Expanding the notion of inconsistency in mathematics: the theoretical foundations of inter-model inconsistency

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Abstract

In this article I introduce a new notion of inconsistency in the philosophy of mathematics that is different from the usual notion of inconsistency via contradictory statements in a formal theory. This notion, inter-model inconsistency, substantially relies on aspects from mathematical practice. I develop this notion via the case study of set theory and discuss why this notion of inconsistency accurately captures the use and function of models of set theory in current set-theoretic practice.

With the introduction of this new notion, I aim at providing a way in which the debate about inconsistency toleration vs. consistency preservation in the sciences can be connected to debates in the philosophy of mathematics. The article closes by discussing further consequences of this notion for the inconsistency debate in the philosophy of mathematics.

1 Introduction

When inconsistencies emerge in scientific theories they present serious challenges which are generally regarded as important triggers for a change in the theories. The standard account of how the scientific communities react towards the discovery of an inconsistency is to aim at the removal of the inconsistency (consistency preservation). More recently, however, alternative ways of dealing with inconsistencies have been discussed, in particular the possibility that scientific communities may opt to *tolerate* inconsistencies instead of trying to resolve them (Martínez-Ordaz and Estrada-González, 2017).

In these discussions, the case studies have mostly been drawn from the natural sciences, whereas cases in which inconsistencies arise in mathematics have been discussed separately in the philosophy of mathematics (see, for example, Colyvan, 2008). One of the reasons for the relative disconnectedness of these two debates is that different forms of inconsistencies seem to be discussed: in science, inconsistencies can arise within a scientific theory, between different scientific theories, and between observations and scientific theories. Moreover, inconsistency in the sciences appear at least partly informally, not being dependent on a formal mathematical theory.[1] In contrast, we usually only speak of one form of inconsistency in standard mathematics. Such an inconsistency appears as a logical contradiction in a formal theory, i.e. there is some mathematical statement P such that $P \wedge \neg P$ holds in the theory. (We call this a *logical inconsistency*.) A theory usually denotes a set of specific sentences, for example, some axioms and their closure under a logic, for example, first-order logic. Taking set theory as an example, the standard theory under consideration is the one stemming from the Zermelo-Fraenkel axioms with the Axiom of Choice, **ZFC**, closed under classical first-order logic.[2] If we were to find a logical inconsistency in this theory, we would be able to prove from the axioms that $P \wedge \neg P$ for some P. On the semantic side, this means that the theory would not have a model, i.e. there is no structure that satisfies all sentences of the theory.

But the differences between the cases of science and mathematics do not seem to be restricted to the *forms* of inconsistency: the *reaction* of the community towards the appearance of an inconsistency seem to differ as well. As mentioned above, in the sciences, a case can be made for inconsistency toleration (see, for example, Meheus, 2002). In mathematics, on the contrary, this is highly controversial: if an inconsistency occurs, it always seems to lead to a concerned effort to remove it. A paradigmatic example for this is the case of Russell's Paradox in early set theory, which was removed by restricting the Axiom of Comprehension. Yet there are cases where scholarship has argued for something like inconsistency toleration in mathematics. One often-cited example is the early calculus and its use of infinitesimals. Colyvan (2009) argues that this is an example of an inconsistent theory that was fruitfully used for quite some time[3], despite the community being aware of the inconsistency. However, it has contested by Vickers (2013) that this case actually constitutes an inconsistency, and similar objections hold for other such cases.[4] So, in mathematics, the focus lies almost exclusively on removing the inconsistencies; as Colyvan (2009, 161) observes: 'Looking for inconsistency so that it might be avoided seems to be the extent of the interest.'

This article is the first in a two-part series of papers that approaches the question of inconsistency toleration from a different perspective, namely, one that focuses on mathematical practice (the second paper in this article series is Kuby, forthcoming).[5] While investigating the modern practice of set theory, I observed that the foundational debate prompted by mathematical results about models of set theory shows features quite similar to ones discussed in the philosophy of science regarding cases of consistency preservation and inconsistency toleration. These similarities range from discussions about the adequacy of concepts to competing

[1] However, they mostly still depend on a general mathematical apparatus. These kinds of inconsistencies are then neither fully formal nor fully informal. I would like to thank the anonymous reviewer for pressing me to clarify this point.

[2] There is also second-order set theory, a theory that is based on second-order logic. Second-order theories have been mostly absent from set-theoretic practice for quite some time, but have seen a bit of a revival in the last decade (see, for example, Williams, 2018; Antos and Friedman, 2017; Gitman et al., 2021). However, set theory in a first-order axiomatization is still the gold standard for set-theoretic practice; therefore we will take this as the basis for our investigations.

[3] A rational reconstruction of the early calculus as an inconsistent theory can be found in (Brown and Priest, 2004). This reconstruction involves a specific kind of para-consistent logic to avoid the occurrence of explosion.

[4] For further discussion, see Bueno (2017).

[5] This article is also published in the present volume.

foundational programs, in which some argue for consistency preservation whilst others imply forms of inconsistency toleration. However, the underlying case study from set theory does not match the notion of *logical* inconsistency, as we do not know of a logical contradiction in set theory. This leads us to believe that there is a different form of inconsistency at play here, and we propose *inter-model* inconsistency as this new notion of inconsistency.

In this paper I want to introduce this new concept and explore its fundamental properties on the basis of the particular situation in set theory. In particular, I will show that while this notion is a weaker form of inconsistency than logical inconsistency, it can still be considered to be a proper form of inconsistency. Kuby (forthcoming) then shows that the pressing current debates in the philosophy of set theory can indeed be interpreted as an answer to the situation of inter-model inconsistency by proposing consistency-preservation and, more interestingly, inconsistency-toleration strategies.

It is important to point out that the aim of this series of papers is a *descriptive* one. I want to show how the current situation in set theory can be described to constitute a case of inter-model inconsistency toleration and explain why this is relevant for the current debate in the philosophy of set theory. We *do not* propose this account via inter-model inconsistency as an alternative to programs in the philosophy of set theory aiming to resolve the situation of inter-model inconsistency such as forms of universism and multiversism do. Indeed, Kuby (forthcoming) will argue that such normative programs react to the problematic situation described via inter-model inconsistency. However, we will see that the notion of inter-model inconsistency provides a challenge to these programs and can be used to argue for the introduction of other such programs, therefore substantially contributing to the debate. In particular, we see that there can be cases of inconsistency toleration based on inter-model inconsistency in mathematics and therefore the discussions in philosophy of science and philosophy of mathematics can be connected more closely after all.

The structure of this paper is as follows. In Section 2, I first present the case study of models of set theory on which I base the introduction of the new notion of inconsistency. To justify its introduction, in Section 2.1, I first provide an overview of the development of set theory since the introduction of large-scale model-building techniques in the 1960s. Then, in Section 2.2, I provide a detailed analysis of how the practice of models of set theory gives rise to what I call the inconsistent practice of set theory. From that, I develop a definition of inter-model inconsistency in set theory (MIST) in Section 3 and I argue that MIST is a notion of inconsistency as it correctly matches the inconsistent practice of set theory and makes it explicit. Section 4 concludes by showcasing how MIST can contribute to several philosophical debates by introducing the possibility of new consistency-preservation and inconsistency-toleration approaches.

2 An inconsistent practice in modern set theory?

In this section we will gather together all the elements we need to define a new notion of inconsistency, which I will call *inter-model inconsistency*. This notion is based on the case study of modern set theory and a major goal of this section is to show that this part of mathematics indeed calls for the introduction of such a notion. We will therefore aim at tracking what I call *inconsistent practice* of set theory and see how it is correctly captured by the notion of inter-model inconsistency. I will proceed in two steps. In Section 2.1, I will outline the historical development of modern set theory in the last decades, specifically detailing how its practice has changed since the introduction of large-scale model-building techniques since the 1960s. Then, in Section 2.2, I give an analysis of this practice by showing how different aspects of it give rise to an inherent inconsistency. However, before we start, let me provide the logical basis of this model-related practice of set theory that will later serve also as the logical basis for the notion of inter-model inconsistency.

