Cassirer's Functional–Based Approach in the Reconstruction of the Early Quantum Theory

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ABSTRACT

I propose a rational reconstruction of the early quantum theory (1900–1913) in terms of the ideas presented by Ernst Cassirer. Specifically, I propose to reconsider the early quantum theory through the lens of the method of conceptual functionalization that Ernst Cassirer laid down in his *Substance and Function* (*S&F*, 1910) and he later refined in *Determinism and Indeterminism in Modern Physics* (*D&I,* 1937). Following Cassirer's functional interpretation of natural sciences, it is my primary concern to reconsider the conceptual evolution of Planck's quantum of action from 1900 to 1913. In this regard, I shall emphasize the importance of the quantum of action (Planck's constant) in the architectonic structure of the early quantum theory, in the role of an element of fundamental continuity between the first quantum theory and the formulation by Niels Bohr of the first atomic theory.

WORK TYPE
Article

ARTICLE HISTORY
Received:
30–March–2018
Accepted:
20–June–2018

ARTICLE LANGUAGE
English

KEYWORDS
Functional–based approach
Neo–kantism
Planck
Einstein
Bohr

NOTES ON CONTRIBUTOR

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HOW TO CITE THIS ARTICLE

Angeloni, Roberto (2018). «Cassirer's Functional–Based Approach in the Reconstruction of the Early Quantum Theory». *Disputatio. Philosophical Research Bulletin* 7, no. 8: a012.

Cassirer's Functional–Based Approach in the Reconstruction of the Early Quantum Theory

Roberto Angeloni

§1. Introduction

I HERE PROPOSE A RATIONAL RECONSTRUCTION of the early quantum theory (1900–1913) in terms of the ideas presented by Ernst Cassirer, in order to clarify some fundamental aspects of his epistemology and philosophy of physics. Specifically, I propose to reconsider the first quantum theory through the method of functionalization of concepts that Ernst Cassirer laid down in his *Substance and Function* (1910) (heretofore S&F), and through the epistemological interpretation of quantum physics that he gave in *Determinism and Indeterminism in Modern Physics* (1937) (heretofore D&I).

To begin with, I would like to invite the reader to reflect on the concept of constant of nature, one of the less explored and less understood aspects of Cassirer's philosophy of physics. As Cassirer declared:

> The historical development of modern physical theory throws a clear light on the progress from "individual constants" to "universal constants", as one of the most important and fruitful factors in the whole process of scientific cognition (Cassirer 1957, 444).

In the third volume of the *Philosophy of Symbolic Forms*, Cassirer dealt with the case of modern spectroscopy, specifically the law that Johann Balmer had established in 1885 for the hydrogen spectrum.

By reconsidering the history of spectroscopy, Cassirer noted that Balmer's formula appeared as a special case of a universal law. In particular, the constant R (Rydberg's constant) that appeared in Balmer's series, at first it was considered as peculiar to the hydrogen, but it later became a universal constant. Cassirer was aware that the role and the relationships between Balmer's law and the constant R (in the formulation given by Rydberg and Ritz) could be fully understood only in the framework of the quantum theory and linked to the universal magnitude h, that is, Planck's quantum of action[1].

[1] Planck introduced the notion of "elementary quantum of action" with the following words: "I want to designate this [i.e. the constant h] as the elementary quantum of action [elementares Wirkungsquantum] or as "element of action" [Wirkungselement], because it has the same dimension as the quantity which owes its name to the Principle of Least Action (Planck 1906, p. 154).

Disputatio. Philosophical Research Bulletin Vol. 7 | No. 8 | Dec. 2018 | a012 | ISSN: 2254-0601 | www.disputatio.eu

In *D&I*, Cassirer explicitly recognized that Planck's constant represents a real 'firm and secure point' of modern physics, and all considerations over the quantum theory presuppose the universal validity of this magnitude:

> The elementary quantum of action constitutes, as it were, the fixed frame, into which all statements of quantum theory are fixed; and the security and firmness of this frame alone ought to be sufficient to protect the indeterminism of the theory against those speculative interpretations to which it was exposed in the transition from physics to general conclusions concerning man's Weltanschauung (D&I, 121– 122).

In this respect, the universal constants of nature appear as the starting point of Cassirer's epistemological reflection on scientific knowledge. As a result, we should take the quantum of action as the starting point of the epistemological reflection on the quantum theory[2]. But, in Cassirer's conception, the quantum of action cannot be regarded as an element of the "world of facts", which would be accessible by immediate sensation; for Cassirer, Planck's quantum is rather a part of the "world of physical objects" (Cassirer 1957, 21). The reason being that a constant of nature as a "physical object" can only be determined through a "highly complex intellectual process of interpretation" that constitutes the essence of a physical theory (ibid).

It is my primary concern in the present paper to reconstruct the intellectual process underlying the interpretation of the elementary quantum in the early stages of its conceptual development. Starting from Cassirer's understanding of Planck's *h*, and in light of Cassirer's functional view, I will lay emphasis on the principle of quantization in the role of principle of coordination of the quantum theory, as a conceptual function springing out from the original universal constant of the quantum of action.

As is well known, Cassirer sought to modify the Kantian transcendental doctrine of pure intuition in response to the developments in mathematics and non–Euclidean geometry during the period ca. 1870–1920. In light of such transformations, Cassirer no longer held to the Kantian *a priori* pure forms of intuitions (space and time), but he recast them along the line of the neo–Kantian doctrine of the Marburg School[3], for which space and time are concepts. In particular, Cassirer reconsidered the process of concept formation by purging intuitive or "visualizable" evidential content, and on that he went to elaborate a new abstract form of relation between the concept and its object: that is, the principle of coordination otherwise said as the relation of *Zuordnung*. Specifically, for Cassirer, the principle of coordination (or functional coordination) is a transcendental relation (in the revised version of Marburg neo–Kantianism) of abstract one–to–one correspondence or coordination between series, in conformity with the formulation of the modern concept of function.

2 Cassirer clearly held that the 'elementary action quantum' is "the fundamental concept of the quantum theory" (Cassirer 1957, 474, ft. 85).

3 In addition to Cassirer, other prominent representatives of the Marburg School were Hermann Cohen and Paul Natorp. In general, neo–Kantians downplayed the role of intuition in favor of concepts.

As far as the principle of quantization is concerned, Cassirer remained obscure on its definition, although the idea is widespread among specialists (Ryckman 2015) that the so–called Planck–Einstein–de Broglie relation $E = h\nu$ of proportionality between energy and frequency (where h is Planck's constant) represents the "Archimedean point" of the quantum theory, to the extent that both Planck's energy quanta and Einstein's light quanta were associated by this same equation that Cassirer identified with the principle of coordination of the theory.

Along the principle of quantization, I will then show that from the introduction of Planck's h in 1900, there followed several stages of functionalization of concepts in the early development of quantum theory: Planck's first radiation law (1900), Einstein's method of discrete quantization (1906), Einstein's theory of the specific heats of solids (1906), and the quantization of the hydrogen atom by Niels Bohr (1913).

In so doing, the early quantum theory will be refashioned into a "functional structure" which Cassirer implicitly referred to in *D&I*, without carrying out, though, a proper analysis of the role that each *statement of the results of measurements*, *statement of law*, and *statement of principle* may play in a functional reconstruction of the early quantum, which, in my view, is necessary to understand *D&I*.

For instance, Cassirer claimed:

> Totally different and apparently heterogeneous groups of phenomena are brought under the [quantum] principle: the theories of heat radiation —of the photo— electric effect and of specific heats —are brought together, and interpreted in a new manner by this principle (D&I, 110).

In another passage, Cassirer observed:

> All particular certainties as they are given for instance in Planck's law of radiation, in the Balmer's series of the hydrogen spectrum, in the formula for the atomic heat, etc. always lead back to this general certainty [the quantum of action] (D&I, 121).

