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AN INEQUALITY FOR THE LEBESGUE MEASURE AND ITS APPLICATIONS

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Abstract. In [*Univ. Beograd Publ. Elektrotehn. Fak. Ser. Math.* **15** (2004), 85–86], the first author of this paper proved a new inequality for the Lebesgue measure and gave some applications. Here, we present a new applications of this inequality.

1. Introduction

If λ is the Lebesgue measure on the set of real numbers $\mathbb{R}$ and $\{A_n\}$ sequences of Lebesgue measurable sets in $\mathbb{R}$, then we have the following inequality:

$$\lambda(\underline{\lim} A_n) \leq \underline{\lim}\, \lambda(A_n).$$

But for the inequality

$$\overline{\lim}\, \lambda(A_n) \leq \lambda(\overline{\lim} A_n)$$

we must suppose that $\lambda(\cup_{i=n}^{+\infty} A_n) < +\infty$ for at least one value of n (see [6, p. 40]).

Example 1.1. For a family of intervals $I_n = [n, n+1)$, $n = 0, 1, \ldots$, we have: $\overline{\lim}\, \lambda(A_n) = 1$ and $\lambda(\overline{\lim} A_n) = 0$.

In [1] the first author presented the following inequality, and as its applications short and simple proofs of two famous results of Steinhaus.

Proposition 1.1. *Let A be a measurable set of a positive measure and $\{x_n\}$ a bounded sequence of real numbers. Then*

$$\lambda(A) \leq \lambda(\overline{\lim}(x_n + A)).$$

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2. Results

In this paper we present some further applications of this inequality. First of them is the following simple proof of the continuity of measurable solution of Cauchy functional equation. The first proof of the this statement was present by M. Fréchet [4], but it depends on the *Axiom of Choice*. Proofs which are independent of this axiom were obtained by S. Banach [2] and W. Sierpinski [9].

Corollary 2.1. (M. Fréchet [4]) *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a measurable function such that*

$$f(x+y) = f(x) + f(y),$$

for any $x, y \in \mathbb{R}$. *Then* f *is a continuous function.*

Proof. Let f be a noncontinuous function. Then there exists $\varepsilon > 0$, $x_* \in \mathbb{R}$, and a convergent sequence $\{x_n\} \subseteq \mathbb{R}$, such that $\lim x_n = x_*$ and $|f(x_n) - f(x_*)| > \varepsilon$. Then, by Lusin's theorem, there exists a compact set of the positive measure K such that f is continuous on K. The sequence $\{x_n\}$ is bounded, because it is convergent. Then, according to Proposition 1.1, we have

$$\lambda(\overline{\lim}(K - x_n)) > \lambda(K) > 0,$$

which implies that

$$(\overline{\lim}(K - x_n)) \neq \emptyset.$$

Thus, there exists $t \in \mathbb{R}$ and a subsequence $\{x_{n_j}\} \subseteq \{x_n\}$ such that $\{t + x_{n_j}\} \subseteq K$. From $x_{n_j} \to x_*$ it follows $t + x_* \in K$, because K is closed. So

$$\lim(f(x_{n_j} + t) - f(x_* + t)) = 0,$$

because f is continuous on K. Hence

$$\lim(f(x_{n_j} + t) - f(x_* + t)) = \lim(f(x_{n_j}) - f(x_*)) = 0,$$

which is a contradiction. $\square$

The second application is the following simple proof of the *Karamata's uniform convergence theorem*, which is one of fundamental results in the theory of regularly varying functions (cf. [3, 5, 8]).

Corollary 2.2. (J. Karamata [7]) *Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function such that*

$$\lim_{s\to+\infty}[f(t+s) - f(s)] = 0,$$

for all $t \in \mathbb{R}$. Then

$$\lim_{s\to+\infty} \sup_{t\in[a,b]} [f(t+s) - f(s)] = 0,$$

for any $a, b \in \mathbb{R}$ such that $a < b$.

Proof. Using Egoroff's theorem, it follows that for any $a, b \in \mathbb{R}$ $(a < b)$, there exists a measurable set $A \subseteq [a, b]$ of a positive measure such that

$$\lim_{s\to+\infty} \sup_{t\in A}[f(t+s) - f(s)] = 0.$$

Assume now that the convergence is not uniform on $[a, b]$. Then there exists $\varepsilon > 0$, $\{x_n\} \subseteq [a, b]$, $\{y_n\} \subseteq \mathbb{R}$, such that $\lim y_n = \infty$ and

$$\lim[f(x_n + y_n) - f(y_n)] > \varepsilon.$$

By Proposition 1.1 we have

$$\lambda(\overline{\lim}(A - x_n)) > \lambda(A) > 0,$$

which implies that there exists $t \in \mathbb{R}$ and the subsequence $\{x_{n_j}\} \subseteq \{x_n\}$ such that $\{t + x_{n_j}\} \subseteq A$. Then

$$|f(x_{n_j}+y_{n_j})-f(y_{n_j})| \le |f(x_{n_j}+t+y_{n_j}-t)-f(y_{n_j}-t)|+|f(y_{n_j}-t)-f(y_{n_j})|.$$

Now

$$\lim |f(x_{n_j} + t + y_{n_j} - t) - (f(y_{n_j} - t))| = 0,$$

because $\{t + x_{n_j}\} \subseteq A$ and $\lim(y_{n_j} - t) = \infty$. From

$$\lim(f(y_{n_j} - t) - f(y_{n_j})) = 0$$

it follows

$$\lim(f(x_{n_j} + y_{n_j}) - f(y_{n_j})) = 0,$$

which is a contradiction. $\square$

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