With a model of set theory we usually mean a model that satisfies some appropriate axiomatization of set theory. The axioms are sentences formulated in the first-order language of set theory[6] containing one binary relation-symbol $\in$ that is standardly interpreted by the model as the usual 'is element of' relation. The axiomatic system normally used for set theory is the Zermelo-Fraenkel axioms with the Axiom of Choice, **ZFC**. Additionally, variations of this system are frequently used in practice: examples are fragments of **ZFC** such as **ZF** (without Choice); or extensions of **ZFC**, where additional axioms such as large cardinal axioms claiming the existence of large cardinals are added. So, there is a variety of axiomatizations that are considered to be set-theoretic in character and at least some of their corresponding models are usually called 'models of set theory'.[7]

Usually, models of a theory will not be elementarily equivalent, meaning that they don't satisfy all of the same first-order sentences.[8] In particular, this comprises situations where the models disagree on a sentence A in the sense that A is true in one model and false in the other. In areas like set theory this is a well-known phenomenon that is of foundational importance to several of its research endeavours: in set theory, a major area of research is the study of mathematical statements that are independent of an appropriate axiomatization of set theory. The research is focused on establishing consistency and independence results as well as investigating in which situations these sentences can become true or false. The most well-known example is that of the Continuum Hypothesis, **CH**, stating that the cardinality of the set of real numbers is the next biggest cardinality after the size of the set of natural numbers. To prove that this statement is independent from some set-theoretic axiomatization, say **ZFC**, one usually builds a model of **ZFC + CH** and one of **ZFC + ¬CH** showing that **ZFC** is consistent with both statements. Gödel (1940) was able to provide such a model L in which

[6]One can also work with models of set theory in a second-order setting; this is especially fruitful when considering questions of categoricity. However, in the set-theoretic practice of the last decades, working within first-order logic has been the common standard as the main axiomatizations are first-order. So when we refer to models of set theory we normally assume a first-order context and explicitly say differently otherwise.

[7]Some set theorists such as Hamkins (2012) understand this even more inclusively, considering also models of much weaker theories than **ZF** to be models of set theory. Here we are not concerned with providing exact boundaries for which models are models of set theory or not; it is enough to know that the notion is not only used to denote models of **ZFC**, but also includes models for variations of it.

[8]I would like to thank the anonymous reviewer for pressing me on this point. It is fundamental to the new notion I aim to introduce that it is not solely based on the existence of such models. We will see what other ingredients this notion needs in due course.

CH holds; twenty years later Cohen (1963) completed the proof of independence by showing that there is a model of **ZFC** + ¬**CH**. Here we find explicit instances of non-elementarily equivalent models of one theory, where in one model **CH** is true and in the other one it is false. In the following we will call such models *incompatible models* for the independent sentence P.

Of course, this does not imply that the theory given by **ZFC** is *logically* inconsistent. On the contrary, as **CH** is independent from **ZFC**, neither **CH** nor ¬**CH** can be proven from **ZFC** (and the logical inconsistency of **ZFC** involving **CH** would mean that both can be proven). However, we can undertake a change in perspective, which will also serve to highlight the significance of this situation for inter-model inconsistency as defined below (Section 3): instead of studying models for a given theory, we can also consider a specific model and study the theories it is a model of. For instance, the model L provided by Gödel is not only a model of the axiomatization **ZFC**, it is also a model of the theory of **ZFC** + **CH**.[9] This also holds if we do not want to name **CH** as an axiom for the extended formal theory, but prefer to regard it as a theorem following from some axiom such as **V** = **L**, **V** = **Ultimate L**, etc. In the same way, the model provided by Cohen is a model of **ZFC** + ¬**CH**. So the two models considered here are not only models of the same theory, they can also be seen as models for two different theories that disagree on (at least) one mathematical sentence. Again, this does not mean much on a purely *logical* level, as we can of course find models of any theory that is not logically inconsistent and models of different theories can coincide. But, on the *mathematical* level, this a point worth highlighting, as this situation has been a motivation for, and subject of, much of set-theoretic research in the last decades.

In general, the fact that the same state of affairs can be of different significance from a mathematical and a logical point of view is not surprising. For example, finding a different proof for an already proven theorem might not be very important, logically speaking[10]. Mathematically, however, an additional proof can have a huge impact, for example, when one proof is constructive and the other abstract, one is more explanatory than the other, one uses only resources from the specific field of research in mathematics and the other connects to previously unconnected fields of research from mathematics, and so on. One reason for this is that formalized theories of mathematics capture only certain aspects of mathematics and leave out others that are closely related to the actual practice of mathematics.[11] In this paper I do not claim that one perspective is 'better' or more 'important' than the other; they all provide valuable information and can address different questions or aspects in and about mathematics. However, for the purpose of this paper, the inconsistency we are aiming at is of mathematical rather than logical significance and relies on the practice-based aspects of the mathematics involved. In

[9]More accurately, the axiomatization considered for L is **ZF**; then it was shown that this is consistent with the Axiom of Choice, **AC**, and the additional axiom of **V=L**, and **CH** then follows from this additional axiom. But, of course, this also means that L is a model of **ZFC** plus **CH**.

[10]This assumes that important logical background information is kept constant or comparable; for example, the proofs should be equal in being correct, formalizable, etc., and of the same logical strength, for example by not requiring different axiomatizations. Also, logical assessments of proof might change when one changes the background logic in which the mathematics is done, for example from classical to intuitionistic.

[11]This is one of the main motivations for the research done in the field of Philosophy of Mathematical Practice (PMP) (for an introduction to this field see Mancosu, 2008).

particular, it is based on the practice of model building in modern set theory and the extensive study of set-theoretic models it has given rise to. This is what we will examine in the next section.

2.1 The development of modern set theory

Set theory, as it is done today, has been shaped by groundbreaking methods and results obtained in the 1960s. What we call the 'modern' development of set theory has not been the object of much investigation in historical studies yet;[12] however, it is essential prerequisite knowledge for understanding current set-theoretic practice and the new concept of inconsistency I introduce in this paper. I will therefore give an overview of the development since the 1960s concentrating on the relevant issues of independence and model building.[13]

The mathematical value of being able to explicitly produce models of set theory and study what holds in them had been theoretically understood since Gödel's incompleteness theorems. Finding models of statements (as well as finding models for their negations) that were conjectured to be independent from some axiomatization of set theory has provided a reliable way of actually showing their independence ever since. Gödel (1940) himself developed a very important model, the so-called constructible universe L, with which he showed that the Axiom of Choice, **AC**, is consistent with **ZF**, and that the Continuum Hypothesis, **CH**, is consistent with **ZF+AC**. However, actually building such incompatible models explicitly, or even only one of them, was hard work. One reason for this is that for a long time the techniques to build models were not very broadly adaptable: Gödel's method of building L, by using constructible sets, cannot be adapted to show complementary results, namely, to show that ¬**AC** is consistent with **ZF** and ¬**CH** is consistent with **ZF+AC**. Other model-building techniques known during that time, such as the construction of models with urelements, exhibited similar limitations.

This situation changed drastically with the introduction of large-scale, flexible model-building techniques which were introduced in the 1960s, with Cohen's technique of forcing leading the way.[14] Cohen introduced forcing to prove the consistency of ¬**AC** with **ZF** and ¬**CH** with **ZFC**, but more importantly, he provided a technique that can be applied to a very wide variety of questions of independence: for a great number of appropriate statements P, one can use forcing to build specific models that satisfy **ZFC+P**. This construction is done along the following lines. We start from a model of **ZFC**[15], called the ground model, and from this

[12]An example for this is Ferreirós (2007); notable exceptions are the work of Moore (1987) and Kanamori (2008).

[13]For a more comprehensive overview of the historical development of set theory from its beginning, see, for example, (Kanamori, 2010).

[14]There are other such model-building techniques, such as Ultraproduct constructions and Scott's Ultrafilter method (see, for example, Scott, 1961; Kunen, 1970; Silver, 1971). This was groundbreaking work for the area of inner model theory, see, for example, (Mitchell, 2010) and (Steel, 2010).

