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Jean van Heijenoort's Contributions to Proof Theory and Its History

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Abstract. Jean van Heijenoort was best known for his editorial work in the history of mathematical logic. I survey his contributions to model-theoretic proof theory, and in particular to the falsifiability tree method. This work of van Heijenoort's is not widely known, and much of it remains unpublished. A complete list of van Heijenoort's unpublished writings on tableaux methods and related work in proof theory is appended.

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1. Introduction

Jean van Heijenoort was best known as a historian of logic. He was famous for his editorial work, and especially for his anthology *From Frege to Gödel* (1967) [164], which, as a representative documentary history of modern logic during the formative period of 1879–1931, has attained, in the words of Feferman [51, p. 5], [53] "the status of a classic." Not very well known, however, is van Heijenoort's non-historical work in logic, which consisted primarily of results in model-theoretic proof theory and which circulated in the narrow circle of his students and colleagues and remain largely unpublished. It is not known why these technical contributions remained unpublished, although van Heijenoort's well-known perfectionism may have played a prominent role in his decision that the writings in which these results appeared did not satisfy his standard of excellence. Many of the papers in this category circulated in manuscript form; and even those which circulated in typescript were frequently

This is a much expanded reprinting of the revised and corrected English version of [13] that originally appeared as [16].

only drafts. It is the object of this paper to survey the technical content of van Heijenoort's contributions to model-theoretic proof theory. The physical description of the unpublished works in which these contributions were made was given in [11].

In order to understand the choice of topics on which van Heijenoort worked, it is necessary to view his technical interests in the history of logic. Trained as a geometer and topologist (his doctoral thesis was in the field of convex sets), van Heijenoort traced questions in metamathematics and foundations of mathematics to problems of axiomatization of mathematics. It should also be kept in mind that, making the transition from geometry and topology to logic, van Heijenoort was guided in his early studies by Georg Kreisel (b. 1923), and one finds, among van Heijenoort's *Nachlaß* (his research notes on Kreisel's considerations of the work of Jacques Herbrand (1908–1931).[1] Van Heijenoort and Kreisel enjoyed an excellent rapport, and van Heijenoort admitted that Kreisel was the logician who influenced him most (see [51, p. 262]). This is not to say that van Heijenoort followed Kreisel's lead in research or adopt Kreisel's views; rather, he listened carefully to Kreisel, and took Kreisel's views seriously, and it was Kreisel, as much as anyone, who aroused van Heijenoort's incipient interests in logic. As William Alvin Howard (b. 1926) [97, pp. 282–283] has noted, Kreisel held that proof theory could be seriously advanced, and required correction, by the application of model-theoretic considerations and of recursion theory.

Van Heijenoort saw the development of quantification theory, beginning with the work of Frege, as the crucial factor in the history of mathematical logic. He understood the primary goal of quantification theory to be the elucidation of the concepts of *consistency, completeness,* and *(being a) proof.* In this sense, van Heijenoort's technical work belongs to the tradition of Hilbertian metalogic, elaborated by David Hilbert (1862–1943) and Paul Isaac Bernays (1888–1977) as *Beweistheorie* as the logical study of the proofs of logic. Let us begin, then, by reviewing van Heijenoort's study of the relevant history. Indeed, for van Heijenoort, proof theory meant precisely the same as it did for Hilbert, who, as Kreisel [112, p. 321] explained in the first installment (1968) of his "Survey of Proof Theory", understood *Beweis* to be the study of formal derivations (proofs) within a formal system. Likewise, in his 1967 "Informal Rigour and Completeness Proofs" Kreisel [111, p. 138] defined formal rigor as consisting in "setting out formal rules and checking that a given derivation follows these rules." In the course of the development of proof theory after Hilbert, proof theory, Kreisel [112, p. 321] averred, also came to take into account an analysis of intuitive proofs, and consideration of the choice of formal systems. Thus, when Kreisel wrote the first part of his "Survey of Proof

[1] Van Heijenoort's notes of 1968 on "Herbrand—Kreisel on Herbrand" [166] is found in Box 3.8/86-33/2 of the van Heijenoort archive; see [160] for the *Nachlaß*. Most of the unpublished typescripts and manuscripts on the tree method and related proof-theoretic methods of various calculi are to be found in Box 3.8/86-33/1. The Stanford University Libraries Department of Special Collections and University Archives includes the Kreisel-van Heijenoort correspondence in Box 21, folder 5 of the Georg Kreisel Papers (SCM0136).

Theory" [112], proof theory considered not only formal derivations, but also intuitive proofs. Van Heijenoort for his part noted [190, p. 185] Hilbert "worked with axioms and rules [of inference]," regarding quantifiers as ranging over restricted, well-defined collections—say points, lines and planes; or over tables, chairs, and beer mugs,[2] but not both at once or, as did Frege and Russell, over "everything"—be it, as Hilbert indirectly imputed to Frege (*q.v.* [60, p. 13]), love, laws, and chimney sweeps; and that whereas second-order logic has its own important place, it increases complexity, such that it is crucial first and foremost, to "see what can be done" in first-order logic." Establishing albeit tacitly, a connection with van Heijenoort's treatment of the role of the syntactic/semantic distinction as an aspect of the difference between the "Booleans" and the "Fregeans", Van Heijenoort characterized Hilbert's position as lying between that of the adherents of logic as calculus (Boole, De Morgan, *et alii*), and those of logic as language (Frege and Russell),[3] and Hilbert himself, in his "Axiomatisches Denken" of 1918 [89], readily acknowledged Russell's axiomatization of logic to be the "crowning work" of axiomatization in general [89, p. 412]. Joong Fang (1923–2010) identified the differences between Russell and Hilbert as rooted in the fact that whereas Russell approached the task of axiomatization from the standpoint "that was "more logical or philosophical than mathematical," Hilbert "marched straightforward into metamathematics directly through his professional studies of mathematics proper" [49, p. 168].

In his letter of 29 December 1895 to Frege, Hilbert considered a universe of discourse indiscriminately populated by love, laws, and chimney sweeps, and anything and everything else (see [60, p. 13]): "Ja, es ist doch selbstverständlich eine jede Theorie nur ein Fachwerk oder Schema von Begriffen nebst ihren nothwendigen Beziehungen zu einander, und die Grundelemente können in beliebiger Weise gedacht werden. Wenn ich unter meinen Punkten irgendwelche Systeme von Dingen, z.B. das System: Liebe, Gesetz, Schornsteinfeger. . ., denke und dann nur meine sämmtlichen Axiome als Beziehungen zwischen diesen Dingen annehme, so gelten meine Sätze, z.B. auch der Pythagoras

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[2] Hilbert, as quoted by Otto Blumenthal (1876–1944) reads [32, p. 403]: ,,Man muss jederzeit an Stelle von Punkten, Geraden und Ebenen Tische, Stühle oder Bierseidel sagen konnen."

[3] We may also include in this group the American postulate theorists and their students, who took their cue, directly or indirectly, from the algebraic tradition of Boole, De Morgan, Peirce, and Schröder on the one hand and Hilbert on the other; among them, Edward Vermilye Huntington (1874–1952); Oswald Veblen (1880–1960); Eliakim Hastings Moore (1862–1932); and George Bruce Halsted (1853–1922). Veblen, for example, produced a critical discussion [204] of the first edition of Hilbert's 1899 *Die Grundlagen der Geometrie* [88] and was a colleague at Princeton of Peirce's former student Allan Marquand (1853–1924). Aspray [19, p. 55] describes the purpose of Veblen, Huntington, Moore and other American postulate theorists as "investigat[ing] the logical adequacy of axiom systems for mathematics" and in the process "develop[ing] the concepts of completeness, independence, and categoricity" for purposes of applying these to axiom systems for algebra, geometry, and analysis. For more background on the American postulate theorists, stressing the work in particular of Huntington and Vebeln, see [149].

auch von diesen Dingen. Mit andern Worten eine jede Theorie kann stets auf unendliche viele Systeme von Grundelementen angewandt werden." Roman Murawski [130] explains that in contrast with Frege's system which requires a linguistic *Universum* underwritten by a run-away metaphysics, Hilbert was attempting to explain to Frege that he, on the contrary, merely sought "to secure the validity of mathematical knowledge by syntactical considerations without appeal to semantic ones." As Murawski [130, pp. 94–95] noted, the program of the Beweistheorie for Hilbert was "never intended as a comprehensive philosophy, of mathematics; its purpose was instead to legitimate the entire corpus of mathematical knowledge"; moreover, he "sought to justify mathematical theories by means of formal systems, i.e., using the axiomatic method." For Hilbert, the goal of *Beweistheorie*, or metamathematics, was to establish the credentials of a mathematical system, and most especially its completeness and consistency. For Hilbert's own summary of his position, see his "Axiomatische Denken" of 1918 [89], where the efforts at axiomatization pertain in each case to "the facts of "a specific more or less comprehensive field of knowledge"—"die Tatsachen eines bestimmten mehr oder minder umfassenden Wissensgebietes" [89, p. 405]. Hilbert forthwith made it clear that the primitve elements of one's axiom system, as well as the axioms themselves, are determined by the [mathematical] theory being axiomatized. His student Bernays took this view one step further, by arguing that the constructability of a piece of mathematics as a theory that entertains no inconsistency counts as sufficient cause to establish the existence of the element of that field. This is the thesis stated by the title of, and presented in, Bernays' "Mathematische Existenz und Widerspruchsfreiheit" [26] which was reviewed by van Heijenoort [161]. In relating existence and consistency, the mathematician is freed from dependence upon appeal to Platonic "ideal entities" ["eines ideal Sein"]. The existence–consistency relation is not, however, transitive, since existence axioms do not necessarily guarantee the consistency of a theory, which is a property of the theory as whole. Thus, there is no absolute requirement for an underlying metaphysics to construct a mathematical theory axiomatically. This, I suggest, is the philosophical import of Hilbert's remark that it should not matter in designing an axiomatic system whether our terms are points, lines and planes or tables, chairs and beer mugs, and that, if our system is deductively sound, then the theorems drawn will be true, i.e. logically valid in accordance with the inference rules, under either interpretation. In undertaking axiomatizations of geometry, of algebra, and in algebra, of group theory, of Boolean algebra, etc., it transpires that for each of these axiomatizations [89, p. 406]: "Diese grundegenden Sätze können von einem ersten Standpunkte aus als die *Axiome der einzelnen Wissenschaftsgebiete angesehen werden*" [Hilbert's emphasis]—thus, one set of axioms for one discipline. The American postulate theorists, jointly influenced to various degrees by both Peirce and by Hilbert, produced sets of postulates for different branches of mathematics, and even for different subfields within these branches. And, like Hilbert and his students, their concern for each of these systems of postulates was to establish its completeness, consistency, independence, and categoricity.[4] What is of central concern for Hilbert (and the Postulate theorists) is the character of the various axiomatic systems and of the mathematical theories which they were constructed to elaborate, and in particular the logical nature of the deductive proofs of the theorems obtained within these axiomatic systems (so that it does not, in a metaphysical sense, matter whether the system is operating with points, lines and planes or with tables, chairs and beer mugs). The concern of *Beweistheorie* is specifically with the nature of*being a proof*, whereas, for Frege and Russell, the general concern of the Begriffsschrift is the elaboration of a logical language for the mechanical development of any and every science and the specific concern is the reduction, or rendition, of all of mathematics in the universal ideal language of formal logic. In the original, unabridged, edition of his *Survey of Symbolic Logic* of 1918 [116], Clarence Irving Lewis (1883–1964) who had served as a teaching assistant in the logic course of Josiah Royce (1855–1916) and had been given the task of cataloguing the Peirce papers when they were delivered to Harvard, and who, like Royce himself had corresponded with Peirce, defined a mathematical system, in a manner highly reminiscent of the formalism of Hilbert, as "any set of strings of recognizable marks in which some of the strings are taken initially and the remainder [are] derived from these by operations performed according to rules which are independent of any meaning assigned to the marks" [116, p. 355].

Introducing Stefan Bauer-Mengelberg's English translation of Löwenheim's 1915 "Über Möglichkeiten im Relativkalkul" [119], Jean van Heijenoort [164, p. 228], noticed that the "problems dealing with the validity, in different domains, of formulas of the first-order predicate calculus and with various aspects of the reduction and the decision problems" which Löwenheim's article treated "had remained alien to the trend that had by then become dominant in logic, that of Frege–Peano–Russell" until Hilbert and the postulate theorists raised the questions of the categoricity and completeness of axiom systems. Jean van Heijenoort's introduction to the collected papers of Herbrand; there van Heijenoort [165, p. 2] wrote:

> Le mémoire de Löwenheim de 1915 ouvre une nouvelle époque dans le développement de la logique moderne. Dans ce mémoire, Löwenheim établit un théorème troublant: si un formule de la théorie de la quantification est satisfiable, elle es satisfiable dans un domaine dénombrable. Mais, plutôt que le résultat même, ce sont les notions et les méthodes employées par Löwenheim qui doivent maintenant retenir notre attention. Löwenheim n'a ni axiomes ni règles d'inference. Il laisse de côté la notion démonstrabilité. Ce qu'il manie,

[4] Thus, for example, in his "A Complete Set of Postulates for the Theory of Absolute Continuous Magnitude" of 1902, Huntington (at [98, pp. 264–268]) introduced the notion of categoricity for interpretable propositions, under the name "sufficiency". Huntington's definition [98, p. 264] of sufficiency was that there is only one set possible for axiomatizing the reals by means of sequences that satisfy a given set of axioms. For background on the American postulate theorists, see [36] and [140, esp. p. 988].

