Purity and Explanation: Essentially Linked?

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PURITY AND EXPLANATION: ESSENTIALLY LINKED?

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Abstract. In his 1978 paper "Mathematical Explanation", Mark Steiner attempts to modernize the Aristotelian idea that to explain a mathematical statement is to deduce it from the essence of entities figuring in the statement, by replacing talk of essences with talk of "characterizing properties". The language Steiner uses is reminiscent of language used for proofs deemed "pure", such as Selberg and Erdős' elementary proofs of the prime number theorem avoiding the complex analysis of earlier proofs. Hilbert characterized pure proofs as those that use only "means that are suggested by the content of the theorem", a characterization we have elsewhere called "topical purity". In this paper we will examine the connection between Steiner's account of mathematical explanation and topical purity. Are Steiner-explanatory proofs necessarily topically pure? Are topically pure proofs necessarily Steiner-explanatory? Answers to these questions will shed light on the general question of the relation between purity and explanatory power.

Keywords. Philosophy of mathematics, mathematical explanation, purity, metaphysics, epistemology

1. Introduction

The *American Mathematics Monthly* is a journal published by the *Mathematical Association of America*, intended for a general audience from high-school students to researchers. Each issue contains a list of problems for readers to solve, as well as solutions of problems published in earlier issues. The following problem was published in 2002:

> Problem 10830. Proposed by Floor van Lamoen, Goes, The Netherlands. A triangle is divided by its three medians into 6 smaller triangles. Show that the circumcenters of these smaller triangles lie on a circle. (Cf. Edgar et al. [2002], pp. 396–397).

The editors then gave the following commentary on the solutions they received from readers:

> The submitted solutions used analytic geometry (or complex numbers) and involved lengthy computations (some done with Maple or Mathematica). The editors felt that a coordinate-free statement deserves a coordinate-free solution; such a solution may shed more light on why the result is true. (*Ibid.*)

Date: October 6, 2022.
Thanks to Adrien Champougny, Christopher Pincock,Tabea Rohr, and the editors of this volume, for their comments on an earlier draft.

Though the editors believed that the analytic solutions they received were correct, they nevertheless sought other solutions "closer" to the problem, avoiding coordinates. Such solutions, they wrote, might better show why the result is correct.

This statement, published in a completely ordinary mathematical journal, indeed one aimed at a wide audience, shows how closely two different properties of mathematical proofs are linked in common discourse. One of these properties is the one known as *purity*, in which a proof of a theorem avoids what is "distant" or "foreign" to what is being proved. Instead, a pure proof uses what is "close" or "intrinsic" to its object. The other property is the one known as *explanation*, in which a proof gives the reason why a theorem is correct. Purity and explanation represent two *ideals* of mathematical activity because both are commonly (though by no means universally) posed as goods: that a pure proof of a theorem is better than an impure proof of it, cetirus paribus, and similarly an explanatory proof of a theorem is better than an unexplanatory proof of it.

Concern for purity has been a part of mathematics since its beginnings. Some mathematicians have judged the application of algebra in geometry to be "rather far" from the problems at hand. For instance, Newton wrote that the application of algebra to geometry is "contrary to the first Design of this Science", giving purely geometric proofs of theorems that Descartes had proved using his algebraic methods (cf. Newton [1967], pp. 119–20). Complex analytic proofs of number theoretic results are another example. In analytic number theory, one uses imaginary numbers to solve arithmetic problems. For instance, the prime number theorem, stating how prime numbers are distributed among the natural numbers, was first proved by complex analytic methods. Mainstream mathematicians judged these methods "very remote from the original problem, and it is natural to ask for a proof of the prime number theorem not depending on the theory of a complex variable" (cf. Ingham [1932], pp. 5–6). Such a proof was found independently by Erdős and Selberg, contributing to Selberg's winning a Fields Medal in 1950. While of course most mathematicians also seek impure proofs, the search for purity remains an important part of mathematical practice.

