $1 < \beta \leq \alpha$, define rooted torsion-free abelian groups $\mathcal{G}(\Sigma^0_\beta)$ and $\mathcal{G}(\Pi^0_\beta)$ (if β is odd) or $\mathcal{G}(\Sigma^0_\beta(m))$ for $m \in \omega$ and $\mathcal{G}(\Pi^0_\beta)$ (if β is even) by recursion as follows.

- For $\beta = 2$, define $\mathcal{G}(\Sigma^0_\beta(m))$ to be the group $\mathbb{Z}$ with root $r = p_1^m$ and define $\mathcal{G}(\Pi^0_\beta)$ to be the group $[\mathbb{Z}]_{p_1}$ with root $r = 1$.

[3] We refer the reader to [2], for example, for definitions and background on computable infinitary formulas.

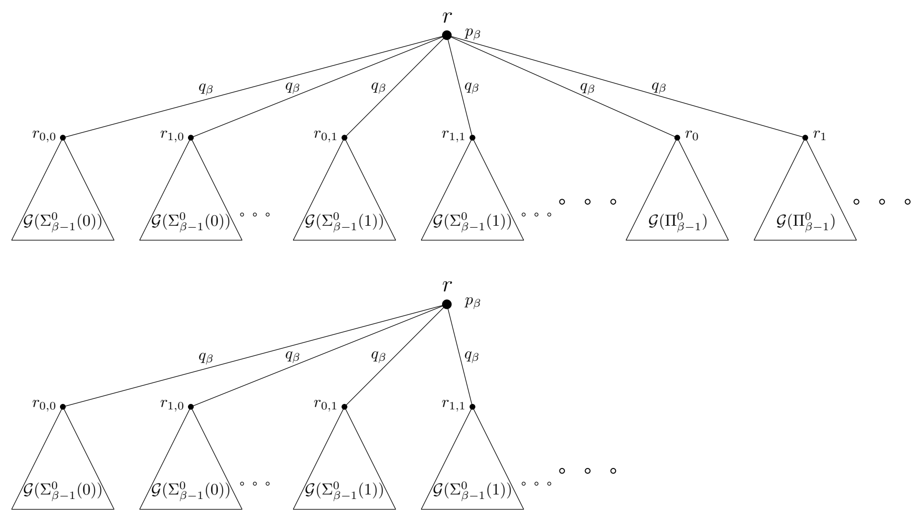

FIGURE 1. $\mathcal{G}(\Sigma^0_\beta)$ (Top) and $\mathcal{G}(\Pi^0_\beta)$ (Bottom) if $\beta = 2\delta + 2\ell + 1 > 1$

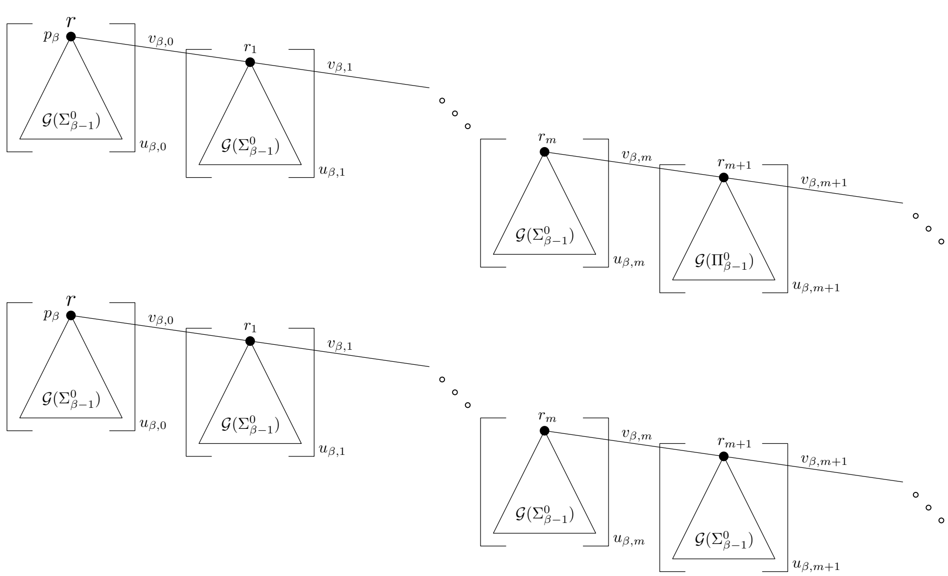

FIGURE 2. $\mathcal{G}(\Sigma^0_\beta(m))$ (Top) and $\mathcal{G}(\Pi^0_\beta)$ (Bottom) if $\beta = \delta + 2\ell + 2 > 2$

- For odd $\beta = \delta + 2\ell + 1 \geq 3$, define $\mathcal{G}(\Sigma^0_\beta)$ to be the group

$$\left\langle [\mathbb{Z}]_{p_\beta} \oplus \bigoplus_{k\in\omega}\bigoplus_{m\in\omega} \mathcal{G}(\Sigma^0_{\beta-1}(m)) \oplus \bigoplus_{k\in\omega} \mathcal{G}(\Pi^0_{\beta-1});\ q_\beta^{-t}(r + r_k), q_\beta^{-t}(r + r_{k,m}) : k, m, t \in \omega \right\rangle,$$

with root $r = 1$ in $[\mathbb{Z}]_{p_\beta}$, where r_k is the root of the kth copy of $\mathcal{G}(\Pi^0_{\beta-1})$ and $r_{k,m}$ is the root of kth copy of $\mathcal{G}(\Sigma^0_{\beta-1}(m))$.

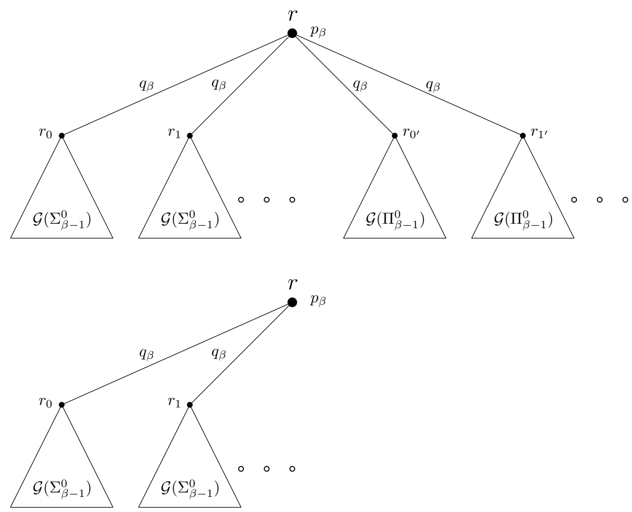

FIGURE 3. $\mathcal{H}(\Sigma^0_\beta)$ (Top) and $\mathcal{H}(\Pi^0_\beta)$ (Bottom) if $\beta = \delta + 2\ell + 2 > 2$

For odd $\beta = \delta + 2\ell + 1 \geq 3$, define $\mathcal{G}(\Pi^0_\beta)$ to be the group

$$\left\langle [\mathbb{Z}]_{p_\beta} \oplus \bigoplus_{k\in\omega} \bigoplus_{m\in\omega} \mathcal{G}(\Sigma^0_{\beta-1}(m));\ q_\beta^{-t}(r + r_{k,m}) : k, m, t \in \omega \right\rangle,$$

with root $r = 1$ in $[\mathbb{Z}]_{p_\beta}$, where $r_{k,m}$ is the root of kth copy of $\mathcal{G}(\Sigma^0_{\beta-1}(m))$.

These are illustrated in Figure 1.

- For even $\beta = \delta + 2\ell + 2 > 2$, define $\mathcal{G}(\Sigma^0_\beta(m))$ to be the group

$$\left\langle \bigoplus_{0\leq k\leq m} \left[\mathcal{G}(\Sigma^0_{\beta-1})\right]_{u_{\beta,k}} \oplus \bigoplus_{k>m} \left[\mathcal{G}(\Pi^0_{\beta-1})\right]_{u_{\beta,k}};\ p_\beta^{-t} r_0, v_{\beta,k}^{-t}(r_k + r_{k+1}) : k, t \in \omega \right\rangle,$$

with root $r = r_0$, where r_k is the root of the kth copy of $\mathcal{G}(\Sigma^0_{\beta-1})$ or of $\mathcal{G}(\Pi^0_{\beta-1})$ depending on whether $k \leq m$ or $k > m$.

For even $\beta = \delta + 2\ell + 2 > 2$, define $\mathcal{G}(\Pi^0_\beta)$ to be the group

$$\left\langle \bigoplus_{k\in\omega} \left[\mathcal{G}(\Sigma^0_{\beta-1})\right]_{u_{\beta,k}};\ p_\beta^{-t} r_0, v_{\beta,k}^{-t}(r_k + r_{k+1}) : k, t \in \omega \right\rangle,$$

with root $r = r_0$, where r_k is the root of the kth copy of $\mathcal{G}(\Sigma^0_{\beta-1})$.

These are illustrated in Figure 2.

- For limit $\beta = \delta > 0$, define the group $\mathcal{G}(\Sigma^0_\beta(m))$ to be

$$\left\langle \bigoplus_{0\leq k\leq m} \left[\mathcal{G}(\Sigma^0_{f_\beta(k)})\right]_{u_{\beta,k}} \oplus \bigoplus_{k>m} \left[\mathcal{G}(\Pi^0_{f_\beta(k)})\right]_{u_{\beta,k}};\ p_\beta^{-t} r_0, v_{\beta,k}^{-t}(r_k + r_{k+1}) : k, t \in \omega \right\rangle,$$

with root $r = r_0$, where r_k is the root of the kth copy of $\mathcal{G}(\Sigma^0_{f_\beta(k)})$ or of $\mathcal{G}(\Pi^0_{f_\beta(k)})$ depending on whether $k \leq m$ or $k > m$.

Define $\mathcal{G}(\Pi^0_\beta)$ to be the group

$$\left\langle \bigoplus_{k\in\omega} \left[\mathcal{G}(\Sigma^0_{f_\beta(k)})\right]_{u_{\beta,k}} ;\, p_\beta^{-t} r_0, v_{\beta,k}^{-t}(r_k + r_{k+1}) : k, t \in \omega \right\rangle,$$

with root $r = r_0$, where r_k is the root of the kth copy of $\mathcal{G}(\Sigma^0_{f_\beta(k)})$.[4]

This completes the formal descriptions of these groups.

For odd $\beta \geq 3$, recall that the group $\mathcal{G}(\Sigma^0_\beta)$ has the form

$$\left\langle [\mathbb{Z}]_{p_\beta} \oplus \bigoplus_{k\in\omega}\bigoplus_{m\in\omega} \mathcal{G}(\Sigma^0_{\beta-1}(m)) \oplus \bigoplus_{k\in\omega} \mathcal{G}(\Pi^0_{\beta-1});\, q_\beta^{-t}(r + r_k), q_\beta^{-t}(r + r_{k,m}) : k, m, t \in \omega \right\rangle.$$

We refer to the subgroups (indexed by k and m) of the form $\mathcal{G}(\Sigma^0_{\beta-1}(m))$ with roots $r_{k,m}$ as the $\mathcal{G}(\Sigma^0_{\beta-1}(m))$ *components* of $\mathcal{G}(\Sigma^0_\beta)$. Similarly, we refer to the $\mathcal{G}(\Pi^0_{\beta-1})$ subgroups (indexed by k) with root r_k as the $\mathcal{G}(\Pi^0_{\beta-1})$ *components* of $\mathcal{G}(\Sigma^0_\beta)$. We use similar language in the case of even β and limit β, as well as for other groups defined inductively within the paper.

We emphasize the components of a group do not detach as direct summands (because of the divisibility introduced by the primes q_β and $v_{\beta,k}$). For clarity, we always refer to direct components (when the directness is an issue) without omitting the word "direct".

When we speak of components, we mean the components which are used in the inductive definition of these groups (or their prime closures), and we do not care if there are alternate ways to present the group. More formally, every such group will be considered as an image of one *canonical copy* given by the definition, and a subgroup is a component if and only if it is an image of a component which was used in the definition of this canonical copy. The isomorphism is chosen once and forever.

The important relationship between $\mathcal{G}(\Sigma^0_\beta)$ and $\mathcal{G}(\Pi^0_\beta)$ for odd β and $\mathcal{G}(\Sigma^0_\beta(m))$ and $\mathcal{G}(\Pi^0_\beta)$ for even β is whether each embeds within the other. For small β, one can see that the groups defined above satisfy the following embeddability relations: if $\beta > 1$ is odd, then $\mathcal{G}(\Sigma^0_\beta) \not\leqq \mathcal{G}(\Pi^0_\beta)$ and $\mathcal{G}(\Pi^0_\beta) \leqq \mathcal{G}(\Sigma^0_\beta)$; if $\beta > 0$ is even, then $\mathcal{G}(\Pi^0_\beta) \not\leqq \mathcal{G}(\Sigma^0_\beta(m))$ and $\mathcal{G}(\Sigma^0_\beta(m)) \leqq \mathcal{G}(\Pi^0_\beta)$ for all $m \in \omega$. For larger ordinals β, the formal proof of these properties is less straightforward. Moreover, stronger properties of such groups are needed to run a successful induction. We avoid these formal difficulties by not using these embedability relations in later proofs, stating them only in order to aid intuition. Though they will not be formally used, the reader may find it useful to keep in mind which groups are "bigger".

The embeddability relations discussed reflect the utility of the coding. Informally, we will ask

Is there a large subgroup attached to x?

about an element x that is infinitely divisible by an appropriate prime. The answer will allow us to extract whether the Σ^0_β outcome or the Π^0_β outcome was the case.

We (informally) justify not using a simpler recursive scheme to define the groups $\mathcal{G}(\Sigma^0_\beta)$ and $\mathcal{G}(\Pi^0_\beta)$ in the following remark.

[4]We emphasize that the definition of $\mathcal{G}(\Sigma^0_\beta(m))$ and $\mathcal{G}(\Pi^0_\beta)$ for $\beta = \delta + 2\ell + 2$ is identical to the case of $\beta = \delta$ as by definition $f_{\delta+2\ell+2}(k) = \delta + 2\ell + 1$ for all k. We separate them, here and in some later proofs, in hopes of not obfuscating the intuition.

Remark 3.2. It would of course be simpler if Definition 3.1 used only the odd recursion schema (for all successor ordinals). Unfortunately, the embeddability relations would not be satisfied in this case, e.g., when $\beta = 4$ it would be the case that $\mathcal{G}(\Sigma^0_\beta) \leqq \mathcal{G}(\Pi^0_\beta)$ and $\mathcal{G}(\Pi^0_\beta) \leqq \mathcal{G}(\Sigma^0_\beta)$. The reason is $\mathcal{G}(\Sigma^0_4)$ would contain infinitely many copies of $\mathcal{G}(\Pi^0_3)$ and infinitely many copies of $\mathcal{G}(\Sigma^0_3)$ whereas $\mathcal{G}(\Pi^0_4)$ only would contain infinitely many copies of $\mathcal{G}(\Sigma^0_3)$. As $\mathcal{G}(\Pi^0_3) \leqq \mathcal{G}(\Sigma^0_3)$, it would follow that $\mathcal{G}(\Sigma^0_4) \leq \mathcal{G}(\Pi^0_4)$. Hence asking if there is a *large* subgroup would not distinguish between the Σ^0_β and the Π^0_β outcomes.

For even successor ordinals $\beta \geq 4$, we will need additional auxiliary groups $\mathcal{H}(\Sigma^0_\beta)$ and $\mathcal{H}(\Pi^0_\beta)$.

Definition 3.3. For each even computable ordinal $\beta = \delta+2\ell+2 \geq 4$, define rooted torsion-free abelian groups $\mathcal{H}(\Sigma^0_\beta)$ and $\mathcal{H}(\Pi^0_\beta)$ as follows.

Define $\mathcal{H}(\Sigma^0_\beta)$ to be the group

$$\left\langle [\mathbb{Z}]_{p_\beta} \oplus \bigoplus_{k\in\omega} \mathcal{G}(\Sigma^0_{\beta-1}) \oplus \bigoplus_{k'\in\omega} \mathcal{G}(\Pi^0_{\beta-1});\ q_\beta^{-t}(r+r_k), q_\beta^{-t}(r+r_{k'}) : k, k', t \in \omega \right\rangle,$$

with root $r = 1$ in $[\mathbb{Z}]_{p_\beta}$, where r_k is the root of the kth copy of $\mathcal{G}(\Sigma^0_{\beta-1})$ and $r_{k'}$ is the root of k'th copy of $\mathcal{G}(\Pi^0_{\beta-1})$.

Define $\mathcal{H}(\Pi^0_\beta)$ to be the group

$$\left\langle [\mathbb{Z}]_{p_\beta} \oplus \bigoplus_{k\in\omega} \mathcal{G}(\Sigma^0_{\beta-1});\ q_\beta^{-t}(r+r_k) : k, t \in \omega \right\rangle$$

with root $r = 1$ in $[\mathbb{Z}]_{p_\beta}$, where r_k is the root of the kth copy of $\mathcal{G}(\Sigma^0_{\beta-1})$.

These are illustrated in Figure 3.

It is now possible to define $\mathcal{G}^\alpha_S$ for $S \subseteq \omega$.

Definition 3.4. For each successor ordinal $\alpha \geq 3$ and set $S \subseteq \omega$, define a torsion-free abelian group $\mathcal{G}^\alpha_S$ as follows.

- If $\alpha = \delta + 2\ell + 1 \geq 3$, define $\mathcal{G}^\alpha_S$ to be the group
$$\mathcal{G}^\alpha_S := \bigoplus_{n\in S} \left[\mathcal{G}(\Sigma^0_\alpha)\right]_{d_n} \oplus \bigoplus_{n\notin S} \left[\mathcal{G}(\Pi^0_\alpha)\right]_{d_n}.$$
- If $\alpha = \delta + 2\ell + 2$, define $\mathcal{G}^\alpha_S$ to be the group
$$\mathcal{G}^\alpha_S := \bigoplus_{n\in S} \left[\mathcal{H}(\Sigma^0_\alpha)\right]_{d_n} \oplus \bigoplus_{n\notin S} \left[\mathcal{H}(\Pi^0_\alpha)\right]_{d_n}.$$

The following definition and associated observation will be exploited in later sections when we wish to express elements as sums of roots of subcomponents.

Definition 3.5. If $\mathcal{G}$ is any group within this section, or any direct product of prime closures of such groups, we let $R_\mathcal{G}$ denote the set of roots of the recursively nested components of $\mathcal{G}$.