[15]Actually, the meta-mathematical set-up of forcing is much more involved. The main approaches used in practice are the Boolean-valued model approach (for more detail, see Jech, 2003) and the countable transitive model approach (for more detail, see Kunen, 1980). The arguments of this article do not rely on the technical details of these approaches and go through in both of them, therefore I refrain from providing further detail here.

build other models, the extension models. These extensions are then the models of statements P or $\neg P$ that are independent of **ZFC**. The fundamental theorem for forcing states that the models built this way are again models of **ZFC**[16] and are definable in the ground model. So, one builds models via forcing that are both models of **ZFC** and models of an extension of this axiomatization while having the possibility to 'access' mathematical truths in the extension models from the ground model.

Here we can see how the change in perspective allowing us to focus on models as relevant informants about independent sentences actually works: with forcing, we start from some model of (a fragment of) **ZFC** and build two extensions, namely one of **ZFC+P** and the other of **ZFC + ¬P**. The extensions are models of **ZFC** but they are also models of extensions of **ZFC** that correspond to theories that decide the sentence P differently. This allows set theorists to study different formal theories within the mathematical field of set theory, where the models provide them with detailed insight into the set-theoretic truth within these models: through the access the forcing method provides us with, one can study these models in detail. Taking models of the negation of **CH** as an example, in addition to knowing that **CH** fails, one can also specify how it fails, i.e. which exact cardinality of an uncountable cardinal greater than $\aleph_1$ the continuum takes.[17]

This change in perspective is actually even more pronounced than this description seems to imply at first: in set-theoretic practice, it is often no longer explicitly considered which theory the model extensions correspond to, apart from trivial observations as to which additional statements hold.[18] Instead, one focuses on the models themselves and the set-theoretic information they provide. As set theorist Joel Hamkins puts it:

> A large part of set theory over the past half-century has been about constructing as many different models of set theory as possible, often to exhibit precise features or to have specific relationships with other models. Would you like to live in a universe where **CH** holds, but $\diamond$ fails? Or where $2^{\aleph_n} = \aleph_{n+2}$ for every natural number n? Would you like to have rigid Suslin trees? Would you like every Aronszajn tree to be special? Do you want a weakly compact cardinal κ for which $\diamond_\kappa(\mathrm{REG})$ fails? Set theorists build models to order. (Hamkins, 2012, 418)[19]

Nowadays, if a set-theorist wants to know if a certain statement holds in set theory, it often requires an answer that is dependent on models along the lines of 'It holds in the following models and not in others...' followed by an explanation how these models can be built and how

[16]As mentioned before, this also holds for fragments of **ZFC**. Forcing only requires a surprisingly weak fragment of **ZFC** to be carried out (see Mathias, 2015).

[17]In general this can be a great number of cardinals. However, there are some restrictions to this, as the continuum cannot take the size of a cardinal of cofinality ω. This follows from König's theorem (König, 1905).

[18]A good example is the model Cohen produced when showing that ¬**CH** is consistent with **ZFC**. It is usually only said that it is a model of **ZFC+¬CH**, although ¬**CH** is not considered to be a good axiom. Of course, logically one can use ¬**CH** (or **CH** for that matter) as an additional axiom to **ZFC**. However giving them the status of axioms is only justified by the desired resolution of their independence and therefore seems quite *ad hoc*.

[19]The exact meaning of the symbols used here are not relevant for the rest of the article. They can be found in any of the usual textbooks of set theory such as (Jech, 2003) or (Kunen, 1980).

they interrelate.[20] To push this further, new research programs have been developed, which study the modal structure of the relation between models (e.g. Hamkins and Löwe, 2008) and how they overlap via definability (e.g. Fuchs et al., 2015). But also research programs in the philosophy of set theory that aim at resolving the issue of the truth value of independent sentences have been impacted by this: Gödel's (1947) call for the search for new axioms was the prevalent approach for trying to decide independent statements, but nowadays programs in this vein concentrate on building a model that has certain desirable properties to make it a good candidate for providing information about independent statements. Which axiomatization this model corresponds to is then only a second step that requires further research. Here, independent statements are decided according to the way in which this desirable model decides the statements. Examples for such programs are (Woodin, 2001a) and (Woodin, 2017); we will discuss this in more detail in the next sections.

Summarizing, we can say that the introduction of model-building techniques such as forcing has had a significant impact on set-theoretic practice in terms of results, methodology, research goals and research topics. Akihiro Kanamori, one of the few set theorists giving a historical account of set theory from the introduction of forcing onward, describes forcing and the study of models it enables as having been a deeply transformative experience for the mathematical field:

> If Gödels construction of L had launched set theory as a distinctive field of mathematics, then Cohen's forcing began its transformation into a modern, sophisticated one. [...] Set theory had undergone a sea-change and with the subject so enriched, it is difficult to convey the strangeness of it. (Kanamori, 2008, 351)

He concludes with the following assessment:

> Forcing has thus come to play a crucial role in the transformation of set theory into a modern, sophisticated field of mathematics, one tremendously successful in the investigations of the continuum, transfinite combinatorics, and strong propositions and their consistency strength. In all these directions forcing became integral to the investigation and became part of their very sense, to the extent that issues about the method became central and postulations in its terms, 'forcing axioms', became pivotal. (Kanamori, 2008, 374)

So, we see that in the mathematical field of set theory the independence phenomenon is not the end point of its overall research endeavour. Instead, it is the starting point of the majority of the research done today. The logical basis of having incompatible models for an independent sentence can be made mathematically specific by being able to target mathematical statements that are of special import to set theory and build models specifically tailored towards making

[20] A nice example on how these mathematical developments impact approaches in the philosophical debate in set theory can be found in (Maddy, 2017), in which she revises one of her foundational goals for set theory from (Maddy, 1997, 26) to accommodate this dependence on models.

them true or false. This is strictly more than knowing the simple fact that a sentence is independent from one formal theory—it shows how the sentence behaves over incompatible models of (possibly different) formal theories of the mathematical field of set theory as well as showing how the models of these (possibly different) formal theories behave and interrelate. Through these constructions we can gain much additional knowledge about these sentences, the models in which they hold and the theories that may decide them or not. What is even more important: to set theorists, all of this knowledge is decidedly *set-theoretic*, meaning that it gives us insight into the structure and theorems of the mathematical field of set theory. So the logical phenomenon of having incompatible models of the same formal theory that decide a sentence P differently gains further mathematical traction: these models are not merely vehicles for carrying diverging truth values for the statement showing its independence; they become the relevant objects of research for the practice.[21] This is a major ingredient for the inter-model inconsistency which will be properly defined in Section 3; I will spell out more closely what this means in the next section.

2.2 The practice of models of set theory

In the last section, we have seen that modern set-theoretic practice[22] very much depends on the practice with models of set theory. This practice pursues two very fundamental research endeavours.[23] First, set theorists want to know if certain set-theoretic statements are true or false. For statements that can be decided in (some sufficient fragment of) **ZFC** this goal is achieved by proving or disproving the statement. For statements independent of **ZFC** this method is not viable (within **ZFC**). However, we have seen in the last section that model-building techniques like forcing provide us with a powerful method to tackle this problem. We can investigate the independent statements in different models of set theory, gaining much knowledge about when they are true inside a model or false inside a different one. Second, set theorists want to gain as much knowledge about their set-theoretic objects as possible, exploring the full extent and variety of their field of research. Again, model-building techniques constitute a most powerful tool, as they provide detailed knowledge about a wide variety of possible 'set theories' as instantiated by models of set theory.

Both research endeavours are therefore tightly connected with the set-theoretic practice dealing with models. In particular, they delineate two dimensions of the practice with models in set theory—what I want to call the *singular-model dimension* and the *multi-model dimension*.

[21]With this, I don't necessarily mean that they have become fundamental objects of research, as for example claimed by Hamkins (2012). Even if we only consider sets to be the fundamental objects of research of set theory, it remains a fact that they are investigated primarily inside some model of set theory, and in general their properties depend on the model in which they are studied.