First, it is my claim that a functional reconstruction of the early quantum theory in light of *S&F* may contribute to clarify the epistemological interpretation of quantum physics that Cassirer offered in *D&I*, with particular reference to the notion of the general principle of causality. Cassirer, indeed, regarded the general principle of causality as the most fundamental concept of his epistemological reflection. As it will be shown, Cassirer's critical conception of causality can only be explained and understood in a functional perspective.

Second, the present historical reconstruction of the quantum theory will provide an effective example of Cassirer's *genetic view* of knowledge[4], according to which a theory's constructive presupposition (the concept) may neither be imposed on its "empirical basis"

[4] This epistemological approach characterized all the members of the Marburg School.

(the object) nor be derived from it. For Cassirer, the so–called "physical object" (in the process of advancement of scientific knowledge) can only be conceived as a "limiting concept" (S&F, 228) that will never be conclusively determined. Following this conception, as Friedman (2000) pointed out, we will arrive at the conclusion that

> There is thus no 'pre–conceptual' manifold of sensations existing independently of pure thought at all. There is only an infinite methodological series in which the forms of pure thought are successively and asymptotically applied (p. 31)[5].

By adopting Cassirer's functional–based approach, the early quantum theory shall be presented as constituted of a rational and progressive series of theories, with respect to which, the discovery of Planck's constant and the first theory of radiation (1900), Einstein's rule of quantization (1906), Einstein's theory of the specific heats of solids (1906), the quantization of the hydrogen atom by Niels Bohr (1913) play the role of functional nodes at which the theory's scientific propositions intersect along the axis of the above mentioned principle of quantization.

Furthermore, a conceptual shift characterizes the transition from the earlier level of the theory onwards, that is —in Cassirer's phrasing— from a statement of the results of measurements (Planck's constant) to a statement of law (Planck's radiation theory), and from this to statements of principle (i.e. Einstein's rule of quantization, the theory of the specific heats of solids, and Bohr's quantized atom), that is, the assumptions concerning the quantization of *energy*, *radiation*, *material entities*, and *atomic structures* which constitute the essence of the early quantum revolution.

Unlike Thomas Kuhn, Cassirer's functional–based approach upholds the claim that conceptual changes do take place in science in a rational and progressive way, in the sense that each new stage, which is connected to the preceding one, constitutes a new advancement towards the limiting physical concept, which will never be fully realized in experience.

Before starting this historical detour, I will give an exposition of Cassirer's functional approach in order to familiarize the reader with his epistemology and philosophy of physics.

§2. Ernst Cassirer's functional–based approach

One important characteristic permeates the whole Cassirer's philosophy of physics: the functional coordination among theory concepts (and among theories). According to Cassirer, functionality is the mutual coordination of one thing to another, and it is a necessary feature of the process of knowledge: if we consider the whole of experience, this whole —Cassirer stated— "is never a mere aggregate of perceptual data", but it is divided and unified according to definite theoretical points of view" (S&F, 267). For Cassirer, such points of view correspond

5 In fact, Cassirer's epistemology is known as "logical idealism".

to a presupposed order among the manifold of experience. Let's see how to arrive at this order.

In *S&F* Cassirer identified two fundamental moments in the process of induction which are worthy of consideration for the construction of scientific concepts: "the gaining of particular 'facts'" (S&F, 265) *and* the connection of these facts into laws[6]. For any kind of assertion concerning facts, we need a presupposition, that is, a hypothesized *relation* among those basic facts. The question of functionality thus concerns the *extent* and the *validity* of these relations.

With regard to the *extent* of the process of functionalization of concepts, in *S&F* Cassirer formulated the so–called method of super–ordination and subordination, which clearly explains the totality of empirical knowledge in terms of a function, which reproduces the characteristic relation according to which we conceive of the individual members as arranged in mutual dependence:

> Two fields of phenomena A and B are first united each according to a particular law $\psi_1(\alpha_1, \alpha_2, \alpha_3)$, $\psi_2(\beta_1, \beta_2, \beta_3)$; these laws are again connected among themselves by a new relation $\phi(\psi_1, \psi_2)$ until we finally reach the most general relation, which ascribes to each individual factor its definite place with regard to the others. The fundamental form F is analyzed for thought into a structure of mutually dependent determinations, which would be symbolically represented, for example, by an expression F [$\phi_1(\psi_1, \psi_2), \phi_2(\psi_3, \psi_4), \phi_3 \ldots$] (S&F, 267).

The function F hence represents the totality of empirical knowledge at any point of time. This expression describes a system of elements (viz. phenomena), arranged in a "mutual relation of super–ordination and subordination" (ibid.), from which arises a "complex whole of overlapping synthesis" (ibid.). The problem concerning the quality or the condition of these relations introduces the second point of our discussion with regard to functionality: the *validity* of relations in the process of functionalization of concepts. The question at issue is: under what conditions a relation among concepts, along the process of subordination and super–ordination, may be (mathematically/empirically) defined.

According to Cassirer, the *validity* of a connection is given by the conformity of this connection to law. This is an issue that links to the most fundamental concept of Cassirer's epistemological reflection: the general principle of causality.

6 Cassirer held that the processes of concepts formation in mathematics and physics have a common origin (which is identified with the "activity of constructing limit concepts"), although such processes are not identical (Mormann 2015, 35). This dual aspect of the concept formation has been remarkably described by Friedman (2000): "Cassirer's *Substance and Function*, for example, retains an element of duality between pure thought and empirical reality, namely, the contrast between the pure relational structures of pure logic and mathematics, on the one side, and the historical sequence of successor theories representing the methodological progress of empirical natural science, on the other" (p. 76).

In Cassirer's view, the general principle of causality does not refer to objects, but rather "to our cognition of objects in general" (D&I, 58). Cassirer conceived of causality neither as a subjective necessity nor as a constitutive *a priori* principle, but as a regulative *a priori* principle[7]. Causality is hence an ideal end for seeking a "never fully realizable complete system of knowledge of nature" (Friedman 2000). This teleological feature characterizes the concept of causality, which in this respect constitutes an overarching *relation* among conceptual objects such as *measurements*, *laws*, and *principles* (D&I, 60), as it refers to the *way* through which these statements should be combined with one another. Therefore, we are entitled to claim that the general principle of causality firstly is a method and a "postulate of empirical thought":

> What the causal principle signifies —and this is the thesis I want to explain and establish in the sequel— is not a new insight concerning content, but solely one concerning method (D&I, 60).
>
> […] With regard to content, it [the general causality principle] does not go beyond what has already been observed; it only confirms it and confers upon it as it were the epistemological *imprimatur*. In this sense it belongs, using the language of Kant, to the modal principles; it is a postulate of empirical thought (ibid).

As is well known, the principle of causality over and over was subjected to different interpretations. In particular, the traditional principle of causality and the principle of continuity had been closely related from time to time. In classical physics, causality and continuity were so closely interwoven that it became difficult to distinguish them.

Let me recall Kant's two fundamental formulations of the principle of causality[8]. Both were spelled out in the first edition of the *Critique of Pure Reason* (1781): one formulation can

[7] As is well known, Cassirer rejected the Kantian distinction between faculty of sensibility and faculty of understanding, as a consequence, the distinction between constitutive *a priori* principles (which arise from the application of the intellectual faculties of understanding and reason to the faculty of sensibility) and regulative *a priori* principles (which arise from the intellectual faculties themselves without such an application) is also untenable (see Friedman 2000). In conformity with the neo–Kantian School of Marburg, Cassirer went to retain only the regulative principles, which remain indeterminate by definition. For this reason, causality, as a regulative principle, also remains indeterminate, because it cannot be fully realized in experience, but it is oriented towards the limit of an unrealizable complete system of knowledge (cf. Friedman 2000, 76).