Of course, some elements serve as the root of more than one component at different ordinal levels. For example, if β is odd, then the root of a $\mathcal{G}(\Sigma^0_{\beta-1}(m))$ component of $\mathcal{G}(\Pi^0_\beta)$ is also the root of a $[\mathcal{G}(\Sigma^0_{\beta-2})]_{u_{\beta-1,0}}$ component. However, this root appears only once in $R_\mathcal{G}$.

Fact 3.6. The set $R_\mathcal{G}$ is a basis for both $\mathcal{G}$ and $D(\mathcal{G})$.

4. Proof of Theorem 1.5

Having defined $\mathcal{G}_S^\alpha$ for each successor ordinal $\alpha \geq 3$, it of course remains to verify the desired properties. We state these explicitly.

Lemma 4.1. *For each successor ordinal $\alpha \geq 3$, there is an effective enumeration $\{\Upsilon_n\}_{n\in\omega}$ of computable Σ_α^c sentences such that $\mathcal{G}_S^\alpha \models \Upsilon_n$ if and only if $n \in S$.*

Lemma 4.2. *For each successor ordinal $\alpha \geq 3$, if $S \in \Sigma_\alpha^0(X)$, then $\mathcal{G}_S^\alpha$ has an X-computable copy.*

Assuming Lemma 4.1 and Lemma 4.2, we prove Theorem 1.5. Lemma 4.1 is demonstrated in Section 4.1 and Lemma 4.2 is demonstrated in Section 4.2. We note the proof of Theorem 1.5 from Lemmas 4.1 and 4.2 is identical to the context of linear orders (see [1]).

Proof of Theorem 1.5. Fix a computable ordinal α, a degree $\mathbf{a} > \mathbf{0}^{(\alpha)}$, and a set $A \in \mathbf{a}$. By Theorem 1.4, we may assume that $\alpha \geq 3$.

If α is a successor ordinal $\beta + 1$, we argue the torsion-free abelian group $\mathcal{G} := \mathcal{G}^\alpha_{A\oplus\overline{A}}$ has proper α^{th} jump degree $\mathbf{a}$. By Lemma 4.1 and Lemma 4.2, we have

$$\begin{aligned}\mathrm{DegSpec}(\mathcal{G}) &= \{X : A \oplus \overline{A} \in \Sigma_\alpha^0(X)\}\\ &= \{X : A \in \Delta_\alpha^0(X)\}\end{aligned}$$

(for α finite, these are $\Sigma^0_{\alpha+1}(X)$ and $\Delta^0_{\alpha+1}(X)$). It follows $\{X^{(\alpha)} : X \in \mathrm{DegSpec}(\mathcal{G})\}$ contains precisely those sets that compute A. Thus $\mathcal{G}$ has α^{th} jump degree $\mathbf{a}$. On the other hand, if $\beta < \alpha$, the set $\{X^{(\beta)} : X \in \mathrm{DegSpec}(\mathcal{G})\}$ has no element of least degree (see Lemma 1.3 of [1]). Thus $\mathcal{G}$ does not have β^{th} jump degree for any $\beta < \alpha$.

If α is a limit ordinal, fix an α-generic set B such that $B^{(\alpha)} \equiv_T B \oplus \emptyset^{(\alpha)} \equiv_T A$. Viewing B as a subset of $\omega \times \omega$, we write $B_n := \{k : (n,k) \in B\}$. We argue the torsion-free abelian group

$$\mathcal{G} := \bigoplus_{n\in\omega} [\mathcal{G}_{B_n}]_{e_n}$$

has proper α^{th} jump degree $\mathbf{a}$, where $\mathcal{G}_{B_n}$ is the group $\mathcal{G}_{B_n}^{f_\alpha(n)}$ associated with the set B_n and the ordinal $f_\alpha(n)$. Making use of the uniformity in both Lemma 4.1 and Lemma 4.2 and that the prime e_n distinguishes the subgroup $\mathcal{G}_{B_n}^{f_\alpha(n)}$ from $\mathcal{G}_{B_{n'}}^{f_\alpha(n')}$ for $n' \neq n$, we have

$$\mathrm{DegSpec}(\mathcal{G}) = \{X : B_n \in \Sigma^0_{f_\alpha(n)}(X) \text{ uniformly in } n\}.$$

It follows $\{X^{(\alpha)} : X \in \mathrm{DegSpec}(\mathcal{G})\}$ contains precisely those sets that compute $B^{(\alpha)}$. Thus $\mathcal{G}$ has α^{th} jump degree $\mathbf{a}$. On the other hand, the set $\{X^{(\beta)} : X \in \mathrm{DegSpec}(\mathcal{G})\}$ has no element of least degree for any $\beta < \alpha$ (see discussion after Lemma 3.1 of [1]). Thus $\mathcal{G}$ does not have β^{th} jump degree for any $\beta < \alpha$. □

4.1. **Proof of Lemma 4.1.**

The definition of the Σ_α^c sentences $\{\Upsilon_n\}_{n\in\omega}$ is done recursively, mirroring the recursive nature of the definition of $\mathcal{G}_S^\alpha$. Before we start constructing formulas $\Phi_\beta(x)$ and $\Psi_\beta(x)$ connected semantically to $\mathcal{G}(\Sigma^0_\beta)$, $\mathcal{G}(\Sigma^0_\beta(m))$, and $\mathcal{G}(\Pi^0_\beta)$, we demonstrate two divisibility lemmas that isolate aspects of the odd and even inductive steps. The proofs of these are similar to proofs of lemmas by Downey and Montalbán (see Lemma 2.3 and Lemma 2.4 of [7]). For the proof of

Lemma 4.4(1), we make explicit whether we are viewing elements of $\mathcal{B}$ as belonging to $\mathcal{B}$ or the divisible closure of $\mathcal{B}$. For later parts of Lemma 4.4 and Lemma 4.5, we do not make it explicit as which it should be is clear from context. Before stating these two divisibility lemmas, we note a number of simple number theoretic facts (without proof) that we will use repeatedly (without mention).

Fact 4.3. The following facts hold of prime closures of $\mathbb{Z}$.

- For any primes p_0 and p_1, $[\mathbb{Z}]_{p_0}+[\mathbb{Z}]_{p_1} = [\mathbb{Z}]_{p_0,p_1}$, where the sum $[\mathbb{Z}]_{p_0}+[\mathbb{Z}]_{p_1}$ denotes the set of all $q \in \mathbb{Q}$ such that $q = a + b$ for some $a \in [\mathbb{Z}]_{p_0}$ and $b \in [\mathbb{Z}]_{p_1}$.
- For all sets of primes P_0 and P_1, $[\mathbb{Z}]_{P_0} \cap [\mathbb{Z}]_{P_1} = [\mathbb{Z}]_{P_0 \cap P_1}$.
- If P_0 and P_1 are disjoint sets of primes, then $([\mathbb{Z}]_{P_0} \setminus \mathbb{Z}) \cap [\mathbb{Z}]_{P_1} = \emptyset$ and $0 \notin ([\mathbb{Z}]_{P_0} \setminus \mathbb{Z}) + [\mathbb{Z}]_{P_1}$.

Lemma 4.4. *Fix pairwise disjoint sets of prime numbers P_0, F_1, and F_2 and a prime number $\rho \notin P_0 \cup F_1 \cup F_2$. For each $i \in \omega$, fix a copy of $[\mathbb{Z}]_{F_1}$ and let x_i denote the element 1 in this copy. For each $i, j \in \omega$, fix a copy of $[\mathbb{Z}]_{F_2}$ and let $y_{i,j}$ denote the element 1 in this copy. Let $\mathcal{B}$ be the group*

$$\mathcal{B} := \left[\left\langle \bigoplus_{i \in \omega} [\mathbb{Z}]_{F_1} \oplus \bigoplus_{i,j \in \omega} [\mathbb{Z}]_{F_2}; \frac{x_i + y_{i,j}}{\rho^k} : i, j, k \in \omega \right\rangle \right]_{P_0}.$$

Then $\mathcal{B}$ has the following properties:

(1) *For any $z \in \mathcal{B}$ and $\sigma_1 \in F_1$, we have $\sigma_1^\infty \mid z$ if and only if $z = \sum_i m_i x_i$ with $m_i \in [\mathbb{Z}]_{F_1,P_0}$.*
(2) *For any $y \in \mathcal{B}$ and $\sigma_2 \in F_2$, we have $\sigma_2^\infty \mid y$ if and only if $y = \sum_{i,j} m_{i,j} y_{i,j}$ with $m_{i,j} \in [\mathbb{Z}]_{\rho,F_2,P_0}$.*
(3) *Fixing an integer ℓ, if $\rho^\infty \mid \sum_j m_{\ell,j} y_{\ell,j}$, then $\sum_j m_{\ell,j} = 0$.*
(4) *If z can be expressed as $z = \sum m_i x_i$ with $m_i \in [\mathbb{Z}]_{P_0}$, then for each $\sigma_1 \in F_1$ and $\sigma_2 \in F_2$, the element z satisfies the formula*

$$(\dagger) \qquad \sigma_1^\infty \,|\, z \;\wedge\; (\exists y \in \mathcal{B}) \left[\rho^\infty \,|\, (z + y) \;\wedge\; \sigma_2^\infty \,|\, y \right].$$

(5) *If $z \in \mathcal{B}$ satisfies (†) with witness $y \in \mathcal{B}$, then $z = \sum_i m_i x_i$ with $m_i \in [\mathbb{Z}]_{P_0}$ and $y = \sum_{i,j} m_{i,j} y_{i,j}$ with $m_{i,j} \in [\mathbb{Z}]_{\rho,F_2,P_0}$ for all i, j and $m_i = \sum_j m_{i,j}$ for all i (noting that $m_i = 0$ is possible).*

Indeed, in any torsion-free abelian group of which $\mathcal{B}$ is a pure subgroup, these facts remain true.

Proof. For (1), the backward direction is immediate. For the forward direction, we express z (in the divisible closure) as $z = \sum_i m_i x_i + \sum_{i,j} m_{i,j} y_{i,j}$ with $m_i, m_{i,j} \in \mathbb{Q}$ (allowing the possibility of a coefficient being zero). If $\sigma_1^\infty \mid z$, as the summation is finite, there is a $[\mathbb{Z}]_{\sigma_1}$-multiple $\hat{z}$ of z in $\mathcal{B}$ with

$$\hat{z} = \sum_i \frac{\hat{m}_i}{\sigma_1^{n_i}} x_i + \sum_{i,j} \frac{\hat{m}_{i,j}}{\sigma_1^{n_{i,j}}} y_{i,j}$$

(with the right hand side expressed in the divisible closure) where $\hat{m}_i, \hat{m}_{i,j} \in \mathbb{Z}$, $\hat{m}_i \neq 0$ implies $\sigma_1 \nmid \hat{m}_i$, $\hat{m}_{i,j} \neq 0$ implies $\sigma_1 \nmid \hat{m}_{i,j}$, and $n_i, n_{i,j} > 0$. Since the coefficient of $y_{i,j}$ in any element of $\mathcal{B}$ (viewed in the divisible closure) is an element of $[\mathbb{Z}]_{\rho,F_2,P_0}$[5]

[5] Though we justify this here, we omit such arguments in the rest of the paper as all are similar to the argument here.

and $([\mathbb{Z}]_{\sigma_1}\backslash\mathbb{Z})\cap([\mathbb{Z}]_{\rho,F_2,P_0})=\emptyset$, it must be that $\hat{m}_{i,j}=0$ for all i,j, and so $m_{i,j}=0$ for all i,j. The reason that the coefficient of $y_{i,j}$ in any element of $\mathcal{B}$ (viewed in the divisible closure) is an element of $[\mathbb{Z}]_{\rho,F_2,P_0}$ is an immediate consequence of the fact that every element of $\mathcal{B}$ is a formal sum $\sum_i a_i x_i + \sum_{i,j} b_{i,j}(x_i+y_{i,j}) + \sum_{i,j} c_{i,j}y_{i,j}$ with $a_i\in[\mathbb{Z}]_{F_1,P_0}$, $b_{i,j}\in[\mathbb{Z}]_{\rho,P_0}$, and $c_{i,j}\in[\mathbb{Z}]_{F_2,P_0}$. Thus, in the divisible closure, the coefficient of any fixed $y_{i,j}$ is an element of $[\mathbb{Z}]_{\rho,P_0}+[\mathbb{Z}]_{F_2,P_0}=[\mathbb{Z}]_{\rho,F_2,P_0}$.

Thus if $\sigma_1^\infty\mid z$, then $z=\sum_i m_i x_i$ (in the divisible closure) with $m_i\in\mathbb{Q}$. From the structure of elements of $\mathcal{B}$, we have $m_i\in[\mathbb{Z}]_{\rho,F_1,P_0}$. Fix i. If $m_i\notin[\mathbb{Z}]_{F_1,P_0}$, then there would be a non-$[\mathbb{Z}]_{P_0}$-multiple of $x_i+y_{i,j}$ in z for some j, in particular a $[\mathbb{Z}]_{\rho,P_0}\backslash[\mathbb{Z}]_{P_0}$-multiple. Then the coefficient of this $y_{i,j}$ in z (in the divisible closure) would be in $[\mathbb{Z}]_{\rho,P_0}\backslash[\mathbb{Z}]_{P_0}+[\mathbb{Z}]_{F_2,P_0}$. However $0\notin[\mathbb{Z}]_{\rho,P_0}\backslash[\mathbb{Z}]_{P_0}+[\mathbb{Z}]_{F_2,P_0}$, yielding a contradiction to the form $z=\sum_i m_i x_i$. Thus $m_i\in[\mathbb{Z}]_{F_1,P_0}$ for all i, completing the proof of (1).

For (2), the argument is similar and we leave the minor change in details to the reader.

For (3), as $\rho^\infty\mid\sum_j m_{\ell,j}y_{\ell,j}$, there is a $[\mathbb{Z}]_\rho$-multiple $\hat{z}$ of $\sum_j m_{\ell,j}y_{\ell,j}$ in $\mathcal{B}$ with

$$\hat{z}=\sum_j \frac{\hat{m}_{\ell,j}}{\rho^{n_{\ell,j}}}y_{\ell,j}$$

where $\hat{m}_{\ell,j}\in\mathbb{Z}^{\neq 0}$, $\rho\nmid\hat{m}_{\ell,j}$, and $n_{\ell,j}>0$. Indeed, we may assume that $\sum_j\frac{\hat{m}_{\ell,j}}{\rho^{n_{\ell,j}}}\in[\mathbb{Z}]_\rho\backslash\mathbb{Z}$ (in particular, that it is not an element of $[\mathbb{Z}]_{P_0,F_2}$) if $\sum_j m_{\ell,j}\neq 0$. From the structure of elements of $\mathcal{B}$, we have

$$\hat{z}=a_\ell x_\ell+\sum_j b_{\ell,j}(x_\ell+y_{\ell,j})+\sum_j c_{\ell,j}y_{\ell,j}$$

with $a_\ell\in[\mathbb{Z}]_{F_1,P_0}$, $b_{\ell,j}\in[\mathbb{Z}]_{\rho,P_0}$, and $c_{\ell,j}\in[\mathbb{Z}]_{F_2,P_0}$. As $\sum_j\frac{\hat{m}_{\ell,j}}{\rho^{n_{\ell,j}}}\notin[\mathbb{Z}]_{P_0,F_2}$, it must be the case that $\sum_j b_{\ell,j}\notin[\mathbb{Z}]_{P_0}$. However this would imply the coefficient of x_ℓ is nonzero as $0\notin[\mathbb{Z}]_{F_1,P_0}+[\mathbb{Z}]_{\rho,P_0}\backslash[\mathbb{Z}]_{P_0}$. This would contradict the form of $\hat{z}$, showing (3).

For (4), we note if $z=\sum_i m_i x_i$ with $m_i\in[\mathbb{Z}]_{P_0}$, then $y=\sum_i m_i y_{i,0}$ is in $\mathcal{B}$. Moreover, by Parts (1) and (2), this y witnesses z satisfying (†), showing (4).

For (5), fix z and y with $\sigma_1^\infty\mid z$, $\rho^\infty\mid z+y$, and $\sigma_2^\infty\mid y$. By Part (1), we have $z=\sum_i m_i x_i$ with $m_i\in[\mathbb{Z}]_{F_1,P_0}$. By Part (2), we have $y=\sum_{i,j}m_{i,j}y_{i,j}$ with $m_{i,j}\in[\mathbb{Z}]_{\rho,F_2,P_0}$. As $\rho^\infty\mid z+y$, there is a $[\mathbb{Z}]_\rho$-multiple $\hat{z}+\hat{y}$ of $z+y$ in $\mathcal{B}$ with

$$\hat{z}+\hat{y}=\sum_i\frac{\hat{m}_i}{\rho^{n_i}}x_i+\sum_{i,j}\frac{\hat{m}_{i,j}}{\rho^{n_{i,j}}}y_{i,j}$$

where $\hat{m}_i,\hat{m}_{i,j}\in\mathbb{Z}$, $\hat{m}_i\neq 0$ implies $\rho\nmid\hat{m}_i$, $\hat{m}_{i,j}\neq 0$ implies $\rho\nmid\hat{m}_{i,j}$, and $n_i,n_{i,j}>0$. As

$$\hat{w}:=\sum_i\frac{\hat{m}_i}{\rho^{n_i}}x_i+\sum_i\frac{\hat{m}_i}{\rho^{n_i}}y_{i,0}$$

is in $\mathcal{B}$ (by virtue of it being a sum of $[\mathbb{Z}]_\rho$-multiples of $x_i+y_{i,0}$) and infinitely divisible by ρ, the element

$$\hat{z}+\hat{y}-\hat{w}=\sum_{i,j}\frac{\hat{m}_{i,j}}{\rho^{n_{i,j}}}y_{i,j}-\sum_i\frac{\hat{m}_i}{\rho^{n_i}}y_{i,0}$$

is in $\mathcal{B}$ and is infinitely divisible by ρ. By Part (3), this implies $\frac{\hat{m}_i}{\rho^{n_i}} = \sum_j \frac{\hat{m}_{i,j}}{\rho^{n_{i,j}}}$ for all i. This is equivalent to $m_i = \sum_j m_{i,j}$ for all i.