[22]When I write about 'practice' here, I refer (mainly, but not exclusively) to the professional mathematical output of the set-theoretic community, like textbooks and research articles. This includes, in particular, the books and articles referred to in this paper. I will therefore refrain from listing more works except for the most recent and very comprehensive *Handbook of Set Theory* (Foreman and Kanamori, 2010).

[23]Of course, I do not claim that these are the only research endeavours of set-theoretic practice. Also, they are not specific to set theory. Interest in the truth of mathematical statements as well as gaining broad knowledge about the respective field of research is part of research in every mathematical discipline.

Before detailing what these dimensions are, let me emphasize that I do not claim that they instantiate two different practices in set theory that are pursued by different parts of the community (or, even stronger, two different communities). My overall argument in this section is that they are aspects of the *same* practice of *one* community, but nonetheless are incompatible in a way that gives rise to what I will call the inconsistent character of the practice of set theory. The rest of this section is dedicated to clarifying and arguing for this idea.

Considering the practice of models of set theory under the singular-model dimension means that we concentrate on the knowledge one model can provide. As an example, let us take the aforementioned model L. It was originally built to show the consistency of **AC** and **CH**, but its study has contributed much more than this to set-theoretic knowledge. Of major impact was, for example, the result that large cardinals that are greater than measurable cardinals do not exist in L (this goes back to Scott, 1961), and that L gives rise to fine-structure theory (see Jensen, 1972; Schindler and Zeman, 2009). So, set-theoretic study of a specific model[24] answers questions about consistency of statements, which set-theoretic entities it accommodates, what properties its elements have, etc.

The multi-model dimension, on the other hand, concentrates on the way in which different models relate: be it how models can arise from other models or how mathematical statements behave over different models. Examples are general investigations into how models can be built from other models (prime examples being the already discussed forcing and inner model program and specific theories that are dedicated to such questions, such as (Fuchs et al., 2015)) or more specific questions like investigating absoluteness,[25] or how sentences change their truth values when going from one model to another.[26] For a concrete example, take the fact that L is the smallest transitive model of **ZF** that contains all ordinals (see, for example, Jech, 2003, 182–183). That tells us much about how models of **ZF** relate: transitive models of **ZF** with the same ordinals and under a standard interpretation of $\in$ always give rise to the same constructible universe L.[27] Another example is that the fine structure of L has been essential in the first comprehensive study of how to generalize the technique of forcing from sets to classes, therefore providing new models to consider and new facts about how they relate.[28]

When pursuing the research endeavours outlined above, the lines of reasoning employed for each of them are deeply incompatible with one another, indeed inconsistent: when searching for the truth value of independent statements, the reasoning rests on providing a good argument for why the way a specific model resolves the independent statement under consideration[29]

[24]This often involves not only one model but a class of models that is similar in construction to the model. In the case of L, these are model like $L[A]$ for a parameter A, meaning that we build the models according to the same construction process as L but relative to the use of A.

[25]Sentences that don't change their truth value over models are called absolute (see, for example, Shoenfield, 1965)

[26]For forcing, there are sentences that can be fixed by some model so it cannot change when going to further models and there are sentences that remain changeable. (See Hamkins and Löwe, 2008).

[27]In particular, the L of L is again L.

[28]This is done in Friedman (2011). Class forcing is the name for notions of forcing where the underlying partially ordered set is class-sized. Although specific class forcings have been used since the seminal Easton (1970), Friedman provides the first general study.

[29]Under the 'search for new axioms' approach, which was prevalent before the focus on models as outlined

is the 'right' way to decide it, meaning that truth in that model informs us about the truth *simpliciter* of the statement. This argument is fundamentally informed by the singular-model dimension of the practice as the success of the argument rests on detailed information about the specific properties of the model and the set-theoretic statements that hold or fail in it. So, the singular-model dimension provides a valuable and straightforward strategy for reasoning about the truth value of independent statements. In contrast, the multi-model dimension of the practice, while still providing a helpful tool for finding model candidates, has no further significance for deciding the truth value of the statements under consideration.

In the case of exploring the full extent of set theory, the situation is reversed. Here, the reasoning of set theorists is fundamentally informed by the multi-model dimension; its aim lies in providing many models of set theory, studying how they can be built (or otherwise reached) from one another and how sentences behave when going from one model to another. So, the multi-model dimension provides a valuable and straightforward strategy to investigate what 'set theories' there are by answering questions like 'What models of set theory exist?', 'How are these models connected (e.g. are they extensions or grounds of one another)?' and 'What does this tell us about how the truth of independent statements varies over different models of set theory?' In contrast, the singular-model dimension of the practice, while still contributing to this research by providing additional knowledge about specific models, is of only instrumental import to this kind of reasoning.

So, while both dimensions play a certain role in both lines of reasoning, their respective role differs fundamentally. If one dimension is particularly informative as the basis for one type of reasoning, the other dimension is reduced to only providing technical tools. If set theorists ask whether **CH** is true or false, they cannot accept equally all the models they obtain in the multi-model dimension as models of set theory, as this would lead to reasoning that justifies a conclusion about the truth *simpliciter* of **CH** as being both true and false (according to its truth value in different models of set theory). If set theorists want to know what other 'set theories' exist and how they are related (therefore asking the questions about models of set theory from above), they cannot accept that these models of set theory inform them on the truth *simpliciter* of **CH**, as this would again lead to reasoning that justifies a conclusion about the truth *simpliciter* of **CH** as being both true and false (according to its truth value in different models of set theory). In this sense, reasoning in set-theoretic practice is deeply incompatible, because set theorists employ contradictory assumptions when following each line of reasoning. If set theorists both assume that all the models are models of set theory *and* they inform us about truth *simpliciter*, we arrive at a contradiction. However, if we reject one of these assumptions, we don't do justice to the actual practice of set theory; as we have seen from the discussion in the last and present sections, current set-theoretic practice jointly pursues *both* of these lines of reasoning by pursuing both of the above research endeavours and contributing to both of the above dimensions of practice.

In the inconsistency debate in the sciences, such an inconsistent situation is described as

in Section 2.1, this would have been the way in which one specific axiomatization decides the independent statement.

pertaining to reasoning (patterns). As correctly noted by Brown (2015, 416-7), most philosophers who argue for the existence of (interesting) cases of inconsistencies and their toleration in the sciences do so with respect to the *reasoning* of scientists, rather than their beliefs. The main issue is not the question of whether scientists held inconsistent beliefs at any given time, but whether the putative presence of inconsistent assumptions in their scientific reasoning gave way to explosion.[30] Until now, I have analogously argued for the claim that the practice of set theory is inconsistent as it employs mutually inconsistent reasoning patterns. Now, I want to defend the stronger but independent claim that the inconsistent practice in set theory even leads to inconsistent *beliefs*.[31]

In current practice, set theorists accept both models that decide **CH** as true *and* models that decide **CH** as false as providing putative evidence about **CH**. This situation occurs because some incompatible models, which decide **CH** differently, not only provide set-theoretic knowledge, but also provide justification for a belief about the truth *simpliciter* of **CH**, in fact a knowledge *claim* about the truth *simpliciter* of **CH**.[32] In other words, set-theoretic practice gives rise to justified belief that **CH** is true and justified belief that **CH** is false. This, again, means that set-theoretic practice is inconsistent not only with regard to reasoning, but also with regard to the collective epistemic situation of the set-theoretic community, as it gives rise to contradictory justified beliefs. But clarification is still needed to meet possible objections.

Firstly, let us come back to a claim I made at the beginning of this section, namely, that we have one practice of set theory that comprises both dimensions, instead of giving rise to two practices. Evidence for that can be found in a most different setting: standard textbooks of set theory treat the dimensions alluded to here as part of the same practice.[33] Practitioners of set theory that can be identified as emphasizing the significance of different dimensions do nonetheless work together frequently and fruitfully. One example is Joel Hamkins and Hugh Woodin (see e.g. Hamkins and Woodin, 2000, 2005, 2018). Deborah Kant recently conducted an interview study with 28 set theorists. She comes to the conclusion that set theorists agree at the level of mathematical practice, therefore providing no evidence for the claim of the existence of practices split along the line of the dimensions introduced above. I thus conclude that we are justified to speak of *the* practice of set theory.