[8] In *S&F* Cassirer placed emphasis on the twofold form of causality: the "scientific" and the "historical construction of concepts":

«[Causality] signifies nothing else than the "necessity in the temporal sequence of the parts of reality"; we must postulate such a necessity also where we are concerned with the succession of purely individual events, which can thus never recur in precisely the same way. The specifically "historical causality" is founded on the application of this point of view» (S&F, 226 –227, ft. 85).

Along with the historical conception, Cassirer exposed the "scientific conception" of causality as conformity to law: «Causality, which natural science affirms and makes the basis of its explanations, be found in the *Second Analogy* of the "Transcendental Doctrine of the Power of Judgment". This can be regarded as the general standard formulation of causality, and it refers to the conformity to law: "Everything that happens, that is, begins to be, presupposes something upon which it follows according to a rule" (Kant 1781, A 189; from D&I, 162).

The second formulation also is stated in the same edition of the *Critique*, precisely in the "Deduction of the Pure Concepts of the Understanding", and it introduces two conditions: the first is the connection of causality with time; the second condition is continuity.

Time is the fundamental condition for the "succession of the manifold" being described in terms of the cause–effect scheme. Further, the condition of time introduces the condition of continuity, which is essential to explain the "metaphysical" causal law that Cassirer rejected, and to better grasp the difference with his methodological use of causality as the general characteristic of order according to law. A critical theory of causality does not contain any references to the connections between "things" and "events" —Cassirer argues— but a statement about "the systematic interrelation of cognitions" (D&I, 163).

By contrast, for Kant, an application of the category of understanding to appearances becomes possible by means of the transcendental time–determination (Kant 1781, A 139, 272), which involves the principle of continuity. Cassirer, on his side, could not accept a time–dependent concept of causality, having well in mind the latest developments in physics, such as the consequences of the formulation of Einstein's theory of special relativity, which undermined space and time as *a priori* pure forms of sensuous intuition, and the discovery of Planck's quantum of action, which undermined the assumption that all physical processes could be reduced to changes occurring continuously in time (Planck 1910, 239).

In Cassirer's view, one can hence give up a space–time representation while maintaining the concept of causality (Sundaram 1972; Ryckman 1991, 2015).

In the wake of the nineteenth century program concerning the formulation of the modern concept of function, Cassirer purged the concept of cause of any visualizable content, expunging time and continuity from his "scientific conception" of causality. In so doing, the relation between concept and sensuously presented content (the *phenomena*, in Kant's phrasing) was replaced by a new abstract relation between theory and experience (concept and its object), which aims to stress the consistency of a conceptual system, that is the relation of functional coordination or *Zuordnung*[9]. To say it with Thomas Ryckman's words:

> […] Once the relation of theory to sense experience was solely one of abstract correspondence or *Zuordnung*, methodological constraints or principles of theory choice gained new prominence, e.g.,

can be reduced to the idea of universal lawfulness. According to this view, to conceive an event causally means to subsume it under general laws» (ibid, 226).

9 According to Ryckman (1991), the concept of *Zuordnung* can perhaps be attributed to Dirichlet (1837), who sought to specify the concept of "arbitrary function". A special case of *Zuordnung* is *ein–eindeutig*, a one–to–one coordination or correspondence.

> *Eindeutigkeit*, i.e., the "univocality" of the assignment of (the system of) symbols to (the system of) sense experience, "simplicity", "unity", and fewest "arbitrary elements" (Ryckman 1991, 60).

This is a very important passage, for the reason that *Zuordnung* is the key concept for explaining the meaning and role of the general principle of causality in terms of conformity to rule. For Cassirer, the role of functional coordination (*Zuordnung*) is twofold: on the one hand, "it characterizes the "law" (*Gesetz*) of a series, connecting its individual members" (Ryckman 1991, 61). Cassirer identified this role with the *rule of progression* of a series when he analyzed the ordinary schema of the construction of concepts:

> The connection of the members of a series by the possession of a common "property" is only a special example of logically possible connections in general. The connection of the members is in every case produced by some general law of arrangement through which a thoroughgoing rule of succession is established. That which binds the elements of the series a, b, c, … together is not itself a new element, that was factually blended with them, but it is the *rule of progression*, which remains the same, no matter in which member it is represented (S&F, 17).

On the other hand, functional coordination is "the relation of dependency or comparison between the members of distinct series, notably including the abstract and non–depictive relation of a system of concepts (or theory) to the manifold of sense perception" (Ryckman 1991, 61). As Cassirer stated:

> Thus if we view the totality of empirical knowledge at any point of time, we can represent it in the form of a function, which reproduces the characteristic relation according to which we conceive the individual members arranged in *mutual dependency* (S&F, 267).

In particular, the relation of functional coordination, in the role of "functional dependency", does help us clarify the nexus between the question regarding the *validity* of relations in the process of functionalization of concepts and the neo–Kantian stance on the general principle of causality. By adopting the conception of causality as conformity to law, we will get that the *validity* of each logical connection at different levels of a series concept —along the process of super–ordination and subordination— is given by the conformity of these connections to the *relation of dependency* of such a series. The "limit result" will be the "asymptotical" combination of the *valid* logical connections within such a system, the system of physical knowledge.

For instance, in the expression $F\ [\Phi_1\ (\psi_1, \psi_2), \Phi_2(\psi_3, \psi_4), \Phi_3]$, the symbols Φ_1, Φ_2, Φ_3 represent the logical connections at the higher level of the function F, while the symbols ψ_1, ψ_2 and ψ_3, ψ_4 stand for the connections at a lower —less general— level of functional relations within the series. The "rule of progression" accounts for the relations among the members within Φ_1, Φ_2, etc., as well as among the members within ψ_1, ψ_2 and ψ_3, ψ_4, for which Φ_1 and

Φ_2 are respectively rules of progression; whilst the "relation of dependency" accounts for the relations among the members of Φ_1 and Φ_2 as well as among those of ψ_1 and ψ_3, ψ_2 and ψ_4.

The resulting connections between the relation of dependency and the rules of progression can be regarded —by using a neo–Kantian phrasing— as "contextually *a priori* statements", as far as each relation is *necessary* only at a certain level of the series concept, through their conformity to the principle of coordination (in the role of relation of dependency) of the series.

Cassirer began to elaborate his functional–based approach to theory development since *S&F*, although, it was in *D&I* that he was able to complete his effort, highlighting the dynamical character of the process of theory development. Indeed, Cassirer in *D&I* clearly characterized the dynamical and holistic process of transformation of scientific concepts in terms of a "hierarchy of scientific propositions":

> If we choose a spatial analogy for the structure of physics, we must not liken this structure to a pyramid resting on a broad base of immediately given and independent "facts", rising gradually from this and ending in a highest point, perhaps in a simple "cosmic formula" […] Physics accordingly is to be compared not to a pyramid but insofar as we regard any spatial symbol as adequate and permissible, to the "well–rounded sphere" with which Parmenides described his "universe" (D&I, 35).

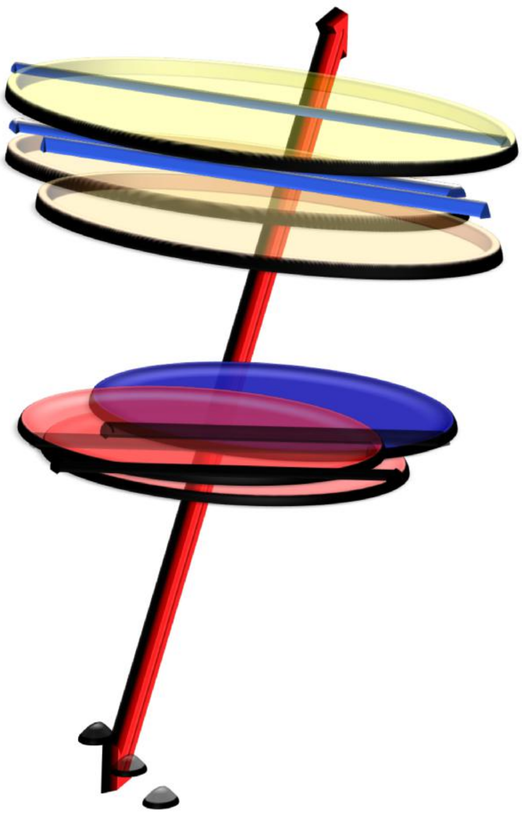
As it is evident, physics is not a static system in which each level remains in isolation from one another, but a conceptual shift or translation characterizes this system of knowledge from "facts of science" (statements of results of measurements) to statements of laws, and from these to statements of principles.