As $m_{i,j} \in [\mathbb{Z}]_{\rho,F_2,P_0}$ for all i,j, fixing i, the sum $\sum_j m_{i,j}$ is in $[\mathbb{Z}]_{\rho,F_2,P_0}$. As $[\mathbb{Z}]_{F_1,P_0} \cap [\mathbb{Z}]_{\rho,F_2,P_0} = [\mathbb{Z}]_{P_0}$, it follows $m_i \in [\mathbb{Z}]_{P_0}$ for all i. This shows (5). □

Lemma 4.5. *Fix pairwise disjoint sets of primes F_i, for $i \in \omega$, and P_0, and fix a sequence of distinct primes ρ_n, for $n \in \omega$, such that $\rho_n \notin (\cup_{i\in\omega} F_i) \cup P_0$ for each n. Let $\mathcal{B}$ be the group*

$$\mathcal{B} := \left[\left\langle \mathcal{F}; \frac{x_{i,j}}{\sigma_i^k}, \frac{x_{i,j} + x_{i+1,j}}{\rho_i^k} : i,j,k \in \omega \text{ and all } \sigma_i \in F_i \right\rangle\right]_{P_0}$$

where $\mathcal{F}$ is the free abelian group on the elements $x_{i,j}$ for $i,j \in \omega$. Then $\mathcal{B}$ has the following properties:

1. *Fixing an integer ℓ, a prime $\sigma_\ell \in F_\ell$, and an element $z \in \mathcal{B}$, if $\sigma_\ell^\infty \mid z$, then $z = \sum_j m_{\ell,j} x_{\ell,j}$ with $m_{\ell,j} \in [\mathbb{Z}]_{F_\ell,P_0}$.*
2. *Fixing an integer ℓ, if $z = \sum_j m_{\ell,j} x_{\ell,j}$ is nonzero, then $\rho_i^\infty \nmid z$ for any i.*
3. *Fixing primes $\sigma_i \in F_i$ for $0 \le i \le k+1$, if $z_0, \dots, z_{k+1} \in \mathcal{B}$ satisfy*
$$\sigma_i^\infty \mid z_i \text{ for all } i \le k+1 \text{ and } \rho_i^\infty \mid (z_i + z_{i+1}) \text{ for all } i \le k$$
then there are constants $m_j \in [\mathbb{Z}]_{P_0}$ such that $z_i = \sum_j m_j x_{i,j}$ for all $0 \le i \le k+1$.

Indeed, in any torsion-free abelian group of which $\mathcal{B}$ is a pure subgroup, these facts remain true.

Proof. For (1), we express z as $z = \sum_{i,j} m_{i,j} x_{i,j}$ with $m_{i,j} \in \mathbb{Q}^{\neq 0}$. As $\sigma_\ell^\infty \mid z$ and the summation is finite, there is a $[\mathbb{Z}]_{\sigma_\ell}$-multiple $\hat{z}$ of z in $\mathcal{B}$ with

$$\hat{z} = \sum_{i,j} \frac{\hat{m}_{i,j}}{\sigma_\ell^{n_{i,j}}} x_{i,j}$$

where $\hat{m}_{i,j} \in \mathbb{Z}^{\neq 0}$, $\sigma_\ell \nmid \hat{m}_{i,j}$, and $n_{i,j} > 0$. Thus the coefficient of $x_{i,j}$ in $\hat{z}$ is an element of $[\mathbb{Z}]_{\sigma_\ell} \backslash \mathbb{Z}$. On the other hand, the coefficient of $x_{i,j}$ in any element of $\mathcal{B}$ is an element of $[\mathbb{Z}]_{F_i,P_0} + [\mathbb{Z}]_{\rho_i,P_0} + [\mathbb{Z}]_{\rho_{i-1},P_0}$[6]. As $([\mathbb{Z}]_{\sigma_\ell} \backslash \mathbb{Z}) \cap [\mathbb{Z}]_{F_i,\rho_i,\rho_{i-1},P_0} = \emptyset$ if $i \neq \ell$, it follows that z can be expressed as $z = \sum_j m_{\ell,j} x_{\ell,j}$ with $m_{\ell,j} \in \mathbb{Q}^{\neq 0}$.

We show $m_{\ell,j} \in [\mathbb{Z}]_{F_\ell,P_0}$ for all j. Fixing j, if $m_{\ell,j}$ were not in $[\mathbb{Z}]_{F_\ell,P_0}$, there would necessarily be a non-$[\mathbb{Z}]_{P_0}$-multiple of either $x_{\ell,j} + x_{\ell+1,j}$ or $x_{\ell-1,j} + x_{\ell,j}$ in $\hat{z}$. This implies the coefficient of $x_{\ell+1,j}$ or $x_{\ell-1,j}$ is nonzero in $\hat{z}$ as $0 \notin [\mathbb{Z}]_{\rho_\ell,P_0} \backslash [\mathbb{Z}]_{P_0} + [\mathbb{Z}]_{\rho_{\ell+1},P_0} + [\mathbb{Z}]_{F_{\ell+1},P_0}$ and $0 \notin [\mathbb{Z}]_{\rho_\ell,P_0} \backslash [\mathbb{Z}]_{P_0} + [\mathbb{Z}]_{\rho_{\ell-1},P_0} + [\mathbb{Z}]_{F_{\ell-1},P_0}$. However this contradicts the form of $\hat{z}$, so it must be that $m_{\ell,j} \in [\mathbb{Z}]_{F_\ell,P}$, showing (1).

For (2), fix an ℓ, a nonzero element $z = \sum_j m_{\ell,j} x_{\ell,j}$, and an integer i towards a contradiction. As we are assuming $\rho_i^\infty \mid z$ for a contradiction, there is a $[\mathbb{Z}]_{\rho_i}$-multiple $\hat{z}$ of z in $\mathcal{B}$ with

$$\hat{z} = \sum_j \frac{\hat{m}_{\ell,j}}{\rho_i^{n_{\ell,j}}} x_{\ell,j}$$

where $\hat{m}_{\ell,j} \in \mathbb{Z}^{\neq 0}$, $\rho_i \nmid \hat{m}_{\ell,j}$, and $n_{\ell,j} > 0$. Fix j and note that the coefficient of $x_{\ell,j}$ in $\hat{z}$ is an element of $[\mathbb{Z}]_{\rho_i} \backslash \mathbb{Z}$. On the other hand, the coefficient of $x_{\ell,j}$ in any element of $\mathcal{B}$ is an element of $[\mathbb{Z}]_{F_\ell,P_0} + [\mathbb{Z}]_{\rho_\ell,P_0} + [\mathbb{Z}]_{\rho_{\ell-1},P_0}$. As $([\mathbb{Z}]_{\rho_i} \backslash \mathbb{Z}) \cap$

[6] We ignore the degenerate case of $i = 0$ as it is actually simpler.

$([\mathbb{Z}]_{F_\ell,P_0}+[\mathbb{Z}]_{\rho_\ell,P_0}+[\mathbb{Z}]_{\rho_{\ell-1},P_0})=\emptyset$ if $\ell\notin\{i,i+1\}$, we must have $\ell\in\{i,i+1\}$. We show that either yields a contradiction.

If $\ell=i$, as $([\mathbb{Z}]_{\rho_\ell}\setminus\mathbb{Z})\cap([\mathbb{Z}]_{F_\ell,P_0}+[\mathbb{Z}]_{\rho_{\ell-1},P_0})=\emptyset$, the term $x_{\ell,j}+x_{\ell+1,j}$ must have a nonzero coefficient in the expression of $\hat{z}$ as an element of $\mathcal{B}$. Indeed, this coefficient must be an element of $[\mathbb{Z}]_{\rho_\ell}\setminus\mathbb{Z}$ as the coefficient of $x_{\ell,j}$ in $\hat{z}$ is in $[\mathbb{Z}]_{\rho_\ell}\setminus\mathbb{Z}$. This implies the coefficient of $x_{\ell+1,j}$ in $\hat{z}$ is nonzero as $0\notin([\mathbb{Z}]_{\rho_\ell}\setminus\mathbb{Z})+[\mathbb{Z}]_{\rho_{\ell+1},P_0}+[\mathbb{Z}]_{F_{\ell+1},P_0}$, contradicting the form of z. If $\ell=i+1$, identical reasoning suffices to contradict the form of z. We have thus shown (2).

For (3), we induct on k. For $k=0$, by Part (1), we have $z_i=\sum_j m_{i,j}x_{i,j}$ with $m_{i,j}\in[\mathbb{Z}]_{F_i,P_0}$ for $i\in\{0,1\}$. As $\rho_0^\infty\mid(z_0+z_1)$, there is a $[\mathbb{Z}]_{\rho_0}$-multiple $\hat{z}$ of $z:=z_0+z_1$ in $\mathcal{B}$ with

$$\hat{z}=\sum_j\frac{\hat{m}_{0,j}}{\rho_0^{n_{0,j}}}x_{0,j}+\sum_j\frac{\hat{m}_{1,j}}{\rho_0^{n_{1,j}}}x_{1,j}$$

with $\rho_0\nmid\hat{m}_{0,j}$, $\rho_0\nmid\hat{m}_{1,j}$, $n_{0,j}>0$, and $n_{1,j}>0$. We rewrite $\hat{z}$ as

$$\hat{z}=\sum_j\frac{\hat{m}_{0,j}}{\rho_0^{n_{0,j}}}(x_{0,j}+x_{1,j})+\sum_j\frac{\hat{m}_{1,j}-\rho_0^{n_{1,j}-n_{0,j}}\hat{m}_{0,j}}{\rho_0^{n_{1,j}}}x_{1,j}.$$

As the first summation and $\hat{z}$ are both in $\mathcal{B}$ and infinitely divisible by ρ_0, so is the second summation. By Part (2), the second summation must be zero. Thus $\hat{m}_{1,j}/\rho_0^{n_{1,j}}=\hat{m}_{0,j}/\rho_0^{n_{0,j}}$ for all j, and so $m_{0,j}=m_{1,j}$ for all j, with this value an element of $[\mathbb{Z}]_{F_0,P_0}\cap[\mathbb{Z}]_{F_1,P_0}=[\mathbb{Z}]_{P_0}$, completing the base case.

Assuming Part (3) for k, we show it true for $k+1$. As in the base case, write $z_i=\sum_j m_{i,j}x_{i,j}$ with $m_{i,j}\in[\mathbb{Z}]_{F_i,P_0}$ for $i\le k+2$ by Part (1). By the induction hypothesis, for each fixed j, the values of $m_{i,j}$ for $0\le i\le k+1$ are equal. Let m_j denote this common value. Since m_j is in $[\mathbb{Z}]_{F_0,P_0}\cap\cdots\cap[\mathbb{Z}]_{F_k,P_0}$, it must be in $[\mathbb{Z}]_{P_0}$. As $z_{k+2}=\sum_j m_{k+2,j}x_{k+2,j}$ with $m_{k+2,j}\in[\mathbb{Z}]_{F_{k+2},P_0}$, the same analysis as in the base case implies $m_{k+2,j}=m_{k+1,j}=m_j$. □

We continue by introducing various formulas that capture structural aspects of the groups. These formulas describe how group elements interact in terms of infinite divisibility by certain primes. When defining these formulas and verifying their properties, we often restrict quantification from ranging over all group elements to ranging only over those elements which are infinitely divisible by certain primes.

To make this notion precise, we define the (computable infinitary) language of infinite divisibility. The signature of this language is the same as the signature of the language of groups except that for each prime p, we add a relation symbol for the relation $p^\infty\mid x$. That is, we treat $p^\infty\mid t$ for each prime p and term t as an atomic statement. We build up formulas in this language in the standard computable infinitary manner.

Definition 4.6. For any formula φ in the infinite divisibility language and any prime q, we define the relativized formula φ^q by recursion as follows:

- If φ is atomic, then $\varphi^q=_{\text{def}}\varphi$.
- If $\varphi:=(\bigwedge_i\beta_i)$, then $\varphi^q=_{\text{def}}\bigwedge_i\beta_i^q$; similarly for $\bigvee$, $\neg$, and $\longrightarrow$.
- If $\varphi:=(\exists x)\,\beta(x)$, then $\varphi^q=_{\text{def}}(\exists x)\,[q^\infty\mid x\wedge\beta^q(x)]$.
- If $\varphi:=(\forall x)\,\beta(x)$, then $\varphi^q=_{\text{def}}(\forall x)\,[q^\infty\mid x\longrightarrow\beta^q(x)]$.

Thus, a formula φ^q restricts all quantification to be over elements which are infinitely divisible by the prime q. The following lemma is a formal statement of this property.

Lemma 4.7. *Let $\mathcal{G}$ be a torsion-free abelian group, let q be a prime, and let $\mathcal{G}_q$ be the subgroup consisting of the elements infinitely divisible by q. If $\mathcal{G}_q$ is a pure subgroup, then for any formula $\varphi(\overline{x})$ in the language of infinite divisibility and any parameters $\overline{a}$ from $\mathcal{G}_q$, we have*

$$\mathcal{G} \models \varphi^q(\overline{a}) \qquad \text{if and only if} \qquad \mathcal{G}_q \models \varphi(\overline{a}). \tag{1}$$

In particular, if $\mathcal{G}$ is $\{q\}$-closed, then $\mathcal{G}_q = \mathcal{G}$ and hence

$$\mathcal{G} \models \varphi^q(\overline{a}) \qquad \text{if and only if} \qquad \mathcal{G} \models \varphi(\overline{a}).$$

Proof. Suppose $\mathcal{G}_q$ is a pure subgroup. We proceed by induction on $\varphi(\overline{x})$. If $\varphi(\overline{x})$ is atomic, then $\varphi^q(\overline{a})$ is the same as $\varphi(\overline{a})$. If $\varphi(\overline{a})$ has the form $t_0(\overline{a}) = t_1(\overline{a})$, then (1) follows because $\mathcal{G}_q$ is a subgroup. If $\varphi(\overline{a})$ has the form $p^\infty \mid t(\overline{a})$, then (1) follows because $\mathcal{G}_q$ is pure. The inductive cases for $\bigwedge$, $\bigvee$, $\longrightarrow$ and $\neg$ follow immediately by definition, leaving only the quantifier cases. It suffices to consider the case for $\exists$.

Suppose $\varphi^q(\overline{a})$ has the form $((\exists x)\beta(x, \overline{a}))^q$ and $\mathcal{G} \models (\exists x)[q^\infty \mid x \wedge \beta^q(x, \overline{a})]$ with a fixed witness x. Since x is infinitely divisible by q, we have $x \in \mathcal{G}_q$. By the inductive hypothesis $\mathcal{G}_q \models \beta(x, \overline{a})$ and hence $\mathcal{G}_q \models (\exists x)\beta(x, \overline{a})$ as required. Conversely, suppose $\mathcal{G}_q \models (\exists x)\beta(x, \overline{a})$ with fixed witness $x \in \mathcal{G}_q$. By the inductive hypothesis, $\mathcal{G} \models \beta^q(x, \overline{a})$, and since every element of $\mathcal{G}_q$ is infinitely divisible by q, we have $q^\infty \mid x$. Therefore, we have $\mathcal{G} \models (\exists x)[q^\infty \mid x \wedge \beta^q(x, \overline{a})]$ as required. □

Because the language of infinite divisibility is infinitary, we can express the relation $p^\infty \mid x$ using the standard formula $\varphi_p(x)$ given by

$$\varphi_p(x) := \bigwedge_{k \in \omega} (\exists y) \left[p^k y = x \right].$$

In any group, the atomic relation $p^\infty \mid x$ and the formula $\varphi_p(x)$ are equivalent in the sense that they are satisfied by the same elements. Thus, we can always translate formulas in the language of infinite divisibility into formulas in the (computable infinitary) language of group theory.[7] Notice, however, that some caution is required because the relativized formulas $(p^\infty \mid x)^q$ and $\varphi_p^q(x)$ are not (always) equivalent: the former is satisfied by those elements infinitely divisible by p, whereas the latter is satisfied by those elements infinitely divisible by p and q.

When we measure the quantifier complexity of a formula in the language of infinite divisibility, we will always mean its complexity as a formula in the language of group theory. Given the remarks in the previous paragraph, we need to be careful how we translate relativized formulas in the language of infinite divisibility into formulas in the language of group theory for the purposes of measuring complexity. Thus, when we say a formula φ^q (in the language of infinite divisibility) is in Σ^c_β or Π^c_β, we mean that the following formula ψ (in the language of group theory) is in the complexity class.

- First, use the recursive definition of relativized quantifiers to write φ^q in an unrelativized form in the language of infinite divisibility.

[7] As all of our languages are computable infinitary languages, we drop explicit reference to this fact from now on.

- Second, replace each occurrence of an atomic formula $p^\infty \mid t$ in this unrelativized formula by the corresponding formula $\varphi_p(t)$ to obtain a formula ψ in the language of group theory.

By performing the translation in this order, we ensure that we do not add additional divisibility conditions on the witnesses for $p^\infty \mid t$ and thus each atomic fact $p^\infty \mid t$ remains Π^c_2 even if it is under the scope of a relativizing prime.

We need one further convention before giving our formulas. Note that this convention does not change the quantifier complexity of any formula.

Convention 4.8. When we quantify over group elements using $(\exists z)$ or $(\forall z)$, the quantification is restricted to nonzero group elements. Hence $(\exists z)\,[\psi(z)]$ is an abbreviation for $(\exists z)\,[z \neq 0 \wedge \psi(z)]$ and $(\forall z)\,[\psi(z)]$ is an abbreviation for $(\forall z)\,[z = 0 \vee \psi(z)]$.

In a similar manner, we regard each of the formulas $A_\beta(x)$, $\Phi_\beta(x)$, $\Psi_\beta(x)$, $B_\beta(x)$, and $\Theta_\beta(x)$ (all defined later) as having an additional conjunct $x \neq 0$. In most cases, we could show that such a conjunct is unnecessary, but it easier to add it and ignore the issue of the zero element. The point of this convention is merely to keep our formulas a reasonable size and to avoid repeatedly stating assumptions that elements are not the zero element.

The formulas $A_\beta(x)$ below capture when an element x is a sum of roots of $\mathcal{G}(\Sigma^0_\beta(m))$ components (for even β). The formulas $\Phi_\beta(x)$ and $\Psi_\beta(x)$ capture when an element x is a sum of roots of $\mathcal{G}(\Sigma^0_\beta)$ components and a sum of roots of $\mathcal{G}(\Pi^0_\beta)$ components, respectively.

Definition 4.9. For each even ordinal β, we let $A_\beta(x)$ be the computable infinitary formula $A_\beta(x) := p^\infty_\beta \mid x \,\wedge\, (\exists w)\left[u^\infty_{\beta,1} \mid w \,\wedge\, v^\infty_{\beta,0} \mid (x+w)\right]$.

Definition 4.10. For each ordinal β with $\beta \geq 2$, we define computable infinitary formulas $\Phi_\beta(x)$ (for odd β) and $\Psi_\beta(x)$ (for even β) by recursion as follows.