Secondly, when I make claims about experience or knowledge in set-theoretic practice,

[30]Brown (2015) uses this observation to conclude that Vickers (2013), in arguing against putative examples of scientists holding inconsistent beliefs, simply misses the point:

> A preservationist study of inconsistency in science need not claim that the scientists involved believed an inconsistent set of sentences to be trueonly that they reasoned with sentences that were inconsistent in a way that preserved some interesting logical property of those sentences. (Vickers, 2013, 417)

[31]That is, *mathematical* beliefs, which are strictly larger than the sentences of some axiomatization of set theory closed under logical consequence; see above, Section 2, p. 5.

[32]Obviously, these are not justified *true* beliefs, because I take knowledge to be veridical and therefore—on pain of assuming dialetheism—cannot be both true and false.

[33]As the notions of singular-model and multi-model dimensions were introduced by the author of this article, one will not find them under this name in textbooks. But one can find the set-theoretic results and methods I subsumed under these dimensions there.

I assume *groups* to be the epistemic agents, on the collective level of the community, not individuals or particular research groups.[34] To illustrate this point: even if research group A spends all their lives working only with models of **ZFC+X** and research group B does the same for models of **ZFC+¬X**, they both contribute to the collective knowledge of the community about models of set theory that do or do not fulfill X. This leads to extensive knowledge and experience of the community with models incompatible with respect to X and in turn to our claim about justified belief in contradictory statements.[35]

Concluding, let me explain the scope and implications of my claim that the practice of set theory is inconsistent. First, my claim is *descriptive*. By diagnosing an inconsistency at the level of set-theoretic practice, I do not intend to suggest or prescribe a change of the practice: it is fully reasonable for set-theoretic practice to pursue the endeavours formulated above. As such, it is fully reasonable for the practice to comprise both the multi-model and the singular-model dimensions. As it is now, set-theoretic practice is in a state of *indecision* about which way to go or even whether such a way has to be chosen at all. The community as a whole does not endorse the additional commitment that only some of these models really are models *of set theory*, nor has it given up on resolving the open questions of independent statements.[36] Importantly, this shows that the inconsistency has not led to suspension of judgement, as is sometimes the case (and often normatively appropriate) when equally good evidence is found for a claim and its negation. On the contrary, the set-theoretic community is in a state of *de facto* inconsistency toleration. Set-theoretic practice can simply go on in this manner, continuing to not resolve the tensions between these endeavours and dimensions of the practice and therefore continuing to tolerate the inconsistency. Following the practice of set theory in the last decades, we have seen that such a situation is not detrimental to set-theoretic progress as it has not led to stagnation, but, on the contrary, has brought about vast and fruitful research producing impressive results.

This state of inconsistency, however, becomes a point of contention when the state of indecision is called into question and the *de facto* inconsistency toleration of the set-theoretic community is either challenged or underwritten by advancing normative philosophical reasons against or for it. From the inconsistency perspective developed so far, I see various philosophical programs in the philosophy of set theory as taking on this role, a claim I will preview in Section 4 and that is argued in Kuby (forthcoming) in full. In that section, I will also show that this new concept of inconsistency has wider consequences for the general inconsistency debate in the philosophy of mathematics. But first, in the next section I will make the concept of inconsistency—which I motivated, developed and defended in the previous sections—more precise by providing a matching definition.

[34] See, for example, Bird (2014) for a distributive model of group knowledge.

[35] Tying the epistemic dimension to the group level makes it unnecessary to show that there are set-theorists that believe both X and $\neg X$ at the same time, therefore making a debate about inconsistent beliefs of the individual, like the line of attack pursued by Vickers (2013), a moot point in our case.

[36] In the interview study by Kant this was expressed multiple times. Quotations include transcribed statements such as: 'I'm kind of optimistic about the things of the level like the continuum hypothesis. I think there is a good chance that has an answer and we'll answer it.'

3 Inter-model inconsistency in set theory

In the last section, I detailed the inconsistency present in set-theory at the level of its practice. In the following, I will present a new concept of inconsistency, *inter-model inconsistency*, which aims at capturing this inconsistent practice. This new concept will be introduced as an inconsistency in set theory. However, from the outset it is unclear what this means.[37] The usual notion of logical inconsistency is defined with respect to a formal theory. In the wider debate about inconsistency in mathematics, this is sometimes transferred to include practices that produce direct contradictions without specifying the formal theory behind this practice (for instance, when such a formal theory was not worked with in the practice). An example of this is the practice of the early calculus that operated amongst others with a parameter that was treated as being equal to zero and greater than zero *in the same calculation* (see, for example, Bueno, 2017). However, the inconsistency we are to define here is conceptually situated at the level of the collective knowledge that arises from the practice of set theory. In this situation, we are not given one proof or calculation in which it arises and that we can simply pick out, but have to take into account different aspects of the practice and the reasoning within it and study how they are interconnected. It is then the body of knowledge arising from there that is targeted by the new concept of inconsistency. This is a crucial point: it could *prima facie* be assumed that by referring to 'set theory' we always mean the formal theory based on the axioms of **ZFC**. However, we have already seen that this is not always the case. The term 'set theory' refers in practice to formal theories arising from different axiomatizations that can be weaker or stronger than **ZFC**. More importantly, there is a use of the term 'set theory' that is not primarily being defined by some specific formal theory, but rather refers to what I have called 'the mathematical field of set theory'. This is the body of knowledge to which set-theoretic practice gives rise and which goes beyond the set of sentences that follow from some axiomatization.[38] For example, it includes knowledge about how sentences behave over different models of set theory (that in turn can correspond to different formal theories): set theorists *know* that **CH** can be forced to hold or to fail over arbitrary models of set theory. This has consequences for how **CH** can be decided (or not) by certain formal theories; for example, it follows that **CH** cannot be decided when adding large cardinal axioms to **ZFC**.[39]

We now define the new concept of inconsistency as an inconsistency in the mathematical field of set theory, meaning the body of knowledge the practice gives rise to. With the following definition, we aim at grasping the inconsistent practice of set theory as closely as possible:

Inter-Model Inconsistency in Set Theory (MIST) The mathematical field of set theory is *inter-model inconsistent* if there are at least two first-order models of some appropriate

[37]I would like to thank the anonymous reviewer for pressing me to clarify this point.

[38]This is not specific to set theory and indeed is comparable to other mathematical fields, such as analysis and differential geometry as well as more formalizable fields like algebra and number theory. Here, despite being able to give axiomatizations and formal theories depending on them, the usual understanding when talking about 'algebra' or 'analysis' is that of a non-axiomatized, informal body of knowledge.

[39]This is due to the fact that **CH** can be forced to hold or fail via small forcing notions that are not impacted by large cardinals, see Lévy and Solovay (1967).

axiomatization of set theory, M_1 and M_2, and at least one statement P such that

1. P holds in one model and $\neg P$ holds in the other, so either $M_1 \models P$ and $M_2 \models \neg P$ or $M_1 \models \neg P$ and $M_2 \models P$ (*Incompatibility Clause*);

2. P is considered to be an important open problem[40] by the set-theoretic community. A resolution of this problem can have fundamental consequences for the entire field, and this is shown by concrete mathematical results. (*Open Problem Clause*); and

3. the models M_1 and M_2 (and possibly others like them) are recognized to represent the mathematical field of set theory. This recognition is grounded in the practice of the field, meaning that extensive research is done which leads to plentiful and fruitful knowledge about M_1 and M_2, recognizing them as models of set theory that decide the statement P differently. (*Models of Set Theory Clause*)

We say that M_1, M_2 and P *witness* the inter-model inconsistency if they fulfill the above clauses.