Purity and explanation have been linked by many mathematicians and philosophers, since antiquity. This link was central to Aristotle's epistemology. Aristotle identified a cognitive attitude he called ἐπιστήμη [*epistêmê*], typically translated as "understanding". This attitude obtains, he wrote, "when we think that we know the cause on which the fact depends, as the cause of that fact and of no other, and, further, that the fact could not be other than it is" (*Post. An.*, 71b8-12). To know the cause of a fact, he held, is to grasp "the 'why' of it (which is to grasp its primary cause)" (*Physics* II.3 194b18–20). Answers to "why" questions are given by deductions he called "scientific demonstrations", which concern essential attributes of what is to be deduced. We may thus summarize Aristotle's account as follows. A scientific demonstration, one that engenders understanding, is a causal demonstration, and in mathematics that means a proof from definitions that state the essences of what composes the conclusion. Thus we may characterize Aristotle as seeking *explanations*.

As we have seen, for Aristotle essence and explanation are fundamentally linked. He drew a further conclusion from this link: "It follows that we cannot, in demonstrating, pass from one genus to another. We cannot, for instance, prove geometrical truths by arithmetic" (*Post. An.*, 75a38). Reasoning which crossed generic lines in this way was termed μετάβασις εἰς ἄλλο γένος [*metábasis eis állo génos*], kind-crossing, and could not provide for understanding. He argued for this by observing that the cause of a fact must be of the same kind as that fact; indeed for Aristotle, this relation is characteristic of a causal relation, as opposed to a merely accidental relation. Since scientific demonstrations must reflect causal relations, and causal relations do not cross kinds, it follows that for Aristotle scientific demonstrations must be, in our terms, *pure*. (For more on Aristotle's argument against kind-crossing, cf. Steinkrüger [2018].)

Aristotle's conception of scientific understanding and demonstration were dominant in antiquity and the Middle Ages. By the early modern period the conception of causality that supported Aristotle's metaphysics and epistemology receded in importance, as conceptions closer to newer experimental practices arose. Yet in mathematics, where these newer causal models seemed inapplicable, the Aristotelian conception continued to be taken seriously. For instance, Bolzano drew upon it in his mathematical writings:

> I must point out that I believed I could never be satisfied with a completely strict proof if it were not *derived from the same concepts which the thesis to be proved contained*, but rather made use of some fortuitous, alien, *intermediate concept* [*zufälligen, fremdartigen Mittelbegriffes*], which is always an erroneous μετάβασις εἰς ἄλλο γένος. In this respect I considered it an error in geometry that all propositions about angles and ratios of straight lines to one another (in triangles) are proved by means of *considerations of the plane* for which there is no cause [*Veranlassung*] in the theses to be proved. (Cf. Bolzano [1999b], p. 173)

He later applied this reasoning to justify the value of his pure proof of his version of the intermediate value theorem (cf. Bolzano [1999a]; for more on the link between these works, see Centrone [2016]).

This link between purity and explanation continued in the twentieth century. In a textbook on linear algebra and geometry, Dieudonné (a member of Bourbaki) offered the following more general remarks on the value of purity:

> An aspect of modern mathematics which is in a way complementary to its unifying tendencies…concerns its capacity for sorting out features which have become unduly entangled…It may well be that some will find this insistence on "purity" of the various lines of reasoning rather superfluous and pedantic; for my part, I feel that one must always try to *understand* what one is doing as well as one

> can and that it is good discipline for the mind to seek not only economy of means in working procedures but also to adapt hypotheses as closely to conclusions as is possible. (Cf. Dieudonné [1969], p. 11)

Dieudonné thus suggested purity provided better understanding of what was proved. Another such view is offered by the mathematicians Gelfond and Linnik. Writing of problems in number theory statable in elementary terms like the prime number theorem, they remark that solving these problems "has often required extremely complicated devices, at first sight remote from the theory of numbers." These transcendental methods "lead in numerous cases to extremely strong and precise results" and as a result "one cannot talk of rejecting transcendental methods in modern number theory." They then add:

> However, it is the natural desire of an investigator to search for a possible more arithmetic route to the solution of problems which have an elementary formulation. Besides the obvious methodological value of such a way, it is also important in that it frequently gives a simple and natural view of the theorem obtained and the reasons underlying its existence. (Cf. Gelfond and Linnik [1966], pp. ix–x)

On this view, a pure proof can reveal the reasons for a theorem: that is to say, an explanation of it.