- If $\beta = 2$, define $\Psi_\beta(x)$ to be the formula $\Psi_2(x) := p^\infty_1 \mid x$.
- If $\beta = 3$, define $\Phi_\beta(x)$ to be the formula
$$\Phi_3(x) := p^\infty_3 \mid x \,\wedge\, (\exists y)\big[q^\infty_3 \mid (x+y) \,\wedge\, \Psi_2(y)\big].$$
- If $\beta = \delta + 2\ell > 2$, define $\Psi_\beta(x)$ to be the formula

$$\Psi_\beta(x) := \bigwedge_{m\in\omega} (\exists x_0,\ldots,x_m)\left[x_0 = x \wedge \bigwedge_{k\leq m} u^\infty_{\beta,k} \mid x_k \wedge \bigwedge_{k<m} v^\infty_{\beta,k} \mid (x_k + x_{k+1}) \wedge \Phi^{u_{\beta,m}}_{f_\beta(m)}(x_m)\right].$$

  Note that when β is a successor ordinal, the last conjunct is $\Phi^{u_{\beta,m}}_{\beta-1}(x_m)$.
- If $\beta = \delta + 2\ell + 1 > 3$, define $\Phi_\beta(x)$ to be the formula
$$\Phi_\beta(x) := p^\infty_\beta \mid x \wedge (\exists y)\left[q^\infty_\beta \mid (x+y) \wedge A_{\beta-1}(y) \wedge \Psi_{\beta-1}(y)\right].$$

Lemma 4.11. *The complexity of $A_\beta(x)$ is Σ^c_3 (independent of β). If $\beta = \delta+2\ell \geq 2$, then $\Psi_\beta \in \Pi^c_\beta$. If $\beta = \delta+2\ell+1 \geq 3$, then $\Phi_\beta \in \Sigma^c_\beta$. Furthermore, the relativization of these formulas to any prime does not change their complexity.*

Proof. These statements follow immediately from $p^\infty \mid x$ being Π^c_2 and induction. □

Fact 4.12. Let ρ_0, ρ_1 and ρ_2 be distinct prime numbers and let $\psi(x)$ be the formula $\rho^\infty_0 \mid x \wedge (\exists y)[\rho^\infty_1 \mid y \wedge \rho^\infty_2 \mid (x+y)]$. The following properties hold for any prime q.

(1) If $\mathcal{G} \models \psi^q(x)$ for a fixed $x \in \mathcal{G}$ with witness y and $\mathcal{H}$ is a pure subgroup of $\mathcal{G}$ with $x, y \in \mathcal{H}$, then $\mathcal{H} \models \psi^q(x)$ with witness y.
(2) If $\mathcal{H} \models \psi^q(x)$ for a fixed $x \in \mathcal{H}$ with witness y and $\mathcal{H}$ is a subgroup of $\mathcal{G}$, then $\mathcal{G} \models \psi^q(x)$ with the same witness.

In particular, these properties hold for $A_\beta(x)$.

More generally, we have the following fact about our formulas as a consequence of them imposing only positive infinite divisibility conditions.

Fact 4.13. Let $\varphi(x)$ be a formula of the form $A_\beta(x)$, $\Phi_\beta(x)$ or $\Psi_\beta(x)$. If $\mathcal{H} \models \varphi(x)$ for some fixed $x \in \mathcal{H}$ and if $\mathcal{H}$ is a subgroup of $\mathcal{G}$, then $\mathcal{G} \models \varphi(x)$.

The next lemma gives the key properties needed to verify that our construction succeeds.

Lemma 4.14. *Fix an odd ordinal $\beta \geq 3$ and a set of primes P disjoint from $\{p_\rho\}_{\rho\leq\beta} \cup \{q_\rho\}_{\rho\leq\beta} \cup \{u_{\rho,m}\}_{\rho\leq\beta, m\in\omega} \cup \{v_{\rho,m}\}_{\rho\leq\beta,m\in\omega}$. Let $\mathcal{G}$ be the group $[\oplus_{i\in\omega}\mathcal{C}_i]_P$, where each $\mathcal{C}_i$ is either isomorphic to $\mathcal{G}(\Sigma^0_\beta)$ or $\mathcal{G}(\Pi^0_\beta)$.* [8]

(1) *If $\beta = 3$, then $\mathcal{G} \models \Psi_2(y)$ if and only if y can be expressed as $y = \sum a_i y_i$ with each y_i a root of a $\mathcal{G}(\Pi^0_2)$ component and $a_i \in [\mathbb{Z}]_{p_1,q_3,P}$.*
(2) *For $\beta = \delta + 2\ell + 1 > 3$:*
   (a) *If $\mathcal{G} \models A_{\beta-1}(z) \wedge \Psi_{\beta-1}(z)$, then $z = \sum a_i z_i$ with z_i a root of a $\mathcal{G}(\Pi^0_{\beta-1})$ component, $a_i \in [\mathbb{Z}]_{P,p_{\beta-2},q_\beta}$ (if $\beta - 1$ is not a limit) and $a_i \in [\mathbb{Z}]_{P,q_\beta}$ (if $\beta - 1$ is a limit).*
   (b) *If $z = \sum a_i z_i$ with z_i a root of a $\mathcal{G}(\Pi^0_{\beta-1})$ component and $a_i \in [\mathbb{Z}]_P$, then $\mathcal{G} \models A_{\beta-1}(z) \wedge \Psi_{\beta-1}(z)$.*
(3) *For $\beta = \delta + 2\ell + 3 > 3$ and $k \geq 0$:*
   (a) *If $\mathcal{G} \models u^\infty_{\beta-1,k} \mid z \wedge \Phi^{u_{\beta-1,k}}_{\beta-2}(z)$, then $z = \sum a_i z_i$ with $a_i \in [\mathbb{Z}]_{u_{\beta-1,k},q_\beta,P}$ and z_i a root of a $[\mathcal{G}(\Sigma^0_{\beta-2})]_{u_{\beta-1,k}}$ component.*
   (b) *If $z = \sum a_i z_i$ with $a_i \in [\mathbb{Z}]_{u_{\beta-1,k},P}$ and z_i a root of a $[\mathcal{G}(\Sigma^0_{\beta-2})]_{u_{\beta-1,k}}$ component, then $\mathcal{G} \models u^\infty_{\beta-1,k} \mid z \wedge \Phi^{u_{\beta-1,k}}_{\beta-2}(z)$.*
(4) *For $\beta = \delta + 1$ and $k \geq 0$:*
   (a) *If $\mathcal{G} \models u^\infty_{\beta-1,k} \mid z \wedge \Phi^{u_{\beta-1,k}}_{f_{\beta-1}(k)}(z)$, then $z = \sum a_i z_i$ with $a_i \in [\mathbb{Z}]_{u_{\beta-1,k},q_\beta,P}$ and z_i a root of a $[\mathcal{G}(\Sigma^0_{f_{\beta-1}(k)})]_{u_{\beta-1,k}}$ component.*
   (b) *If $z = \sum a_i z_i$ with $a_i \in [\mathbb{Z}]_{u_{\beta-1,k},P}$ and z_i a root of a $[\mathcal{G}(\Sigma^0_{f_{\beta-1}(k)})]_{u_{\beta-1,k}}$ component, then $\mathcal{G} \models u^\infty_{\beta-1,k} \mid z \wedge \Phi^{u_{\beta-1,k}}_{f_{\beta-1}(k)}(z)$.*

Moreover, the same is true if $\mathcal{G}$ is a finite sum of such groups $\mathcal{C}_i$.

Before proving Lemma 4.14, we establish some notation and some basic facts which will be useful in the proof. By Fact 3.6, we can write any element of $\mathcal{G}$, as an element of $D(\mathcal{G})$, in the form $\sum q_i x_i$ where each $q_i \in \mathbb{Q}^{\neq 0}$ and $x_i \in R_\mathcal{G}$. We will often use various divisibility conditions to narrow which roots x_i can occur in such a sum for particular elements and then use Lemma 4.4 and Lemma 4.5 to restrict the possible values for the coefficients q_i.

[8] The astute reader will note the upcoming statements are almost, but definitely not, biconditionals as a consequence of differences of elements within distinct $\mathcal{C}_i$. Though it is not too difficult to formulate (stating precisely is a bit more difficult) exact conditions for an element to satisfy the appropriate conjunction, we do not need them for our purposes.

Definition 4.15. If $X \subseteq R_{\mathcal{G}}$, denote by $\mathrm{Span}_{\mathcal{G}}(X)$ the set of all elements $g \in \mathcal{G}$ such that, in $D(\mathcal{G})$, $g = \sum q_i x_i$ where $q_i \in \mathbb{Q}$ and $x_i \in X$ for all i.

Lemma 4.16. *For any set $X \subseteq \mathcal{G}$ (in particular any set $X \subseteq R_{\mathcal{G}}$), the set $Span_{\mathcal{G}}(X)$ is a pure subgroup of $\mathcal{G}$.*

Proof. The set $\mathrm{Span}_{\mathcal{G}}(X)$ is clearly a subgroup of $\mathcal{G}$. To see that it is pure, fix $g \in \mathrm{Span}_{\mathcal{G}}(X)$, $n > 0$, and $h \in \mathcal{G}$ such that $nh = g$ in $\mathcal{G}$. We need to show that $h \in \mathrm{Span}_{\mathcal{G}}(X)$. Write $g = \sum q_i x_i$ (in $D(\mathcal{G})$) with $q_i \in \mathbb{Q}$ and $x_i \in X$. Because $\mathcal{G}$ is torsion-free, the element h is the unique element satisfying $nh = g$. Therefore, in $D(\mathcal{G})$, we have $h = \sum (q_i/n) x_i$ and hence $h \in \mathrm{Span}_{\mathcal{G}}(X)$. □

Note that in the context of torsion-free abelian groups, the subgroup $\mathrm{Span}_{\mathcal{G}}(X)$ need not separate as a direct summand of $\mathcal{G}$. Nevertheless, in the proof of Lemma 4.14, we will often be able to describe the isomorphism types of such subgroups. The next lemma pertains to any torsion-free abelian group.

Lemma 4.17. *Let $\mathcal{H}$ be a torsion-free abelian group which is P-closed for a set P of primes. Let ρ be a prime and let $h \in \mathcal{H}$ be infinitely divisible by ρ. Then for any $q \in [\mathbb{Z}]_P$, the element qh is infinitely divisible by ρ.*

Proof. Let $g \in \mathcal{H}$ satisfy $\rho^k g = h$. Since $\mathcal{H}$ is $[\mathbb{Z}]_P$-closed and $q \in [\mathbb{Z}]_P$, we can multiply this equation by q in $\mathcal{H}$ to obtain $(q\rho^k)g = qh$. Thus, the element qg witnesses that qh is divisibly by ρ^k. □

We return to the proof of Lemma 4.14. We work both within $\mathcal{G}$ and $D(\mathcal{G})$ during this proof and often rely on context to indicate which group we are working in.

Proof of Lemma 4.14. Before establishing Lemma 4.14, we say a word about its proof. For $\beta = 3$, we demonstrate (1) directly. For $\beta > 3$, we demonstrate (2), (3), and (4) by simultaneous induction on β. The base case of the induction is the case $\beta = 5$ for (3). The induction cases proceed as follows. To prove (2) for β, we use that (3) and (4) hold for values less than or equal to β; to prove (3) for β, we use that (2) holds for values less than β; and to prove (4) for β, we use that (3) holds for values less than β. Because (3) includes our base case, we begin with the proof of (3) after showing (1).

(1) For $\beta = 3$, we show y can be so expressed if $\mathcal{G} \models \Psi_2(y)$, i.e., if $\mathcal{G} \models p_1^\infty \mid y$. Working in $D(\mathcal{G})$, we express y as $y = \sum a_i y_i$ where $a_i \in \mathbb{Q}$ and y_i is the root of a $\mathcal{G}(\Sigma^0_2(m))$ component, a $\mathcal{G}(\Pi^0_2)$ component, or a $[\mathbb{Z}]_{p_3}$ component. We note that it is impossible that any y_i is the root of a $\mathcal{G}(\Sigma^0_2(m))$ component. For if one were, with y_j the root of a $\mathcal{G}(\Sigma^0_2(m_j))$ component, there would be a $[\mathbb{Z}]_{p_1}$-multiple $\hat{y}$ of y in $\mathcal{G}$ with

$$\hat{y} = \sum_i \frac{\hat{a}_i}{p_1^{n_i}} y_i$$

where $\hat{a}_i \in \mathbb{Z}^{\neq 0}$, $p_1 \nmid \hat{a}_i$, and $n_j > m_j$. However, this is impossible as the coefficient of the root of any $\mathcal{G}(\Sigma^0_2(m_j))$ component in $\mathcal{G}$ has the form a/p_1^k where $a \in [\mathbb{Z}]_{q_3,P}$ and $k \leq m_j$.

Thus, we have that $y = \sum a_i y_i$ where each y_i is the root of a $[\mathbb{Z}]_{p_3}$ component or a $\mathcal{G}(\Pi^0_2)$ component. In other words, we have $y \in \mathcal{B}$ where $\mathcal{B} := \mathrm{Span}_{\mathcal{G}}(X)$ and X

is the set of roots of $\mathcal{G}(\Pi^0_2)$ components and $[\mathbb{Z}]_{p_3}$ components of $\mathcal{G}$. Hence $\mathcal{B}$ can be written as a direct sum of subgroups

$$\left[\left\langle [\mathbb{Z}]_{p_3} \oplus \bigoplus_{k\in\omega} [\mathbb{Z}]_{p_1} ; q_3^{-t}(r + r_k) : k, t \in \omega \right\rangle\right]_P$$

since $\mathcal{G}(\Pi^0_2) \cong [\mathbb{Z}]_{p_1}$.[9] Since $\mathcal{G} \models p_1^\infty \,|\, y$ and $\mathcal{B}$ is a pure subgroup of $\mathcal{G}$ (by Lemma 4.16), we have that $\mathcal{B} \models p_1^\infty \,|\, y$. Applying Lemma 4.4(2) to $\mathcal{B}$ (with $F_1 = \{p_3\}$, $F_2 = \{p_1\}$, $\rho = q_3$, and $P = P$) yields that each y_i is the root of a $\mathcal{G}(\Pi^0_2)$ component and each $a_i \in [\mathbb{Z}]_{p_1,q_3,P}$.

Conversely, suppose $y = \sum a_i y_i$ with $a_i \in [\mathbb{Z}]_{p_1,q_3,P}$ and y_i the root of a $\mathcal{G}(\Pi^0_2)$ component. Since y is the sum of roots of $\mathcal{G}(\Pi^0_2)$ components, we have $y \in \mathcal{B}$ (where $\mathcal{B}$ is as in the other direction). Since each $a_i \in [\mathbb{Z}]_{p_1,q_3,P}$, Lemma 4.4(2) implies that $p_1^\infty \,|\, y$.

(3) For the base case when $\beta = 5$, we first show (3)(a). Fix $k \in \omega$ and suppose that $\mathcal{G} \models u_{4,k}^\infty \,|\, z \wedge \Phi_3^{u_{4,k}}(z)$, recalling $u_{4,k}^\infty \,|\, z \wedge \Phi_3^{u_{4,k}}(z)$ is

$$(\ddagger) \qquad u_{4,k}^\infty \,|\, z \;\wedge\; p_3^\infty \,|\, z \;\wedge\; (\exists y)[u_{4,k}^\infty \,|\, y \;\wedge\; q_3^\infty \,|\, (z+y) \;\wedge\; p_1^\infty \,|\, y].$$

We need to show that $z = \sum a_i z_i$ with each $a_i \in [\mathbb{Z}]_{u_{4,k},q_5,P}$ and each z_i a root of a $[\mathcal{G}(\Sigma^0_3)]_{u_{4,k}}$ component.

Since $u_{4,k}^\infty \,|\, z$, the element z must be a sum $z = \sum w_i$, where each w_i comes from a $[\mathcal{G}(\Sigma^0_3)]_{u_{4,k}}$ or $[\mathcal{G}(\Pi^0_3)]_{u_{4,k}}$ component (which we denote by $\mathcal{G}_i$). Indeed, since $p_3^\infty \,|\, z$ by hypothesis, each w_i is a multiple of the root of $\mathcal{G}_i$. Hence, the element z must be a sum $z = \sum a_i z_i$ where $a_i \in \mathbb{Q}$ and each z_i is the root of $\mathcal{G}_i$. We endeavor to show that, in fact, each $a_i \in [\mathbb{Z}]_{u_{4,k},q_5,P}$ and each $\mathcal{G}_i$ is a $[\mathcal{G}(\Sigma^0_3)]_{u_{4,k}}$ component.

Fix a witness y for $(\ddagger)$. Since $u_{4,k}^\infty \,|\, y$, the element y must also be contained within the $[\mathcal{G}(\Sigma^0_3)]_{u_{4,k}}$ and $[\mathcal{G}(\Pi^0_3)]_{u_{4,k}}$ components. Furthermore, since $p_1^\infty \,|\, y$, the element y must have the form $y = \sum b_j y_j$ where each $b_j \in \mathbb{Q}$ and y_j is the root of a $\mathcal{G}(\Pi^0_2)$ component. Since the $[\mathcal{G}(\Pi^0_3)]_{u_{4,k}}$ components do not contain $\mathcal{G}(\Pi^0_2)$ components, each y_j is the root of a $\mathcal{G}(\Pi^0_2)$ subcomponent of a $[\mathcal{G}(\Sigma^0_3)]_{u_{4,k}}$ component. Thus $z, y \in \mathcal{B}$ where $\mathcal{B} := \mathrm{Span}_{\mathcal{G}}(X)$ and X contains the roots of the $[\mathcal{G}(\Pi^0_3)]_{u_{4,k}}$ components, the roots of the $[\mathcal{G}(\Sigma^0_3)]_{u_{4,k}}$ components, and the roots of the $\mathcal{G}(\Pi^0_2)$ components of the $[\mathcal{G}(\Sigma^0_3)]_{u_{4,k}}$ components. By Lemma 4.16, the group $\mathcal{B}$ is a pure subgroup of $\mathcal{G}$.

To describe the isomorphism type of $\mathcal{B}$, we need to analyze which primes infinitely divide the roots occurring in X. The point is that a particular element of X may be the root of components at more than one level and each level will introduce different infinite divisibilities. Because of these considerations, we split into cases depending on whether $k > 0$ or $k = 0$.