> c'est la notion de validité (ou, ce qui revient au même, de satisfiabilité). Sa logique est, comme le dira plus tard Skolem (1938, p. 25) [145] «une logique du prédicat fondée sur la théorie des ensembles» . Cette manière d'aborder les problèmes logique, Löwenheim l'a trouvée, avec la notation qu'il emploie, chez Schröder, lui-même inspiré par Boole et Peirce.

For van Heijenoort, proof theory remained the investigation of formal derivations of proofs and the elucidation of the model-theoretic properties of formal proof-theoretic methods, in particular their soundness and completeness, and the bulk of his work in proof theory was devoted to the investigation of the various procedures for formal derivations in various calculi, propositional calculus, first-order calculus, second- and higher-order calculi, the comparison of the various systems, the axiomatic method, natural deduction, the Gentzen sequent calculus, Herbrand's method, and especially tableaux methods, most particularly the falsifiability tree method, i.e. Smullyan's analytic tableaux in which proofs are carried out by contradiction, and of the completeness and soundness of these methods for various classical and non-classical calculi. This work by van Heijenoort included general considerations of the falsifiability tree method for sentential calculus and classical quantification theory and their properties, in particular the soundness and completeness of the method (e.g. [163,167,174,176,180,182,183,188,189]); intuitionistic propositional calculus and quantification theory (e.g. [178,179,185,186]); and the study of related methods, including Beth tableaux (e.g. [175,179,189]); Grzegorczyk's method [177]; Gentzen's system of natural deduction (e.g. [181,189]); and Herbrand's method (e.g. [165–167,171,187,189,191–193,197,198,201,202]); and application of the falsifiability tree method to modal logics (e.g. [194,195]).[5]

Kreisel's conception of model-theoretic logical consequence, as given in "Informal Rigor..." [111], according to which, whenever an argument (of a standard first-order language) exhibits the condition of model-theoretic logical consequence, the completeness theorem guarantees that its conclusion is derivable (according to well known and provably sound rules of inference) from its premises, also played a role in van Heijenoort's proofs of the completeness and soundness of the falsifiability tree method for various calculi. More specifically, Kreisel [111] distinguished between $Val\alpha$ or intuitive validity, and $V\alpha$, or *formal validity*, where $V\alpha \rightarrow \alpha_\varepsilon$ requires that α be true in the structure consisting of all sets (with the membership relation), that is, must be true in every model, and for some property P applied all sets α in $\alpha_\varepsilon P\alpha_\varepsilon$ is provable and thus holds in every set-theoretic structure.

Defining *clashing* in terms of some formula F of a theory $\mathfrak{T}$ in which both F and $\sim F$ are derivable, let $\mathfrak{L}$ be the language of $\mathfrak{T}$ (say Zermelo set theory) and $\mathfrak{A}$ be the set of elementary axioms of $\mathfrak{T}$, and assuming that $\mathfrak{A}$ has finite models; Kreisel [111, p. 173] states, and then proves, the following theorem:

[5] See the Appendix "Proof-theoretic and Related Writings of van Heijenoort's in the *Nachlaß* [160]; Box 3.8/86-33/1] (exclusive of research notes and unfinished work)" attached to the references for a more comprehensive listing.

Theorem. *Suppose $F(X_1, \ldots, X_r)$ is a formula of $\mathfrak{L}_{\mathscr{P}}$ built up from prime formulae $t_\eta X_i, (i \leq r)$ using the operations of $\mathfrak{L}$ only, where t denotes the terms of $\mathfrak{L}$. Let A be a set of formulae of $\mathfrak{L}$ itself. Then $\mathfrak{U}, \mathfrak{P}_{\mathfrak{L}} \vdash \exists X_1, \ldots, X_r \Phi(X_1, \ldots, X_r)$ iff there are formulae $F_i(x, u_1, \ldots, u_p)(i < r)$ of $\mathfrak{L}$ such that $\mathfrak{U} \vdash \exists u_1 \ldots u_p \Phi(F_1, \ldots, F_r)$ where $\Phi(F_1, \ldots, F_r)$ is obtained from $(F_1, \ldots, F_r)$ by replacing $t_\eta X_i$ by: $F_i(t, u_1, \ldots, u_p)$; it is supposed that all the variables in F_i are distinct from all the variables in $\Phi(F_1, \ldots, F_r)$ to avoid clashing.*

From this theorem the two corollaries are obtained:

Corollary 1. *If Φ is a formula of $\mathfrak{L}$ and $\mathfrak{U}, \mathscr{P}_{\mathfrak{L}} \vdash \Phi$, then also $\mathfrak{U} \vdash \Phi$.*

Corollary 2. *Suppose $\Psi(X)$ is built up from prime formulae $t_\eta X$ by using the operations of $\mathfrak{L}$ only. Then, for Φ in $\mathfrak{L}, \mathfrak{U}, \forall X \Psi(X) \vdash \Phi$ iff there is an F in $\mathfrak{L}$ such that $\mathfrak{U} \vdash \forall u_1 \ldots u_p \Psi(F) \to \Phi$.*

2. Van Heijenoort's View of the History of Proof Theory

In *El desarrollo de la teoría de la cuantificación* (1976) [189] van Heijenoort gave an exposition and historical analysis of the theory of quantification from a metamathematical perspective. He argued that quantification theory is a "family of formal systems" ("*la teoría de la cuantificación es una familia de sistemas formales*" [189, p. 7], the creator of which was Gottlob Frege (1848–1925) in 1879 in the *Begriffsschrift* [59]. The family members of quantification theory are the axiomatic method, Herbrand quantification, the Gentzen sequent calculus, and natural deduction as developed by Stanisław Jaśkowski (1906–1965) and Gerhard Karl Erich Gentzen (1909–1945). The axiomatic method has two distinct branches, Frege-type systems and Hilbert-type systems. A *Hilbert-type system* is distinguished by being simply a set of well-formed formulae (wffs), including a list of axioms, a set of "rules of passage", that is derivation rules, for which a proof is a sequence of wffs, the last wff of the sequence being the formula which is proven. A *Frege-type system* is a formal language ("Begriffsschrift") containing an arbitrary set of axioms, a set of equivalence and inference rules, and in which nothing exists outside of proofs. These represent for van Heijenoort the four principal methods of approach to first-order predicate calculus. Van Heijenoort's research notes and uncompleted work include undated writings on the axiomatic method [203], on Herbrand [202] and considerations of Herbrand's method for non-classical logics [201]. In *Desarrollo,* van Heijenoort traced the mutual relations among these four approaches, and traced their histories. He examined each in its classical, intuitionistic, and minimal versions, and pointed out the strengths and weaknesses of each.[6] Thus, for example, he noted that Herbrand's method is particularly suited for use with computers, but is not easily generalized

[6] As presented in 1937 by Ingebrigt Johansson (1904–1987) [103], a "minimal version" is a reduced intuitionistic formalism, in which only one of (1) $\vdash (\neg a \vee b) \to (a \to b)$or(2) $\vdash (a \to b) \to (\neg a \vee b)$ holds.

to second-order logic. However, he showed in his 1975 paper on "Herbrand" [187, p. 6] that we can establish the constructive equivalence of a second-order Herbrand formula to a classical second-order formula, provided the formula $(\forall x)(\vdash_Q F \Leftrightarrow \vdash_{\mathrm{QH}} F)$, properly gödelized, that is having an infinite list $\{\underline{0}, \underline{1}, \underline{2}, \ldots\}$ of variables of Q not occurring in F, can be shown to be provable in primitive recursive arithmetic, where Q is classical quantification theory and $\mathrm{Q_H}$ is Herbrand quantification theory.

In *Desarrollo,* van Heijenoort treated the principal methods of quantification theory proof-theoretically. The axiomatic method attains results based on the concept of *formal system* and provides an analysis of theorems, but is not yet itself a study of proofs. For Herbrand's system, quantifier-free formulae can be obtained effectively from quantified formulae, such that these quantifier-free formulae are sententially valid, by using Herbrand expansions. Thus, Herbrand helped introduce a new conception of *validity* into logic, where for Leopold Löwenheim (1878–1957) the essential consideration was still *satisfiability,* or validity invariant with respect only to a particular model.

Gentzen's work in the sequent calculus rests on the results given by Herbrand. Herbrand's Fundamental Theorem, for example, can be understood to be a special case of Gentzen's *verschärfter Hauptsatz,* and Gentzen's *Mittelsequenz* corresponds to Herbrand's valid disjunction D_k. Beyond that, Gentzen also gives an analysis of the sentential parts of the proof of validity. Thus, van Heijenoort was particularly interested in the ways in which proofs are carried out in the axiomatic, Herbrand, Gentzen sequential, and natural deduction methods. Indeed, it could be said that, for van Heijenoort, quantification theory is a family of methods of logical deduction.

The various methods for quantification theory, according to van Heijenoort, taken together, represent an evolution or development (*desarrollo*) of quantification theory, starting from the definition of the Hilbert program and Herbrand quantification, through the Gödel incompleteness results, to the Gentzen sequences and natural deduction, rather than provide an opportunity for possible conflict. The Hilbert program of metamathematical study of proofs arose just because the axiomatic method fails to study proofs even while it provides an analysis of theorems. It was Hilbert who undertook to define and carry out systematically the construction of mathematics within his system. This required a fully developed concept of proof, for the Hilbert program had two aspects: to define the technical apparatus which would permit a finitist construction of all of mathematics, and to ensure that the set of (mathematical) sentences derived within the axiomatic system was consistent. Frege's *Begriffsschrift*-program, however, focused almost exclusively on only part of the first aspect of the Hilbert program, namely the attempt to derive all of mathematics from the *Begriffsschrift*'s logical apparatus. The focus of much of van Heijenoort's work centered in particular on Herbrand and his method, and his edition of Herbrand's logical writings [85] was second only to *From Frege to Gödel* as his most significant contribution to his editorial enterprises.

Consistency and completeness were raised as questions for quantification theory as soon as the universality of logic was proclaimed *and* deductive validity replaced satisfiability as the defining characteristic of *being a proof.* Universality, raised as an issue by Frege, required that all of mathematics should be constructed within the logical theory presented by the *Begriffsschrift.* It also required that the *Begriffsschrift* theory deal with what we now call "metamathematical" problems, such as completeness and consistency, since there is no extrasystematic or metasystematic apparatus to be distinguished from the system. Frege also required that every function in the system be defined for every argument in the universe of the system's syntax, that is, that every object in the semantics of the system be in the range of every function. Thus, there was no longer a question of restricting a theory to a select model. Thanks to Frege, the model was the universal domain—*the Universe*—rather than some arbitrary domains that acted as interpretations for submodels of the universal domain. This universal domain in effect contained only two objects, the True and the False. In practice, every object of the universe was, according to the assignment of truth-values, an element of either the True or the False. Now, as understood semantically by van Heijenoort, a formula (or theory) is satisfiable if there is some (at least one) assignment of truth-values which makes the formula (or theory) true, and valid if the formula (or theory) is true for every assignment of truth-values. These definitions by van Heijenoort of satisfiability and validity were based on Löwenheim's work, and in particular the realization that (a) a formula may be valid in some domain but false in another, and that (b) a formula may be valid in every finite domain but not valid in every domain.

Van Heijenoort's definition of proof was in essence model-theoretic. If Φ is a sequence of formulae $F_0, F_1, \ldots, F_n, Q$,then Φ is a proof of Q provided the deduction theorem holds, according to which $\Phi \vdash Q$ if and only if $\Phi \to Q$ and

$$
\begin{gathered}
(F_0 \wedge F_1 \wedge \ldots \wedge F_{n-1} \to (F_n \to Q) \\
\vdots \\
(F_0 \to F_1 \to \cdots \to F_n) \to Q
\end{gathered}
$$

The proof is *satisfiable* if there exists a model of $\Phi \vdash Q$ for which $\models \overline{\Phi} \to Q$ for at least one assignment of truth-values to Φ, and the proof is *valid* if $\Phi \to Q$ for every such assignment of truth-values to Φ. Defining a *proof* as an extended formula built up from a sequence of formulae, van Heijenoort's model-theoretic approach makes no distinction between deductive validity, that is the validity of proofs, and the validity of formulae. Extending these definitions, he was able to assert that *satisfiability* can be understood as *validity with respect to a specific model*, while *validity* can be understood as *satisfiability invariant with respect to any particular model.* The contrapositive gives us validity invariant with respect to a particular model as our definition of satisfiability. This definition is closely akin to the definition of validity introduced by Kreisel as an unpublished appendix to the paper "Set-theoretic Problems Suggested by the Notion of Potential Totality" in *Infinitistic Methods* [110] presented in Warsaw in 1959. It is a generalization of Alfred Tarski's (1901–1995) calculus of systems as extracted from his paper "Über den Begriff der logischen Folgerung" of 1936 on the concept of logical consequence ([158, p. 7]; see also [159, p. 415]) according to which:

If, in the sentences of the class K and in the sentence X, the constants—apart from purely logical constants—are replaced by any other constants (like signs being everywhere replaced by like signs), and if we denote the class of sentences thus obtained from K by 'K'', and the sentence obtained from X by 'X'', then the sentence X' must be true provided only that all sentences of the class K' are true.

In the no-counterexample interpretation of validity due to Kurt Friedrich Gödel (1906–1978) in "Über eine bisher noch nicht benützte Erweiterung des finiten Standpunktes" of 1958 [67] that Kreisel [9], [112, pp. 332–333] described, we have:

(i) *to each formula A is associated an interpretation of the form $\exists s \forall t A_1(s,t)$, A_1 quantifier-free, and* (ii) *the laws of (classical) logic are valid for this interpretation when applied to suitably restricted classes of formulae, providedenoort correspondence is found the closure conditions formulated in* [67] *are satisfied by the functions considered (for negations of prenex formulae in both interpretations, for formulae built up from $\neg, \wedge, \forall$.)*

This is the solution which Gödel propounded when he first detected the error in Herbrand's lemma 3.3 [81, p. 101] connection with Herbrand's proof of the Fundamental Theorem. Gödel did not publish his findings until the issue was raised in 1963 by Dreben.[7] Kreisel [9] [112, p. 333] notes that he first heard of it from Gödel in 1955, and spoke with him about it in 1957 at the Summer Institute of the Association for Symbolic Logic on the campus of Cornell University, and he adds that Gödel informed him that he first presented the technical details in lectures at Princeton University as early as 1941. The details, including those of its relation to Herbrand, are elaborated by S, Feferman [54, esp. pp. 250–254].