On these accounts surveyed so far, there is a link between explanation and purity. Some recent work on explanation has tried to drive a stake between explanation and purity. Pincock introduced a particular kind of explanation he called "abstract mathematical explanation" and argued that explanatory proofs of this kind are generally impure (cf. Pincock [2015]). Rather than dwell on Pincock's well-reasoned argument, we observe that it focuses on one particular construal of mathematical explanation. In this paper, we will follow its lead in this regard, and examine the link between another particular construal of mathematical explanation, that of Mark Steiner. We will also focus on one particular construal of purity, which we have elsewhere called the "topical" conception (cf. Detlefsen and Arana [2011]). We will thus continue by briefly presenting these two conceptions, and then turn to the relationship between purity and explanation so construed.

2. Topical purity

As a rough characterization, a proof is *pure* if it draws only on what is "close" or "intrinsic" to what is being proved, rather than on what is "extraneous", "distant", "remote", "alien", or "foreign" to it. This characterization suffers from imprecision because of the variety of ways in which these distance measures between proof and theorem may be understood. While there are many such ways to precisify such measures of distance (cf. Arana [2022] for an overview), in this paper we will focus on one that we take to be central to mathematical practice, what we have called the “topical” conception of purity (cf. Detlefsen and Arana [2011]). This conception was presented by Hilbert, remarking on the apparent impurity of the spatial proof of the *planar* Desargues theorem:

> Therefore we are for the first time in a position to put into practice *a critique of means of proof*. In modern mathematics such criticism is raised very often, where the aim is to preserve *the purity of method* [*die Reinheit der Methode*], i.e. to prove theorems if possible using means that are suggested by [*nahe gelegt*] the content [*Inhalt*] of the theorem. (Cf. Hilbert [2004], p. 315–6)

What is critical for a proof’s being pure or not, according to Hilbert, is whether the means it draws upon are “suggested by the content of the theorem” being proved.

We construe Hilbert’s criterion in the following way. We call the *topic* of a theorem the collection of commitments that determine the understanding of this theorem, relative to an agent α. These are the definitions, axioms and inferences such that if α stopped accepting one of them, then she would no longer understand this theorem. For example, if one stopped accepting that every natural number has a successor, then one would no longer understand the theorem of the infinitude of primes—that for every natural number there is a greater prime number—nor other theorems of elementary arithmetic. That each natural number has a successor is a part of the ordinary conception of the natural numbers as an indefinitely extended sequence. Thus the axiom that every natural number has a successor belongs to the topic of the infinitude of primes.

We then say that a proof of a theorem is *topically pure* if it draws only on what belongs to the topic of the theorem. For example, the classical Euclidean proof of the infinitude of primes can be shown to be topically pure (cf. Arana [2014], Section 3.1), though there are some concerns about its use of induction that we will address later. By contrast, the classical spatial proof of the planar Desargues theorem mentioned above is topically impure, as one can understand this theorem without understanding spatial geometry (cf. Arana and Mancosu [2012], Section 4).

How are we to determine what belongs to a topic? Consider for instance the notion of a straight line. What definitions, axioms and inferences are necessarily implicated in our understanding of this notion? Discussing a theorem of incidence geometry, that is, one concerning only points, lines and their incidence relations, Coxeter wrote that “distance [is] essentially foreign”; on this view metric notions are excluded from this theorem’s topic (cf. Coxeter [1948], p. 27; for more on this, cf. Arana [2009], pp. 4-5). Instead, Coxeter defined straight lines in terms of the notion of betweenness. Other mathematicians have instead taken metric notions as essential to the definition of straight line; for instance Legendre wrote that “the line is the shortest path from one point to another” (cf. Legendre [1794], p. 1). We can try to determine which definition is “correct”, but this presents deep philosophical problems (cf. Tappenden [2008]). We thus treat topic determination for the time being in the naive way we have done so here; doing so is consistent with the way mathematicians have treated purity in practice.