First, consider the case when $k > 0$ and let r be the root of a $[\mathcal{G}(\Pi^0_3)]_{u_{4,k}}$ component or a $[\mathcal{G}(\Sigma^0_3)]_{u_{4,k}}$ component. The root r is infinitely divisible by p_3 (since it is a root at level 3), by $u_{4,k}$ (by the prime closure of the component added at level 4) and by all the primes in P (by the prime closure of $\mathcal{G}$). Because $k > 0$, the element r is not the root of a component at level 4 and because the level 3 (at which r is a root) is odd, the element r is not the root at level 2. Similarly, if r is the root of a $\mathcal{G}(\Pi^0_2)$ subcomponent of a $[\mathcal{G}(\Sigma^0_3)]_{u_{4,k}}$ component, then r is infinitely

[9]Of course, here we mean r to be the root of the $[\mathbb{Z}]_{p_3}$ component and r_k to be the root of the kth copy of $[\mathbb{Z}]_{p_1}$. When obvious, we omit such explanation.

divisible by p_1 (since it is the root of $\mathcal{G}(\Pi^0_2)$), by $u_{4,k}$ (by the prime closure) and by all the primes in P. Again, the element r is not the root at any other level. Thus, when $k > 0$, the group $\mathcal{B}$ is isomorphic to a direct sum of infinitely many copies of $[\mathbb{Z}]_{p_3,u_{4,k},P}$ (coming from the roots of the $[\mathcal{G}(\Pi^0_3)]_{u_{4,k}}$ components) and infinitely many copies of

$$\left[\left\langle [\mathbb{Z}]_{p_3} \oplus \bigoplus_{k\in\omega} [\mathbb{Z}]_{p_1}; q_3^{-t}(r + r_k) : k, t \in \omega \right\rangle\right]_{u_{4,k},P} \tag{2}$$

(coming from the roots of the $[\mathcal{G}(\Sigma^0_3)]_{u_{4,k}}$ components and the roots of their $\mathcal{G}(\Pi^0_2)$ subcomponents).

We show that each z_i in the sum $z = \sum a_i z_i$ is the root of a $[\mathcal{G}(\Sigma^0_3)]_{u_{4,k}}$ component. If not, then we can suppose without loss of generality that z_0 is the root of a $[\mathcal{G}(\Pi^0_3)]_{u_{4,k}}$ component, that is, the element z_0 is the element 1 in a direct summand of $\mathcal{B}$ of the form $[\mathbb{Z}]_{p_3,u_{4,k},P}$. Since $q_3^\infty \mid (\sum a_i z_i + \sum b_j y_j)$, there is a $[\mathbb{Z}]_{q_3}$-multiple $\hat{w}$ of $z + y$ in $\mathcal{B}$ such that

$$\hat{w} = \sum_i \frac{\hat{a}_i}{q_3^{k_i}} z_i + \sum_j \frac{\hat{b}_j}{q_3^{\ell_j}} y_j$$

where $\hat{a}_i, \hat{b}_j \in \mathbb{Z}$, $q_3 \nmid \hat{a}_i$, $k_i > 0$, $q_3 \nmid \hat{b}_j$ and $\ell_j > 0$ (assuming $\hat{a}_i, \hat{b}_j \neq 0$). However, the coefficient of z_0 in any element of $\mathcal{B}$ must be from $[\mathbb{Z}]_{p_3,u_{4,k},P}$. Hence, we have $\hat{a}_0 = 0$ and therefore $a_0 = 0$.

Having established that each z_i is the root of a $[\mathcal{G}(\Sigma^0_3)]_{u_{4,k}}$ component, it follows that $z, y \in \mathcal{B}'$ where $\mathcal{B}' := \mathrm{Span}_{\mathcal{G}}(X')$ with $X' \subseteq X$ containing only the roots of the $[\mathcal{G}(\Sigma^0_3)]_{u_{4,k}}$ components and the roots of their $\mathcal{G}(\Pi^0_2)$ subcomponents. That is, the group $\mathcal{B}'$ is the subgroup of $\mathcal{B}$ consisting of the direct sum of the infinitely many copies of the group in (2). Since $\mathcal{B}'$ is a pure subgroup of $\mathcal{G}$, we have by Fact 4.12(1)

$$\mathcal{B}' \models p_3^\infty \mid z \wedge (\exists y)[q_3^\infty \mid (z + y) \wedge p_1^\infty \mid y]$$

(with our fixed element $y \in \mathcal{B}'$ as witness). Therefore, we can apply Lemma 4.4(5) (with $F_1 := \{p_3\}$, $F_2 := \{p_1\}$, $\rho := q_3$ and $P := P \cup \{u_{4,k}\}$) to conclude that $z = \sum a_i z_i$ with $a_i \in [\mathbb{Z}]_{u_{4,k},P}$.

Second, consider the case when $k = 0$. In this case, we have $z = \sum a_i z_i$ where each z_i is the root of a $[\mathcal{G}(\Sigma^0_3)]_{u_{4,0}}$ component since there are no $[\mathcal{G}(\Pi^0_3)]_{u_{4,0}}$ components. A root r of a $[\mathcal{G}(\Sigma^0_3)]_{u_{4,0}}$ component is infinitely divisible by p_3 (since it is a root at level 3), by $u_{4,k}$ (by the prime closure), by p_4 (since $k = 0$ and hence r is also the root of a $\mathcal{G}(\Pi^0_4)$ or $\mathcal{G}(\Sigma^0_4(m))$ component) and by all the primes in P. Furthermore, if r, r' are roots of (distinct) $\mathcal{G}(\Pi^0_4)$ or $\mathcal{G}(\Sigma^0_4(m))$ components within the same $\mathcal{C}_i$, then $r - r'$ is infinitely divisible by q_5. (This divisibility does not add to the infinite divisibility of either r or r', but it does effect the isomorphism type of $\mathcal{B}$.) However, if r, r' are roots of such components in different $\mathcal{C}_i$, then $r - r'$ is not divisible by q_5. To smooth out this difference in divisibility and to simplify the calculations, we work in $[\mathcal{B}]_{q_5}$.

The group $[\mathcal{B}]_{q_5}$ is isomorphic to the direct sum of infinite many copies of

$$\left[\left\langle [\mathbb{Z}]_{p_3,p_4} \oplus \bigoplus_{k\in\omega} [\mathbb{Z}]_{p_1}; q_3^{-t}(r + r_k) : k, t \in \omega \right\rangle\right]_{u_{4,0},q_5,P}$$

(coming from the roots of the $[\mathcal{G}(\Sigma^0_3)]_{u_{4,0}}$ components and their $\mathcal{G}(\Pi^0_2)$ subcomponents). Since $\mathcal{B}$ is a pure subgroup of $\mathcal{G}$ and $\mathcal{B}$ is a subgroup of $[\mathcal{B}]_{q_5}$, we have by

Fact 4.12(1) and Fact 4.12(2)

$$[\mathcal{B}]_{q_5} \models p_3^\infty \,|\, z \,\wedge\, (\exists y)[q_3^\infty \,|\, (z+y) \,\wedge\, p_1^\infty \,|\, y]$$

with our fixed element $y \in \mathcal{B}$ as the witness. Applying Lemma 4.4(5) (with $F_1 := \{p_3, p_4\}$, $F_2 := \{p_1\}$, $\rho := q_3$ and $P := P \cup \{u_{4,0}, q_5\}$), we conclude that $a_i \in [\mathbb{Z}]_{P,u_{4,0},q_5}$. This completes the proof of (3)(a) when $\beta = 5$.

To prove (3)(b) when $\beta = 5$, assume $z = \sum a_i z_i$, where $a_i \in [\mathbb{Z}]_{u_{4,k},P}$ and each z_i is the root of a $[\mathcal{G}(\Sigma_3^0)]_{u_{4,k}}$ component. Let $\mathcal{G}_i$ denote the $[\mathcal{G}(\Sigma_3^0)]_{u_{4,k}}$ component containing z_i. We need to show that $\mathcal{G}$ satisfies

$$u_{4,k}^\infty \,|\, z \,\wedge\, p_3^\infty \,|\, z \,\wedge\, (\exists y)[u_{4,k}^\infty \,|\, y \,\wedge\, q_3^\infty \,|\, (z+y) \,\wedge\, p_1^\infty \,|\, y].$$

Since z_i is the root of $\mathcal{G}_i$, we have $p_3^\infty \,|\, z_i$. By Lemma 4.17 and the fact that $\mathcal{G}_i$ is $P \cup \{u_{4,k}\}$-closed, it follows that $p_3^\infty \,|\, a_i z_i$ and $u_{4,k}^\infty \,|\, a_i z_i$. Hence, we have $p_3^\infty \,|\, z$ and $u_{4,k}^\infty \,|\, z$.

Let y_i be the root of a $\mathcal{G}(\Pi_2^0)$ component inside $\mathcal{G}_i$ and let $y := \sum a_i y_i$. Since $\mathcal{G}(\Pi_2^0) \cong [\mathbb{Z}]_{p_1}$, we have $p_1^\infty \,|\, y_i$. As $\mathcal{G}_i$ is $P \cup \{u_{4,k}\}$-closed, it follows that $p_1^\infty \,|\, a_i y_i$ (by Lemma 4.17) and that $u_{4,k}^\infty \,|\, a_i y_i$. Hence, both p_1 and $u_{4,k}$ infinitely divide y. By the definition of $\mathcal{G}(\Sigma_3^0)$, we have $q_3^\infty \,|\, (z_i + y_i)$ and applying Lemma 4.17 one more time, we obtain $q_3^\infty \,|\, (a_i z_i + a_i y_i)$. Therefore $\mathcal{G}$ satisfies $\Phi_3^{u_{4,k}}(z)$ with witness y.

This completes the base case of $\beta = 5$.

Next, we show (3) for $\beta > 5$ supposing (2) holds for $\beta - 2$. To prove (3)(a), we suppose $\mathcal{G} \models u_{\beta-1,k}^\infty \,|\, z \,\wedge\, \Phi_{\beta-2}^{u_{\beta-1,k}}(z)$, recalling $u_{\beta-1,k}^\infty \,|\, z \,\wedge\, \Phi_{\beta-2}^{u_{\beta-1,k}}(z)$ is

(‡‡) $u_{\beta-1,k}^\infty \,|\, z \wedge p_{\beta-2}^\infty \,|\, z \wedge (\exists y)[u_{\beta-1,k}^\infty \,|\, y \wedge q_{\beta-2}^\infty \,|\, (z+y) \wedge A_{\beta-3}^{u_{\beta-1,k}}(y) \wedge \Psi_{\beta-3}^{u_{\beta-1,k}}(y)]$.

We need to show that $z = \sum a_i z_i$ with each $a_i \in [\mathbb{Z}]_{u_{\beta-1,k},q_\beta,P}$ and each z_i a root of a $[\mathcal{G}(\Sigma_{\beta-2}^0)]_{u_{\beta-1,k}}$ component.

As in the $\beta = 5$ case, together $u_{\beta-1,k}^\infty \,|\, z$ and $p_{\beta-2}^\infty \,|\, z$ imply that $z = \sum a_i z_i$, where $a_i \in \mathbb{Q}$ and each z_i is the root of a $[\mathcal{G}(\Sigma_{\beta-2}^0)]_{u_{\beta-1,k}}$ or $[\mathcal{G}(\Pi_{\beta-2}^0)]_{u_{\beta-1,k}}$ component. We endeavor to show that, in fact, each $a_i \in [\mathbb{Z}]_{u_{\beta-1,k},q_\beta,P}$ and each z_i is the root of a $[\mathcal{G}(\Sigma_{\beta-2}^0)]_{u_{\beta-1,k}}$ component.

Fix a witness y for (‡‡). Our first goal is to show that y is a sum of roots of $\mathcal{G}(\Pi_{\beta-3}^0)$ components. Since $u_{\beta-1,k}^\infty \,|\, y$, the element y lies within the $[\mathcal{G}(\Sigma_{\beta-2}^0)]_{u_{\beta-1,k}}$ and $[\mathcal{G}(\Pi_{\beta-2}^0)]_{u_{\beta-1,k}}$ components. Thus, we have $y \in \mathcal{H} := \mathrm{Span}_{\mathcal{G}}(X)$ where X contains the roots of the $[\mathcal{G}(\Sigma_{\beta-2}^0)]_{u_{\beta-1,k}}$ and $[\mathcal{G}(\Pi_{\beta-2}^0)]_{u_{\beta-1,k}}$ components as well as the roots of all the components nested via the recursive construction inside these components. Note that $\mathcal{H}$ is the subgroup of $\mathcal{G}$ consisting of the elements infinitely divisible by $u_{\beta-1,k}$. Because $\mathcal{G}$ satisfies $A_{\beta-3}^{u_{\beta-1,k}}(y) \wedge \Psi_{\beta-3}^{u_{\beta-1,k}}(y)$, we have that $\mathcal{H}$ satisfies $A_{\beta-3}(y) \wedge \Psi_{\beta-3}(y)$ by Lemma 4.7.

We describe the isomorphism type of $\mathcal{H}$ in two cases: when $k > 0$ and when $k = 0$. If $k > 0$, then the roots of the $[\mathcal{G}(\Sigma_{\beta-2}^0)]_{u_{\beta-1,k}}$ and $[\mathcal{G}(\Pi_{\beta-2}^0)]_{u_{\beta-1,k}}$ components are not roots of components at any level other than $\beta - 2$. Thus, the group $\mathcal{H}$ is an infinite direct sum of $[\mathcal{G}(\Sigma_{\beta-2}^0)]_{u_{\beta-1,k},P}$ and $[\mathcal{G}(\Pi_{\beta-2}^0)]_{u_{\beta-1,k},P}$ groups.

If $k = 0$, then note that there are no $[\mathcal{G}(\Pi_{\beta-2}^0)]_{u_{\beta-1,0}}$ components. Each root of a $[\mathcal{G}(\Sigma_{\beta-2}^0)]_{u_{\beta-1,0}}$ component is also the root of a $\mathcal{G}(\Sigma_{\beta-1}^0(m))$ or a $\mathcal{G}(\Pi_{\beta-1}^0)$ component. Thus, each such root is infinitely divisible by $p_{\beta-1}$ (in addition to the divisibility imposed at level $\beta - 2$). Furthermore, if r, r' are roots of $\mathcal{G}(\Sigma_{\beta-1}^0(m))$ or $\mathcal{G}(\Pi_{\beta-1}^0)$ components from the same $\mathcal{C}_i$, then $q_\beta^\infty \,|\, (r - r')$. If they are roots from different $\mathcal{C}_i$, then we have no such q_β divisibility. To incorporate the extra divisibility by $p_{\beta-1}$ and to smooth out this divisibility difference by q_β, we study $[\mathcal{H}]_{q_\beta,p_{\beta-1}}$. The group $[\mathcal{H}]_{q_\beta,p_{\beta-1}}$ is isomorphic to an infinite direct sum of $[\mathcal{G}(\Sigma^0_{\beta-2})]_{P'}$ groups where $P' := P \cup \{u_{\beta-1,0}, q_\beta, p_{\beta-1}\}$.

In each of the $k > 0$ and $k = 0$ cases, we can apply Part (2)(a) for $\mathcal{H}$ or $[\mathcal{H}]_{q_\beta,p_{\beta-1}}$ and $\beta - 2$ to conclude that $y = \sum b_j y_j$ is a sum of roots y_j of $\mathcal{G}(\Pi^0_{\beta-3})$ components in $\mathcal{G}$ (with appropriate coefficients, which depend on which case we are in). Thus, we have established our first goal.

Our second goal is to show that in the sum $z = \sum a_i z_i$, where each z_i is the root of a $[\mathcal{G}(\Sigma^0_{\beta-2})]_{u_{\beta-1,k}}$ component (as opposed to a $[\mathcal{G}(\Pi^0_{\beta-2})]_{u_{\beta-1,k}}$ component) and each coefficient a_i lies in $[\mathbb{Z}]_{p_{\beta-1},q_\beta,P}$. We have $z, y \in \mathcal{B} := \operatorname{Span}_{\mathcal{G}}(X)$ where X contains the roots of the $[\mathcal{G}(\Pi^0_{\beta-2})]_{u_{\beta-1,k}}$ components, the roots of the $[\mathcal{G}(\Sigma^0_{\beta-2})]_{u_{\beta-1,k}}$ components, and the roots of their $\mathcal{G}(\Pi^0_{\beta-3})$ components. We split into cases depending on whether $k > 0$ or $k = 0$ and proceed with an analysis of the infinite divisibilities as in the $\beta = 5$ case.

First, suppose that $k > 0$. A root r of a $[\mathcal{G}(\Pi^0_{\beta-2})]_{u_{\beta-1,k}}$ or $[\mathcal{G}(\Sigma^0_{\beta-2})]_{u_{\beta-1,k}}$ component is infinitely divisible by $p_{\beta-2}$ (being a root at level $\beta - 2$), by $u_{\beta-1,k}$ (by prime closure), and by the primes in P (by prime closures). Since $\beta - 2$ is odd, the element r is not a root at a lower level; since $k > 0$, the element r is not a root at a higher level.

A root r of a $\mathcal{G}(\Pi^0_{\beta-3})$ subcomponent of a $[\mathcal{G}(\Sigma^0_{\beta-2})]_{u_{\beta-1,k}}$ component is infinitely divisible by $p_{\beta-3}$ (being a root at level $\beta - 3$), by $u_{\beta-1,k}$ (by prime closure), and by the primes in P (by divisible closures). In addition, if $\beta - 3$ is not a limit ordinal, then r is also the root of a $[\mathcal{G}(\Sigma^0_{\beta-4})]_{u_{\beta-3,0}}$ component and hence is infinitely divisible by $p_{\beta-4}$ and $u_{\beta-3,0}$. Notice that the recursion stops at this point because $\beta - 4$ is an odd ordinal and hence r is not the root at any lower level. If $\beta - 3$ is a limit ordinal, then r is also the root of a $[\mathcal{G}(\Sigma^0_{f_{\beta-3}(0)})]_{u_{\beta-3,0}}$ component and hence is infinitely divisible by $p_{f_{\beta-3}(0)}$ and $u_{\beta-3,0}$. Again, the recursion stops at this point because $f_{\beta-3}(0)$ is an odd ordinal. Recall that if $\beta - 3$ is not a limit, then $f_{\beta-3}(0) = \beta - 4$. Thus, we can also describe the infinite divisibility by $p_{\beta-4}$ (in the case when $\beta - 3$ is not a limit) as infinite divisibility by $p_{f_{\beta-3}(0)}$. In future analyses, we will combine these cases in this manner.