The point —Cassirer remarked— is to identify the logical process through which "the transition between the various types of physical statements take place" (D&I, 39) as well as "the rule by which thought may be guided from one law to the next" (D&I, 45).

Cassirer pointed out that this process (of transition or "change of dimension", *D&I*, 42) is a "new fundamental form of induction" that cannot be confused with the inductive process "from particulars to particulars" (D&I, 45). This new form of induction is but a "methodological demand", for a thoroughgoing unity of the physical view of the universe" (D&I, 47), which underlies the formulation of any law and principle of nature. But it is the functional coordination among all the elements of physical knowledge that allows scientific statements to take part in the advancement of knowledge, ascribing to them a *structural* unity. Since there is a lower and a higher level in such a structure, these levels are nonetheless separated as from one another, provided that there is an essential interconnection among scientific concepts, which is guaranteed by the functional coordination of the series. As I see it, if we want to grasp the essence of the logic underlying the process of theory development prospected by Cassirer in *D&I*, we have to take up the procedure of functional coordination among theory concepts as it was originally laid down in *S&F* but that Cassirer omitted in *D&I*.

Several attempts have been proposed to endow Cassirer's conceptual scheme with a geometrical representation, but even the one offered by Cassirer himself (the "well–rounded" *Parmenidean* sphere) does not seem to help us understand such a complex structure. For the sake of clarity, I here propose a new geometrical representation:

Diagram 1.



Statements of principles.

(*Different levels of generality*).

Statements of laws.

Statements of results of Measurements.

The vertical axis represents the "relation of dependency" *among* the levels of a series concept. The horizontal axes represent "the rules of progression" *at* each level of a series concept. The intersection between the "relation of dependency" and the "rules of progression" gives rise to (*valid*, that is, contextually necessary) logical connections at each level along the process of super–ordination and subordination of the series. The lower stage represents the "statements of results of measurements". The statements of measurements are individual, as they all possess a definite point of space and time to which they are bound. They can also be designated as statements of the "first level", as far as they constitute the basis of the system of knowledge:

> The statements of the results of measurements may indeed be designated as the alpha and omega of physics, its beginning and the end. From them all its judgments take their departure and to them they must all lead back again (D&I, 36).

Nevertheless, we cannot think of the basis of the system of physical knowledge as constituted of a mere aggregate of facts. This system is primarily based on "concrete determinations provided by the statements of the results of measurements" (ibid.). What gives objective validity and significance to a "physical object" is the very act of measurement, that is, the attribution of a precise numerical value to an object:

> Insofar as we determine in this way the pressure, volume, and temperature of a gas, the potential or kinetic energy of a system, the electric or magnetic field strength, we have in these determinations precisely what physics understands by its various objects (D&I, 36).

Working upwards, statements of measurements turn into statements of laws (second stage of Diagram 1), which are general statements. It is worth noting that a conceptual shift from a lower to a higher level does not exhaust itself in the process of "expansion of the basis of knowledge" (D&I, 44), that is, in a mere *generalization* of a higher level with respect to a lower one, this process rather demands a "change of viewpoint". Cassirer speaks of a change of dimension "that distinguishes the statements of laws from the mere statements of the results of measurements" (D&I, 42). As a matter of fact, any conceptual shift is characterized by a "jump" from a "type" of statements to another, that is, a certain type of statement can shift into the contiguous one by means of a "change of genus", although this process goes on according to a "fundamental form of induction", which is characterized by a tension towards the unity of the physical view of the universe. Cassirer pointed out that this conceptual shift is bidirectional, so that statements of laws may turn into principles, and the reversal is also possible. For this reason, a constant of nature is the starting point and ultimately given datum of epistemological reflection on scientific knowledge.

What precisely is the difference between statements of measurements and statements of laws?

Cassirer clearly identified the statements of laws with those statements which show the capacity to "characterize 'matter' by properties and relations invariant with regard to changes of particular positions in space and time" (D&I, 42), and he regarded them as statements of "higher order" (D&I, 35).

Working upwards once more, statements of laws move to statements of principles (third stage of Diagram 1), that is, to fundamental laws of nature, which show the following methodological characteristics: i) they are universal; ii) they have a heuristic value: they are rules "for seeking and finding laws" (D&I, 52); iii) they do not refer directly to phenomena, but to the forms of the laws through which we order phenomena; iv) they have different grades of generality[10].

[10] «Thus the difference between statements of principles, no matter how general, and the causal principle itself cannot be wiped out » (D&I, 57). In my view, Cassirer was arguing that statements of principle might lie on different levels, depending on their grades of generality in the process of development of scientific knowledge.

The principle of causality is on the whole a statement of principle, although its level of universality is so high that we can arrive at that only by means of a further "jump". Cassirer defined the general principle of causality not as a statement about objects, but as a statement about our cognition of objects. More precisely, the general principle of causality constitutes the overall relational structure among measurements, laws and principles, by showing *how* all these statements should be related and combined with one another: that is, through the conformity of each statement to the principle of coordination of a theory. In this sense the scientific principle of causality represents the principle of coordination of the entire system of knowledge in its constant tension towards the construction of limit concepts.

§3. The functional character of Planck's first radiation law

The first level of the series of functional relations with regard to the quantum theory was set in 1900, when Planck succeeded in obtaining the law for the energy distribution of blackbody radiation by introducing a new constant of nature. This constant allowed establishing a relation between the energy of a resonator (a hypothetic device that Planck used to discuss the problem of equilibrium radiation in a blackbody) and the intensity of radiation for a given wavelength[11]. The episode is crucial as it offers a glimpse of the fundamental role played by a particular law in establishing a relation among phenomena: specifically, a relation for the distribution of energy among N resonators of frequency ν. This fact reminds us of the very early stage of the procedure of super–ordination and subordination, in which, according to Cassirer "two fields of phenomena A and B are first united each according to a particular law" (S&F, 267).

Moreover, the introduction of the quantum of action was carried out in the context of a background of knowledge (i.e. Boltzmann's statistical method) that according to Cassirer's terminology represents the "logical presupposition" which precedes any operation of enumeration and measurement. On 14 December 1900, Planck presented his new radiation law at the German Physical Society in Berlin. He started his speech by emphasizing the necessity of using Boltzmann's probability arguments in the blackbody theory, and he soon after dealt with the distribution of a given amount of energy E among N cavity resonators with frequency ν.

> If E is considered as an infinitely divisible quantity, the distribution can be made in an infinite number of ways. However, we consider —and this is the most important point of the entire calculation— E as being composed of a completely definite number of finite, equal parts, and make use for that purpose of the natural constant $h = 6.55 \cdot 10^{-27}$ (Erg · Sec^2). This constant, when multiplied by the common frequency ν of the resonators, yields the energy element ε in ergs; and by dividing E by ε we obtain P, the n umber of energy elements which have to be distributed among the N resonators. If the quotient

11 Planck approached this domain of research also in consequence of the metrological importance of the radiation problem, which was a means to measure the temperature of a blast furnace.

> (E/ε) thus calculated does not happen to be an integral number, then one has to take for P an integer close to it (the quotient) (Planck 1900, 239–240).