In his comments on the work of Löwenheim and Herbrand, van Heijenoort stated (in *From Frege to Gödel* and elsewhere) that Herbrand's work on elucidating the concept of *proof* for Hilbert's axiomatic system was inspired by questions raised by the Löwenheim–Skolem theorem. To the two theses presented by van Heijenoort in *Desarrollo* that quantification theory is a family of formal systems, and that the four principal theories, rather than being in competition, represent a natural development, I add a third (introduced in [7,8] and detailed in [15]), namely that the technical developments in Hilbert-type systems, including the development of *Beweistheorie* by Hilbert and Bernays, and the development of alternative theories of quantification, are primarily due to questions raised about the Löwenheim–Skolem theorem.

[7] See [72, pp. 9–10]. For the details of how this came about, see [6]. A summary is given below, in Sect. 4.

It is clear from Herbrand's own comments in his thesis of 1930 *Recherches sur la théorie de la démonstration* [81] that his investigations were undertaken to clarify the concept of *being a proof* for a Hilbert-type quantification system. This, as van Heijenoort recorded in one of his manuscript notes on Herbrand [202], led Burton Spencer Dreben (1927–1999) and John Stanton Denton [46, p. 419] to remark that "the Herbrand approach is perhaps best viewed as a reformulation of Hilbert's evaluation method, a reformulation that frees that method from its customary (and in our opinion obfuscating) dependence on the ε-calculus." On the next page of the manuscript notes on Herbrand, van Heijenoort raised the possibility that "dans sa signification, le théorème de Herbrand ne feut être considere comme un substitut de celui de Löwenheim, dans les applications...." The link between Herbrand and Hilbert is emphasized by van Heijenoort in a manuscript page of the notes on Herbrand [202] where it is remarked that, although Herbrand early on read the *Principia Mathematica* (1910–1913) [205], if only the first volume, he took from Bertrand Russell (1872–1970) an interest only in the first-order predicate calculus, and was left unaffected by and uninterested in the theory of types. Referring to Goldfarb's introduction [72, p. 1] to the English translation [86] of Herbrand writings, van Heijenoort, in an isolated note of [202] quotes Goldfarb to the effect that "one of the methods Whitehead and Russell use to construct quantificational logic seems to be the source of two key concepts in Herbrand, namely, those of normal identity and property A." One is then sent to Goldfarb's introduction [72, p. 5],[8] where a formula F is said to have Property A if a quantifier-free tautology results from given instantiations. Elsewhere in his manuscript notes on Herbrand [202], van Heijenoort remarked that, whereas Herbrand studied the *Principia Mathematica*, or at least the first volume of *Principia*, it had little influence on him, and that he took from it only the first-order predicate calculus; rather, it was Hilbert who "gave Herbrand a view of logic and of its role in the foundations of mathematics...," and evinced little interest otherwise in the work of Russell in general or in the theory of types in particular.

In the introduction to his *Recherches* [81], Herbrand spoke of the recursive method to "prove that every true proposition has a given property A" (see the translation by Goldfarb [86, p. 49]), and he immediately tied this to the finitist limit on recursive proofs enunciated by Hilbert. For Herbrand, this finitist limit challenges the transfinitist proofs of Löwenheim, in terms of $\aleph_0$-satisfiability, and requires that Löwenheim's infinite conjunction be reinterpreted as Herbrand expansion, the basis for Herbrand's method of quantification. Thus, as van Heijenoort stated [164, p. 526] in his introduction to Herbrand's *Recherches,* "Herbrand's work can be viewed as a reinterpretation, from the point of view of Hilbert's program, of results of Löwenheim and Skolem," and that Herbrand's fundamental theorem is, as Herbrand himself stated [83, p. 4] in his paper "Sur la non-contradiction de l'arithmétique" (1931), "a more

[8] This is clearly a typographical error in [72, p. 1], since Goldfarb's characterization of Property A occurs at [72, p. 5].

precise statement of the Löwenheim–Skolem theorem."[9] In an isolated comment in his notes on Herbrand, van Heijenoort [202] remarked that: "La critique que Herbrand fait de Löwenheim est sévère mais il n'est pas toujours quels sont précisément les points que Herbrand attaque." Elsewhere in the same set of manuscript notes, van Heijenoort, referring to Skolem's "Sur la portée de théorème du Löwenheim–Skolem" of 1938 [145], notes that Thoralf Albert Skolem (1887–1963) 'gives Herbrand as "praising" Löwenheim"; to which van Heijenoort emphatically replies: "no" [van Heijenoort's underlining — I.A.]. Van Heijenoort [202] elaborated upon the comment on Herbrand's critique of Löwenheim in his notes on Herbrand's article "Le bases de la logique hilbertienne" in the *Revue de Métaphysique et de Morale* of 1930 [82], writing that Herbrand:[10]

> states Löwenheim's fundamental result thus: If a formula of quantification theory is valid, it is provable. What Löwenheim's Theorem 2 in fact says (if we leave aside the question of identity) is: If a formula of quantification theory is $\aleph_0$-valid, it is valid. Löwenheim could not bring in the question of provability because he has no proof procedure (where through axioms and rules of inference or through any other means that would single out/mark off a recursively enumerable set of (provable) formulas). Löwenheim has no explicit definition of 'valid', but the notion is perfectly clear to him and he handles it without difficulties [81, p. 110], but cannot be asking for a definition of 'valid"', which is there to be seen in Löwenheim's paper, but that the notion of validity used by Löwenheim belongs to naïve set theory. This is in fact what Herbrand means [81, p. 118] when he writes that Löwenheim gives to the notion 'true in an infinite domain' an intuitive meaning. What he reproaches Löwenheim [with] is the use of a notion that belongs to naïve set theory.

Regarding 'true in an infinite domain', van Heijenoort adds, by way of a footnote, that that notion

> belongs to Herbrand theory and Löwenheim never uses it. If a formula is true in an infinite domain, it is, by a non-finitistic argument, $\aleph_0$-satisfiable, and conversely. But 'true in an infinite domain' and '$\aleph_0$-satisfiable' have entirely different definitions. Here, Herbrand uses 'true in an infinite domain' for '$\aleph_0$-satisfiable' because, according to him, the first notion, as understood by him, is the finitistic reformulation of the second.

Recalling, then, that is *Desarrollo* [189] van Heijenoort stated that for Herbrand's system, quantifier-free formulae can be obtained effectively from quantified formulae, such that these quantifier-free formulae are sententially

[9] For a more detailed consideration of the role of the Löwenheim–Skolem theorem on the work of Herbrand and on the rise of quantification theory from the proof-theoretic perspective, see [15].

[10] On the next page of the notes, this passage is partially rewritten; I have here taken the liberty, for the sake of clarity, of integrating the two versions.

valid, by using Herbrand expansions, and that Herbrand thereby helped introduce a new conception of *validity* into logic, where for Löwenheim the essential consideration was still *satisfiability,* or validity invariant with respect only to a particular model. It was, then, Herbrand, working with applications of Löwenheim's concepts to Hilbert's system, who initiated the shift from satisfiability to validity, and Hilbert who explicitly made *Beweistheorie* a fundamental task for the logician.

The details of van Heijenoort's treatment for repairing Herbrand's error, which remained unpublished and are to be found in the paper "Herbrand" of 1975 [187], is the subject of Claus-Peter Wirth, who presents the correction and an analysis of it [207]. Van Heijenoort's approach is to dispense with Herbrand's inference rules in favor of a generalized quantification rules, thus removing the problematic results that Herbrand's application of *modus ponens* introduced. Wirth, quite understandably, wonders why van Heijenoort failed to publish his correction. The answer to that question is, I suggest, the same as the answer to the question of why so much of van Heijenoort's work remained unpublished: that he viewed it as "work-in-progress", continually perfectable, but still imperfect, and hence, "unpublishable". It is van Heijenoort's approach to Herbrand quantification as Herbrand expansion—Herbrand disjunction for existentialoid quantifiers and Herbrand conjunction for universaloid quantifiers—that enables his repair of Herbrand's error.

3. Herbrand Quantification

For van Heijenoort, Herbrand is a major figure in the history of logic, and he devoted much attention to the work of Herbrand, defending Herbrand against such critics as Roland Fraïssé (1920–2008), whose criticisms in "Réflexions sur la complétude selon Herbrand," [58] of Herbrand's concept of validity had already been dealt with by van Heijenoort in his edition of Herbrand's *Écrits logiques* [85] and in the introduction to Herbrand's "Sur la non-contradiction de l'arithmétique" [83] (in [164, pp. 618–620]; see also [73], for example).

Fraïssé's claims ([58]; see [73], which also summarizes Jean-Pierre Bénéjam's [24]) included the incorrect assertions that Herbrand was concerned with establishing the equivalence of quantificational validity and the property A and that Herbrand had no concept of syntactic truth apart from that property, whereas Herbrand's goal was precisely to establish the equivalence of syntactic truth in the usual sense of derivability in a standard axiomatic formulation of quantification theory. Moreover, Herbrand's property C, which a formula F has in case there is a truth-functionally valid Herbrand expansion of F, is the basis of Herbrand's work.

In his "Préface" to Herbrand's *Écrits logiques,* van Heijenoort [165, pp. 1–12] briefly traced the history of the development of quantification theory, with special emphasis on Herbrand's role as the focal point in that history. Van Heijenoort pointed out in particular that Herbrand studied Löwenheim's treatment of satisfiability for Hilbert's axiomatic system and in *Recherches* generalized the results in Löwenheim's 1915 paper "Über Möglichkeiten im

Relativkalkül" [120] to validity by showing how to obtain, from the satisfiable quantified formulae of Hilbert's system, quantifier-free formulae that are sententially valid. Thus, if F is a formula in Hilbert's system which is satisfiable, then it is provable. Using methods developed by Löwenheim and strengthened by Skolem in his "Logisch-kombinatorische Untersuchungen über die Erfüllbarkeit oder Beweisbarkeit mathematischer Sätze nebst einem Theoreme über dichte Mengen" of 1920 [144], F is rewritten in Skolem normal form as a new formula F'. Employing the method now known as Herbrand expansion, quantifiers are eliminated from F' to obtain a quantifier-free formula F_Q, where $F=$ Exp$[F', D]$, i.e. F_Q is the Herbrand expansion of F', and D is the domain containing the elements that are terms of Hilbert's quantification theory.

Let us consider the details.

We may begin with Kreisel's 1958 [108, p. 161] account in "Mathematical Significance of Consistency Proofs" of Herbrand's Fundamental Theorem:

> Herbrand's theorem provides an interpretation of the classical predicate calculus (with or without equality) by the elementary calculus with free variables (with or without equality): the latter is obtained from the former by suppressing quantifiers.
>
> Consider $(x)(Ey)(\mathrm{z})A(x, y, z)$: with it are associated disjunctions of the form $A[x, \lambda_0(x), z] \vee A[x, \lambda_1(x, z), z_1] \vee \ldots \vee A[x, \lambda_k(x, z, \ldots, z_{k-1}]$ where λ_i are terms made up of the function symbols occurring in A and individual variables, not containing variables z_p, with $p \geq i$ (Informally: the 'function' $\lambda_0(x)\ldots$ satisfies $A[x, \lambda_0(x), z]$ for all z or, if it does not for $z = \bar{z}$, then $\lambda_1(x, \bar{z})$ satisfies $A[x, \lambda_1(x, \bar{z}), z_1]$ for all z_1, etc.). From each of these disjunctions $(x)(Ey)(z)A(x, y, z)$ can be proved.

(Kreisel had already, in 1951 in "On the Concepts of Completeness and Interpretation of Formal Systems" [107], established that the *concept* of *interpretation* presupposes a relation between two systems.)

A quantifier of a classical formula F is *existentialoid* (*universaloid*) if it would become existential (universal) if F were put in prenex form. In the matrix of F, each existentialoid variable y is replaced by a functional term whose arguments are the universaloid variables that are superior to y, where a variable x is superior to a variable y in case the quantifier binding y is in the scope of the quantifier binding x. In his survey "Historical Development of Modern Logic", composed in 1974 and posthumously published in 1992 ([200]; also this issue), van Heijenoort illustrated the procedure for creating the matrix table and enumerated the rules for its construction.

In the manuscript notes on Herbrand [202], on a page labeled "A survey of logic: Herbrand", the quantification rules for Herbrand's system are stated as follows:

Rule of universalization. If F_2 contains a subformula $\mathrm{Q}yG$ occurring within the scope of no quantifier in F_2, and Qy is general, and F_1 is obtained from F_2 by replacing QxG by $G(y/z)$, the F_2 may be inferred from F_1.

Rule of existentialization. If F_2 contains a subformula $\mathrm{Q}xG$, and $\mathrm{Q}x$ is restricted, and F_1 is obtained from F_2 by replacing $\mathrm{Q}xG$ by $\mathrm{G}(x/z)$,[11] then F_2 may be inferred from F_1.