From this analysis, when $k > 0$, the group $\mathcal{B}$ is isomorphic to the direct sum of infinitely many copies of $[\mathbb{Z}]_{p_{\beta-2},u_{\beta-1,k},P}$ (from the roots of $[\mathcal{G}(\Pi^0_{\beta-2})]_{u_{\beta-1,k}}$ components) and infinitely many copies of

$$\left[\left\langle [\mathbb{Z}]_{F_1} \oplus \bigoplus_{j\in\omega} [\mathbb{Z}]_{F_2}; \frac{x + y_j}{\rho^k} : j, k \in \omega \right\rangle\right]_{P,u_{\beta-1,k}} \tag{3}$$

where $F_1 := \{p_{\beta-2}\}$, $F_2 := \{p_{\beta-3}, u_{\beta-3,0}, p_{f_{\beta-3}(0)}\}$ and $\rho := q_{\beta-2}$ (from the roots of $[\mathcal{G}(\Sigma^0_{\beta-2})]_{u_{\beta-1,k}}$ components and their $\mathcal{G}(\Pi^0_{\beta-3})$ subcomponents). A divisibility argument almost identical to the one used in the $\beta = 5$ case (using the fact that $q^\infty_{\beta-2} \mid (z+y)$) shows that none of the z_i elements can come from the $[\mathbb{Z}]_{p_{\beta-2},u_{\beta-1,k},P}$ summands. Therefore, each z_i is the root of a $[\mathcal{G}(\Sigma^0_{\beta-2})]_{u_{\beta-1,k}}$ component.

Let $\mathcal{B}'$ be the subgroup of $\mathcal{B}$ consisting of the direct sum of infinitely many copies of the group in Equation (3). Since $\mathcal{B}'$ is a pure subgroup of $\mathcal{G}$ containing y and z

and $\mathcal{G}$ satisfies

$$p^{\infty}_{\beta-2} \,|\, z \,\wedge\, q^{\infty}_{\beta-2} \,|\, (z+y) \,\wedge\, p^{\infty}_{\beta-3} \,|\, y,$$

we have that this formula is also satisfied in $\mathcal{B}'$ (by Fact 4.12(1)). Applying Lemma 4.4(5) to $\mathcal{B}'$ with the above values for F_1, F_2, and ρ yields that each $a_i \in [\mathbb{Z}]_{P,u_{\beta-1,k}}$, completing the case when $k > 0$.

Second, suppose $k = 0$. The analysis of the isomorphism type of $\mathcal{B}$ is almost identical to the case when $k > 0$ except for three points. First, there are no components of the form $[\mathcal{G}(\Pi^0_{\beta-2})]_{u_{\beta-1,0}}$ and hence no argument is needed to conclude that each z_i is the root of a $[\mathcal{G}(\Sigma^0_{\beta-2})]_{u_{\beta-1,0}}$ component. Second, the root of a $[\mathcal{G}(\Sigma^0_{\beta-2})]_{u_{\beta-1,0}}$ component is also the root of a $\mathcal{G}(\Sigma^0_{\beta-1})$ or $\mathcal{G}(\Pi^0_{\beta-1})$ component and hence is infinitely divisible by $p_{\beta-1}$ in addition to the infinite divisibilities given above. Third, to smooth out the fact that q_β infinitely divides $r - r'$ when r, r' are roots of $\mathcal{G}(\Sigma^0_{\beta-1})$ or $\mathcal{G}(\Pi^0_{\beta-1})$ components from the same $\mathcal{C}_i$, we work with $[\mathcal{B}]_{q_\beta}$. With these observations, the group $[\mathcal{B}]_{q_\beta}$ is isomorphic to the direct sum of infinitely many copies of

$$\left[\left\langle [\mathbb{Z}]_{F_1} \oplus \bigoplus_{j\in\omega} [\mathbb{Z}]_{F_2}; \frac{x+y_j}{\rho^k} \,:\, j,k \in \omega \right\rangle\right]_{P,u_{\beta-1,k},q_\beta}$$

where $F_1 := \{p_{\beta-2}, p_{\beta-1}\}$, $F_2 := \{p_{\beta-3}, u_{\beta-3,0}, p_{f_{\beta-3}(0)}\}$ and $\rho := q_{\beta-2}$. Since $\mathcal{B}$ is a pure subgroup of $\mathcal{G}$ and $\mathcal{G}$ satisfies

$$p^{\infty}_{\beta-2} \,|\, z \,\wedge\, q^{\infty}_{\beta-2} \,|\, (z+y) \,\wedge\, p^{\infty}_{\beta-3} \,|\, y$$

this formula is also satisfied in $\mathcal{B}$ (by Fact 4.12(1)). Since $[\mathcal{B}]_{q_\beta}$ is an expansion of $\mathcal{B}$, it remains true in $\mathcal{B}'$ (by Fact 4.12(2)). We apply Lemma 4.4(5) to conclude that each $a_i \in [\mathbb{Z}]_{P,u_{\beta-1,k},q_\beta}$.

To prove (3)(b) when $\beta > 5$, fix an element $z = \sum a_i z_i$ with each $a_i \in [\mathbb{Z}]_{u_{\beta-1,k},P}$ and each z_i the root of a $[\mathcal{G}(\Sigma^0_{\beta-2})]_{u_{\beta-1,k}}$ component (which we denote by $\mathcal{G}_i$). We have $u^{\infty}_{\beta-1} \,|\, z_i$ and $p^{\infty}_{\beta-2} | z_i$ as a consequence of the structure of $[\mathcal{G}(\Sigma^0_{\beta-2})]_{u_{\beta-1,k}}$ components and z_i being the root. As $a_i \in [\mathbb{Z}]_{u_{\beta-1,k},P}$ and $\mathcal{G}_i$ is $\{u_{\beta-1,k}, P\}$-closed, it follows that $u^{\infty}_{\beta-1} \,|\, a_i z_i$ and $p^{\infty}_{\beta-2} | a_i z_i$ and hence that $u^{\infty}_{\beta-1} \,|\, z$ and $p^{\infty}_{\beta-2} | z$.

Let $y := \sum a_i y_i$, where y_i is the root of a $\mathcal{G}(\Pi^0_{\beta-3})$ subcomponent of $\mathcal{G}_i$. Since each $\mathcal{G}_i$ is a $[\mathcal{G}(\Sigma^0_{\beta-2})]_{u_{\beta-1,k}}$ component, it follows from the structure of these components that $u^{\infty}_{\beta-1,k} \,|\, y$ and $q^{\infty}_{\beta-2} \,|\, (z+y)$. It remains to show that $\mathcal{G}$ satisfies $A^{u_{\beta-1,k}}_{\beta-3}(y)$ and $\Psi^{u_{\beta-1,k}}_{\beta-3}(y)$.

Let $\mathcal{B} := \mathrm{Span}_{\mathcal{G}}(X)$ where X contains the roots of the $[\mathcal{G}(\Sigma^0_{\beta-2})]_{u_{\beta-1,k}}$ components and the roots of any component nested via the recursive construction inside such a component. Note that $y, z \in \mathcal{B}$ and that $\mathcal{B}$ is the subgroup of $\mathcal{G}$ consisting of the elements which are infinitely divisible by $u_{\beta-1,k}$. Applying Part (2)(b) to $\mathcal{B}$ with $\beta - 2$ and $P = P \cup \{u_{\beta-1,k}\}$, we get that $\mathcal{B}$ satisfies $A_{\beta-3}(y) \,\wedge\, \Psi_{\beta-3}(y)$. Since $\mathcal{B}$ consists of the elements of $\mathcal{G}$ which are infinitely divisible by $u_{\beta-1,k}$, we have that $\mathcal{G}$ satisfies $A^{u_{\beta-1,k}}_{\beta-3}(y) \,\wedge\, \Psi^{u_{\beta-1,k}}_{\beta-3}(y)$ by Lemma 4.7 as required.

(2) We show (2) for β supposing (3) and (4) hold for values less than or equal to β. To show (2)(a), we suppose $\mathcal{G} \models A_{\beta-1}(z) \,\wedge\, \Psi_{\beta-1}(z)$, recalling $A_{\beta-1}(z)$ is

$$p^{\infty}_{\beta-1} | z \wedge (\exists w) \left[u_{\beta-1,1} | w \wedge v^{\infty}_{\beta-1,0} | (z+w) \right].$$

Since $p_{\beta-1}^{\infty} \,|\, z$, we can express z as $z = \sum a_i z_i$ where $a_i \in \mathbb{Q}$ and z_i is the root of a $\mathcal{G}(\Sigma^0_{\beta-1}(m))$ or $\mathcal{G}(\Pi^0_{\beta-1})$ component (which we denote by $\mathcal{G}_i$). Since z_i is also the root of the $[\mathcal{G}(\Sigma^0_{f_{\beta-1}(0)})]_{u_{\beta-1,0}}$ component inside this $\mathcal{G}(\Sigma^0_{\beta-1}(m))$ or $\mathcal{G}(\Pi^0_{\beta-1})$ component, we have that z_i is infinitely divisible by $p_{f_{\beta-1}(0)}$ and $u_{\beta-1,0}$. As above, the recursion stops here since $f_{\beta-1}(0)$ is an odd ordinal and hence z_i is the element 1 in a copy of $[\mathbb{Z}]_{p_{f_{\beta-1}(0)}}$.

Fix an element w witnessing $\mathcal{G} \models A_{\beta-1}(z)$. The condition $u^{\infty}_{\beta-1,1} \,|\, w$ implies that w is a sum of elements from $[\mathcal{G}(\Sigma^0_{f_{\beta-1}(1)})]_{u_{\beta-1,1}}$ and $[\mathcal{G}(\Pi^0_{f_{\beta-1}(1)})]_{u_{\beta-1,1}}$ components. The condition $v^{\infty}_{\beta-1,0} \,|\, (z+w)$ implies (by divisibility arguments similar to those already given many times) that $w = \sum b_i w_i$, where each $b_i \in \mathbb{Q}$ and each w_i is the root of a $[\mathcal{G}(\Sigma^0_{f_{\beta-1}(1)})]_{u_{\beta-1,1}}$ or $[\mathcal{G}(\Pi^0_{f_{\beta-1}(1)})]_{u_{\beta-1,1}}$ component. Since $f_{\beta-1}(1)$ is an odd ordinal, the root of such a component is the element 1 in a copy of $[\mathbb{Z}]_{p_{f_{\beta-1}(1)}}$. Thus, the element w_i is infinitely divisible by $p_{f_{\beta-1}(1)}$ and $u_{\beta-1,1}$ but the recursion stops at this point. (Note that for the same reason, the roots of $[\mathcal{G}(\Sigma^0_{f_{\beta-1}(k)})]_{u_{\beta-1,k}}$ and $[\mathcal{G}(\Pi^0_{f_{\beta-1}(k)})]_{u_{\beta-1,k}}$ components for $k \geq 2$ are infinitely divisible only by $p_{f_{\beta-1}(k)}$ and $u_{\beta-1,k}$.)

To find the coefficients a_i in $z = \sum a_i z_i$, let $\mathcal{B} := \mathrm{Span}_{\mathcal{G}}(X)$ where X contains the roots of the $[\mathcal{G}(\Sigma^0_{f_{\beta-1}(k)})]_{u_{\beta-1,k}}$ and $[\mathcal{G}(\Pi^0_{f_{\beta-1}(k)})]_{u_{\beta-1,k}}$ components for $k \in \omega$. As in the proof of Part (3), we work in $[\mathcal{B}]_{q_\beta}$ since, for roots r, r' of $[\mathcal{G}(\Sigma^0_{f_{\beta-1}(0)})]_{u_{\beta-1,0}}$ components (which are also roots of $\mathcal{G}(\Sigma^0_{\beta-1}(m))$ or $\mathcal{G}(\Pi^0_{\beta-1})$ components), it is the case that q_β infinitely divides $r - r'$ if and only if these roots come from the same $\mathcal{C}_i$ summand of $\mathcal{G}$.

The group $[\mathcal{B}]_{q_\beta}$ is isomorphic to the $P' := P \cup \{q_\beta\}$ closure of

$$\left\langle \mathcal{F}; \frac{x_{k,j}}{\sigma_k^\ell}, \frac{x_{k,j} + x_{k+1,j}}{\rho_k^\ell} : j, k, \ell \in \omega \text{ and all } \sigma_k \in F_k \right\rangle$$

where $\mathcal{F}$ is the free abelian group on $x_{k,j}$ (for $k, j \in \omega$), $F_0 := \{p_{f_{\beta-1}(0)}, p_{\beta-1}, u_{\beta-1,0}\}$, $F_k := \{p_{f_{\beta-1}(k)}, u_{\beta-1,k}\}$ (for $k > 0$) and $\rho_k := v_{\beta-1,k}$. In this presentation, for each fixed j, the element $x_{k,j}$ is the root of a $[\mathcal{G}(\Sigma^0_{f_{\beta-1}(0)})]_{u_{\beta-1,k}}$ component or a $[\mathcal{G}(\Pi^0_{f_{\beta-1}(0)})]_{u_{\beta-1,k}}$ component within a fixed $\mathcal{G}(\Sigma^0_{\beta-1}(m))$ or $\mathcal{G}(\Pi^0_{\beta-1})$ component of $\mathcal{G}$. As j varies, we range over all $\mathcal{G}(\Sigma^0_{\beta-1}(m))$ and $\mathcal{G}(\Pi^0_{\beta-1})$ components of $\mathcal{G}$. If $\beta - 1$ is a limit ordinal, then $p_{f_{\beta-1}(k)} \neq p_{f_{\beta-1}(k')}$ for $k \neq k'$. If $\beta - 1$ is not a limit ordinal, then $p_{f_{\beta-1}(k)} = p_{\beta-2}$ for all k. In this case, we can remove the primes $p_{f_{\beta-1}(k)}$ from F_k and add $p_{\beta-2}$ to P' (since $[\mathcal{B}]_{q_\beta}$ is $p_{\beta-2}$ closed). This change has the effect of including infinite divisibility by $p_{\beta-2}$ for our coefficients a_i.

Since $\mathcal{B}$ is a pure subgroup of $\mathcal{G}$, the group $\mathcal{B}$ is a subgroup of $[\mathcal{B}]_{q_\beta}$, and $z, w \in \mathcal{B}$, we have (applying both Fact 4.12(1) and Fact 4.12(2)) that $[\mathcal{B}]_{q_\beta}$ satisfies $A_{\beta-1}(z)$ with our element w as witness. Therefore, by Lemma 4.5(3), we obtain that $z = \sum a_i z_i$ with $a_i \in [\mathbb{Z}]_{P, p_{\beta-2}, q_\beta}$ (if $\beta - 1$ is not a limit ordinal) or $a_i \in [\mathbb{Z}]_{P, q_\beta}$ (if $\beta - 1$ is a limit ordinal).

Next, we use the fact that $\mathcal{G} \models \Psi_{\beta-1}(z)$ to show each $\mathcal{G}_i$ is a $\mathcal{G}(\Pi^0_{\beta-1})$ component. For if one were not, then some $\mathcal{G}_\ell$ would be a $\mathcal{G}(\Sigma^0_{\beta-1}(m_0))$ component for some $m_0 \in \omega$. With $m := m_0 + 1$, we fix a sequence $g_0, g_1, \ldots, g_m$ witnessing that $\mathcal{G}$ satisfies the m-th conjunct of $\Psi_{\beta-1}(z)$. Since $\mathcal{G}$ satisfies $u^{\infty}_{\beta-1,m} \,|\, g_m$ and $\Phi^{u_{\beta-1,m}}_{f_{\beta-1}(m)}(g_m)$, we have by Part (3) or Part (4) (depending on the form of $f_{\beta-1}(m)$)

that $g_m = \sum c_j y_j$ where each y_j is the root of a $[\mathcal{G}(\Sigma^0_{f_{\beta-1}(m)})]_{u_{\beta-1,m}}$ component. Since $g_0 = z = \sum a_i z_i$ and $v^\infty_{\beta-1} \,|\, (g_k + g_{k+1})$ for $0 \le k < m$, one of the y_j roots in the summand for g_m must lie in the component $\mathcal{G}_\ell$. However, the group $\mathcal{G}_\ell$ is a $\mathcal{G}(\Sigma^0_{\beta-1}(m_0))$ component with $m_0 < m$, so it does not contain a $[\mathcal{G}(\Sigma^0_{f_{\beta-1}(m)})]_{u_{\beta-1,m}}$ component, yielding the desired contradiction. This completes the proof of (2)(a).

To prove (2)(b), fix an element $z = \sum a_i z_i$ with $a_i \in [\mathbb{Z}]_P$ and z_i a root of a $\mathcal{G}(\Pi^0_{\beta-1})$ component of $\mathcal{G}$ (which we denote $\mathcal{G}_i$). We need to show that $\mathcal{G} \models A_{\beta-1}(z)$ and $\mathcal{G} \models \Psi_{\beta-1}(z)$. For the former, we need to show that $\mathcal{G}$ satisfies

$$p^\infty_{\beta-1}|z \wedge (\exists w)[u^\infty_{\beta-1,1} \,|\, w \wedge v^\infty_{\beta-1,0} \,|\, (z+w)].$$

From the structure of $\mathcal{G}(\Pi^0_{\beta-1})$ components, we have each z_i is the root r_0 of the $\left[\mathcal{G}(\Sigma^0_{f_{\beta-1}(0)})\right]_{u_{\beta-1,0}}$ component of $\mathcal{G}_i$. By Lemma 4.17, the condition $p^\infty_{\beta-1}|z$ is satisfied since $a_i \in [\mathbb{Z}]_P$ and $p^\infty_{\beta-1}|z_i$ for each z_i.

To generate the witness w, for each i, let w_i be the root r_1 of the $\left[\mathcal{G}(\Sigma^0_{f_{\beta-1}(1)})\right]_{u_{\beta-1,1}}$ component of $\mathcal{G}_i$. The conditions $u^\infty_{\beta-1,1} \,|\, w_i$ and $v^\infty_{\beta-1,0} \,|\, (z_i + w_i)$ are satisfied since $z_i = r_0$ and $w_i = r_1$ in $\mathcal{G}_i$. As each $a_i \in [\mathbb{Z}]_P$, it follows from Lemma 4.17 that $\mathcal{G} \models A_{\beta-1}(z)$ with witness $w := \sum a_i w_i$.

To see that $\mathcal{G} \models \Psi_{\beta-1}(z)$, we reason as follows. Fix $m \in \omega$. We show how to pick the witnessing elements $g_0, \ldots, g_m$ for the m-th conjunct. For each z_i, pick a sequence of elements $g_{i,0}, g_{i,1}, \ldots, g_{i,m}$ in $\mathcal{G}_i$ by setting $g_{i,0} := z_i$ (which is the r_0 root in $\mathcal{G}_i$) and $g_{i,k} := r_k$ (the root of the $[\mathcal{G}(\Sigma^0_{f_{\beta-1}(k)})]_{u_{\beta-1,k}}$ component of $\mathcal{G}_i$) for $0 < k \le m$. Since $a_i \in [\mathbb{Z}]_P$ and each $\mathcal{G}_i$ is P-closed, we have (from the structure of $\mathcal{G}(\Pi^0_{\beta-1})$) that $u^\infty_{\beta-1,k} \,|\, a_i g_{i,k}$ for $k \le m$ and $v^\infty_{\beta-1,k} \,|\, (a_i g_{i,k} + a_i g_{i,k+1})$ for $k < m$.