Planck introduced the constant *h* in the framework of Boltzmann's method, which was used to get the relation between entropy and probability. Boltzmann's method hence constitutes the "logical presupposition" of the establishment of a new relation; that is, Planck's condition for the oscillator, which in this "turn of events" formally became a "statement of law", although this movement would be fully developed throughout an historical process which led to the formulation of the Einstein–de Broglie relation, that is, the principle of coordination of the theory[12].

I now want to return to the relationship between the formulation of Planck's radiation law and Boltzmann's method in order to clarify the reason for which Boltzmann's method is the logical presupposition of Planck's hypothesis of natural radiation.

In the years between 1897 and 1899, Planck set forth his program for a theory of radiation. This program was in continuity with his early work in thermodynamics for the irreversible approach of radiation to equilibrium. Planck aimed to show that a conservative system consisting of electromagnetic radiation in a blackbody, interacting with a collection of harmonic oscillators, could approach an equilibrium state without the need of any assumption —unlike Boltzmann had done— beyond the laws of electromagnetism. Specifically, Planck's idea was that electromagnetic radiation interacting with a system of electric resonators (oscillators) at every possible frequency should irreversibly evolve towards blackbody radiation. Nevertheless, as Boltzmann himself pointed out to Planck, the laws of electromagnetism did not determine the irreversible approach of radiation to equilibrium. Additional assumptions were needed: statistical assumptions about the disordered character of the initial state such as Boltzmann had made in the theory of gases. It was in this framework of ideas that Planck introduced the hypothesis of natural radiation *h*, according to which the partial vibrations of the radiation and the oscillator remained completely incoherent. The introduction of natural radiation made the large–scale evolution of the system an irreversible process. This procedure became known as Planck's electromagnetic *H–theorem*, because in its demonstration Planck used functions of the oscillator and radiation energy that had the same form as Boltzmann's *H–function*, that is

$$S = K \ln W. \tag{3.1}$$

12 As is well known, light quanta have an energy given by the Planck–Einstein relation $E = h\nu$. Louis de Broglie, in his 1924–PhD dissertation, argued that if light–quanta show both wave–like and particle–like properties, electrons too show wave–like properties. By means of the Planck–Einstein relation, it turned out that the relationship between the wave length, λ, associated with an electron, and its momentum, *p*, through Planck's constant, is: $\lambda = h/p$.

(Where S is the entropy, K is Boltzmann's constant, ln is the natural logarithm, and W is the probability density of the energy over a set of oscillators: the so–called complexion, a definition used also by Boltzmann in 1877).

In this sense, Planck's revolutionary work requires as a precondition Boltzmann's method, with the difference that: for Planck, a "complexion" was given by a correspondence between an oscillator and the number of the energy–elements possessed by this oscillator. For Boltzmann, the correspondence was between a molecule and the discrete energy of this molecule.

We have now arrived at the very crucial question regarding Planck's revolutionary work of 1900, which essentially concerns the method that Planck used for deriving his distribution law. Planck made a fundamental departure from Boltzmann's method (Klein 1962), which precisely consists in the variety of complexions considered in the calculation of W. On the one side, Planck took for his W the total number of complexions for all sets $\{w_r\}$ of resonators. On the other side, Boltzmann limited the choice concerning the complexion to the number of molecules carrying the energy. This inappropriate generalization of Boltzmann's method was the way to obtain Planck's distribution law (Darrigol 1991).

Planck's relation seems to suggest an undue generalization from Boltzmann's method. The relation that here arises is a solution that Planck had to adopt or an *ad hoc* hypothesis that Planck would have formulated in whatever manner which had referred to Boltzmann's expression in order to obtain the relation between entropy and probability. In my view, the introduction of Planck's hypothesis recalls the procedure of the fundamental form of induction, which allows a conceptual system to develop on. As a matter of fact, Planck's solution is a radical change of view that characterizes the advent of a new phase. The introduction of the quantum of action, a new constant of nature, is the starting point of a "chain" of relations inherent in the structure of a new theory (the quantum theory), and, with respect to our diagram, it would lie on the lower stage: as a "statement of results of measurements". However, to say it with Cassirer's words, an individual constant means nothing in itself, its meaning is established with respect to a background knowledge. Boltzmann's method represents the necessary background within which Planck could introduce a new constant of nature, and from which a new chain of relations could start off. But the introduction of the quantum of action marked a radical departure from such a background, as far as there is a substantial difference between gas theory and radiation theory, and this difference rests on the "significance of the constant *h*":

> It [constant *h*] marks an essential difference from the expression for the entropy of a gas. In the latter, the magnitude of an elementary region which we call dω disappears from the final result since its only effect is on the physically meaningless additive constant…The thermodynamics of radiation will therefore not be brought to an entirely satisfactory conclusion until the full and universal significance of the constant *h* is understood (Planck 1906, 153 f.).

This quotation links also to Planck's later perception of his own discovery of 1900, Planck being certainly aware of having introduced a new way of calculating entropies that bridged two domains of physics: radiation theory and gas theory. Specifically, the constant K in the equation (3.1) had to be the same in both realms. However, in 1900–1901 Planck did not make any mention of the meaning of the constant *h*. Specifically, he did not say that the energy of a resonator could only be an integral multiple of *hν*, because in so doing, he would have introduced the concept of discontinuity in physics, as most historians and even scientists have incorrectly argued (Darrigol 2000). Planck himself defined the introduction of the new constant *h* as a "lucky intuition" or an "act of desperation", and in that context (of discovery) he ascribed to it only a formal character, whose real meaning would be clarified later on.

Returning to the present attempt to give a functional reconstruction of the early development of quantum theory, Planck's radiation law here represents a rule of progression of the theory, as far as it constitutes a "common property" (S&F, 17) among the members of the series, by establishing a relation for the energy distribution among the frequencies in blackbody radiation. In this respect, this statement of law agreed with all the experimental data both at low frequencies and at high frequencies —where the "Rayleigh–Jeans" law failed— offering a unified description of all blackbody radiation over N resonators. Furthermore, I would not dispense with the fundamental fact that the introduction of the *h* contains in embryo all the stages of theory development prospected by Cassirer in *D&I*: i) at the very beginning, Planck regarded it as a mathematical device[13]; ii) Planck's condition for the oscillator plays the role of a statement of law, i.e. Planck's distribution law; iii) finally, the formal use that Planck made of the energy–elements, applicable only to the mechanism regulating the interaction between matter and radiation in the resonators, led to a fundamental conceptual development. As Cassirer pointed out:

> [...] What was originally the quantum *law* became the quantum *principle*. Its applications were no longer confined to individual branches of physics, however extended; it came to be understood and used rather as a general point of view, as a postulate of scientific knowledge (D&I, 110).

If in 1900 Planck's theory of radiation (the so–called quantum law) did not yet turn into principle, it already gave rise to the early stage of the principle of coordination (in the role of "relation of dependency") of the quantum theory. I am referring to the well–known equation $E = h\nu$, which went to associate both the interaction between radiation and matter (Planck's energy quanta) and the radiation in transit (Einstein's light quanta) (cf. Jammer 1966, 36). As has been mentioned, the so–called Einstein–de Broglie equation was formally laid down in

13 For Cassirer, the introduction of a constant of nature in a formula allows to establish a relation in the manifold of experience. Thanks to this new relation, the manifold of experience "gains that fixed and definite structure, that makes 'nature'" (S&F, 230). The introduction of the *h* as a mathematical device is an aspect of the constitution of the so–called "physical object", which status will be reached through the very act of measurement.

1900, although it was subjected to a conceptual development when it was associated with the concept of energy quantization in 1905–1906 (thanks to the independent contributions of Albert Einstein and Paul Ehrenfest) and with the particle–wave properties of the electron in 1924 (thanks to de Broglie). For this reason, in view of the present reconstruction, the Einstein–de Broglie equation, as it was adumbrated in 1900, cannot be yet considered as a principle of scientific knowledge.