To show that MIST matches the inconsistent practice, let us look more closely at its different clauses. The Incompatibility Clause describes the basic logical situation of non-elementarily equivalent models of set theory and the fact that this can be made mathematically specific by being able to actually build such models and investigate the mathematics within them (see the discussion at the beginning of Section 2). As such, this clause also provides the logical and mathematical basis for MIST. However, as we have already discussed in Section 2, this clause alone does not provide us with an inconsistency: neither a logical one (as we don't have a contradiction within a formal theory) nor one related to the inconsistent practice of set theory. Taken by itself, the Incompatibility Clause only says that the sentence P can be shown to be undecidable in the base theory of M_1 and M_2, therefore simply matching the notion of independence in set theory.

The Open Problem Clause now encapsulates different aspects of the practice that we identifies as relevant for its inconsistent character in Section 2.2. Primarily, this is the research endeavour to resolve the question of truth values for independent statements and the way in which this emphasizes the singular-model dimension of the practice. The clause further provides a sharpening of the general situation given in the Incompatibility Clause. Instead of simply grasping facts about independence, this clause addresses the research that is done to resolve them. It also clarifies that MIST does not hold for independent statements generally, but only for ones that are considered to be of special mathematical interest for set theory. This special interest is determined by the way these independent statements are treated in set-theoretic practice, depending on how much research has been done about them, which role they might play in further developments in set theory, how many and which results they produce, etc. In

[40] This means that it is unknown whether P is true or false. If P is solvable in **ZFC** this is simply the question of whether it can be proven or refuted. However, as we have seen in Section 2.1, many of the sentences currently considered in set-theoretic practice are not of this kind. Here, the question becomes more complicated, because we first have to clarify the mathematical premises to resolve this question (such as under which axiomatization or in which model we consider them).

general, which P satisfies these conditions has to be clarified on a case-by-case basis. Examples are, amongst others, sentences such as **CH**, the generalized Continuum Hypothesis **GCH**,[41] the Borel Conjecture,[42] or Suslin's Hypothesis.[43] Statements that do not fulfil the Open Problem Clause are, amongst others, the Axiom of Choice (as it is generally regarded to be decided), the statement that **ZF(C)** is consistent (as this is generally assumed to be true; also, it being false would imply a logical inconsistency, therefore putting it outside the purview of MIST), or the original Gödel sentence used in his Incompleteness Theorem (it was constructed *ad hoc* to be independent, but is of no further set-theoretic importance or interest).[44]

The Models of Set Theory Clause captures the aspects of the practice of set theory opposed to the ones in the Open Problem Clause. As detailed in Section2.2, these aspects are connected to the research endeavour aiming at exploring the full extent of set theory and the associated multi-model dimension. It emphasizes the essential practice of recognizing those models as models of set theory which significantly contribute to set-theoretic knowledge. Like the Open Problem Clause, this clause also places constraints on the entities whose existence fulfills the Incompatibility Clause, but this time on the *models* that give rise to independence (instead of the sentence P under consideration, as in the Open Problem Clause). Again, a decision about which models fulfill these requirements is dependent on case-by-case studies. Candidates for models that will not give rise to MIST are ones that do not fulfil enough of **ZFC**, for example, missing the Powerset Axiom and/or the Axiom of Foundation.[45]

Taking all of the clauses of MIST together, I claim that the concept of inter-model inconsistency matches the inconsistency we described in the practice of set theory. Recall, the inconsistency in the practice of set theory as described in Section 2.2 arose out of the way the practice reasons about the truth of a suitable P and the truth of $\neg P$ either as truth *simpliciter* in answer to the open problem of P or as truth in a model under the perspective of acquiring knowledge about the breadth of set theory. This situation is mirrored in MIST: taking the Incompatibility Clause as the common logical and mathematical basis, the inconsistency arises out of jointly acknowledging what is described in both the Open Problem Clause and the Models of Set Theory Clause as fundamental parts of the mathematical field of set theory. Specifically, this includes the knowledge about how models can make some suitable P true or

[41]**CH** says that the size of the continuum $2^{\aleph_0} = \aleph_1$ and **GCH** generalizes this to the other cardinalities $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for every ordinal α.

[42]The Borel Conjecture states that all strong measure zero sets are countable. For more detail see, for example, (Jech, 2003, 564)

[43]Suslin's Hypothesis states that there are no Suslin Trees. For more detail see (Jech, 2003, 274).

[44]It might happen that there is a sentence P suitable for MIST that has logically equivalent formulations that give rise to mathematically different statements (if the Axiom of Choice would still be considered to be open, that would, for example, hold for Zorn's Lemma). If these statements are not considered as important open questions in set theory, they will not give rise to MIST: although fulfilling logical requirements such as the Incompatibility Clause, they do not fulfil the additional requirements by practice. However, this is not really problematic as every discussion about their truth value will be connected back to the discussion about the truth value of P. Therefore the set-theoretically relevant situation of MIST for P is used as a proxy for the logically equivalent statements.

[45]Note that we always regard models that decide a statement P satisfying the Open Problem Clause as ones that differ substantially. This follows from the significance these sentences have for set theory according to the Open Problem Clause. A possible objection along the lines of *intertranslatability* of models as discussed in (Steel, 2004, 7) therefore does not hold for sentences which fulfill the clauses of MIST.

false, the fact that set theorists recognize this as set-theoretic knowledge in their practice, and that this knowledge provides possible answers to the question about the truth *simpliciter* of P.

This matches the inconsistency in the practice not only in the way the inconsistency holds, but also in the way in which it can fail: if we have a statement P for which the Open Problem Clause fails yet corresponding models M_1 and M_2 satisfy the Models of Set Theory Clause, P does not witness inter-model inconsistency. As the search for the truth *simpliciter* of P is not pursued, the knowledge provided by the Models of Set Theory Clause is just knowledge about truth relative to these models and nothing more. If, on the other hand, we have M_1, M_2 and P that satisfy the Open Problem Clause but the Models of Set Theory Clause fails, the truth *simpliciter* of P can be decided according to the one model of set theory that decides P (as the other is not viewed as providing evidence about the truth *simpliciter* of P). Again, P does not witness inter-model inconsistency.[46] But for each of these scenarios, there is no inconsistency in practice, and in both cases, practice does not provide justified belief that P *and* that $\neg P$ in the sense specified in Section 2.2.

4 Consequences and applications

Over the last sections, we have introduced a new concept of inconsistency in set theory. The main motivation for this introduction is to relate the debate about inconsistency in mathematics more closely to the inconsistency debate in the sciences. To address this, let me relate MIST to the bigger picture of various concepts of inconsistency in the different debates.

As mentioned in Section 1, inconsistencies in the sciences are said to arise within a theory, between different theories, or between theories and observations. Here, inconsistencies are treated in an informal manner, as they do not occur in formal theories. In contrast to the science context, until now, inconsistency in mathematics was tied to the existence of a contradiction in a formal theory. In my view, the situation in set theory shows that there is a need for grasping the phenomenon of inconsistency in mathematics in a more fine-grained manner. This does not mean that MIST is informal in the same sense as inconsistencies in the sciences, as it has a logical basis as expressed in the Incompatibility Clause. However, it is also not completely formal, as it relies heavily on informal aspects of the practice. Specifically, I see MIST as describing a situation from a mathematical practice that is analogous to the way *inconsistencies between theories* occur in the sciences. The current situation between the Standard Model and General Relativity is usually taken as an example for such a case. Here, the inconsistency does not occur in one theory but comes up because two well-confirmed (collections of) theories disagree on the fundamental nature of spacetime: quantum field theories describe a fixed background and a preferred splitting of spacetime into space and time, while General Relativity describes a dynamical spacetime, with no preferred reference frame. This becomes a problem when

[46]Note that we do not consider the case of sentences and models for which the Incompatibility Clause fails. Such sentences are not independent and, as we have seen, they are handled differently by simply searching for proofs of them or their negation.