For each $0 \le k \le m$, let $g_k := \sum_i a_i g_{i,k}$. By the divisibility conditions above, we have $u^\infty_{\beta-1,k} \,|\, g_k$ for $k \le m$ and $v^\infty_{\beta-1,k} \,|\, (g_k + g_{k+1})$ for $k < m$. Furthermore, $g_0 = z$. Therefore, it only remains to show that $\Phi^{u_{\beta-1,m}}_{f_{\beta-1}(m)}(g_m)$. We already have $u^\infty_{\beta-1,m} \,|\, g_m$. Since $g_m = \sum_i a_i g_{i,m}$ where $a_i \in [\mathbb{Z}]_P$ and $g_{i,m}$ is the root of a $[\mathcal{G}(\Sigma^0_{f_{\beta-1}(m)})]_{u_{\beta-1,m}}$ component, it follows from Part (3)(b) or Part (4)(b), depending on the form of $f_{\beta-1}(m)$, that $\mathcal{G}$ satisfies $\Phi^{u_{\beta-1,m}}_{f_{\beta-1}(m)}(g_m)$ and hence $\mathcal{G} \models \Psi_{\beta-1}(z)$.

(4) As $f_{\beta-1}(k)$ is an odd ordinal and $f_{\beta-1}(k) < \beta - 1$ for all $k \in \omega$, the proof of Part (4) is essentially the same as the proof of Part (3) with the appropriate notational changes to reflect that $\beta - 1$ is a limit ordinal.

□

Lemma 4.18. *Let $\beta = \delta + 2\ell + 1 \ge 3$. Then for $\mathcal{G} = \bigoplus_{n\in\omega} \mathcal{G}_n$, where $\mathcal{G}_n$ is either $[\mathcal{G}(\Sigma^0_\beta)]_{d_n}$ or $[\mathcal{G}(\Pi^0_\beta)]_{d_n}$, the following holds:*

$$\mathcal{G} \models [(\exists x)\Phi_\beta(x)]^{d_n} \qquad \text{if and only if} \qquad \mathcal{G}_n \cong [\mathcal{G}(\Sigma^0_\beta)]_{d_n}.$$

Proof. Since $\mathcal{G}_n$ is the subgroup of elements of $\mathcal{G}$ which are infinitely divisible by d_n, we have by Lemma 4.7 that

$$\mathcal{G} \models [(\exists x)\Phi_\beta(x)]^{d_n} \qquad \text{if and only if} \qquad \mathcal{G}_n \models (\exists x)\Phi_\beta(x).$$

Therefore, it suffices to show that $\mathcal{G}(\Sigma^0_\beta) \models (\exists x)\Phi_\beta(x)$ and $\mathcal{G}(\Pi^0_\beta) \not\models (\exists x)\Phi_\beta(x)$.

First, we show that $\mathcal{G}(\Sigma^0_\beta) \models \Phi_\beta(r)$ where r is the root of $\mathcal{G}(\Sigma^0_\beta)$. That is, we show that $\mathcal{G}(\Sigma^0_\beta)$ satisfies

$$p^\infty_\beta \,|\, r \,\wedge\, (\exists y)[q^\infty_\beta \,|\, (r+y) \,\wedge\, A_{\beta-1}(y) \,\wedge\, \Psi_{\beta-1}(y)].$$

Since r is the root of $\mathcal{G}(\Sigma^0_\beta)$, we immediately obtain $p^\infty_\beta \,|\, r$. We claim that the root r_k of a $\mathcal{G}(\Pi^0_{\beta-1})$ component works for the choice of y in Φ_β. From the definition of $\mathcal{G}(\Sigma^0_\beta)$, we have $q^\infty_\beta \,|\, (r+r_k)$ and by Lemma 4.14(2)(b), we have that $\mathcal{G}(\Sigma^0_\beta)$ satisfies both $A_{\beta-1}(r_k)$ and $\Psi_{\beta-1}(r_k)$ as required.

Second, assume for a contradiction that $\mathcal{G}(\Pi^0_\beta) \models (\exists x)\Phi_\beta(x)$ and fix the witness x. The condition $p^\infty_\beta \,|\, x$ implies that x is a multiple of the root of $\mathcal{G}(\Pi^0_\beta)$. Fix the witness y such that $q^\infty_\beta \,|\, (x+y) \,\wedge\, A_{\beta-1}(y) \,\wedge\, \Psi_{\beta-1}(y)$. By Lemma 4.14(2)(a), the condition $A_{\beta-1}(y) \wedge \Psi_{\beta-1}(y)$ implies that y is a sum of multiples of the roots of $\mathcal{G}(\Pi^0_{\beta-1})$ components. However, the group $\mathcal{G}(\Pi^0_\beta)$ has no such components, giving the desired contradiction. □

We continue by constructing sentences connected semantically to $\mathcal{H}(\Sigma^0_\beta)$ and $\mathcal{H}(\Pi^0_\beta)$. We first give lemmas similar to Lemma 4.14 for the groups $\mathcal{H}(\Sigma^0_\beta)$ and $\mathcal{H}(\Pi^0_\beta)$.

Lemma 4.19. *Let $\beta = \delta + 2l + 2 \geq 4$ and let $\mathcal{H} \cong \mathcal{H}(\Pi^0_\beta)$. Let $y \in \mathcal{H}$ be a sum $y = \sum b_j y_j$ where each $b_j \in \mathbb{Z}$ and each y_j is the root of a (distinct) $\mathcal{G}(\Sigma^0_{\beta-1})$ component of $\mathcal{H}$. Then $\mathcal{H} \models \Phi_{\beta-1}(y)$.*

Proof. We need to show that $\mathcal{H}$ satisfies

$$p^\infty_{\beta-1} \,|\, y \wedge (\exists w)[q^\infty_{\beta-1} \,|\, (y+w) \wedge A_{\beta-2}(w) \wedge \Psi_{\beta-2}(w)].$$

By the structure of $\mathcal{G}(\Sigma^0_{\beta-1})$, we have $p^\infty_{\beta-1} \,|\, y_j$ for all j. Since $b_j \in \mathbb{Z}$, we have $p^\infty_{\beta-1} \,|\, b_j y_j$ and hence $p^\infty_{\beta-1} \,|\, y$. For each j, let w_j be the root of a $\mathcal{G}(\Pi^0_{\beta-2})$ component within the $\mathcal{G}(\Sigma^0_{\beta-1})$ component with root y_j and let $w := \sum b_j w_j$. It follows from the structure of $\mathcal{G}(\Sigma^0_{\beta-1})$ that $q^\infty_{\beta-1} \,|\, (y+w)$. Therefore, it remains to show that $\mathcal{H}$ satisfies $A_{\beta-2}(w) \wedge \Psi_{\beta-2}(w)$.

The group $\mathcal{H}$ is built by taking a direct sum of the groups $[\mathbb{Z}]_{p_\beta}$ (with root r) and $\bigoplus_{k\in\omega} \mathcal{G}(\Sigma^0_{\beta-1})$ (with roots r_k) and then adding extra elements (from the divisible closure of this sum) to witness $q^\infty_\beta \,|\, (r+r_k)$. Since $w = \sum b_j w_j$ with each $b_j \in \mathbb{Z}$, we can view w as an element of the group $\bigoplus_{k\in\omega} \mathcal{G}(\Sigma^0_{\beta-1})$ in this construction of $\mathcal{H}$. By Lemma 4.14(2)(b) applied to w as an element of $\bigoplus_{k\in\omega} \mathcal{G}(\Sigma^0_{\beta-1})$, we have that $\bigoplus_{k\in\omega} \mathcal{G}(\Sigma^0_{\beta-1})$ satisfies $A_{\beta-2}(w) \wedge \Psi_{\beta-2}(w)$. Since $\bigoplus_{k\in\omega} \mathcal{G}(\Sigma^0_{\beta-1})$ is a subgroup of $\mathcal{H}$, Fact 4.13 implies that $\mathcal{H}$ satisfies $A_{\beta-2}(w) \wedge \Psi_{\beta-2}(w)$ as required. □

Lemma 4.20. *Let $\beta = \delta + 2l + 2 \geq 4$ and let $\mathcal{H} \cong \mathcal{H}(\Sigma^0_\beta)$. Let r be the root of a $\mathcal{G}(\Pi^0_{\beta-1})$ component of $\mathcal{H}$. Then $\mathcal{H} \not\models \Phi_{\beta-1}(r)$.*

Proof. We show that there is no $w \in \mathcal{H}$ such that $\mathcal{H}$ satisfies

$$q^\infty_{\beta-1} \,|\, (r+w) \wedge A_{\beta-2}(w) \wedge \Psi_{\beta-2}(w).$$

For a contradiction, fix such an element $w \in \mathcal{H}$. To simplify dealing with q_β divisibility in $\mathcal{H}$, we work in the prime closure $[\mathcal{H}]_{q_\beta}$ and note that if w satisfies this formula in $\mathcal{H}$, then by Fact 4.13, it also satisfies the formula in $[\mathcal{H}]_{q_\beta}$.

The group $[\mathcal{H}]_{q_\beta}$ decomposes as a direct sum

$$[\mathbb{Z}]_{p_\beta,q_\beta} \oplus \bigoplus_{i\in\omega}[\mathcal{C}_i]_{q_\beta}$$

where each $\mathcal{C}_i$ is isomorphic to $\mathcal{G}(\Sigma^0_{\beta-1})$ or $\mathcal{G}(\Pi^0_{\beta-1})$. The divisibility condition $p^\infty_{\beta-2} \mid w$ (from the fact that $[\mathcal{H}]_{q_\beta} \models A_{\beta-2}(w)$) implies that $w = \sum a_i w_i$ where each $a_i \in \mathbb{Q}$ and each w_i is the root of a $\mathcal{G}(\Sigma^0_{\beta-2}(m))$ or $\mathcal{G}(\Pi^0_{\beta-2})$ component. Therefore, as an element of $[\mathcal{H}]_{q_\beta}$, we have $w \in \bigoplus_{i\in\omega}[\mathcal{C}_i]_{q_\beta}$. In addition, by arguments similar to previous ones, the condition $q^\infty_{\beta-1} \mid (r+w)$ implies that at least one w_i is the root of a $\mathcal{G}(\Sigma^0_{\beta-2}(m))$ subcomponent of the $\mathcal{G}(\Pi^0_{\beta-1})$ component with root r (as this component has no $\mathcal{G}(\Pi^0_{\beta-2})$ subcomponents).

Assume for a moment that $\bigoplus_{i\in\omega}[\mathcal{C}_i]_{q_\beta}$ satisfies $A_{\beta-2}(w) \wedge \Psi_{\beta-2}(w)$. Under this assumption, Lemma 4.14(2)(a) implies that w is a sum of roots of $\mathcal{G}(\Pi^0_{\beta-2})$ components, contradicting the fact that at least one w_i is the root of a $\mathcal{G}(\Sigma^0_{\beta-2}(m))$ component. Therefore, to complete our proof, it suffices to show that $\bigoplus_{i\in\omega}[\mathcal{C}_i]_{q_\beta}$ satisfies $A_{\beta-2}(w) \wedge \Psi_{\beta-2}(w)$.

To show $\bigoplus_{i\in\omega}[\mathcal{C}_i]_{q_\beta}$ satisfies $A_{\beta-2}(w)\wedge\Psi_{\beta-2}(w)$, we use the fact that $\bigoplus_{i\in\omega}[\mathcal{C}_i]_{q_\beta}$ is a pure subgroup of $[\mathcal{H}]_{q_\beta}$ (since it is a direct summand) along with the following observation. Because $[\mathcal{H}]_{q_\beta}$ is a direct sum, any element $z \in [\mathcal{H}]_{q_\beta}$ can be written (uniquely) in the form $z = z_0 + z_1$ where $z_0 \in [\mathbb{Z}]_{p_\beta,q_\beta}$ and $z_1 \in \bigoplus_{i\in\omega}[\mathcal{C}_i]_{q_\beta}$. If ρ is a prime and $\rho^\infty \mid z$, then $\rho^\infty \mid z_0$ and $\rho^\infty \mid z_1$. Therefore, if $\rho^\infty \mid z$ and ρ is not p_β or q_β, we can conclude that $z \in \bigoplus_{i\in\omega}[\mathcal{C}_i]_{q_\beta}$.

Using this observation, we show that the following implications hold for all γ with $2 \leq \gamma \leq \beta-2$. Let $\varphi(x)$ be either $A_\gamma(x)$ or $\Psi_\gamma(x)$ (if γ is even) or $\Phi_\gamma(x)$ (if γ is odd), and let ρ be any prime number. For any $x \in \bigoplus_{i\in\omega}[\mathcal{C}_i]_{q_\beta}$, we have

$$[\mathcal{H}]_{q_\beta} \models \varphi(x) \qquad \text{implies} \qquad \bigoplus_{i\in\omega}[\mathcal{C}_i]_{q_\beta} \models \varphi(x) \text{ and}$$

$$[\mathcal{H}]_{q_\beta} \models \varphi^\rho(x) \qquad \text{implies} \qquad \bigoplus_{i\in\omega}[\mathcal{C}_i]_{q_\beta} \models \varphi^\rho(x).$$

Notice that establishing this property finishes our proof as $w \in \bigoplus_{i\in\omega}[\mathcal{C}_i]_{q_\beta}$ and $[\mathcal{H}]_{q_\beta} \models A_{\beta-2}(w)\wedge\Psi_{\beta-2}(w)$, so by the property $\bigoplus_{i\in\omega}[\mathcal{C}_i]_{q_\beta} \models A_{\beta-2}(w)\wedge\Psi_{\beta-2}(w)$.

First, consider the case when $\varphi(x)$ is $A_\gamma(x)$ and assume $[\mathcal{H}]_{q_\beta} \models A_\gamma(x)$. In this case, the existential witness y in $A_\gamma(x)$ is infinitely divisible by $u_{\gamma,1}$. As $u_{\gamma,1} \notin \{p_\beta, q_\beta\}$, we have $y \in \bigoplus_{i\in\omega}[\mathcal{C}_i]_{q_\beta}$. Since $\bigoplus_{i\in\omega}[\mathcal{C}_i]_{q_\beta}$ is a pure subgroup containing x and y, the group $\bigoplus_{i\in\omega}[\mathcal{C}_i]_{q_\beta}$ satisfies $A_\gamma(x)$. The same proof works for $A^\rho_\beta(x)$.

Second, consider the cases when $\varphi(x)$ is $\Phi_\gamma(x)$ (for odd γ), $\Psi_\gamma(x)$ (for even γ) or a prime relativization of one of these formulas. We proceed by induction on γ and note that in each case the proof for the relativized formula is identical to the proof for the unrelativized formula. In each case, we assume $x \in \bigoplus_{i\in\omega}[\mathcal{C}_i]_{q_\beta}$ and $[\mathcal{H}]_{q_\beta} \models \varphi(x)$.

The first base case is when $\beta = 2$. Since $\Psi_2(x)$ is $p_1^\infty \mid x$, $x \in \bigoplus_{i\in\omega}[\mathcal{C}_i]_{q_\beta}$, and $\bigoplus_{i\in\omega}[\mathcal{C}_i]_{q_\beta}$ is a pure subgroup, we have $\bigoplus_{i\in\omega}[\mathcal{C}_i]_{q_\beta} \models \Psi_2(x)$.

The second base case is $\beta = 3$. The existential witness y in the formula $\Phi_3(x)$ satisfies $p_1^\infty \mid y$ (from $\Psi_2(y)$). As $p_1 \notin \{p_\beta, q_\beta\}$, we have $y \in \bigoplus_{i\in\omega}[\mathcal{C}_i]_{q_\beta}$. Since $\bigoplus_{i\in\omega}[\mathcal{C}_i]_{q_\beta}$ is a pure subgroup containing x and y, we have $\bigoplus_{i\in\omega}[\mathcal{C}_i]_{q_\beta} \models \Phi_3(x)$.

For the inductive cases, suppose γ is even and $4 \leq \gamma \leq \beta-2$. Consider the m-th conjunct of $\Psi_\gamma(x)$. The witnesses $x_0, \ldots, x_m$ satisfy $u^\infty_{\gamma,k} \mid x_k$ and thus are in

$\bigoplus_{i\in\omega}[\mathcal{C}_i]_{q_\beta}$ as $u_{\gamma,k}\notin\{p_\beta,q_\beta\}$. Since $\bigoplus_{i\in\omega}[\mathcal{C}_i]_{q_\beta}$ is a pure subgroup and it satisfies $\Phi^{u_{\gamma,m}}_{f_\gamma(m)}(x_m)$ by the inductive hypothsis, we have that $\bigoplus_{i\in\omega}[\mathcal{C}_i]_{q_\beta}\models\Psi_\gamma(x)$.

If γ is odd and $4<\gamma\le\beta-2$, then the existential witness y in $\Phi_\gamma(x)$ satisfies $p^\infty_{\gamma-1}\,|\,y$ from $A_{\gamma-1}(y)$. Thus, as $p_{\gamma-1}\notin\{p_\beta,q_\beta\}$, we have $y\in\bigoplus_{i\in\omega}[\mathcal{C}_i]_{q_\beta}$. Since $\bigoplus_{i\in\omega}[\mathcal{C}_i]_{q_\beta}$ is a pure subgroup and satisfies $A_{\gamma-1}(y)$ and $\Psi_{\gamma-1}(y)$ by the inductive hypothesis, we have $\bigoplus_{i\in\omega}[\mathcal{C}_i]_{q_\beta}\models\Phi_\gamma(x)$. □

Definition 4.21. If $\beta\ge 4$ is not a limit ordinal, define $B_\beta(x)$ to be the formula

$$B_\beta(x):=p^\infty_\beta\,|\,x\;\wedge\;(\exists w)\left[p^\infty_{\beta-1}\,|\,w\wedge q^\infty_\beta\,|\,(x+w)\right].$$

Definition 4.22. If $\beta=\delta+2\ell+2\ge 4$, define Θ_β to be the formula

$$\Theta_\beta(x):=(\forall y)\left[\left(B_\beta(x)\wedge B_{\beta-1}(y)\wedge q^\infty_\beta\,|\,(x+y)\right)\to\Phi_{\beta-1}(y)\right].$$

Lemma 4.23. *The complexity of $B_\beta(x)$ is Σ^c_3 (independent of β).*

If $\beta=\delta+2\ell+2\ge 4$, then $\Theta_\beta(x)\in\Pi^c_\beta$.