The issue here at stake is the same one that Kuhn (1978) regarded as a fundamental aspect of the scientific revolution brought about by the introduction of quantum discontinuity: the assumption of the quantization of energy and radiation. But there is a difference between the present view and Kuhn's account.

For Kuhn, Einstein's and Ehrenfest's derivations of Planck's radiation law amounted to a scientific revolution that led to the establishment of a new paradigm; the reason being that, according to this standpoint, Einstein and Ehrenfest introduced quantum discontinuity in the distribution of energy.

According to the present view, Einstein's endeavor of 1906 (I will not deal with Eherenfest's contribution) is a *generalization* of higher level of Planck's h along the relation of dependency $E = h\nu$, which was achieved through a fundamental form of induction, which is rational and progressive (see sections 2 and 4).

In particular, the formulation of Einstein's rule of quantization and the quantum–light hypothesis represent the transition of the h into a statement of principle. In the present functional perspective, this conceptual change did not determine a paradigm shift, but it rather constituted a conceptual extension demanding a "change of view" with respect to Planck's theory of radiation: from the procedure of quantization, applied to material oscillators (Planck's radiation theory) to the quantization of radiation.

To conclude this section, I point out that Planck introduced the constant h and he associated finite energy–elements $h\nu$ with harmonic oscillations at frequency ν without changing his views on continuity and irreversibility with regard to electromagnetic radiation, but this happened in connection with a radical departure from the "logical presupposition" of the quantum theory, that is, Boltzmann's statistical method.

In view of Cassirer's reading, I claim that if a revolution occurred, it happened on December 14, 1900, when Planck announced that he had found a new way of calculating entropies with respect to Boltzmann's method. Therefore, our functional–based enquiry into the early quantum has now a starting point: the introduction of Planck's h, which constitutes the very act through which the relation of functional coordination between a concept and its object first was posed in the context of the quantum theory.

§4. The conceptual functionalization of Einstein's first derivation of Planck's radiation law

When in the mid–1906 Einstein derived Planck's blackbody law, he brought into quantum theory a fundamental step along the *relation of dependency* of the theory.

Both Einstein with his "Theory of the Emission and Absorption of Light" (1906b), and Ehrenfest with "On Planck's Radiation Theory" (1906) were convinced that Planck's law correctly described the equilibrium distribution of radiant energy, and time was hence ripe for a fundamental break with classical physics, because the basis on which Planck's radiation theory lay were incompatible with those of Maxwell's electromagnetic theory.

Einstein, on his side, obtained Planck's distribution law by supposing that the energy E of a resonator can assume only values that are integral multiples of *hν*.

It should be mentioned that Einstein's revolutionary step was already implicit in the so–called heuristic principle laid down in the 1905–article on the "photoelectric effect", in which Einstein introduced the concept of light–quantum[14]:

> If, in regard to the volume dependence of the entropy, monochromatic radiation (of sufficiently low density) behaves as a discrete medium consisting of energy quanta of magnitude Rβν/N, then this suggests an inquiry as to whether the laws of the generation and conversion of light are also constituted as if light were to consist of energy quanta of this kind (Einstein 1905, 143–144).

Einstein's hypothesis of light–quantum is an assertion on the significance of the quantum of action, which, through the heuristic principle, was extended to the very nature of the interaction between radiation and matter, that is, regarding not only the average energy of a material oscillator of frequency ν.

We should now enquire into the intellectual process underlying the conceptual development of the *h*, which led Einstein to formulate the concept of quantization of radiation.

Let us consider that in 1905 Einstein introduced a quantization related to (the light–quantum hypothesis) without using the –U relation, where U is the energy of Planck's oscillator. The point is to show how Planck's quantization is related to U and Einstein's quantization is related to ρ.

[14] Einstein saw his derivation of Planck's law as the confirmation of the existence of the light–quanta introduced in his theory on the photoelectric effect in 1905, although the author of the theory of special relativity had lots of hesitations in writing down the relation p = *h* ν/ c "side by side" with E = *h* ν. In fact, Einstein (1916) explicitly introduced photon momentum only when he was able to re–derive Planck's radiation law (Pais 1982, 408–409). Moreover, in 1916 Robert Millikan gave experimental confirmation of the photoelectric equation, as a consequence Einstein's light–quantum hypothesis was endowed with physical significance.

According to Einstein (1906b), the connection between his own quantization for ρ and Planck's quantization for U was possible by introducing a new assumption: "the energy E of a resonator cannot possess every arbitrary value, but only values which are integral multiples of ε, with $\varepsilon = (R/N_0) \cdot \beta\nu$ [with $\beta = h/k$ and $R/N_0 = k$ in terms of Planck's constant]" (Einstein 1906b, 202).

When Einstein reexamined the derivation of Planck's blackbody law in light of this new assumption, he concluded:

> We must consider the following theorem to be the basis of Planck's radiation theory: the energy of a (Planck's oscillator) can take on only those values that are integral multiples of hν (Einstein 1906b, 202).

According to the procedure of super–ordination and subordination, we might say that if Planck's radiation law can be represented by the relation ψ_1 (α_1, α_2, α_3) Einstein's rule of quantization represents a new relation Φ (ψ_1, ψ_2) that connects Planck's law with other empirical laws. In view of the present reconstruction, I argue that Einstein's discrete quantization (and the light–quantum hypothesis) represents a relation of super–ordination with respect to Planck's radiation theory, along the Einstein–de Broglie relation of proportionality between energy and frequency. As a matter of fact, in 1905–1906, Einstein succeeded in *extending* the field of application of Planck's hypothesis to radiation by shedding new light on the *meaning* of the quantum of action in terms of light–quanta, whereas Planck had shown his predilections for setting apart the problem of the radiation field.

The formulation of Einstein's method of discrete quantization is an example of a conceptual transition from an empirical law to a statement of principle, the reason being, not that a principle can be derived from a statement of law —in fact, Einstein's method was not derived from Planck's law, as Einstein gave independent reasoning with respect to Planck— but that Planck's law was subjected to a conceptual extension, which required a "change of genus": from the procedure of quantization applied to material oscillators (Planck's radiation theory) to a new conception of radiation field (Einstein's light–quantum and his method of quantization). As it is evident, Einstein's solution demands a change of perspective that Planck himself, indeed, could not accept at that time, because it would have contradicted the foundations of classical electrodynamics. As Einstein pointed out:

> It is easy to see in what ways the foundations of Planck's theory must be modified in order that the Planck's radiation law becomes really a consequence of those theoretical foundations […]. A structure which may vibrate with the frequency ν and which, because it possesses an electron charge, can convert radiation energy into energy of matter and vice versa, may not occupy vibration states with any arbitrary energy, but only those states with energies which are an integral multiple of hν (Einstein 1909, 182).

Einstein showed that Planck's law could be derived only if resonator energies were restricted to integral multiples of hν. This was a revolutionary fact that brought discontinuity into the tradition of physical sciences. What was a mere radiation law finally became a principle with a higher degree of generality.

In a second instance, Einstein's method has a heuristic value, as it points to rules "for seeking and finding laws" (D&I, 52): such as the explanation of the Doppler effect of canal rays by Johannes Stark, who in 1907 provided a new theoretical interpretation of the Doppler effect on the basis of Planck's energy quantum (Stark 1907); the explanation of the properties of series and band spectra (Stark 1908), and the fluorescence of organic substances (Stark and Steubing 1908).

As a statement of principle, Einstein's method plays the role of a rule of progression among several empirical laws, which are connected by and can be derived from that: as it was for Planck's radiation law, the Doppler's effect of canal rays etc. This fact supports Cassirer's claim that statements of laws may turn into principles, and the reversal is also possible.

Finally, Einstein's rule of quantization is a causal principle, as far as the validity of its functional connection is guaranteed by its conformity to the essence of the Einstein–de Broglie relation: the concept of quantization, which we can substantiate with Einstein's claim that "the energy of a Planck's oscillator can take on only those values that are integral multiples of *hν*" (Einstein 1906b).