In his informally published paper "Herbrand" of 1975 [187], in which excerpts from the "Préface" [165] to his edition of Herbrand's writings appeared, van Heijenoort detailed very carefully the technical apparatus which Herbrand developed and which was briefly outlined in the "Préface". In addition, this 1975 "Herbrand" paper [187] discusses the "rules of passage" in Herbrand quantification theory; in particular, van Heijenoort discussed in detail the rules of existentialoid quantification and universaloid quantification that are the basis for Herbrand expansion.

A formula of Herbrand's system $\mathrm{Q_H}$ is called *rectified* if it contains no vacuous quantifiers, no variable has free or bound occurrences in it, and no two quantifiers bind occurrences of the same variable. A quantifier in a rectified formula is called *existentialoid* if it existential and in the scope of an even number of negation symbols, or universal and in the scope of an odd number of negation symbols; otherwise, it is *universaloid.* A variable in a rectified formula is an *existentialoid variable* or is *restricted* if and only if it is bound by an existentialoid quantifier, and is a *universaloid variable* if and only if it is bound by a universaloid quantifier. Thus, a quantifier of a classical formula F is existentialoid (universaloid) if it would become existential (universal) if F were put in prenex form. In the matrix of F, each existentialoid variable y is replaced by a functional term whose arguments are the universaloid variables that are superior to y, where a variable x is *superior* to a variable y in case the quantifier binding y is in the scope of the quantifier binding x.[12]

In his paper on the historical development of modern logic, [200, p. 248], van Heijenoort constructed this matrix as follows.

Let $\Gamma = \{\underline{0}, \underline{1}, \underline{2}, \ldots\}$ be an infinite list of variables of Q not occurring in F. Assume that the restricted variables of F are x_1 and x_2, that its non-restricted variables are y_1, y_2, and y_3, y_1 being free, the quantifier binding y_2 being in the scopes of the quantifiers binding x_1 and x_2, and the quantifier binding y_3 being in the scope of the quantifier binding x_2 (where we call x_1 and x_2 the 'arguments' of x_1 and call x_2 the 'argument' of y_3). Then we may write the table:

Line	x_1	x_2	y_1	$y_2(x_1, x_2)$	$y_3(x_2)$
1	$\underline{0}$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$
2	$\underline{0}$	$\underline{1}$	$\underline{1}$	$\underline{4}$	$\underline{5}$
3	$\underline{1}$	$\underline{0}$	$\underline{1}$	$\underline{6}$	$\underline{3}$
4	$\underline{1}$	$\underline{1}$	$\underline{1}$	$\underline{7}$	$\underline{5}$
5	$\underline{0}$	$\underline{2}$	$\underline{1}$	$\underline{8}$	$\underline{9}$

[11] Van Heijenoort here wrote, and then crossed out: "where z is a variable not occurring in F_2."

[12] See [187, p. 1], [197, p. 58], [199, p. 100].

according to the rules:

(1) Under the restricted variables we shall write elements of Γ occurring on the previous lines of the table, except for the first line, on which we write $\underline{0}$ under each restricted variable; on a given line these elements form an ordered pair, and the order in which these ordered pairs are written is determined by the following rules: if $\max(i, j) < \max(k, l)$, $\langle i, j\rangle$ precedes $\langle k, l\rangle$; if $\max(i, j) = \max(k, l)$, the relative order of $\langle i, j\rangle$ and $\langle k, l\rangle$ is their lexicographic order.

(2) On a given line, the element of Γ written under an unrestricted variable is different from any element of Γ previously written in the table (on a line above or on the same line but to the left) except in the following case: if on line q the 'arguments' of the nonrestricted variable u have been assigned the same symbols as on line p, with $p < q$, then under u on line q we write the symbol occurring under u on line p (if the nonrestricted variable v has no 'argument', then on every line the same symbol is written under v).

Each line of the table determines a substitution instance of F. We delete all quantifiers in F and replace each variable by the symbol assigned to it in the line. Let A_k, with $k = 1, 2, 3, \ldots$, be the substitution instance obtained by the use of line k. The formula D_k obtained by this operation on F is a quantifier-free formula of Q called the *kth Herbrand disjunction* of F, i.e. $\mathrm{Exp}[F, D_k]$.

Next, van Heijenoort, following Bernays in his "Über den Zusammenhang der Herbrand'schen Satzes mit den neueren Ergebnissen von Schütte und Stenius" of 1954 [27], concluded that Herbrand's "Fundamental Theorem," according to which *a formula F is provable in any classical quantification theory without identity if and only if there is a k such that Exp* $[F, D_k]$ *is sententially valid*, is the central theorem of quantification theory. Gentzen [61, p. 409], for example, thought that Herbrand's Fundamental Theorem was simply a special case of his own *verschärfter Hauptsatz*. But while the *Hauptsatz* presents cut-elimination and is analogous to Herbrand's elimination of *modus ponens* for $\mathrm{Q_H}$, and his Fundamental Theorem applies to a sequent whose antecedent is empty and whose succedent is a single prenex formula (while Gentzen's *verschärfter Hauptsatz* applies to any prenex formulae), Gentzen's cut-elimination for prenex formulae is neither as general nor as strong as Herbrand's elimination of *modus ponens* for $\mathrm{Q_H}$, since Herbrand's result is not necessarily restricted in its application to prenex formulae. One of the most important contributions of Herbrand's Fundamental Theorem—if not *the* most important contribution—is that it gives us a truth-functional test of the validity of quantified formulae. It is precisely in this sense that Herbrand understood his Fundamental Theorem to be an improvement of the Löwenheim–Skolem Theorem.

With this apparatus, we are now able to give an example of the sentential validity of the Herbrand expansion of a provable formula of classical quantification theory. Suppose that $\mathrm{Exp}[F, D_k]$ is a Herbrand expansion of F over

D_k (where D_k is the *kth* element of the sequence D containing the individual constants and universaloid variables of F and the elements $D_0, \ldots, D_{k-1}$, where initial D_0 contains the individual constants and the existentialloidally free universaloid variable of F, and where, if F contains none of these, then the one element of D_0 is an arbitrary constant u_0), then the formula F is provable in classical quantification theory if and only if there exists a k such that $\mathrm{Exp}[F, D_k]$ is sententially valid. Thus, for the formula F_Q of classical quantification theory without identity, for the *kth* Herbrand expansion F_{QH_k} for any $k, \vdash F_Q \rightarrow F_{\mathrm{QH}_k}$, and F_{QH_k} is sententially valid.

The connection between Löwenheim and Herbrand was made explicit for van Heijenoort precisely through Herbrand's Fundamental Theorem. "Herbrand's work can be viewed," van Heijenoort, recall, wrote [164, p. 526] "as a reinterpretation, from the point of view of Hilbert's program, of the results of Löwenheim and Skolem," adding that "of his fundamental theorem, Herbrand writes (in "Sur la non-contradiction de l'arithmétique" [83, p. 4] 'that it is a more precise statement of the Löwenheim–Skolem theorem'." In an unmarked page of his manuscript notes on Herbrand [202], van Heijenoort remarked that expansion in Löwenheim is a notion which "may have been suggested, vaguely, by the expansion of quantifiers, but it [is] much more sophisticated." On the following page, van Heijenoort referenced Goldfarb, remarking that a "key notion in Löwenheim's 1915 paper [120] is that of a quantifier-free expansion of a quantified formula."

A much more extensive and systematic account of Herbrand's work and its historical significance was given by van Heijenoort in his 1982 paper "L'ouevre logique de Jacques Herbrand et son contexte historique" [197], also translated into English, with revisions [199], where he said of Herbrand's Fundamental Theorem and its converse—that, given a Herbrand formula $\mathrm{Exp}[F, D_k]$, we can work backwards to obtain the original formula F of classical quantification theory from whence $\mathrm{Exp}[F, D_k]$ came—that they are "resultats de grande importance pour la théorie de la quantification" [197, p. 57], "results of great importance for quantification theory" [199, p. 99]. A series of *nachgelassene* notes develop the details of various aspects of Herbrand's work, including in particular "The Herbrand Approach to Predicate Logic" (1976) [191] and "On Herbrand's Systems" (1980) [196]. In the latter, van Heijenoort sketches the three distinct systems which Herbrand presented in his papers "Sur la théorie de la démonstration" (1928) [77], "Non-contradiction des axiomes arithmétiques" (1929) [78], and "Sur quelques propriétés des propositions vraies et leurs applications" (1929) [79]. In the *nachgelassene* typescript "Proof of the 'Semantic' Herbrand Theorem" (1976) [192], van Heijenoort went beyond exposition and presented a proof of the equivalence between the semantic approach to quantification theory concerning the validity of a classical formula if it is sententially valid and the Herbrand approach, according to which $\vdash F_Q \rightarrow F_{\mathrm{QH}_k}$, and F_{QH_k} is sententially valid. Moreover, in this paper in particular and in "Herbrand" of 1975 [187], van Heijenoort explored the details of Herbrand's proof of his Fundamental Theorem and sketches the error in the proof.

Several of van Heijenoort's reviews in 1970 and 1971 (e.g. [168–170, 172, 173]) centered on the work of Grigori Efroimovich Mints (Gregory Minc) (e.g. [15, 99, 124, 126, 127]) which examine the relations of Herbrand's work to that of Gentzen and give generalizations of Herbrand's Fundamental Theorem using Gentzen's *Hauptsatz*. Moreover, using the extension of this method from Gentzen's classical sequent calculus **LK** to Gentzen's intuitionistic calculus **LJ**, Mints [125] was able, as van Heijenoort [172] showed in his review of one of Mints's papers, to obtain an analogue of Herbrand's Fundamental Theorem for intuitionistic predicate calculus. In his appendix on Herbrand's theorem [126] to the Russian translation of various papers in proof theory prepared by Alexander Vladimirovich Idel'son and Mints [99], Mints undertakes, as summarized by van Heijenoort [169, pp. 323–324], to "formulate and prove a generalization of Herbrand's [fundamental] theorem for the classical predicate calculus with identity and function symbols", using Gentzen's *Hauptsatz* for classical sequent calculus **LK**.[13] Van Heijenoort's *nachgelassene* notes "Herbrand— Non-classical" [201] deal with the intuitionistic interpretation of Gentzen's sequence calculus through the apparatus provided by Herbrand, and in particular with Mints's extension to Gentzen's **LJ** of Herbrand's Fundamental Theorem. These notes include excerpts from Kreisel's [109] paper "Elementary Completeness Properties of Intuitionistic Logic…" as well as the complete English version of Mints's paper "Disjunctive Interpretation of the **LJ** Calculus" [127] reviewed by van Heijenoort [173]. Mints's work relies upon Kreisel's [109, pp. 326–327, 328–329] proofs of the theorems that *The negation of a prenex formula is provable intuitionistically if and only if it is provable in the classical predicate calculus* and that *there is a Herbrand type theorem for negations of prenex formulæ of the predicate calculus*. In fact, as van Heijenoort [162, p. 351] pointed out in 1957 his review of Robert Feys'(1889–1961) preface [55] to Jean Ladrière's (1921–2007) French translation [62] of Gentzen's "Untersuchungen über das logische Schließen" (1934) [61], Feys' "preface underlines the fact that Gentzen's methods lead 'naturally' to intuitionistic, and not to classical, logic." Mints's analogue, however, as van Heijenoort noted both in his [172, p. 526] review of Mints's paper [125] and again in his informally published paper [187, p. 9] "Herbrand," is not without difficulty and does not apply to arbitrary formulae of intuitionistic logic. There is some confusion regarding Herbrand's use of "intuitioniste", however, as van Heijenoort in his manuscript notes on Herbrand remarked, in that Herbrand does not use the term in every case in precisely the same sense, although it is evident that he intends by it the same meaning as Hilbert in his use of "finit";[14] we find this sense clearly articulated in those of van Heijenoort's manuscript notes on Herbrand [202] labeled "On Herbrand's Finitism" (see also [143, p. 6]). Elsewhere, Herbrand sometimes uses "intuitioniste" as synonymous with Hilbert's use of "metamathematics", or in some cases, van Heijenoort, in his notes on Herbrand [202] considers even

[13] Van Heijenoort [170] dismissed [124], in one sentence, as a brief and earlier version of [126].

[14] Herbrand's Hilbertian conception of "intuitioniste" as finitism is discussed in [208, pp. 204–205].

the suggestion of Arend Heyting (1898–1980) in his 1955 *Les fondements des mathématiques; intuitionnisme, théorie de la démonstration* [87, p. 61] that finitism pertains to informal, or intuitive, mathematics.

There is nothing in either van Heijenoort's notes on Herbrand or in his review of Mints's studies on Herbrand, for example in his [172] review of the paper in which Mints was able to obtain an analogue of Herbrand's Fundamental Theorem for intuitionistic predicate calculus, or in particular in own his later papers on Herbrand (e.g. [199], to justify the assertion by Jean-Yves Girard [63, p. 257], [64, p. 10], that van Heijenoort ever presented Herbrand's system as a synthesis of Brouwer and Hilbert. On the contrary, van Heijenoort made it clear once more (e.g. in [199, pp. 114–116]) that Herbrand was largely indebted to Hilbert, and that Herbrand meant by "intuitionism" precisely, as Hilbert did, "finitism".

What is of interest respecting Ladrière's [62] translation of Gentzen's [61] "Untersuchungen über das logische Schließen" is that, in addition to the preface by Feys [55], there are abundant notes, by Feys and by Ladrière. Note C, by Feys, "Méthodes N de Jaśkowski, Bernays et Johansson" [56] in particular compares Gentzen's sequent calculus and his method of natural deduction with alternative techniques, including method of the natural deduction that Jaśkowski [100] developed and published at precisely the same time (1934) as Gentzen published his "Untersuchungen…". Another note by Feys, Note F, "Signification des sequences et de schemas de structure" [57], demonstrates how to translate Gentzen's sequences using only implication (and Λ for empty sequences).