Note that in the classical Euclidean proof of the infinitude of primes that we judged pure, either addition by one or successor is used, to obtain a product of primes plus one. But neither addition nor successor appears explicitly in the statement of the infinitude of primes. Here we distinguish topical purity from more syntactic types of purity (cf. Arana [2009] for more on the latter). In order to understand the notion of natural number as an indefinitely extended sequence, we must have some way to generate the sequence, to pass from one natural number to the next. For this either successor or addition by one may be used (here we suppose also that the unit one must be understood in order to understand the natural numbers). Thus either successor or addition by one belongs to the topic of the theorem, and may be used in a topically pure proof of it. By contrast, if we take a stricter, more syntactic view of purity, the fact that neither operation is mentioned in the statement of the theorem will bar them from being used in a pure proof of it. We believe topical purity is closer to the way mathematicians typically employ purity in practice than is syntactic purity, permitting as it does this Euclidean proof to count as pure.

3. Steiner-explanation

We now give a brief presentation of Steiner's account of mathematical explanation. Like Aristotle, Steiner analyzes explanation in essentialist terms: "My view exploits the idea that to explain the behavior of an entity, one deduces the behavior from the essence or nature of the entity" (cf. Steiner [1978b], p. 143). Taking essences to involve modalities not available in mathematics, Steiner replaces them with the notion of a "characterizing property. . . by which I mean a property unique to a given entity or structure within a *family* or domain of such entities of structures" (*Ibid.*, p. 143). As an example he gives the property that a right triangle is the only triangle decomposable into two triangles similar both to each other and to the whole triangle (*Ibid.*, p. 144). A *Steiner-explanatory* proof of a proposition then shows how the proposition *depends* on a characterizing property of something figuring in the proposition. He makes this notion of dependence precise in the following way.

> It must be evident, that is, that if we substitute in the proof a different object of the same domain, the theorem collapses; more, we should be able to see as we vary the object how the theorem changes in response. In effect, then, explanation is not simply a relation between a proof and a theorem; rather, a relation between an array of proofs and an array of theorems, where the proofs are obtained from one another by [a] "deformation". . . . (*Ibid.*. 143)

Thus in varying the entity characterized by the property in question, we should be able to generate new theorems and proofs in a systematic way.

To illustrate his notion of explanation, Steiner presents as an example the Pythagorean theorem. Using the characterizing property mentioned above, he presents a proof he attributes to Pólya. It starts with the observation that the areas of similar triangles are proportional to the squares of their corresponding sides (*Elements* VI.19). For the triangle in Figure 1, if we had three similar triangles on its sides a, b, c such that the sum of the areas of the triangles on sides a and b were equal to the area of the triangle on side c, then by this observation $ka^2 + kb^2 = kc^2$ for some constant k, and we would thus have proved the Pythagorean theorem. Now we may observe that the triangles in Figure 1 denoted by I and II share a side and two congruent (right) angles, and so are similar to each other. Additionally, each of I and II is similar to the whole triangle since each shares a side and an angle with it. Finally, the whole triangle is itself a triangle on side c. The sum of the areas of I and II is evidently the area of the whole triangle, so we have found the three similar triangles on a, b and c that we sought.