Proof. These statements follow immediately from $p^\infty\,|\,x$ being Π^c_2 and Lemma 4.11. □

Lemma 4.24. *Let $\beta=\delta+2\ell+2\ge 4$. Let $x\in\mathcal{H}(\Pi^0_\beta)$ satisfy $B_\beta(x)$ with fixed witness w. Then $x=ar$ where $a\in\mathbb{Z}$ and r is the root of $\mathcal{H}(\Pi^0_\beta)$ and $w=\sum b_jw_j$ where $b_j\in[\mathbb{Z}]_{p_{\beta-1},q_\beta}$ and each w_j is the root of a $\mathcal{G}(\Sigma^0_{\beta-1})$ component of $\mathcal{H}(\Pi^0_\beta)$.*

Proof. Since $p^\infty_\beta\,|\,x$, the element x must have the form $x=ar$ where $a\in\mathbb{Q}$ and r is the root of $\mathcal{H}(\Pi^0_\beta)$. Since $p^\infty_{\beta-1}\,|\,w$, the element w must have the form $w=\sum b_jw_j$ where $b_j\in\mathbb{Q}$ and w_j is the root of a $\mathcal{G}(\Sigma^0_{\beta-1})$ component of $\mathcal{H}(\Pi^0_\beta)$.

Let $\mathcal{B}:=\mathrm{Span}_{\mathcal{H}(\Pi^0_\beta)}(X)$ where X contains the root of $\mathcal{H}(\Pi^0_\beta)$ and the roots of the $\mathcal{G}(\Sigma^0_{\beta-1})$ components of $\mathcal{H}(\Pi^0_\beta)$. Then $x,w\in\mathcal{B}$, $\mathcal{B}$ is a pure subgroup of $\mathcal{H}(\Pi^0_\beta)$, and $\mathcal{B}$ is isomorphic to

$$\left\langle[\mathbb{Z}]_{p_\beta}\oplus\bigoplus_{k\in\omega}[\mathbb{Z}]_{p_{\beta-1}};q_\beta^{-t}(r+r_k):k,t\in\omega\right\rangle.$$

Since $\mathcal{B}$ satisfies $p^\infty_\beta\,|\,x$, $p^\infty_{\beta-1}\,|\,w$, and $q^\infty_\beta\,|\,(x+w)$, we can apply Lemma 4.4(5) (with $P=\emptyset$) to conclude that $a\in\mathbb{Z}$ and each $b_j\in[\mathbb{Z}]_{p_{\beta-1},q_\beta}$. □

Lemma 4.25. *Let $\beta=\delta+2\ell+2\ge 4$. If $x,y\in\mathcal{H}(\Pi^0_\beta)$ satisfy*

$$B_\beta(x)\wedge B_{\beta-1}(y)\wedge q^\infty_\beta\,|\,(x+y),$$

then $x=ar$ and $y=\sum b_jy_j$ where $a,b_j\in\mathbb{Z}$, r is the root of $\mathcal{H}(\Pi^0_\beta)$, and y_j is the root of a $\mathcal{G}(\Sigma^0_{\beta-1})$ component of $\mathcal{H}(\Pi^0_\beta)$.

Proof. By Lemma 4.24, the fact that $B_\beta(x)$ holds implies $x=ar$ with $a\in\mathbb{Z}$ and r the root of $\mathcal{H}(\Pi^0_\beta)$. Since $B_{\beta-1}(y)$ implies $p^\infty_{\beta-1}\,|\,y$ and since $q^\infty_\beta\,|\,(x+y)$, the element y works as a witness w in the formula $B_\beta(x)$ for our fixed element x. Therefore, by the previous lemma $y=\sum b_jy_j$ where $b_j\in[\mathbb{Z}]_{p_{\beta-1},q_\beta}$ and y_j is the root of a $\mathcal{G}(\Sigma^0_{\beta-1})$ component. It remains to show the stronger conclusion that $b_j\in\mathbb{Z}$.

Fix a witness w for $B_{\beta-1}(y)$. Since $p^\infty_{\beta-2}\,|\,w$, the element w must have the form $w=\sum c_iw_i$ where $c_i\in\mathbb{Q}$ and w_i is the root of a $\mathcal{G}(\Sigma^0_{\beta-2}(m))$ component inside a $\mathcal{G}(\Sigma^0_{\beta-1})$ component of $\mathcal{H}(\Pi^0_\beta)$. Therefore $y,w\in\mathcal{B}$ where $\mathcal{B}:=\mathrm{Span}_{\mathcal{H}(\Pi^0_\beta)}(X)$

where X contains the roots of the $\mathcal{G}(\Sigma^0_{\beta-1})$ components and the roots of their $\mathcal{G}(\Sigma^0_{\beta-2}(m))$ subcomponents.

To determine the isomorphism type of $\mathcal{B}$, we consider which primes infinitely divide the roots of such components. The root of a $\mathcal{G}(\Sigma^0_{\beta-1})$ component is infinitely divisible by $p_{\beta-1}$. The roots of $\mathcal{G}(\Sigma^0_{\beta-2}(m))$ components are infinitely divisible by $p_{\beta-2}$ and $u_{\beta-2,0}$ from the definition of $\mathcal{G}(\Sigma^0_{\beta-2}(m))$. Each of these roots is also the root of a $\mathcal{G}(\Sigma^0_{f_{\beta-2}(0)})$ component (inside $\mathcal{G}(\Sigma^0_{\beta-2}(m))$) and hence is also infinitely divisible by $p_{f_{\beta-2}(0)}$. However, the recursion stops at this point since the root of $\mathcal{G}(\Sigma^0_{f_{\beta-2}(0)})$ is the element 1 in a copy of $[\mathbb{Z}]_{p_{f_{\beta-2}(0)}}$. Therefore, the group $\mathcal{B}$ is isomorphic to

$$\left[\left\langle \bigoplus_{i\in\omega}[\mathbb{Z}]_{F_1} \oplus \bigoplus_{i,j\in\omega}[\mathbb{Z}]_{F_2}; \frac{s_i + t_{i,j}}{q^l_{\beta-2}} : i,j,l \in \omega \right\rangle\right]$$

where $F_1 = \{p_{\beta-1}\}$, $F_2 = \{p_{\beta-2}, u_{\beta-2,0}, p_{f_{\beta-2}(0)}\}$, the s_i elements generate the copies of $[\mathbb{Z}]_{F_1}$ (representing the roots of the $\mathcal{G}(\Sigma^0_{\beta-1})$ components) and the $t_{i,j}$ elements generate the copies of $[\mathbb{Z}]_{F_2}$ (representing the roots of the $\mathcal{G}(\Sigma^0_{\beta-2}(m))$ subcomponents of the $\mathcal{G}(\Sigma^0_{\beta-1})$ component with root s_i). Since $y,w \in \mathcal{B}$ and $\mathcal{B}$ is a pure subgroup of $\mathcal{H}(\Pi^0_\beta)$ satisfying $p^\infty_{\beta-1} \,|\, y$, $p^\infty_{\beta-2} \,|\, w$, and $q^\infty_{\beta-1} \,|\, (y+w)$, we can conclude from Lemma 4.4(5) (with $P = \emptyset$) that the coefficients in the sum $y = \sum b_j y_j$ come from $\mathbb{Z}$. □

Lemma 4.26. *Let $\beta = \delta + 2\ell + 2 \geq 4$. Then for $\mathcal{H} = \bigoplus_{n\in\omega}\mathcal{H}_n$, where $\mathcal{H}_n$ is either $[\mathcal{H}(\Sigma^0_\beta)]_{d_n}$ or $[\mathcal{H}(\Pi^0_\beta)]_{d_n}$, the following holds:*

$$\mathcal{H} \models [(\forall x)\Theta_\beta(x)]^{d_n} \qquad \textit{if and only if} \qquad \mathcal{H}_n \cong [\mathcal{H}(\Pi^0_\beta)]_{d_n}.$$

Proof. Since $\mathcal{H}_n$ is the subgroup of elements of $\mathcal{H}$ which are infinitely divisible by d_n, we have

$$\mathcal{H} \models [(\forall x)\Theta_\beta(x)]^{d_n} \Leftrightarrow \mathcal{H}_n \models (\forall x)\Theta_\beta(x)$$

Therefore, it suffices to show that $\mathcal{H}(\Pi^0_\beta) \models (\forall x)\Theta_\beta(x)$ and $\mathcal{H}(\Sigma^0_\beta) \not\models (\forall x)\Theta_\beta(x)$.

First, we show that $\mathcal{H}(\Pi^0_\beta) \models (\forall x)\Theta_\beta(x)$. Fix elements $x, y \in \mathcal{H}(\Pi^0_\beta)$ satisfying $B_\beta(x) \wedge B_{\beta-1}(y) \wedge q^\infty_\beta \,|\, (x+y)$. By Lemma 4.25, we can write $y = \sum b_j y_j$ where each $b_j \in \mathbb{Z}$ and y_j is the root of a $\mathcal{G}(\Sigma^0_{\beta-1})$ component. By Lemma 4.19, the element y satisfies $\Phi_{\beta-1}(y)$ as required.

Second, we show that $\mathcal{H}(\Sigma^0_\beta) \not\models (\forall x)\Theta_\beta(x)$ by proving that $\mathcal{H}(\Sigma^0_\beta) \not\models \Theta_\beta(r)$ where r is the root of $\mathcal{H}(\Sigma^0_\beta)$. Let y be the root of a $\mathcal{G}(\Pi^0_{\beta-1})$ component of $\mathcal{H}(\Sigma^0_\beta)$. It is immediate that $\mathcal{H}(\Sigma^0_\beta) \models B_\beta(r) \wedge B_{\beta-1}(y) \wedge q^\infty_\beta \,|\, (r+y)$. However, by Lemma 4.20, the group $\mathcal{H}(\Sigma^0_\beta)$ does not satisfy $\Phi_{\beta-1}(y)$. □

Finally, we are in a position to define the sentences $\{\Upsilon_n\}_{n\in\omega}$ required for Lemma 4.1 and to demonstrate their correctness.

Definition 4.27. Define sentences Υ_n for $n \in \omega$ as follows.

- If $\alpha = \delta + 2\ell + 1 \geq 3$, let $\Upsilon_n := [(\exists x)\Phi_\alpha(x)]^{d_n}$.
- If $\alpha = \delta + 2\ell + 2 \geq 4$, let $\Upsilon_n := \neg\, [(\forall x)\Theta_\alpha(x)]^{d_n}$.

Proof of Lemma 4.1. By Lemma 4.18 and Lemma 4.26, the sentences Υ_n have the desired semantic properties. As a consequence of Lemma 4.11 and Lemma 4.23, the formulas Υ_n have the desired quantifier complexity. Moreover, all the (sub)formulas are computable with all possible uniformity, so Υ_n is uniformly computably Σ^c_α. □

4.2. **Proof of Lemma 4.2.** The construction of an X-computable copy of $\mathcal{G}^\alpha_S$ if $S \in \Sigma^0_\alpha(X)$ is also done by recursion. We treat only the case when $X = \emptyset$, the more general case following by relativization.

Lemma 4.28. *For every even ordinal $\beta = \delta > 0$ or $\beta = \delta + 2\ell + 2 \geq 2$ and Σ^0_β set S, there is a uniformly computable sequence $\{\mathcal{G}_n\}_{n\in\omega}$ of rooted torsion-free abelian groups such that $\mathcal{G}_n \cong \mathcal{G}(\Sigma^0_\beta(m))$ for some $m \in \omega$ if $n \in S$ and $\mathcal{G}_n \cong \mathcal{G}(\Pi^0_\beta)$ if $n \notin S$.*

For every odd ordinal $\beta = \delta + 2\ell + 1 \geq 3$ and Σ^0_β set S, there is a uniformly computable sequence $\{\mathcal{G}_n\}_{n\in\omega}$ of rooted torsion-free abelian groups such that $\mathcal{G}_n \cong \mathcal{G}(\Sigma^0_\beta)$ if $n \in S$ and $\mathcal{G}_n \cong \mathcal{G}(\Pi^0_\beta)$ if $n \notin S$.

Moreover the passage from an index for the set S to an index for the sequence is effective.

Proof. The proof is done by induction on β. We treat the cases $\beta = 2$, $\beta = \delta + 2\ell + 2 \geq 4$, $\beta = \delta + 2\ell + 1 \geq 3$, and $\beta = \delta > 0$ separately. In all cases, we fix a predicate $(\exists s)\,[R(n,s)]$ describing membership of n in S, where $R(n,s)$ is $\Pi^0_{f_\beta(k)}$ for some k. Without loss of generality, we suppose $R(n,s_0)$ implies $(\forall s \geq s_0)\,[R(n,s)]$. Indeed, we suppose this property of all existential subpredicates.

For $\beta = 2$, it suffices to start with the group $\mathbb{Z}$ with root $r_n = 1$ for $\mathcal{G}_n$. When we see $\neg R(n,s)$ for a new existential witness s, we introduce the element $1/p^s$ into the group. It is easy to see the sequence $\{\mathcal{G}_n\}_{n\in\omega}$ has the desired properties.

For $\beta = \delta + 2\ell + 1 \geq 3$, it suffices to start with the group $[\mathbb{Z}]_{p_\beta}$ with root $r_n = 1$ for $\mathcal{G}_n$. For each integer s, we construct (via induction as $\neg R(n,s)$ is $\Sigma^0_{\delta+2\ell}$) a rooted torsion-free abelian group $\mathcal{G}_{n,s}$ with root $r_{n,s}$ and introduce elements $(r_n + r_{n,s})/q^t_\beta$ for all $t \in \omega$. For each integer m, we construct infinitely many copies of $\mathcal{G}(\Sigma^0_{\beta-1}(m))$ with root $r_{n,k,m}$ (where k is the copy number) and introduce elements $(r_n + r_{n,k,m})/q^t_\beta$ for all $t \in \omega$. Again, it is easy to see the sequence $\{\mathcal{G}_n\}_{n\in\omega}$ has the desired properties.

For $\beta = \delta + 2\ell + 2 \geq 4$, we construct (via induction as $\neg R(n,s)$ is $\Sigma^0_{\delta+2\ell+1}$[10]) rooted torsion-free abelian groups $\mathcal{G}_{n,s}$ with root $r_{n,s}$. Within $\mathcal{G}_{n,0}$, we introduce elements $r_{n,0}/p^t_\beta$ for all $t \in \omega$. Within each group $\mathcal{G}_{n,s}$, we introduce elements $x/u^t_{\beta,s}$ for all $t \in \omega$ and $x \in \mathcal{G}_{n,s}$. For each integer s, we introduce elements $(r_{n,s} + r_{n,s+1})/v^t_{\beta,s}$ for all $t \in \omega$. Again, it is easy to see the sequence $\{\mathcal{G}_n\}_{n\in\omega}$ has the desired properties.

For $\beta = \delta$, we construct (via induction) rooted torsion-free abelian groups $\mathcal{G}_{n,s}$ with root $r_{n,s}$, where $\mathcal{G}_{n,0} \cong \mathcal{G}(\Sigma^0_{f_\beta(0)})$ and where, for $s > 0$, $\mathcal{G}_{n,s} \cong \mathcal{G}(\Pi^0_{f_\beta(s)})$ if $\emptyset^{f_\beta(s)}$ suffices to witness $n \in S$ and $\mathcal{G}_{n,s} \cong \mathcal{G}(\Sigma^0_{f_\beta(s)})$ otherwise. Within $\mathcal{G}_{n,0}$, we introduce elements $r_{n,0}/p^t_\beta$ for all $t \in \omega$. Within each group $\mathcal{G}_{n,s}$, we introduce elements $x/u^t_{\beta,s}$ for all $t \in \omega$ and $x \in \mathcal{G}_{n,s}$. For each integer s, we introduce elements

[10]More precisely, we use the $\Sigma^0_{\delta+2\ell+1}$ predicate $\neg R(n,s)$ to control the construction of $\mathcal{G}_{n,s+1}$ and build $\mathcal{G}_{n,0} \cong \mathcal{G}(\Sigma^0_{\beta-1})$. This index shift is necessary as $\mathcal{G}(\Sigma^0_\beta(m))$ has $m+1$ (rather than m) subcomponents of type $\mathcal{G}(\Sigma^0_{\beta-1})$.

$(r_{n,s} + r_{n,s+1})/v^t_{\beta,s}$ for all $t \in \omega$. Again, it is easy to see the sequence $\{\mathcal{G}_n\}_{n\in\omega}$ has the desired properties. □

Lemma 4.29. *For every even ordinal $\beta = \delta + 2\ell + 2 \geq 4$ and Σ^0_β set S, there is a uniformly computable sequence of rooted torsion-free abelian groups $\{\mathcal{H}_n\}_{n\in\omega}$ such that $\mathcal{H}_n \cong \mathcal{H}(\Sigma^0_\beta)$ if $n \in S$ and $\mathcal{H}_n \cong \mathcal{H}(\Pi^0_\beta)$ if $n \notin S$.*

Proof. We fix a predicate $(\exists s)\,[R(n, s)]$ describing membership of n in S, where $R(n, s)$ is $\Pi^0_{\beta-1}$. Without loss of generality, we again suppose $R(n, s_0)$ implies $(\forall s \geq s_0)\,[R(n, s)]$. Indeed, we suppose this property of all existential subpredicates.

It suffices to start with the group $[\mathbb{Z}]_{p_\beta}$ with root $r_n = 1$ for $\mathcal{H}_n$. For each integer s, we (via Lemma 4.28) construct a rooted torsion-free abelian group $\mathcal{G}_{n,s}$ with root $r_{n,s}$ and introduce elements $(r_n + r_{n,s})/q^t_\beta$ for all $t \in \omega$. We also construct infinitely many copies of $\mathcal{G}(\Sigma^0_{\beta-1})$ with root $r_{n,k}$ (where k is the copy number) and introduce elements $(r_n + r_{n,k})/q^t_\beta$ for all $t \in \omega$. Again, it is easy to see the sequence $\{\mathcal{G}_n\}_{n\in\omega}$ has the desired properties. □

Proof of Lemma 4.2. Fix a Σ^0_α set S. From Lemma 4.28 (if α is odd) or Lemma 4.29 (if α is even), there is a uniformly computable sequence $\{\mathcal{G}_n\}_{n\in\omega}$ of groups given by the Σ^0_α predicate. Since it is possible to pass from the group $\mathcal{G}_n$ to $[\mathcal{G}_n]_{d_n}$ uniformly in an index for the group $\mathcal{G}_n$ and d_n, the group $\mathcal{G}^\alpha_S$ is computable. □

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