The existential quantifiers are eliminated by *Herbrand disjunction*, so that

$$\exists x F x = (F(x/t_1, D) \vee (F(x/t_2), D) \vee \ldots \vee (F(x/t_k), D)$$

where D is a k-ary model and $t_1, t_2, \ldots, t_k$ are the terms of D that are arguments for the functions of the formula of Hilbert's system; and universal quantifiers are eliminated by *Herbrand conjunction,* so that

$$\forall x F x = F(x/t_1, D) \wedge (F(x/t_2), D) \wedge \ldots \wedge (F(x/t_k), D)$$

for the k-ary model D and its terms. Van Heijenoort's account of Herbrand conjunction and disjunction is also given in an unpublished manuscript from 1976 specifically devoted to "Herbrand Expansions and Herbrand Disjunctions" in which the details of Herbrand's method are treated [193]. As a result, we obtain a sentential formula to which appropriate tree decomposition rules are applicable. The resolution method, as first presented in detail by in 1965 in "A Machine-oriented Logic based on the Resolution Principle" [136] by John Alan Robinson (b. 1930), applied to clauses rather than to the terms of either sentential or first-order classical calculus, is a direct descendant of Herbrand's method.[15] The inverse method of Sergei Yure'evich Maslov (1939–1982), first presented in 1968 [122], is a close relative of the resolution method, as Maslov

[15] See [2–4] and [17]; [see also especially [115].

himself [123] recognized in 1969.[16] Indeed, Maslov established that there is a one-to-one correspondence between his inverse method and Robinson's resolution ([123]; see also [113]). Gennadi Valentinovich Davydov [37], in fact, was able to obtain a synthesis of the two methods. The application of the resolution method to computerized proof-finding was enhanced by Robinson's development in "Automatic Deduction with Hyper-resolution" [137] of a type of resolution called *hyper-resolution.* This is a resolution whose search space is considerably more sparse than the search space of general resolution. Here, we have a sequence of connected search trees which converge to a ground resolution; these converging trees connect several levels of resolutions. Resolution is described by Jean-Pierre Jounnaud and Claude Kirchner in their paper for the Robinson *Festschrift* as the "first really effective mechanization of first-order logic" [104, p. 257]. They add the historical remark that solving equations on first-order terms emerged with Herbrand's work on proof theory" in his doctoral thesis *Recherches sur la théorie de la démonstration* (1930) [81], "and was coined unification by Alan Robinson" in his first major paper [136]. Robinson [138, p. 290] defined Herbrand in *Recherches...* as essentially having demonstrating that: "the provability of any provable formula A by a proof that does not introduce any formulas that are not *subformulas*, in a suitable sense, of the formula A," which is Robinson's conception of Herbrand's Fundamental Theorem, and which is, as he asserts [38, p. 290] equivalent to Gentzen's result that true sequents can always be given cut-free proofs.

Despite their clear connections with Herbrand's method, van Heijenoort never took account either of Robinson's resolution method or Maslov's inverse method.

Whereas Hilbert's universe was finite, Herbrand's need not necessarily be so (although, in fact, Herbrand himself supported Hilbert's finitism and sought to define validity as satisfiability strictly of finite models, that is, as satisfiability invariant with respect to finite models). Thus, Löwenheim accepted Hilbert's finitism, according to which every interpretation that gives the truth-value $\mathbf{t}$ (true) to the lth expansion of a classical formula F gives the value $\mathbf{t}$ to the kth expansion, where $k < l$. If F is provable in (classical) quantification theory, then that proof is finitary. Löwenheim thus obtained a finite notion of satisfiability which is invariant with respect to one's model and where the cardinality of that model helps determine whether formulæ of that model are satisfiable or not. But since Herbrand expansion may be finite or infinite (despite Herbrand's own rejection of the infinite), satisfiability is no longer model-dependent; instead, Herbrand expansion allows us to obtain formulae which are *sententially valid*, and therefore valid.

In an undated one-page manuscript in the Herbrand file [202] labeled "Relation between the Falsifiability Tree Method and the Herbrand Method" related to his 1968 "On the Relation between the Falsifiability Tree Method and the Herbrand Method in Quantification Theory" [167], van Heijenoort

[16] See [12], see also [17, p. 172] and [117, pp. 18–19; 78–97] gives a detailed exposition of Maslov's inverse method and its applications.

remarked that "[i]n the falsifiability tree method the $\exists$-rule or the $\forall$-rule is applied to a quantifier according as the quantifier is existentialoid or universaloid in $\sim F$." He proceeds to explicate the procedure, as follows:

Let $\mathrm{Q}y_1, \ldots, \mathrm{Q}y_{n_0}$ *be the accessible existentialoid quantifiers of* $\sim F$, *and* $\mathrm{Q}x_1, \ldots, \mathrm{Q}x_{m_0}$ *its accessible universaloid quantifiers. Delete* $\mathrm{Q}y_1, \ldots, \mathrm{Q}y_n$ *and replace* $y_1, \ldots, \mathrm{y}_{n_0}$ *by* $b_1, \ldots, b_{n_0}$, *which are* n_0 *distinct individual constants new to* $\sim F$.

Delete $\mathrm{Q}x_1, \ldots, \mathrm{Q}x_m$ *and replace* $x_1, \ldots, x_m$ *by* $a_1, \ldots a_{m_0}$ *which are* m *constants (distinct or not) taken among* $b_1, \ldots, b_{n_0}$.

Let $\sim F_1$ *be the formula thus obtained.*

Let $Qy_{n_0+1,\ldots}, \mathrm{Q}y_{n_1}$ *be the accessible existentialoid quantifiers of* $\sim F_1$, *and* $Qx_{m_0+1,\ldots}, \mathrm{Q}x_{m_1}$ *its accessible universaloid quantifiers. Delete* $\mathrm{Q}y_{n_0+1,\ldots}$, $\mathrm{Q}y_{n_1}$ *and replace* $y_{n_0+1,\ldots,}y_{n_1}$ *by* $b_{n_0+1}, \ldots, b_{n_1}$, *which are* $n_1 - n_0$ *distinct individual constants new to* $\sim F_1$.

As in the case of Löwenheim's 1915 paper [120], this expansion of quantified formulae in a k-ary universe as formulae in propositional calculus as logical conjunctions and disjunctions in terms of logical sums and logical products was directly borrowed from the *Vorlesungen über die Algebra der Logik (Exakte Logik)*, Bd. II/I [141, §30, p. 35] of 1890 of Ernst Schröder (1841–1902), as

$$\sum_{1}^{n} \lambda a_\lambda = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n$$

$$\prod_{1}^{n} \lambda a_\lambda = a_1 a_2 a_3 \ldots a_{n-1} a_n$$

who in turn borrowed it directly from Charles Sanders Peirce's (1839–1914) translation in his algebra of relatives of quantified equations as logical sums and logical products. (Löwenheim had already done previous work on the logic of relatives [119], and in the first decade of the twentieth century, it remained a topic of continued work; thus, for example, between 1900 and 1910, Karl Eugen Müller (1865–1932), Schröder's editor, undertook to systematize the Gebietkalkul [128,129], and Olga Hahn (later Oga Hahn-Neurath; 1882–1937) undertook to axiomatize the system in 1909 [76]). In "The Logic of Relatives" (1883) [134], Peirce defined the existential and universal quantifiers, designated by 'Σ_i' and 'Π_i' respectively, as logical sums and products, e.g., $\Sigma_i x_i = x_i + x_j + x_k + \cdots$, and $\Pi_i x_i = x_i \exists x_j \exists x_k$, and individual variables, $i, j, \ldots$, are assigned both to quantifiers and predicates. Thus, van Heijenoort [164, p. 228]), in his introduction to Löwenheim's 1915 [120] paper, acknowledges that "Löwenheim's work links up with that of Peirce and Schröder," adding that Löwenheim took his notation directly from Schöder's *Algebra der Logik*, and in discussing, for example, universes of discourse, Löwenheim explicitly refers to Schröder's *Gebietkalkül* in 1895 in Bd. III/I of the *Algebra der Logik* [142, pp. 122–123].[17] It was left, however, to Geraldine Brady in the

[17] 'Gebeitkalkül' is translated by Stefan Bauer-Mengelberg [164, p. 234] as "'abacus of relatives"'.

expository *From Peirce to Skolem* [34] to fill in the historical details and Calixto Badesa in *El teorema de Löwenheim en el marco de la teoría de relativos* [20] and *The Birth of Model Theory: Löwenheim's Theorem in the Frame of the Theory of Relatives* [21] to elaborate the technical developments.[18]

When he came to develop the apparatus for falsifiability tree proofs, van Heijenoort used this distinction between satisfiability and validity and showed how these concepts can be applied both to formulae and to proofs.

Thus, in *Desarrollo* [189], van Heijenoort studied the members of the quantification theory family of formal systems as attempts to elucidate the concepts of *validity* and satisfiability and to develop the technical apparatus for carrying out valid proofs of logic. Thus, he not only dealt there with the historical development of quantification theory, but also made comparisons of the relative strengths and weaknesses of the various family members.

It was also in *Desarrollo* that van Heijenoort presented, in the context of his evaluations of the family members of quantification theory, a defense of the tree method. Indeed, nearly all of van Heijenoort's technical, that is to say non-historical and non-philosophical, writings were devoted to developing the tree method as a powerful method of logical deduction and validity checking. It is these papers that largely remain unpublished (although they were distributed to students and colleagues).

In considering the question of *provability* apropos the study of Herbrand, van Heijenoort [202] remarked that provable means a finite search, and he indicated the various ways of searching, listing two, namely, (1) by Gödel numbers of proofs; and (2) by "demolition" of formulas. By this, he means that "the negation enthält ein Widerspruch, is not satisfiable, hence the formula is valid." He adds that "all the ways of assigning t to the negation end up with assigning t *and* f to no subformula of the formula." On the verso of the page in which this conception of provability is enunciated, we find the equivalents of the rules for Gentzen sequents for introduction and elimination of *modus ponens* that may be applied to the construction of falsifiability trees, along with the definition for the valuation of a formula F, defined for a domain with property P signed by the function φ, and with terms $a_1, a_2, \ldots, a_n$, so that we have F: $Pa_1a_2\ldots a_n$ and the value of an assignment α to F is true, i.e. $\mathrm{v}[\alpha, F] = \mathrm{t}$ where $\mathrm{v}[\alpha, F] = t \Leftrightarrow \langle\varphi(a_1), \varphi(a_2), \ldots, \varphi(a_n)\rangle$, and the formula F is k-satisfiable if $\mathrm{v}[\alpha, F] = \mathrm{t}$ holds for any k-many $\varphi(a_i), \varphi(a_j), \ldots, \varphi(a_k)$ of F: $Pa_1a_2\ldots a_n$.

4. Herbrand's Errors

Several years before Dreben in 1962 detected an error in a lemma employed by Herbrand's to set forth his Property A as part of the proof of his Fundamental Theorem [40,41] and he and Peter B. Andrews and Stål Aanderaa [43,44] found counterexamples demonstrating that Herbrand's lemma was false, and Dreben [40] and Dreben, Andrews, and Aanderaa [44] provided a

[18] See [18] for a discussion of [34].

repair, published in 1963, and Dreben and John Denton obtained an improved repair, published in 1966 [45], taking advantage of tools provided by Dreben and Aanderaa [42] in 1964. In 1958, if not earlier, Gödel noticed the problem and offered his solution, as John Dawson discovered (see [38] for an account, and [74] for details) when he was cataloguing the Gödel Nachlaß.[19] Goldfarb [74] includes a comparison of his treatment and Gödel's, which are not significantly different, although Gödel never published his account, and brought up the issue with Alonzo Church (1903–1995) when apprised by Church that Dreben, and Andrews, then Church's Princeton University doctoral student, had discovered the error and a correction; Samuel R. Buss [35] offer details; and Andrews [6] gives an historical account of the background to the error and of his role in dealing with it.[20]

Andrews [6] explained that, in 1962, he had difficulty following Herbrand's original proof, and told his doctoral dissertation advisor, Church, that he thought he detected a gap in Herbrand's proof. Church consequently recommended to Andrews that he contact Dreben, which he did in a letter of 9 April 1962, in which Andrews referred explicitly to Herbrand's erroneous lemma 3.3, initiating correspondence and collaboration with Dreben. Dreben replied in a letter of May 18, 1962 that the cause of the difficulty was an ambiguity in Herbrand's argument, in particular regarding lemma 3.3 of Chapter 5 of the *Recherches*, focusing on [74, pp. 101–104], and most especially p. 103. On 18 May 1962, Dreben points out that the root of the difficulty is the ambiguity in Herbrand's argument, proposing in particular that it is necessary to distinguish Herbrand's "*réduite*" or*cut*, of an expression over a domain, in which there are, in general, no quantifiers but which contains function letters, and the *evaluation* over the *réduite* in which functional expressions no longer appear.[21] Andrews brought up the question with Gödel, noting that he had understood that Gödel had also discovered an error and propounded a correction. Gödel then mentioned to Church that he had known of errors in Herbrand and had offered a repair. When in 1963 Kreisel visited Princeton, Andrews told

[19] Most of what Gödel wrote on Herbrand is found in pp. 14–79 of the fifth of Gödel's "Arbeitshefte" notebooks in the Gödel *Nachlaß*; see [74, p. 107], with a detailed account of Gödel's correction in [74, pp. 107–112] and an analysis of Gödel's work and comparison with that of Dreben and his colleagues [74, pp. 112–117].