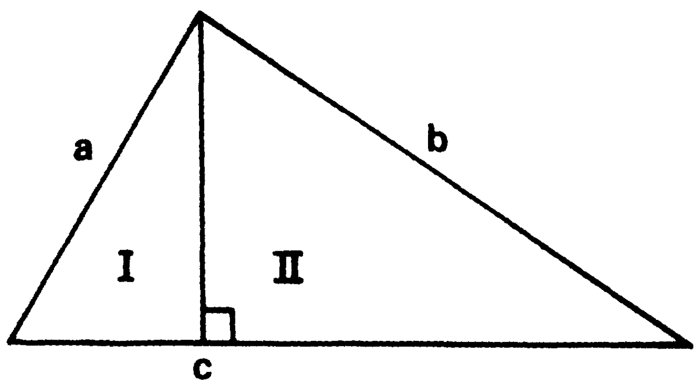

FIGURE 1. Pólya's proof of the Pythagorean theorem

This proof is Steiner-explanatory because it uses a characterizing property of right triangles. To show the dependency of the proof upon this property, Steiner observes that if we start with a non-right triangle and decompose it as before into two similar triangles that are similar to the whole triangle, then there will be a part remaining (see Figure 2). He notes in Steiner [1978a] (p. 137) that by finding the area of this remainder, the same technique shows that $c^2 = a^2 + b^2 - 2ab\cos C$, where C is the angle opposite side c (the full proof is given in Edwards [1994] without any apparent knowledge of Steiner's work). This is the law of cosines, a generalization of the Pythagorean theorem that is usually proved using the Pythagorean theorem (Heath speculates that Pythagoras knew this proof, cf. Euclid [1956], pp. 353-4). Thus we have precise information about how a deformation of the given triangle yields a new theorem. The characterizing property of a right triangle given above, that right triangles are the only triangles decomposable into two triangles similar both to each other and to the whole triangle, can be refined to identify right triangles as those in which the sides x and y of the remainder triangle in Figure 2 coincide. For a given non-right triangle, then, this characteristic property can be used in a proof of the particular case of the law of cosines applying to that triangle. We thus obtain the desired deformation of the original proof for the Pythagorean triangle, using the original characterizing property, to proofs of the resulting theorems for various non-right triangles, using the more general characterizing property.

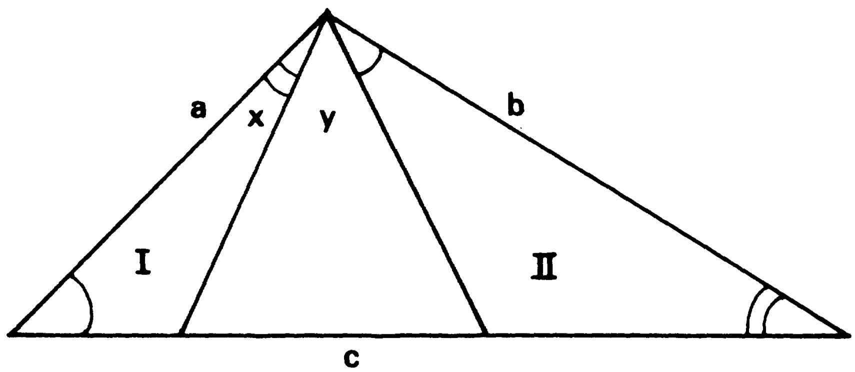

FIGURE 2. Deforming Pólya's proof of the Pythagorean theorem

Let us consider a second example of Steiner-explanation. Firstly, we can prove that the sum $S(n)$ of the first n positive integers equals $\frac{n(n+1)}{2}$ by mathematical induction. Steiner says that "no mathematician will regard this as an explanatory proof" because "we do not see what *about* the sum is "responsible" for this theorem" (cf. Steiner [1978a], p. 134; also Lange [2009]). Instead, we can use what Steiner calls the "symmetric" property of $S(n)$ to prove this, following Gauss:

$$1 + 2 + 3 + \cdots + n = S(n)$$
$$n + (n-1) + (n-2) + \cdots + 1 = S(n)$$

We can add the two lines, column by column, obtaining

$$\underbrace{(n+1) + (n+1) + (n+1) + \cdots + (n+1)}_{n \text{ times}} = 2S(n)$$

so that $n(n+1) = 2S(n)$, and the result follows. This symmetry of $S(n)$, Steiner says, is a characteristic property of it as a summation operation, and thus this proof is Steiner-explanatory. Furthermore, this proof can be deformed to generate new theorems: for instance, that the sum of the first n odd numbers is n^2 can also be shown using this characteristic property (cf. Steiner [1978a], p. 136): adding $1 + 3 + 5 + \cdots + (2n-1)$ and $(2n-1) + (2n-3) + (2n-5) + \cdots + 1$ by column again gives $\underbrace{2n + 2n + 2n + \cdots + 2n}_{n \text{ times}} = 2n^2$.

No such characteristic property is used by the inductive proof, Steiner adds: "induction, it is true, characterizes the *set* of all natural numbers; but this *set* is not mentioned in the theorem" (cf. Steiner [1978b], p. 145). Thus the inductive proof is not Steiner-explanatory.

4. Comparing topical purity and Steiner-explanation

This last remark about induction suggests that topical purity and Steiner-explanation may be closely related. Since induction only characterizes an entity not mentioned in the theorem, the proof in which it figures is not Steiner-explanatory; but at first glance its lack of mention in the theorem gives the impression that it is not "suggested by the content of the theorem" and thus cannot be part of a topically pure proof. On this impression topical purity and Steiner-explanation seem to align.