§5. The functional character of Einstein's theory of the specific heats of solids

A further level of functionalization of concepts, along the relation of dependency of the quantum theory (the Einstein–de Broglie relation), can be traced back to December 1906, when Einstein published the article "Planck's Radiation Theory and the Theory of the Specific Heats of Solids" (1906a).

The problem of the specific heats inaugurated a new chapter in the long history of quantum physics, although, since the appearance of Einstein's pioneering work, it took four years before it went to displace the blackbody problem as one of the main concerns for quantum physicists.

We can reasonably argue that even Einstein's theory of the specific heats of solids is a further *generalization* of Planck's radiation law, as far as Einstein's rule of quantization was extended to the quantization of material entities. Yet the theory did not gain immediately unanimous consensus within the physicists' community, for two main reasons: i) only few specialists were inclined to follow the intuitions of a young and almost unknown scientist, ii) only few physicists were involved in problems concerning the quantum theory.

To summarize the situation regarding the problem of the specific heats prior to Einstein's paper, the classical kinetic theory perfectly accounted for the value given by the Dulong–Petit rule[15] (1819). But it should be noted that almost all the measurements, aiming at verifying such a rule, were carried out at room temperature (300^0 K) or above, "since analogous experiments at lower temperature would have posed insurmountable technical difficulties" (Jammer 1966, 46). Einstein knew of these difficulties and he was aware of the "contradictions" arising from the theory of specific heats, especially if viewed in light of the modern electron theory of matter[16]. In commenting on Planck's results that the mechanism of energy transfer admits only of energy values that are integral multiples of $h\nu$, Einstein declared:

> I believe we should not content ourselves with this result […]. If Planck's theory of radiation goes to the heart of the matter, then we must also expect to find contradictions between the present [i.e. classical] kinetic theory and experiment in other areas of the theory of heat —contradictions that can be resolved by following this new path. In my opinion, this expectation is actually realized (Einstein 1906a, 184).

In particular, Einstein went to apply Planck's radiation law, which gave to any object oscillating in space with the frequency ν at temperature T the mean energy

$$\bar{\bar{\bar{E}}} = 3\frac{R}{N_0}\frac{\beta\nu}{\exp\left(\frac{\beta\nu}{T}\right)^{-1}}, \tag{5.1}$$

From which he determined the specific heat per mole

$$c = 5.94\sum_i \frac{(h\nu_i/kT)^2 e^{h\nu/kT}}{\left(e^{h\nu_i/kT}-1\right)^2}. \tag{5.2}$$

where β = h/k, a formula which shows that C_v decreases with T but is equal to the Dulong–Petit rule at sufficiently high temperature. Equation (5.2) provided an explanation for the observed deviations of the atomic heats from the Dulong–Petit rule as well as their temperature dependence. As Einstein pointed out:

> If T/βν > 0.9, the contribution of the [oscillating constituent] to the molecular [atomic] specific heat does not deviate appreciably from the value 5.94, which also follows from the hitherto accepted

15 The Dulong–Petit rule states that the product of the atomic molecular weight and the specific heat is a constant for all solids (approximately 6 cal/mole).

16 As a matter of fact, the oscillating electron, which caused the dispersion of light, gave rise to extra degree of freedom. Therefore, a specific heat even larger than one foreseen by the Dulong–Petit rule had to be expected for monoatomic solids.

> molecular–kinetic theory of heat. The smaller ν is, the lower will be the temperature for which this case is already achieved. If, however, $T/\beta\nu < 0.1$, then the [oscillating] object under consideration does not contribute appreciably to the specific heat. In between [these two cases] the expression (5.2) increases [with increasing temperature], first rapidly, then more slowly (Einstein 1906a, 186–187).

Some considerations now impose upon us if we seek to view Einstein's achievement in light of Cassirer's functional perspective.

First, Einstein's work on specific heats is a further *extension* of Planck's results along the relation of proportionality between energy and frequency, in the sense that Einstein succeeded in showing that Planck's ideas could be applied beyond the domain of radiation problem.

Second, Einstein's endeavor represents a relation of super–ordination with respect to both Planck's theory of radiation and the method of discrete quantization. The reason being that Einstein's contribution to the specific heats of solids does represent a statement of principle, but with a difference with respect to the method of discrete quantization from 1906: the former was provided with a higher level of generality with respect to the latter and with a new *meaning*, because it made it possible the application of quantum conceptions to the molecular kinetic theory, endowing Planck's ideas on natural radiation with "physical significance" (Jammer 1966, 45). In Cassirer's view, it is worth noting that a "change of genus" is needed that allows a further extension of the concept of quantization to molecular kinetic theory, which is guided by the "methodological demand" (D&I, 47) of a unified explanation of physical phenomena.

Third, the transition from Einstein's discrete quantization into the theory of the specific heats put forward a new heuristic, as far as from 1907 considerations on radioactive phenomena became secondary with respect to considerations on atomic vibrations. As a matter of fact, Einstein's paper on specific heats paved the road for a new line of research. In this connection, it should be noted that Einstein's extension of Planck's results plays the role of a new rule of progression, which is a new "common property" (S&F, 17) among other empirical laws. For example, in 1911 Walther Nernst wrote a paper in collaboration with Frederick Lindemann, entitled "Specific heat and quantum theory", in which they stated:

> In a recently published investigation one of us [Nernst] has given a representation of quantum theory which, following Einstein, considers radiative phenomena as only secondary circumstances ('begleitende Umstände') and takes as its immediate point of departure the *atomic vibrations* (Nernst–Lindemann 1911, from Jammer 1966, 48; Italics by the present author).

That is to say that Nernst, following Einstein's conception on the specific heats, inaugurated a trend of studies, which aimed to approach quantum theory from the molecular kinetic theory.

Fifth, I point out that Einstein's theory of the specific heats of solids conforms to the relation of proportionality between energy and frequency. As Einstein claimed, also the behavior of "oscillating objects" could be explained by assuming that the radiation consists of quanta:

> From the foregoing [results] it is not enough to assume that oscillator energy with the frequency ν can only be emitted and absorbed in quanta of this size [i.e. hν], that one is only concerned with a property of emitting and absorbing matter. The [above] considerations … show that *also the fluctuations in the spatial distribution of radiation and of radiation pressure behave as if radiation consisted of quanta of the size indicated* (Einstein 1909, 191; Italics by the present author).

By applying Cassirer's reading, we can conclude this section by claiming that Einstein's theory of the specific heats is a causal principle, as it conforms to the Einstein–de Broglie relation.

§6. Niels Bohr and the quantization of the hydrogen atom

Cassirer argued that the two fundamental assumptions (stationary states[17] and frequency rule[18]) upon which Bohr's theory of the hydrogen atom is based, were derived by presupposing the quantum postulate (D&I, 133), which in the present notation represents the principle of coordination in the role of relation of dependency of the quantum theory: the Einstein–de Broglie relation.

In this section, I will be demonstrating that even Bohr's endeavor represents a step along the process of functionalization of concepts that started in 1900 with the formulation of the concept of natural radiation. Cassirer, on his side, placed emphasis on the relations between Bohr's atomic theory and Einstein's frequency condition, which is straightforwardly connected to Planck's conception of radiation, along the relation of proportionality between energy and frequency, as it has been shown in section 4.

> Bohr's theory replaces the picture of the atom as a thing–like entity perceived by our senses, by the characteristic frequency condition, according to which the change in energy of an atomic system is related to the frequency of a light wave by a definite equation, $h\nu = E_1 - E_2$. Its success in calculating the energy levels of the hydrogen atom, so that this condition led to the hydrogen spectrum in exact accordance with Balmer's formula, provided empirical proof for the fruitfulness of the theory (D&I, 133–134).