A too casual reading of [38] could lead to the inference that Gödel's discovery can be traced as far back as 1931 when he writes, about Gödel on Herbrand's error in the midst of a discussion of the Gödel-Herbrand correspondence of 1931, which otherwise dealt entirely with the the two issues of the extent of the applicability of finitism and the impact of the incompleteness theorems on the Hilbert program, that he located in the Gödel *Nachlaß* a folder marked "Errors in Herbrand" that provided evidence that Gödel had detected, and found a repair for, Herbrand's flawed lemma "years before" Andrews, Dreben, and Aanderaa. See [71, pp. 3–25] for the Gödel-Herbrand correspondence and Sieg's introduction [143] to the Gödel-Herbrand correspondence, and [75, pp. 389–391] for Goldfarb's introduction to Gödel's correspondence with Dreben. The Gödel-Dreben correspondence is found at [70, pp. 391–396]. Some of the Gödel-van Heijenoort correspondence is found in [71, pp. 307–325]

[20] [208, §3.9, pp. 216–218,] also offers a survey.

[21] Facsimiles of these letters, and the subsequent Andrews–Dreben correspondence, are included in [6, pp. 6–14].

him of the error, and Kreisel brought it up with Bernays, who informed him that Gödel had mentioned it to him in 1958. Andrews' encounter with Kreisel is the source of the dating of Gödel's 1958 treatment of Herbrand's error and no-counterexample solution as found in Gödel's 1958 paper [67]. Andrews [6, p. 15], however, cites the "early 1940s" as the period for Gödel's discovery of Herbrand's error; Goldfarb [74, p. 103] places Gödel's discovery of the flaw in Herbrand's proof "twenty years earlier" than Dreben's, thus, in 1943. Dreben pursued the matter with Gödel in several letters. In March 6, 1963 (see [75, p. 391]), Dreben wrote to Gödel in this connection, remarking that he understood from Andrews that he [Gödel] had informed Church that he [Gödel] had known of errors in Herbrand, and, sending enclosing for Gödel the abstract "Errors in Herbrand" [43] and the preprint of the paper published as "False Lemmas in Herbrand" [44], asking whether the same error considered in "Errors in Herbrand" was among those found by Gödel. Gödel answered none of Dreben's letters, Goldfarb [75, p. 389] noted, however, that Gödel replied to none of the Dreben's letters. Meanwhile van Heijenoort raised the matter with Gödel as well, mentioning the error in Herbrand in a personal conversation that took place in late September 1963 (see [75, p. 389]), as evidenced by a letter from van Heijenoort to Gödel of 14 October 1963 (quoted by Goldfarb [75, p. 389]), in which van Heijenoort told Gödel that he mentioned to Dreben and Hao Wang (1921–1995) that they had discussed the proof of Herbrand's lemma during their conversation, and both had expressed the hope that Gödel would permit publication of his comments and corrections to Herbrand's lemma in *From Frege to Gödel*, as, if his "earlier corrections were made available, this would enhance the historical value of the book." Goldfarb [75, p. 389] notes that there is no indication of how much Gödel told van Heijenoort about his earlier corrections, and failed to respond to van Heijenoort's proposal to publish them. What is known is that Gödel studied Herbrand's *Recherches...* [81] in the early 1940s and found an error in Herbrand's proof, but did not find a counterexample to demonstrate that the lemma, in its original guise was false; that he did, however, propound a correction to Herbrand, which provided a weaker version of the lemma than Herbrand had given, and that Gödel's proposed solution was, "in all essentials" the same as that given by Dreben and Denton in 1966 [45] (see [75, pp. 389–390] and [74, pp. 107–112] is an account of Gödel's treatment, and [74, pp. 112–117] an analysis and comparison between Gödel's and his own solution).

Considering obfuscations in Herbrand's treatment that Dreben and Denton [46, p. 419] indicated, van Heijenoort [165, pp. 11–12], in addition to grammatical obscurities and ambiguity of meanings, took especial note that: "Les textes contiennent, outre les erreurs de raisonnement...."

In a manuscript page included in the notes on Herbrand [202] labeled "Errors in Herbrand", van Heijenoort lists the following rules of passage:

1. $\sim \forall x W \Rightarrow \exists x \sim W$
2. $\sim \exists x W \Rightarrow \forall x \sim W$
3. $\forall x W \vee Z \Rightarrow \forall x(W \vee Z)$
4. $Z \vee \forall x W \Rightarrow \forall x(Z \vee W)$

5. $\exists x W \vee Z \Rightarrow \exists x(W \vee Z)$
6. $Z \vee \exists x W \Rightarrow \exists x(Z \vee W)$

and refers to, but does not list, the converse of each. He then notes, referring to Stephen Cole Kleene's (1909–1994) *Introduction to Metamathematics* [105, pp. 162–163] that $A \vee \forall x B(x) \equiv \forall x(A \vee B(x))$ is not intuitionistically valid and that $A \vee \forall x B(x) \supset \forall x(A \vee B(x))$ is intuitionistically valid.

In a note on Herbrand's "Sur le problème fondamental des mathématiques" [84, p. 555, line 23], van Heijenoort [202] took note of Herbrand's errata [84, p. 570], replacing

$$f_i(b_{j_1} b_{j_2} \ldots b_{j_{n_i}} = f_i, (c_{j_1}, c_{j_2}, \ldots c_{j_{n_i}})$$

with

$$f_i\left(b_{j_1} b_{j_2} \ldots b_{j_{n_i}}\right) = c_m \quad \text{or} \quad f_1\left(c_{j_1}, c_{j_2}, \ldots c_{j_{n_i}}\right) = b_m.$$

Elsewhere, van Heijenoort [202] notes another difficulty in connection with "Sur quelques propriétés des propositions vraies et leurs applications" (1929) [79], remarking that Herbrand's use of "effectivement" is "misleading". Referring to (2), a generalization of the problem of polynomial equations, van Heijenoort remarks that, if this generalization were provable, then there are numbers x, y, z, n, with $n > 2$, such that $x^n + y^n = z^n$; but that the search for these numbers is no more effective than the search for a proof of the generalization. (It is noteworthy in this regard that, in his class lectures, van Heijenoort, as much as had Gödel (in his [61], [65, p. 196] undecidability paper of 1931 and his discussion on the foundations for mathematics of 1931 [66, p. 148]), regarded Fermat's Last Theorem as an example of a problem of number theory that was in principle undecidable).

The root of the difficulty in Herbrand's proof of his Fundamental Theorem concerned Lemma 3.3. Referring to that lemma as it appears in *From Frege to Gödel* [164, p. 544], van Heijenoort, on a page of his manuscript notes [202] remarked that, for the "proposition derived by the rules of passage," the "order is the same". He then refers to B[urton] D[reben]'s demonstration that this is "false for certain cases of property C." He next remarks, with respect to Dreben's Note E [164, p. 571] that the "[p]assage from $\forall x \Phi(x) \vee Z$ to $\forall x[\Phi(x) \vee Z]$ is a negative occurrence." Thus, we have

$$\begin{array}{ccc} P_1 & \rightarrow & P_2 \\ p & & q \end{array}$$

Van Heijenoort himself had suspicions regarding Herbrand's presentation of his proof of the Fundamental Theorem as it appeared in the full text of Herbrand's 1931 "Sur le problème fondamental des mathématiques" [84],[22] where it is designated as Theorem 2 [84, p. 48]. Referencing Herbrand's [84, p. 53] consideration of the decision problem in connection with classes and relations and utilizing the Multiplicative Axiom, van Heijenoort [202] writes that a point that "bothers" him is that "Herbrand does not give any indication

[22] [80] is the abstract of [84].

on how to fill the gap of the law of infinite conjunction.” The problem results from applying *modus ponens* to Herbrand expansions of infinite length when replacing $\forall x(\Phi(x) \vee P)$ with $\forall x\Phi(x) \vee P$; that is, as we have already noted, $\forall x\Phi(x) \vee P \equiv \forall x\Phi(x) \vee P$, is invalid, whereas $\forall x\Phi(x) \vee P \supset x(\Phi(x) \vee P)$ is valid. Reformulating the offending lemma in terms of its converse, we have that, *for some G obtained from F by replacing a positively occurring subformula* $\forall x(\Phi(x) \vee P)$ *with* $\forall x\Phi(x) \vee P$, *where x does not occur in Z, then for any p, if Exp[F, p] of order p is truth-functionally satisfiable, so is Exp[G, p], where Exp[G,p]*, the satisfiability expansion of F of order p is the conjunction of all instances of F^* obtained by substituting the terms of that expansion for its free variables, and $p \leq max(k, l)$. As Goldfarb [74, p. 106] thus explained: “The difficulty arises from the fact that the functional forms F^* and G^* differ; the terms that replace restricted variables whose quantifiers lie in P will have one more argument place in F^* than they have in G^*, since they are governed by the general variable x in F but not in G. Consequently, there will be instances of G^* in which the subformulas corresponding to P are the same; while in the analogous instances of F^* those subformulas are different, due to a difference in the term that supplants x.”

In the pages immediate following that in which van Heijenoort expressed his discomfort with Herbrand’s lack of explaining how to fill the gap of the law of infinite conjunction, he formulates the procedure for construction of a sequence of formulas the terms of the expansions for accessible universaloidally and accessible existentialoidally quantified formulas, and propounds several theorems:

Theorem. *There is a number k such that* G_k *is quantifier-free.*

Now, denoting G_k by F_1^*,

Theorem. F_1^* *is a validity instance of F.*

The details of the expansion are then given on the next pages, as follows:

Given a formula G, let D be the set containing the free variables and individual constants of G; let $y_1, \ldots, y_m$ *be the existentialoidally free universaloid variables of G. In G delete the quantifiers* $Qy_1, \ldots, Qy_m$ *and replace* $y_1, \ldots, y_m$ *by distinct individual variables not in D.* (1) *Given a formula G and its lexicon D, if Qy is an accessible universaloid quantifier of* G_i, *then* G_{i+1} *is G(QyH/H(y/b)) where b is not in* D_i; (2) *If Qx is an accessible existentialoid quantifier of G, the successor is G(QxH/H(x/a)), where* $a \in D$.

On the following page, the expansion is obtained:

Let D' *be the set* $D \cup \{b_1, \ldots, b_n\}$. *Given a formula* G_j, *let D be the set of free variables and individual constants of G. Let* $Qy_1, \ldots, Qy_n$ *be the n accessible universaloid quantifiers of G and* $Qx_1, \ldots, Qx_m$ *its m accessible existentialoid quantifiers. Delete* $Qy_1, \ldots, Qy_n$ *and replace* $y_1, \ldots, y_n$ *by distinct individuals* $b_1, \ldots, b_n$ *not in D. In the formula thus obtained, for each i replace* $Qx_iAx_i by Aa_1Aa_2 \ldots Aa_k$, *where* $a_1, a_2, \ldots, a_k$ *are all the members of* D^{Φ}. *Call the formula obtained* G'.

Theorem. *In the sequence $G, G', G'', \ldots$, there is a $G^{(k)} = G^{(k+1)}$.*

For some G^*, we have:

Theorem. $G^* = \mathrm{Exp}[F, D]$.

(In his notes, van Heijenoort placed a question mark above the "D" in this theorem).

The tree that is drawn for this last theorem is:

$$\begin{array}{cc} y_1 & y_2 \\ | & | \\ x_1 & x_2 \\ | & \\ y_3 & \\ | & \\ x_3 & \\ | & \\ y_4 & \end{array}$$

There is, unfortunately, no means for dating these particular pages.

Claus-Peter Wirth, Jörg Siekmann, Christoph Benzmüller, and Serge Autexier, in the context of their general survey of Herbrand [208], offer a parallel account of the "Gödel-Dreben correction" and the "Heijenoort correction" in their discussion of *modus ponens elimination* [208, §3.11, pp. 221–224], using primarily van Heijenoort's informally published "Herbrand" paper of 1975 [187], some passages from the English translation of his published article on Herbrand [199], and the critical note in Sect. 5 on the Fundamental Theorem in chapter 5 of Herbrand's *Recherches...* [81] the as the basis for their account of van his correction.[23]

5. Brief History of the Tree Method

The falsifiability tree method, sometimes also called the confutation tableau, is typically traced to the work of Jaakko Hintikka [91–96] and his model sets, and to Raymond Merrill Smullyan's (b. 1919) analytic tableau.[24] Both have their historical roots in the deductive tableaux and semantic tableaux of Evert Willem Beth (1908–1964) [28–31], as noted in the brief canonical histories of the method, for example by Robinson [138, p. 290] and Richard Carl Jeffrey (1926–2002) [101, p. 227]. Slightly earlier than the work of Smullyan, which began appearing in 1963 [147–156] is the work of Zbigniew Lis (1960) [118], whose tableau is undifferentiable from that of Smullyan. However, because Lis, wrote his article presenting the tableau method in Polish, his work exercised virtually no influence. Lis's name made no appearance in van Heijenoort's

[23] See [60], [86, p. 171] and [77], [173, p. 555]. It is clear from Goldfarb's preface [86, p. VI] that the note at [60], [86, p. 171] is in fact his, neither van Heijenoort's nor Dreben's; [77], [164, p. 555] is identical with [60], [86, p. 171], on which basis we may suppose that the note as it appeared in *From Frege to Gödel* [164] was originally Goldfarb's.

[24] For a detailed account, see [14]. See also the account by Bondecka-Krzykowska [33].

writing on the tableaux methods. Smullyan's textbook *First-order Logic* [154] of 1968 presented the complete and fully-worked out presentation of his analytic tableaux. Robinson [138, p. 290] described Smullyan's *First-order Logic* as exploring the "nooks and crannies" of the analytic tableau method with elegance.