This impression is too quick. The topic of a theorem is composed of the definitions, axioms and inferences implicated in our understanding of the theorem. Merely being unmentioned in the theorem does not exclude something from belonging to its topic. In this way topical purity is different than syntactic purity. On this criterion of purity, the use of induction need not pose a problem. Poincaré argued credibly that induction was essential to reasoning about the natural numbers (cf. Poincaré [1902]). It would then be a part of the topic of all elementary arithmetic propositions.

The point can be sharpened further. We can classify the strength of the induction used in a proof by the arithmetical complexity of the induction formula. In so-called "full induction", we may induct over formulas of any complexity, whereas we can restrict the induction formulas to some particular complexity such as Δ_0 (known as "bounded induction", since all the quantifiers of a Δ_0 formula are bounded), or even formulas with no quantifiers at all (known as "open induction").

We may then pose the question of what induction is part of the topic of the theorem that the sum $S(n)$ of the first n positive integers equals $\frac{n(n+1)}{2}$: is it full induction, or rather only some limited version of it? The answer to this question determines what proofs are topically pure. In Arana [2014] Section 3.1.2, we surveyed how one might decide this question. For instance, one might understand the theorem that the sum $S(n)$ of the first n positive integers equals $\frac{n(n+1)}{2}$ as belonging to "feasible arithmetic", in which the definable predicates and functions are capable of being evaluated in a practical way by computers like those available today. This theory was analyzed by Parikh in Parikh [1971] and was identified with the formal theory $I\Delta_0$, in which the usual definitions of addition and multiplication hold but induction is limited to Δ_0 formulas. Under this understanding of the theorem, the theorem's topic would include bounded but not full induction, and accordingly a proof of it using full induction would be topically impure. In fact, inspection shows that the proof we gave above uses only bounded induction, so that proof is indeed topically pure for this understanding of the theorem.

We thus have an example of a proof that is topically pure but not Steiner-explanatory. Lange provides other such examples by observing that some theorems can be proved by "brute force", and that these proofs are often pure but are generally not explanatory (cf. Lange [2019], Section 2). By "brute force" Lange means a proof in which an elaborate calculation leads to the theorem.

He gives as an example (p. 8n13) a proof of Desargues' theorem in coordinate geometry, where all the sides of the triangles and other lines are represented by equations and the intersections are calculated by solving systems of these equations. For an agent who conceives of geometrical objects as representable in coordinate geometry, equations of lines and the algebraic methods used in manipulating them will belong to the topic of Desargues' theorem and so this proof will be topically pure; yet its calculations will be opaque to the reader and will explain nothing in any ordinary sense. Furthermore these equations express no characterizing property of triangles and thus cannot give rise to a Steiner-explanatory proof of Desargues' theorem.

A characterizing property of a thing not mentioned in a statement cannot give rise to a Steiner-explanatory proof of that statement. But that does not prevent that property being used in a topically pure proof of that statement. This conclusion may be obscured by the fact that according to Steiner, characterizing properties used in Steiner-explanatory proofs may mention things not mentioned in the theorem. The things characterized by such properties must be mentioned in the theorem, but the things mentioned in such properties need not be. As examples of this Steiner notes the use of complex analysis in proofs about real-valued equations (cf. Steiner [1978c], pp. 18-19) and about number theory (cf. Steiner [1978b], p. 146), and the use of topology in proving Euclid's theorem on the Platonic solids (*Ibid.*, pp. 146–147).

Steiner recently emphasized this aspect of explanation in applying Manya Raman's notion of "fit" in mathematics (cf. Raman [2012]). He writes that in this type of explanation, we are "redescribing mathematical objects and phenomena and then 'fitting' them into a different mathematical scheme from the one in which they were originally presented" (cf. Steiner [2014], p. 5). In doing so, " we begin with a proof that does *not* fit into its ostensible surroundings, but we then find the appropriate theory for the theorem to fit into" (*Ibid.*, p. 13). This transformation of context permits us to "explain away" the theorem as originally understood, by fitting it into a context where the reasons for the theorem can be properly seen.