By 1910s a few specialists had already started to use the quantum of action for the purpose of discrete selection among the possible states of the classical models of atoms, as they had realized that some kind of discontinuity had to be used for explaining the behavior of microphysical entities. Among them: Arthur Erich Haas, William Nicholson, Niels Bjerrum, and Niels Bohr.

In particular, Bohr was the only among them who accepted the challenge to carry on a theory notwithstanding the violation of a direct relation between mechanical and optical frequencies.

17 Any of several energy states an atom may occupy without emitting electromagnetic radiation.

18 Einstein's method of quantization or discrete selection of mechanical states.

If the assumption of the frequency rule (as it was called in the Bohr atom) was not new, the one concerning the stationary states was not unexpected either, as Johannes Stark had already discussed this hypothesis in his work on atomic dynamics from 1911 (which Bohr had read), and the possibility of their existence was also prospected in Nicholson's papers. In this regard, I point out that the application by Bohr of the frequency rule assumption to Rutherford's nuclear planetary model of the atom constitutes a conceptual shift, a change of perspective, which does not mean a conceptual rupture. Most fundamentally, it represents a further functional connection along the relation of proportionality between energy and frequency. In fact, in analogy with the quantization of Planck's harmonic oscillator, Bohr assumed that the single electron of the hydrogen atom could exist only in a series of stationary states, determined by the "quantum rule":

$$E_n = -\alpha n h \bar{\nu}_n, \qquad (6.1)$$

where E_n is the binding energy of the electron, $\bar{\nu}_n$ the orbital frequency, h is Planck's constant, α a numerical constant, n a positive integer. Eq. (6.1) is hence a generalization of Planck's theory of radiation, which states that the emission and absorption of the energy concerns only certain states of the harmonic oscillator, whose energy for each of these states is:

$$E_n = nh\omega. \qquad (6.2)$$

Since Eq. (6.2) cannot describe an atomic model involving a particle motion, the generalization of Planck's theory of radiation is required for the frequency of the oscillator, ω, be replaced by the orbital frequency $\bar{\nu}_n$. As a further step, in order to obtain Balmer–Rydberg's formula, Bohr had to assume Einstein's frequency condition, $h\nu = E_1 - E_2$, which allowed to calculate the frequency of the emitted radiation when an electron passes from a stationary state to another.

In spite of the violation of the causal nexus between the motion of the electron and the emitted radiation (which represents the very "change of view" in this respect), I point out that even the first atomic theory represents a valid logical connection along the process of super–ordination and subordination, as also Bohr's theory conforms to the Einstein–de Broglie relation of proportionality between energy and frequency. In fact, Bohr succeeded in incorporating in his hydrogenoid system previous "levels of connection" of the quantum theory: he started from the analogy with Planck's oscillator, then he proceeded to apply Einstein's frequency condition. In a functional perspective, Bohr's work constitutes the most advanced statement of principle along the "relation of dependency" of the first quantum theory. Retrospectively, Bohr seemed to recognize it, when he admitted that his work was the rational development of Planck's condition for the oscillator, whose generalization led to the formulation of the two fundamental assumptions (stationary states and frequency rule). In a lecture before the German Physical Society in Berlin, Bohr explicitly claimed that Planck's condition for the oscillator was the starting point from which he had managed to obtain "in a purely formal manner" a spectral theory, "the essential elements of which may be considered as a simultaneous rational development of the two ways of interpreting Planck's result" (Bohr 1920, 22; NBCW 3, 244).

Further, Bohr's first atomic theory lies on a position of super–ordination with respect to both Einstein's contributions (discrete quantization and specific heats) and Planck's theory of radiation, to the extent that the first theory of the hydrogen atom is an extension of Einstein's and Planck's conceptions on quantization of energy, radiation, and material entities (molecular kinetic theory) to a new domain: the quantization of atomic structures. Bohr's work on the hydrogen atom is a new rule of progression of the quantum theory, as it represented the "common property" (S&F, 17) among empirical laws in the realm of atomic physics.

For instance: Balmer's equation, Rydberg's constant, and Ritz' combination principle were derived from Bohr's theory and they could be connected by that. Moreover, it rendered possible the construction of the periodic system of elements by setting the number of electrons in the atom equal to the atomic number. Finally, Bohr's quantized model of the atom contains in embryo a heuristic power, which would be developed by the correspondence principle[19], which was formulated by Bohr in 1917 to extend the quantum theory by calculating, in the limit of large quantum numbers, the intensities of the spectral lines by means of the amplitudes of the harmonic components of the electron's classical orbit.

§7. Concluding remarks

As I showed, causality and functionality are interdependent concepts in Cassirer's epistemological reflection. In S&F Cassirer stressed two fundamental issues in the process of functionalization of concepts: the extent and the validity of functional relations. It should be also noted that in Cassirer's system of scientific knowledge the new abstract relation of coordination between the concept and the object replaced the traditional conception of causality, which was related to time and continuity. That is, Cassirer explained the question of the validity of a functional connection within a conceptual system in terms of conformity to law, in which the conception of causality was purged of any visualizable content.

[19] In order to determine the energy of the stationary states, Bohr had to introduce a "quantum rule": the energy of the nth stationary state is $n \cdot h/2$ times the orbital frequency of the electron in this state. Assuming that the orbit of the electron is circular, this assumption is equivalent with the assumption that the angular momentum of the electron round the nucleus is equal to an entire multiple of $h / 2\pi$ (Darrigol1992).

In particular, by setting the frequency of the emitted radiation equal to half the final orbital frequency, Bohr renounced the classical relation between motion and emitted radiation, although he still required an analogous relation to subsist at the quantum level. This requirement is the conceptual embryo of what Bohr would later call the correspondence principle.

Cassirer's conception of critical causality aims at providing the system of scientific knowledge with a "conceptual structure", which is not an independent metaphysical reality, but it is a regulative idea, which is directed towards the construction of limit concepts, which will never be fully realized in experience.

We are now arrived at the core issue of Cassirer's genetic conception of knowledge, which takes the historical development of scientific knowledge as its ultimate given datum (Friedman 2016). On my side, I focused on the concept of constant of nature for shedding new light on Cassirer's epistemological reflection on the early quantum theory. In particular, I sought to explain the relationships between Balmer's law and Rydberg's constant, and how these are related to Planck's constant, by applying Cassirer's functional view to the early development of the quantum theory. For this purpose, I identified Planck's elementary quantum with the starting point of the early quantum theory. In a second instance, I integrated Cassirer's functional view from S&F with the dynamic view that he laid down in D&I. On that I showed that a conceptual shift characterized the transition from the earlier level of the quantum theory onwards, that is: from a statement of the results of measurements (Planck's constant) to a statement of law (Planck's radiation theory), and from this to statements of principle (i.e. Einstein's rule of quantization, the theory of the specific heats of solids, and Bohr's quantized atom), that is, the assumptions concerning the quantization of energy, radiation, material entities, and atomic structures which constitute a conceptual series of progressive stages along the advancement of knowledge, oriented towards the construction of limit concepts. As I showed, the quantum of action's conceptual structure already contains in embryo all the stages of its development (mathematical device, statement of law, statement of principle). And each stage of its successive conceptual development is also comprised of the preceding ones. As I see it, this aspect of theory development fits nicely the "well– rounded *Parmenidean* sphere" to which, for Cassirer, physics is to be compared.

To conclude, I think I have demonstrated the existence of a straightforward logical connection among the following scientific relations: the introduction of the quantum hypothesis that Cassirer identified as a firm and stable frame of the quantum theory, Planck's first theory of radiation, Einstein's light–quantum hypothesis and the formulation of the discrete quantization, the theory of the specific heats of solids, and the formulation by Bohr of the first atomic theory, along the so–called principle of coordination —in the role of relation of dependency of the quantum theory— that Cassirer identified with the quantum postulate.

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