Comparatively recently, Francine Abeles [1] discovered that falsifiability trees appeared in the unpublished, and hitherto lost Part II of the *Symbolic Logic* dating from 1894 (and posthumously published [39]) of Charles Lutwidge Dodgson (Lewis Carroll; 1832–1898).[25] Applying these trees to polysyllogisms, i.e. soritises or chains of syllogisms, Dodgson's inspiration for devising the method derived from the antilogisms, or inconsistent triads, developed by Charles Peirce's student Christine Ladd-Franklin (1847–1930) and the logic machines of Peirce's student Allan Marquand (1853–1924), which he discovered in their respective contributions "On the Algebra of Logic" [114] and "A Machine for Producing Syllogistic Variations" [121], to Peirce's 1833 *Studies in Logic* [133]. Thus, we have, in Ladd's work, *reductio ad absurdum* proofs, with two premises and the contradictory of the conclusion. In his account of Dodgson's work in Part II of Dodgson's *Symbolic Logic*, its discoverer, William Warren Bartley, III (1934–1990) [22] described it as a precursor of, and essentially equivalent to, Beth's semantic tableaux; but only Abeles's treatment [1] clarifies and demonstrates its equivalence to the falsifiability tree. Abeles [5] also demonstrated the soundness and completeness of Dodgson's trees. Whereas van Heijenoort undertook to explore the tree method with other mechanical decision procedures, he did not, and, given its late appearance could not, take account of Dodgson's work.

It is probable that van Heijenoort first learned about Smullyan trees around 1964–1965, at precisely the same time that Raymond Smullyan was beginning his work developing the tree method. The earliest datable unpublished composition which he have in van Heijenoort's hand concerning the tree method goes back to 1966 with the unpublished "Interpretations, Satisfiability, Validity" [163], in which van Heijenoort articulated the truth-value semantic interpretation of satisfiability and validity to first-order formulae, and the latest was composed in 1975. In "Notes on the Tree Method" (1971) [174], van Heijenoort applied the truth-value semantic to truth trees and falsifiability trees in his paper and it appeared in his paper "Falsifiability Trees" (1972) [176], its penultimate version, of 1974 [183], and its final version, of 1975 [188].

Richard Jeffrey reported [102] that he first encountered van Heijenoort in 1964–1965, at a time when he and Jeffrey both were in New York City, Jeffrey teaching at City College of New York and van Heijenoort at New York University and Columbia University. The two men met several times during this period as members of an informal group that convened occasionally to discuss logic and philosophy. It was also during this period, perhaps in 1964, that Jeffrey met Smullyan in New York and possibly also attended Smullyan's

[25] Russinoff [139] traces Ladd-Franklin's work in evolving the refinement of antilogism, but does not connect it with Dodgson's application of it to his falsifiability tree method.

lectures at Princeton University. Jeffrey immediately became an enthusiastic supporter and proselytizer for the tableau method, and in particular of the so-called "Smullyan tree" as a one-sided Beth tableau.[26] It is safe to suppose then that this was when van Heijenoort, too, first learned about Smullyan trees, either directly from Smullyan or through Jeffrey.[27] In any event, the first important document we have available in van Heijenoort's hand on the tree method dates from 1968.

In the paper "On the Relation Between the Falsifiability Tree Method and the Herbrand Method in Quantification Theory" of 1968 [167], van Heijenoort showed how the falsifiability tree method could easily be adapted to Herbrand expansion to test the validity of quantified formulae whether those formulae were in prenex form or not. Van Heijenoort [171] is the published abstract of this result. Thereafter, van Heijenoort was a strong proponent of the falsifiability tree method, a method which, like its immediate precursors, united model theory with proof theory. Van Heijenoort's technical writings were all aimed at broadening and deepening the scope and capabilities of the tree method.

The tree method was first developed by Lis and presented by him in its first form in 1960 [118], based directly upon Beth's deductive and semantic tableaux. However, Lis's paper, published in Polish with brief English and Russian summaries that gave no hint that a new and simpler method than that found in Beth's semantic tableaux was being presented, has largely been ignored. The method was reinvented by Hintikka and Smullyan from Beth's semantic tableaux. This reincarnation differed from Lis's development by being based on the method of model sets developed by Hintikka [91–96] as well as Beth's semantic tableaux [28–31]. Hintikka's work on the model set method in propositional logic [97] already mentions the basic idea of the tree method.

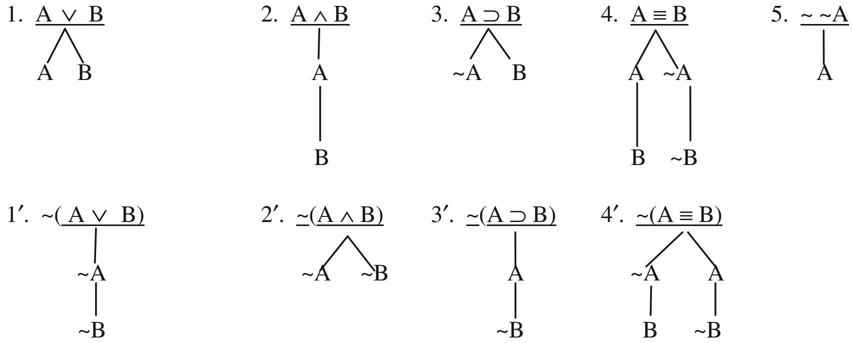
Hintikka's model set is a set of formulae that can be interpreted as a partial description of a model in which all formulae are true. A proof of a formula F in Hintikka's theory is a failed attempt to build a countermodel $\sim F$ to F. Beth's tableau method does much the same thing, except that, for the tableau method, a proof of a formula $F \rightarrow G$ is a failed attempt to build a countermodel of $F \rightarrow G$ by describing a model in which F is true but G is not.

For Beth's method, it is necessary to keep track of both true formulae and non-true formulae. Formulae are listed in tabular form; all true formulae and their derivations are collected in the left-hand column of the table, all non-true formulae and their derivations in the right-hand column. A *tree* is a one-sided (left-sided) tableau in which all formulae are true. There are a small number of tree decomposition rules, one for each of the truth-functional

[26] So-called because Zbigniew Lis developed a full-fledged analytic tableau or tree method by 1962 independently of Smullyan, and published his method in 1960 [118] at a time when Smullyan was just beginning his work. Bondecka-Krzykowska [33, p. 17] argues that it is difficult to tell whether Lis or Smullyan should receive credit for priority in developing analytic tableau (note that Bondecka-Krzykowska [33, pp. 17, 25] misspells my name as "Annelis".

[27] Smullyan [157] states that he had met van Heijenoort several times, but does not state whether they ever discussed the tree method.

connectives which we choose for our base, and one each for the universal and existential quantifiers.[28] Thus, for example, if we follow Jeffrey and provide decomposition rules for (disjunction, conjunction, material implication, material equivalence), along with the corresponding rules for the negation of these connectives, and double negation, we obtain the following rules:



For the universal and existential quantifiers, we introduce the rules:

$$\frac{\forall x G x}{G(x/\mu_0)}$$
$$\vdots$$
$G(x/\mu_k)$, where $\mu_0 \ldots, \mu_k$ are mutants (permissible substitution instances) of the bound variable,

and

$\dfrac{\exists x G x}{G(x/\nu)}$, where ν is a mutant of the bound variable provided ν is new to the path in which it occurs.

Tree decomposition rules may be applied to any formula which is nonbasic. A formula is called *basic* if it is atomic or the negation of an atomic formula, that is, if it contains no subformulae, and hence no connectives to which decomposition rules can be applied.

Let Φ be a set of formulae at the initial node of a tree. By application of tree decomposition rules to the nonbasic formulae of Φ, we obtain successor nodes, each containing some subformulae of (one of) the nonbasic formulae of Φ. A path of a tree, or sequence of such nodes, is *terminated* or *finished* if tree decomposition rules have been applied to every nonbasic formula in the path. A path is *closed* if there appears a formula F at some node n in the path and its negation $\sim F$ occurs at some successor node n' of n in the same path; otherwise the path is *open (nonclosed).* A tree is closed if each of its paths

[28] A *base* is the smallest set of connectives chosen for a deductive system and in terms of which the remaining connectives are defined. For example, the base in the first edition of the *Principia mathematica* [205] is $\{\sim, \vee\}$, the Sheffer stroke in the second edition [206]; and the base for Frege's *Begriffsschrift* [59] is $\{\sim, \supset\}$.

is closed. The tree for Φ is a *proof* of each formula at a terminal node of an open path in the tree for Φ. The set of all formulae in the open paths of the tree for Φ is a *satisfiability model* for Φ. By downward induction on the tree, if each of the formulae of Φ at the initial node of the tree is true, then so are all of the subformulae at each of the successor nodes of the tree, and so are each of the formulae at the terminal nodes. Moreover, by upward induction on the tree, if each formula at the terminal nodes of the tree are true, then so are all formulae at their predecessor nodes, and so too are the formulae at the initial node of the tree. Thus, if we obtain a tree in which some path contains both a formula F and its negation $\sim F$, so that the path closes, then we have derived a contradiction. We will make use of this fact to consider falsifiability trees that allow us to determine whether a formula or set of formulae is valid.

A falsifiability tree is a tree or sequence of trees in which we attempt to find a falsifying assignment for (a set of formulae) F. Let $F_0, F_1, \ldots, F_n, Q$ be the formulae of Φ, and let the sequence $F_0, F_1, \ldots, F_n$ be a proof of Q. Construct a new tree for either $F_0, F_1, \ldots, F_n, \sim Q$ or the negation of the entire sequence $F_0, F_1, \ldots, F_n, Q$ such that we have either the formula $\Phi' = F_0 \wedge F_1, \wedge \ldots \wedge F_n, \wedge \sim Q$ or the formula $\overline{\Phi} = \overline{F_0 \wedge F_1 \wedge \ldots \wedge F_n \wedge Q}$ at the initial node of the tree (thus, a proof can be understood as an "extended" formula obtained by the conjunction of each of the formulae, including the "Endformula" or conclusion, of the sequence; and a formula can be said to be valid or not, in this system, in precisely the same way that a proof is said to be valid or not). If, after application of tree decomposition rules to each of these formulae (and any other of their decomposable subformulae), each path of this new tree closes, then Φ' or $\overline{\Phi}$ is *inconsistent* and Φ is *valid.*

An *assignment* for a set S of formulae is a a function which, when we are given a nonempty set U called the universe of the assignment, associates either (a) an element of U to some atomic term of S; (b) a k-ary function ($k > 0$) of S to some k-ary functional symbol of U; (c) an element of the set of truth-values {t, f} to some propositional symbol of S; or (d) a k-ary function ($k > 0$) of U in {t, f} to some k-ary predicate symbol. We say that [the value of] an assignment α of truth-values to Φ is true (written $\mathrm{v}[\alpha, \Phi] = \mathrm{t}$) is *valid* if $\mathrm{v}[\alpha', \Phi] = \mathrm{f}$ for each related assignment α' and $\mathrm{v}[\alpha'', F_i] = \mathrm{t}$ for each formula F_i of Φ.

The falsifiability tree, then, is a mechanization of proof by contradiction. It is likewise, as van Heijenoort [167] called it, "the dual of that [method] presented in Jeffrey's *Formal Logic* [101]." Moreover, it is a test for validity of proofs.

The falsifiability tree method is *sound* if each provable formula in the system is valid and can be proven to be valid by the falsifiability tree method, i.e. if a formula is provable, then it is valid. The falsifiability tree method is *complete* if each valid formula or set of formulae of the system is provable by the method, i.e. if a formula is valid, then it is provable. In his unpublished papers distributed to his students, van Heijenoort proved the completeness and soundness of the falsifiability tree method for classical quantification theory and for intuitionistic logic.

6. Van Heijenoort's Work on the Falsifiability Tree Method

In 1973, van Heijenoort gave a proof of the soundness and completeness of the falsifiability tree method for sentential logic, that is, for classical propositional calculus [182]. This was followed by the paper "Falsifiability Trees" of 1974 [183], which gives a proof of the soundness and completeness of the falsifiability tree method for quantification theory, that is, specifically for classical first-order calculus (without identity). One consequence of van Heijenoort's completeness and soundness proof is the Löwenheim–Skolem theorem, for which van Heijenoort therefore was able to give a one-line proof. Van Heijenoort's proof makes use of König's lemma, also called König's infinity lemma, and the paper [183] also contains a proof of this lemma. For König's lemma, due to Dénes König (1884–1944) [106], a tree with a finite number of branches at each fork and with a finite number of leaves (or nodes) at the end of each branch is called a *finitely branching tree*. König's lemma then states that *a finitely branching tree is infinite iff it has an infinite path*. This lemma is used in completeness proofs. In connection with his proof of the soundness and completeness of the falsifiability tree method for quantification theory, van Heijenoort [184] also gave a proof of Willard Quine's (1908–2000) Law of Lesser Universes (see [135, pp. 146–147], Theorem 3.05 of van Heijenoort's typescript), according to which, *if* α and β *are two cardinal numbers such that* $0 < \alpha \leq \beta$, then if a formula is α-satisfiable, then it is β-satisfiable; and if a formula is β-valid, then it is α-valid (note that this Law holds only for those theories in which all formulae are in disjunctive normal form and in which there are neither universal quantifiers nor identity; hence, it fails to apply for full first-order functional calculus with identity). Several years earlier, in 1972, van Heijenoort had already given a proof of the soundness and completeness of the falsifiability tree method for the simple theory of types with extensionality [180].