Steiner takes this transformative aspect of explanation quite seriously. He writes that "Euclid's theorem is 'really' topological in character, and…any geometrical proof is 'irrelevant' " (cf. Steiner [1978b], p. 146). What a theorem is "really" about is a different matter; we have discussed the difficulties regarding the subject of tacit or "hidden" content in mathematics at length in Arana and Mancosu [2012], Section 4. What directly concerns us here is the fact that on this account, a topological proof of this theorem of Euclid can be Steiner-explanatory but not topically pure. For one can understand what a Platonic solid is without any topological knowledge, and so topology does not belong to the theorem's content (Pincock makes a similar point in Pincock [2015], p. 17, though to a different end). Similar remarks apply to the complex-analytic proofs mentioned above.

Another example of a proof that is Steiner-explanatory but not topically pure is Pólya's proof of the Pythagorean theorem discussed in the previous section. This proof employs similarity, but one can understand the Pythagorean theorem without knowing anything of similarity. Indeed, Euclid seems to have worked hard to develop a proof avoiding similarity in Book I of the *Elements*, as Proclus remarks (cf. Proclus [1992], p. 338; cf. also Heath's commentary in Euclid [1956], pp. 353–354), and leaves similarity until Book VI. While Euclid's proof of the Pythagorean theorem in I.47 may be topically pure, the proof of Pólya (and perhaps, if Heath is correct, Pythagoras himself) is not, though it is Steiner-explanatory.

We thus have presented examples of proofs that are topically pure but not Steiner-explanatory, and of proofs that are Steiner-explanatory but not topically pure. We may thus conclude that topical purity and Steiner-explanatory are indeed different properties of proofs.

5. Diagnosing the difference between topical purity and Steiner-explanation

The topic of a theorem, relative to an agent α, is the collection of the definitions, axioms and inferences such that if α stopped accepting one of them, then she would no longer understand this theorem. A characterizing property of an entity figuring in a theorem is a property unique to that entity within a family of such entities. We can ask if a characterizing property of an entity figuring in a theorem must belong to the topic of that theorem. For instance, does the property of right triangles used in Pólya's proof of the Pythagorean theorem, that right triangles are the only triangles decomposable into two triangles similar both to each other and to the whole triangle, belong to the topic of the Pythagorean theorem? In particular, is it a *definition* of right triangle?

We have observed above that Euclid avoids developing similarity until Book VI of the *Elements*, while of course triangles are defined in Book I as three-sided figures. It is hard to see why the more developed notion of similarity would play a defining role for triangles over the much simpler definition in Book I. Here we emphasize the role of definition as informing someone who does not know what a thing is.

One might object that this role of definition is inappropriate for explanatory proofs, "for there is a difference between what is prior and better known in the order of being and what is prior and better known to man" (cf. *Post. An.*, 71b34-72a1, translation from Aristotle [1941]). On this Aristotelian account, a proposition may be ordered before another if the former is a cause of the latter; or if human beings come to know the former more readily than the latter. These are different orderings, and Aristotle's scientific demonstrations track the first, "objective" ordering rather than the second, "subjective" ordering. Steiner's account of explanation is a modern update of Aristotle's account of scientific demonstrations, replacing Aristotle's notion of cause with his notion of dependence (as indicated by deformability of proof). As with Aristotle, Steiner's characteristic properties may be definitions, but definitions that function by saying what something is with respect to the order of being rather than the order of knowing. A characteristic property need not define a thing in such a way that a human being would find that definition best adapted to her learning. Instead, it should define the thing in a way that makes clear the explanatory dependence of statements on this definition. This is why for Aristotle, definition and explanation necessarily go together (cf. Charles [2014]).

Thus our question of whether a characterizing property of an entity figuring in a theorem must belong to the topic of that theorem comes down to what is meant by a definition. If definitions reflect the order of human knowing, then the answer is no; if they reflect the explanatory order of being, then the answer is yes. Depending on how we construe the role of definitions, then, Steiner-explanation and topical purity will be more or less closely related.

The epistemic significance of Steiner-explanatory proofs is accordingly different than that of topically pure proofs. A Steiner-explanatory proof expresses a special kind of dependence, of what Steiner says is "responsible" for the theorem (cf. Steiner [1978a], p. 134). This dependence is related to logical implication but is not reducible to it; as we have seen, this kind of dependence is characterized by the "deformability" of proofs using a characteristic property to new proofs with varied characteristic properties. Knowledge of a Steiner-explanatory proof of a proposition is knowledge of this kind of dependence, between theorem and characteristic property.