unification attempts across the whole body of knowledge of fundamental physics are made.[47] Inter-model inconsistency addresses an analogous situation in the formal sciences. Here, the mutually inconsistent theories are instantiated by models of set theory that disagree on a relevant statement from the body of knowledge of set theory: remember that the models under consideration are not only models of the base theory (e.g. **ZFC**) but also models of extensions such as **ZFC+CH**. At the same time, just as physicists working in either General Relativity theory or Electroweak theory and Quantum Chromodynamics are not impeded in their work by the inconsistency when working within the confines of each theory, set theorists are not impeded in their work when working within the confines of this or that model.[48]

So, with inter-model inconsistency we have broadened the topic of inconsistency in mathematics in a way that brings it closer to the more varied treatment of inconsistencies in the philosophy of science. This opens a wide field of further questions and possible transfer operations between the inconsistency debate in the sciences and in mathematics. The most important point in this regard is whether and how MIST gives rise to inconsistency-toleration strategies. There is indeed a philosophy that can be interpreted to address the situation described by MIST. This is the so-called universe/multiverse debate, which has been one of the main points of discussion in the philosophy of set theory in the last decades. Here, *universism* claims that there is an intended model of set theory that provides us with the truth *simpliciter* of the statements independent from axiomatizations of set theory. The view can be spelled out in different ways, for example, by discussions about the exact ontological status of this intended model, how we can approximate it, etc.[49] Recent examples of programs developing universist positions are Woodin (2001b) and Woodin (2017).[50]

Multiversism is, roughly, the counterposition to universism and, in its broadest form, denies that there is one intended model of set theory.[51] As their positive goal, multiversist programs aim at recognizing the variety of set theory provided by the different models available and explore their philosophical significance: what the exact ontological status of these models is, and what this tells us about the truth value of independent statements, a question which is handled very differently in diverse multiversist approaches.[52]

This situation is discussed in detail in the accompanying article of Kuby (forthcoming). He

[47]See Carlip (2001) for a discussion of various approaches aiming at unification by quantizing gravity. Note that mine is a very loose characterization of this situation, but in any case the details do not impact the analogy I want to draw here, which is based only on the fact that there are mutually inconsistent theories for a broader field of research in the sciences.

[48]The reason why they are not impeded might be different, however, as they are, for instance, tied to the different way the practices in mathematics and the sciences (do not) rely on empirical information.

[49]Non-realist universist views are sometimes called absolutist views (see, for example, Koellner, 2013). An example is the position of Arealism in Maddy (2011).

[50]Note that these really are different programs: while both were pursued by Woodin at different times, they differ in their concrete methods and results. Amongst others, if the program based on Ω-logic (Woodin, 2001b) is successful, **CH** will be decided negatively; however, if the 'Ultimate L' program (Woodin, 2017) is brought to successful conclusion, **CH** will be decided positively. Both programs are currently in progress as major conjectures are still open.

[51]Nowadays, a range of different versions of multiversism have been developed; see, e.g., the overview in Antos et al. (2015).

[52]These programs range from full-blown Platonistic pluralism like Hamkins (2012) to non-realist pictures like Arrigoni et al. (2013) or the more methodologically oriented views of Steel (2014).

shows that various research programs in set theory and its philosophy, such as the universist and multiversist programs, implicitly respond to MIST by proposing resolutions to inter-model inconsistency. (Kuby, forthcoming) investigates in detail how MIST gives rise to a variety of consistency-preserving and inconsistency-tolerating strategies. To give a preview of the fruitfulness of introducing MIST, I will give a brief overview of the results of (Kuby, forthcoming) and refer to it for the complete argument.

Kuby (forthcoming) discusses three programs:[53] the Ultimate L program by Woodin (2017), the set-theoretic multiverse program by Hamkins (2012) and the hyperuniverse program by Friedman (see Arrigoni et al., 2013; Friedman and Ternullo, 2018). First, the Ultimate L program is taken as an example for an inter-model consistency-preserving strategy: the program aims to eliminate the Models of Set Theory Clause in a way that recognizes just one of the incompatible models from the Incompatibility Clause as the model representing the universe of set theory (i.e. the intended model of set theory). This model is called Ultimate L, as it is similar to Gödel's model L in many respects (for example, by being impervious to forcing), but still allows for large large cardinals (which Gödel's L does not).[54] If we acknowledge Ultimate L to represent (an approximation of) the set-theoretic universe, not only does the Models of Set Theory Clause fail, but in turn the Open Problem Clause is answered: in Ultimate L, **CH** holds and therefore it is not an open problem anymore. This is a inter-model consistency-preservation strategy because it eliminates the inter-model inconsistency in set theory completely by getting rid of the two MIST Clauses that match the inconsistent practice of modern set-theory. More precisely, using the classification of Bueno (2017), it is a case of consistency preservation via information restriction as we eliminate the models inconsistent with the truth of **CH** by excluding them from the set-theoretic universe.

The set-theoretic multiverse program goes a very different route: it argues that we should fully embrace the Models of Set Theory Clause by recognizing that there are incompatible *universes* of set theory, which Hamkins assumes to exist in a Platonistic manner. More than that, Hamkins (2012) argues that the different models stand for different conceptions of sets, each giving rise to its own set-theoretic universe, therefore making the incompatibility of the models more fundamental than being only instantiated by deciding independent statements differently. However, the multiverse program also partially resolves the inconsistency described in MIST: according to Hamkins, the question of **CH** is not an open problem, because it has already been answered—in, however, a very peculiar way: instead of deciding **CH** in one way or the other, Hamkins sees the problem as resolved by the detailed knowledge set theorists have about the behaviour of **CH** over the different models. If one accepts this as an answer to **CH**, then the Open Problem Clause is resolved. Kuby (forthcoming) argues that this instantiates a weak form of inter-model inconsistency toleration: the Open Problem Clause gets resolved when assuming a changed meaning of what an answer to the question of **CH** is. However, the

[53]In the following, we will only discuss the situation where the undecidable sentence P from MIST is the Continuum Hypothesis. Note, however, that all three programs work for other undecidable sentences as well.

[54]Large large cardinals are generally the large cardinals above and including the measurable cardinals, separating them from the smaller large cardinals such as inaccessible cardinals.

Models of Set Theory Clause is strengthened.

The hyperuniverse program can be interpreted to employ a full-blown inter-model inconsistency toleration strategy. In essence, the hyperuniverse program proposes a compartmentalization strategy to deal with the situation of MIST. Each compartment consists of a collection of models that satisfy some intrinsically justified axiom (additional to **ZFC**) that in turn resolves some of the independent sentences, such as **CH**. So, there are different compartments that satisfy these sentences differently and we have reason to adopt the models in these compartments as genuine models of set theory because they correspond to a well-justified axiomatization. In this sense, we can see that the hyperuniverse program does not reject or resolve any of the clauses of MIST. In particular, it retains both the Open Problem Clause[55] and the Models of Set Theory Clause. The program progresses by 'thinning out' the models that count as models of set theory because it only recognizes intrinsically justified models. However, in general, this still leaves a wide variety of models to be considered as set-theoretic and we are well justified to acknowledge all of them as set-theoretic. It also organizes the models (and corresponding theories) in a way which makes it easier to track and compare the different versions of the body of knowledge of set theory the compartments give rise to. All of this is aimed at providing an answer to the open question of **CH** and similar important independent statements. So, in the end, it provides a way of doing further set-theoretic research while retaining the underlying inter-model inconsistency (strengthening it to something like an 'inter-compartment' inconsistency).

According to Kuby (forthcoming), the last two programs provide inter-model inconsistency toleration strategies. To be sure, none of the proponents of these programs use (or mention the need for) para-consistent logics, which seems to divert from the usual discussion in the philosophy of science (see, for example, Bueno, 2017). To explain this difference, one has to keep in mind that the proponents, as logicians, only classify 'contradictions in formal theories' as inconsistencies—and no logical inconsistency has been detected in the cases discussed. Furthermore, from a logical point of view, set theorists are still justified to work with classical and have no need to adopt a para-consistent logic: after all, MIST describes an inconsistency in the body of knowledge of set theory and this is not the logical notion of inconsistency. We can conclude that, even if philosophical programs in set theory started to adopt para-consistent logics to explain how their compartmentalization strategies work, we expect from a MIST perspective that the mathematical practice will continue to use classical logic.[56]

However, the question of how to accommodate inter-model inconsistency toleration in the philosophical programs remains open. Kuby (forthcoming) elaborates how Hamkins manages to avoid the use of para-consistent logics by re-defining what it means for an open question in set theory to be answered. However, he also points out that a similar strategy is not available for Friedman's hyperuniverse program. Here, MIST provides a challenge to the program: by

[55] Indeed, one main motivation of the program is to find answers to the truth of **CH** and similar statements.