In 1975, van Heijenoort gave his first proof of the soundness and completeness of the tree method for intuitionistic logic. In the case of intuitionistic propositional logic, we are shown in his paper on "The Tree Method for Intuitionistic Sentential Logic" [185] that a tree for an intuitionistic formula A is consistent if and only if A is classically provable; and that every nonconsistent ramified branch of a finished tree for an intuitionistic formula A yields a Kripke model which fails to satisfy A. The same reasoning is applied in the paper "The Tree Method for Intuitionistic Quantification Theory" [186], in which a proof of the soundness and completeness of the tree method for first-order intuitionistic logic is carried out by adding to the proof for intuitionistic propositional logic the three cases of T$\forall$ (true universal quantification), T$\exists$ (true existential quantification), and F$\exists$ (false existential quantification).[29] In his 1979 book *Introduction à la sémantique des logiques non-classiques* [195]

[29] In his lectures on foundations of mathematics, van Heijenoort declared that intuitionistic logic does not permit universal denial; therefore he did not have to consider the case of F$\forall$ (false universal quantification).

van Heijenoort gave more elegant proofs of the soundness and completeness of the tree method for intuitionistic logic.

Also in 1975, van Heijenoort once again turned his attention to the work of Herbrand (in [187]). We noted that in *Desarrollo* [189] van Heijenoort considered Herbrand quantification primarily from an historical context, although he there also made comparisons of the relative strengths and weaknesses of the various members of the family of formal systems called quantification theory. Among the weaknesses of Herbrand quantification were that it was not easily generalized to second-order logic, and that there are no simple results which allow us to obtain an analogue of the Herbrand Fundamental Theorem for arbitrary formulae of intuitionistic quantification theory. One of the main strengths of Herbrand quantification was that it permitted reduction, through Herbrand expansion, of quantified formulae, whether in prenex form or not, to propositional formulae. We recall that a major result of van Heijenoort's 1968 paper "On the Relation Between the Falsifiability Tree Method and the Herbrand Method in Quantification Theory" [167] was that Herbrand quantification could readily be adapted to validity tests by the tree method precisely because quantified Herbrand formulae could be rendered quantifier-free. Now in 1975, van Heijenoort, in his paper on "Herbrand" [187] examined in detail the technical apparatus of Herbrand expansion and gave a proof of Herbrand's Fundamental Theorem. A proof is also given in van Heijenoort's *nachgelassene* paper "Proof of the 'Semantic' Herbrand Theorem" [192] of 1976.

The Fundamental Theorem states that:

Given a formula F of classical quantification theory, we can effectively generate an infinite sequence of quantifier-free formulae $F_1, F_2, \ldots,$ such that F is provable in (any standard system of) quantification theory if and only if there is a k such that F_k is (sententially) valid; and moreover, F_k can be recovered from F through certain special rules.

Now if we analyze Herbrand's theorem, we notice that its main connective is the biconditional, so that it can therefore be reduced to two independent statements: that F is provable in standard quantification theory; and that the Herbrand expansion of a formula F to an infinite sequence of quantifier-free formulae is valid for each formula F_k of that sequence. Van Heijenoort uses this analysis to obtain his proof of the soundness and completeness of Herbrand quantification. Given the statements:

(1) *F is valid*

and

(2) *there is a k such that the kth Herbrand expansion of F is (sententially) valid*

van Heijenoort shows that the implication from statement (2) to statement (1) is the soundness of Herbrand's proof procedure, and that the implication from statement (1) to statement (2) is its completeness. The conjunction of (1) and (2) is called the "semantic" Herbrand theorem. Elsewhere in the manuscript notes on Herbrand [202], on a page labeled H-12, van Heijenoort remarked that "implications is [sic] sometimes spoken of as the 'semantic' Herbrand Theorem".

Herbrand expansion, according to which we obtain a quantifier-free formula by obtaining a k-length conjunction from a universally quantified formula and a k-length disjunction from an existentially quantified formula, where we have a k-ary universe, is in fact just an enlargement of the Löwenheim–Skolem infinite conjunction presented by Skolem in his normalform translations of Hilbert's quantified formulae. What led Herbrand to develop his method was precisely his dissatisfaction with the Löwenheim–Skolem theorem, which asserts (if I may express it in its simplest terms) that if a formula of classical quantification theory is k-satisfiable for every finite k, then that formula is $\aleph_0$-satisfiable. What disturbed Herbrand, as we hinted earlier, was that this theorem was restricted to satisfiability. Herbrand's Fundamental Theorem was, in the words of Herbrand in "Sur la non-contradiction de l'arithmétique" ([83], p. 4, quoted in English translation by van Heijenoort [164, p. 526] in his "Introduction" to Herbrand's *Recherches* [81]), a "more precise statement of the... Löwenheim–Skolem theorem," and thus can be viewed, as van Heijenoort [164, p. 526] noted, "as a reinterpretation, from the point of view of the Hilbert program, of the results of Löwenheim and Skolem." In fact, what Herbrand did was to permit us to state that if a formula is $\aleph_0$-valid, then it is k-valid for every finite k, provided there exists no countermodel to that formula. Consequently, it is thanks to van Heijenoort's work on Herbrand that we are able to argue that the technical developments in Hilbert-type systems, including the development of proof theory by Hilbert and Bernays, as well as of the development of alternative theories of quantification, are due primarily to questions raised by Herbrand about the Löwenheim–Skolem theorem from the point of view of the Hilbert program.

Van Heijenoort's historical interests and his technical work in expanding and developing the tree method coincide, and focus on the need to define and explore the concepts of *satisfiability, validity,* and *being a proof.*

In 1978, van Heijenoort further extended his results of 1975 ([185] and [186]) in which he applied the falsifiability tree method to intuitionistic logic and proved the soundness and completeness of the tree method for intuitionistic logic. To this he added an application of the tree method to propositional and first-order modal logic, together with a proof of the soundness and completeness of the tree method for modal logic. At the same time, he suggested, but did not carry out, the possibility of applying the tree method to two variants of three-valued logic. These new results on intuitionistic logic, modal logic, and three-valued logic were published in 1979 in van Heijenoort's booklet *Introduction à la sémantique des logiques non-classiques* [195]. It represents the only formal publication of a proof by van Heijenoort of the soundness and completeness of the tree method. The first chapter of this work, which considers classical logic, contains, in much more sinewy form, parts of the same materials found in van Heijenoort's earlier, unpublished, technical papers, in particular from the paper on "Falsifiability Trees" of 1974 [183], although the material and their presentation are far from identical.

7. Sketch of Van Heijenoort's Proofs of the Soundness and Completeness of the Falsifiability Tree Method

Van Heijenoort's proofs of the soundness and completeness of the tree method are far more rigorous than the intuitive, informal proofs given in Jeffrey's textbook *Formal Logic* [101]. Van Heijenoort's proofs employ the same concepts and follow the same patterns as do the proofs presented by John Lane Bell and Moshé Machover in their textbook, *A Course in Mathematical Logic* (1977) [23], although van Heijenoort's proofs are somewhat longer and require more bookkeeping, in part because van Heijenoort proofs, unlike those of Bell and Machover, do not make explicit use of Hintikka sets. Of course, van Heijenoort's proofs, although for the most part unpublished, may predate by several years the published proofs found in Bell and Machover's [23] book or van Heijenoort's own published proofs in his *Introduction à la sémantique des logiques non-classiques* [195].[30]

By way of example, we can give a simplified sketch of van Heijenoort's 1973 proof of the soundness and completeness of the falsifiability tree method for propositional logic [182].

Assume that we have both downward and upward induction on the tree. Suppose the following theorem has already been proven.

Theorem 1. *If T is a falsifiability tree for a formula F and there is an assignment α of truth-values to the sentential variables of F such that $v\,[\alpha, \sim F] = t$, then there is a formula G occurring at a node of a branch β of T such that $v[\alpha,\ G] = t$.*

We now prove *soundness.*

Theorem 2. *If there is a closed falsifiability tree for a formula, then that formula is valid.*

Proof. Let T be a closed falsifiability tree for a formula F. Assume that there is an assignment α such that $\mathrm{v}[\alpha, F] = \mathrm{f}$ or $\mathrm{v}[\alpha, \sim F] = \mathrm{t}$. By the previous theorem, there is then a branch β of T such that, if formula G is at a node of β, then $\mathrm{v}[\alpha, G] = \mathrm{t}$. Since T is closed, so is β, and there is a formula H such that both H and $\sim H$ are at nodes of β, so that we have obtained the contradiction $\mathrm{v}[\alpha, H] = \mathrm{t}$ and $\mathrm{v}[\alpha, \sim H] = \mathrm{t}$. □

Now we prove *completeness.*

Consider the following theorem.

[30] Paul Ernest, who was a student of Moshé Machover, pointed out [48, p. 124] that in fact Machover provided his own students with typescripts of what would become his and John Lane Bell's [23] textbook before the end of 1973 and through March 1974, and that these included the completeness and soundness proofs for the confutation tableaux. Hence they are at least contemporaneous with van Heijenoort's proof of 1973 [182], dated 23 September 1973; but Ernest goes on to speculate that Machover's and Bell's work was "presumably written over the preceding years, reaching their final form before or at about the time I received them," thereby suggesting that their proofs predated van Heijenoort's.

Theorem 3. *If there is a nonclosed finished falsifiability tree for a formula F, then there is an assignment α such that $v\ [\alpha, F] = f$.*

Proof. The nonclosed finished tree for F has a nonclosed branch β. Let α be an assignment of truth-values to the sentential variables of F such that if p is a basic formula occurring at a node of β, then $\mathrm{v}[\alpha, p] = \mathrm{t}$, and if $\sim q$ is a basic formula occurring at a node of β, then $\mathrm{v}[\alpha, q] = \mathrm{f}$. By upward induction on the tree, we then have either $\mathrm{v}[\alpha, \sim F] = \mathrm{t}$ or $\mathrm{v}[\alpha, F] = \mathrm{f}$.

If there is a node n of β at which the formula is not basic but T is finished, then we apply the appropriate tree decomposition rule to the formula at n. By doing so, we obtain one or two successor nodes of n containing subformulae of the formula at n. By induction on the tree, these subformulae are true for assignment α. □

Now by contraposition on Theorem 3, we obtain

Theorem 4. *If a formula F is valid, then there is a closed tree for F,*

which is the completeness theorem for trees. □

8. Conclusion

The most obvious instance of van Heijenoort's interest in Herbrand is his edition of 1968 Jacques Herbrand, *Écrits logiques* [85]. An English translation of van Heijenoort's 1968 edition of Herbrand's writings was edited by Warren D. Goldfarb [86], one of van Heijenoort's closest collaborators. Goldfarb's book received a good review at the hands of Jean-Pierre Bénéjam [25], who, in 1975, in particular also noted van Heijenoort's sound critical comments. The interest in Herbrand carried through to the end of van Heijenoort's life, and when he died he was working with Claude Imbert (b. 1933), Professor of Philosophy at the École Normale Supérieur, and with Burton Dreben on a complete edition of Herbrand's logical and mathematical works.[31]

It had been hoped that a collection of van Heijenoort's technical papers on quantification theory and the tree method, whether unpublished or merely distributed to students and a few colleagues, would have been published by this time, so that his work will become better known and receive the attention which it deserves and which has already been accorded to his historical writings. Brief evaluations of his work, especially of his work on quantification theory and the falsifiability tree method, have already been given in [9,10] and more detailed treatments are found in [13,14,16], and [51,52] provide a general overview; [50, pp. 371–390] is a survey of van Heijenoort's academic work and career.

[31] See Feferman [46, p. 235; 44, p. 383]. This project, sponsored by the French Ministry of Culture, and behind which van Heijenoort was the driving force, has apparently been completely discontinued since van Heijenoort's death.

Appendix: Proof-Theoretic and Related Writings of van Heijenoort's in the *Nachlaß* [160, Box 3.8/86-33/1] (Exclusive of Research Notes and Unfinished Work)

Interpretations, satisfiability, validity, 1966
The set-theoretic approach to logic, 1966
The subformula approach to logic, 1966
Note on Herbrand, January 28, 1967
Un procédé de démonstration en calcul des predicats, January 15, 1968
Finitisme et infinitisme en logique, January 18, 1968
On the relation between the falsifiability tree method and the Herbrand method in quantification theory, November 20, 1968
Notes on the tree method, October 15, 1971
Notes on logic, October 15, 1971
Notes on logic, November 1, 1971
Herbrand, 1972
Proof of the completeness of the method of Beth tableaux for intuitionistic quantification theory, 1972
Proof of the completeness of the method of Beth tableaux, March 1, 1972
Falsifiability trees, March 15, 1972
The tree method for intuitionistic quantification theory, March 18, 1972
The tableaux method for Grzegorczyk quantification theory, March 18, 1972
The tree method for intuitionistic quantification theory, March 19, 1972
The falsifiability-tree method for the simple theory of types with extensionality, July 23, 1972
Comparison between the falsifiability-tree method and the Gentzen system, May 3, 1973
A nonnormalizable system, June 8, 1973
Soundness and completeness of the falsifiability-tree method for sentential logic, September 23, 1973
Falsifiability trees, 30 September, 1974
Monadic translation of sentential modal logic, October 31, 1974
The tree method for sentential modal logics, November 3, 1974
Translation of modal quantification theory into nonmodal quantification theory, November 15, 1974
The tree method for intuitionistic quantification theory, May 9, 1975
Notes on logic, 1975
The falsifiability-tree method for sentential intuitionistic logic, May 15, 1975
Herbrand, May 18, 1975
Falsifiability trees, September 30, 1975
Translation of modal quantification theory into classical second-order logic, November 20, 1975
The Herbrand approach to predicate logic, 1976
Proof of the 'semantic' Herbrand theorem, 1976

Herbrand expansions and Herbrand disjunctions, 1976
Axiomatization and formalization, 1976
Méthode des arbres de falsifiabilité pour les logiques modales propositionnelles, November 20, 1976
Gödel, 1977
Hilbert, 1977
Monadic translation of sentential modal logic, February 12, 1977
Modèles pour les logiques modales quantifiées, March 13, 1977
Introduction à la sémantique des logiques non-classiques, 1979
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On Herbrand's systems, November 29, 1980

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