Jaakko Kuorikoski has recently analyzed this type of knowledge, which he calls "formal understanding". In grasping the relevant dependence, he writes, we obtain an "increased inferential ability to answer what-if questions concerning the properties of related mathematical structures", and more precisely, concerning "analogical inferential connections between slightly different but related systems of mathematical inference" (cf. Kuorikoski [2021], p. 206). Kuorikoski observes that this"ability to make inferential connections beyond the immediate property, structure, or theorem" concerns "primarily our systems of reasoning and representation, not directly what we are representing and reasoning about" (*Ibid.*, p. 207).

This latter point underscores the differences between Steiner-explanatory proof and topically pure proof. A topically pure proof directly concerns what we are reasoning about, in contrast with Steiner-explanatory proofs. This entails a difference between the knowledge engendered by a Steiner-explanatory proof and a topically pure proof. Rather than what Kuorikoski called the formal understanding engendered by a Steiner-explanatory proof, topically pure proofs give knowledge that a proposition has been proved that is more "stable" than that engendered by a topically impure proof, in the sense that it perdures through changes in our epistemic situation. This stability is a consequence of a different kind of dependence between proof and theorem than that illuminated by Steiner's account of explanation. We explained this notion of stability in Detlefsen and Arana [2011], Section 3, and will summarize it here.

We seek to prove a proposition as we currently understand it, because we are ignorant concerning it. We seek to prove it in order to relieve this particular, specific ignorance. How enduring that relief of ignorance is depends, though, on whether our proof is topically pure or not. Suppose we change our belief in some element of a proof: for instance, we no longer accept one of its premises or inference rules. We thus no longer take to be a proof what we had taken to be a proof. If that proof is topically pure, that change is also a change in our understanding of the proposition we are proving, since every element of a topically pure proof belongs to the topic of the proposition being proved. Such a change thus redirects us away from that original proposition, and in so doing, represents a relief of ignorance concerning that proposition. Thus a change in belief concerning an element of a topically pure proof ensures that that proof continues to relieve our ignorance of the original proposition (even if it opens a new, different ignorance). This enduring relief is not necessarily engendered by a topically impure proof, since our change of belief might concern a non-topical element and thus might not change the original proposition. In this case, the original proposition would remain unchanged and thus would again be an ignorance demanding our attention. This dependence between a topically pure proof and its "directing" proposition thus ensures that the ignorance relief brought about by this proof is stable with respect to changes in our epistemic situation.

The dependence identified by Steiner and the dependence operative in topical purity are thus distinct. The topic of a theorem is agent-relative: it is the family of commitments that determine a particular agent's understanding of that theorem. By contrast, characteristic properties are not agent-relative: they are properties unique to an entity within a family of such entities. Topics concern an agent's semantic grasp of a theorem; characteristic properties concern the relation of an entity to other entities of a certain type. Topics play a semantic role, while characteristic properties play a metaphysical role. In some cases these may be identical, but they are not so necessarily.

6. Conclusions

The difference between topical purity and Steiner-explanation may then, in broad, be seen as the difference between an "epistemology first" approach versus a "metaphysics first" approach to our understanding of proof. An epistemology-first approach to proof, which can be identified as that of Lakatos [1976] and Manders [1989] for instance, aims to understand how proof contributes to mathematical knowledge, and how that knowledge can be maintained or extended over time. Topical purity's aim of stable mathematical knowledge fits within this approach. A metaphysics-first approach, by contrast, aims to identify the right "order of being" and holds that a better kind of knowing resulting from ordering our beliefs in a way that mirrors this rational order. This latter approach was that of Aristotle, but also of later rationalists such as Leibniz, Bolzano and Frege (cf. Detlefsen [1988]). Steiner-explanation fits within this approach. That topical purity and Steiner-explanation do not coincide is occasion to rethink these two approaches. A lesson of the persistence of interest in purity among mathematicians even after the decline of Aristotelian metaphysics is that we need not believe in an "order of being" in order to pursue purity rationally. As a consequence, one might today instead seek other notions of mathematical explanation that fit better within the epistemology-first approach.

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