[56] This addresses a worry by Vickers (2013, 238) as to what can be gained by reconstructing a practice employing classical logic by means of a para-consistent logic framework. In the case of MIST, the answer is clear: the gain is a better understanding of global views on the practice, while leaving the actual mathematical work with the logic used by its practitioners intact.

pointing out that the program employs an inconsistency-tolerating strategy, it requires an answer to how such a toleration can be accommodated on a philosophical level *even if* it does not require the use of para-consistent logic on the mathematical level. This shows that reinterpreting existing programs in the philosophy of set theory from the MIST perspective provides new insight: first, we see that it is possible to present inconsistency-toleration strategies that do not have to lead to the use of para-consistent logics; and second, it requires programs like Friedman's to provide further clarification on how to philosophically justify its strategy.

Further, MIST might also provide motivation to develop new programs. The multiverse programs discussed above both gave rise to inconsistency-toleration strategies. Kuby (forthcoming) supplements this by outlining a possible program that is multiversist in nature and yet provides a consistency-preservation strategy, which he calls the *algebraic multiverse*. Such a multiverse program rejects the very notion that there is an open problem regarding **CH** in the first place, therefore no need arises to provide an answer. Such a view is reminiscent of the way in which algebraic fields like group theory handle questions of the kind 'Does the commutative property hold?' This is not regarded as an open question in group theory. In fact, it is not even a well-formulated question, as it can only be answered with respect to a certain domain, e.g. along the lines of 'Is the relation of a certain group commutative or not?' In the same way, this multiversist position argues that there is no open question regarding **CH**, i.e. there is no fact of the matter about the truth *simpliciter* of **CH**. Instead, we just work with some models where **CH** holds and some where it does not hold. So here, the Open Problem Clause does not get answered as is the case in Hamkins' multiverse, it simply gets rejected outright. This, in turn, can be seen as analogous to a compartmentalization strategy to secure consistency in the sense of Bueno (2017, 240). Such a consistency-preservation strategy is not possible for a formal theory that is logically inconsistent in the usual sense, as such a theory has no models with which compartmentalization can be achieved. Therefore, not only does MIST allow formulation of this multiverse program in precise terms; the resolution strategy it employs is unique to the setting of inter-model inconsistency.

After having seen various applications the notion of MIST can have, we can ask how to generalize it. The notion of inter-model inconsistency was developed to account for an actual case in the mathematical field of set theory, thereby making sure that there is at least one actual instance of this kind of inconsistency in mathematical practice. A natural further step is to ask where else we can find an inconsistency in the spirit of MIST in mathematics. As the notion of MIST specifically targets set theory, we have to consider how one could patch this definition to be applicable to other areas in mathematics.

A likely case of a similar type of inter-model inconsistency might be found in the specific development stage of geometry when non-Euclidean geometries began to appear. To make a full transfer of inter-model inconsistency to geometry, the details of (the history of) non-Euclidean geometries would have to investigated, which is outside the scope of this paper. But let me outline the underlying idea: with non-Euclidean geometries, statements like the Parallel Postulate are decided differently in theories arising from different axiom systems (satisfying an analogy to the Incompatibility Clause of MIST). The discovery of these geometries posed both philosophical and mathematical quandaries and, for some time, it was an open question how to deal with the problem of relating this discovery to the concept of physical space (a variant of the Open Problem Clause). The massive and fruitful mathematical work by Gauss, Bolyai, Lobachevskii, Riemann and others contributed significantly to the view that these systems should be counted as part of geometry (creating a similar situation to the one described by the Models of Set Theory Clause) (for more details on this development, see the overview in Gray, 2019, Ch.4, Ch.5).

Supposing that the underlying inconsistency here is similar to inter-model inconsistency, the case of non-Euclidean geometry is also interesting because the putative *de facto* inconsistency toleration was resolved very quickly:

> [The] acceptance of non-Euclidean and Riemannian geometries went beyond the presentation of a consistent formalism. It marks the acceptance of the abstract view that geometry is whatever can be described in the Riemannian formalism: one has a very general framework, allowing for a dizzying number of concrete specifications. Thus the door was opened to the view that there are many geometries, each of which must be consistent, and none of which need to refer to Euclidean space, however intuitive this may be. (Gray, 2019, Ch.5.2)

With this reaction, the community rejected the Open Problem Clause of (a geometry-related version of) inter-model inconsistency, therefore defusing the underlying inconsistency.[57] If we transfer this resolution to the *de facto* inconsistency witnessed in the historical development of geometry back to the case of set theory, it strengthens the viability of the algebraic multi-verse approach that Kuby (forthcoming) discusses as a multiverse program with a consistency-preservation strategy. This approach offers a solution to inter-model inconsistency by rejecting that any of the models of set theory refer to an intended model (or other forms of realism). It can be argued that this is a likely future for set theory as we have seen that something similar took place in other areas of mathematics (i.e. geometry).[58]

5 Conclusion and outlook

The main aim of my paper was to introduce the new concept of inter-model inconsistency. If successful, this introduction broadens the notion of inconsistency in mathematics to encompass (at least) two different kinds, one based on the existence of a contradiction in a formal

[57]There are several works in the philosophy of set theory that compare the contemporary situation in set theory with the situation in geometry addressed here (see, for example, Hamkins, 2012). To be sure, I suggest that this analogy might only work at a very abstract level in that both situations are occurrences of a MIST-style inconsistency, but the historical details are quite different and we can expect the practices to be very different as well. See also the related discussion in Kuby (forthcoming).

[58]Note that—*pace* Hamkins—this is a different solution than the one proposed by Hamkins (2012). He ties a very strong ontological claim to his multiverse, according to which the set theories represented by the models all refer to genuine universes of set theory.

theory and the other by contradictory directions in the practice that can be made exact via models. I also argued that this notion should be considered to be a proper form of inconsistency by detailing the situation of modern set-theoretic practice and arguing that MIST grasps the inconsistency in this practice correctly. Lastly, I presented three avenues in which MIST can contribute to existing debates. First, MIST connects the debate about inconsistency in mathematics more closely to the philosophy of science by providing a more varied picture of inconsistency in mathematics and showing that there actually are inter-model inconsistency-tolerating strategies in set theory, a point that has been contested for logical inconsistency. Second, these points also contribute to the inconsistency debate in mathematics. In particular, MIST shows that inconsistency toleration does not always have to necessitate the introduction of para-consistent logics. Third, I have sketched how the debate in the philosophy of set theory can be reinterpreted in the framework of MIST and developed further. I therefore conclude that the introduction of MIST (and its possible generalizations) can have a fruitful impact on several debates by contributing new directions of research and a unique point of view in already existing discussions.

Summarizing, the main motivation for introducing this new notion of inconsistency is to provide a framework for *describing* a state of *de facto* inter-model inconsistency in a mathematical field and a foundation for ways in which inconsistency toleration-like strategies can be diagnosed in mathematics. This paper presents the theoretical foundations for the latter as it introduces a type of inconsistency that will not necessarily trigger consistency-preservation strategies usually found in response to logical inconsistency. Whether and how inter-model inconsistency does give rise to toleration is discussed in Kuby (forthcoming), which is the second part in the series of papers dedicated to the topic of inter-model inconsistency. Kuby (forthcoming) goes back to the roots of inter-model inconsistency, the situation of undecidable statements in set theory, showing how programs in the philosophy of set theory can be framed as reactions to MIST and be categorized on the range leading from consistency-preservation to inconsistency-toleration strategies.

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