POSSIBLE m-DIAGRAMS OF MODELS OF ARITHMETIC

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Abstract. In this paper we investigate the complexity of m-diagrams of models of various completions of first-order Peano Arithmetic (PA). We obtain characterizations that extend Solovay's results for open diagrams of models of completions of PA. We first characterize the m-diagrams of models of True Arithmetic by showing that the degrees of m-diagrams of nonstandard models $\mathcal{A}$ of TA are the same for all $m \geq 0$. Next, we obtain a more complicated characterization for arbitrary completions of PA. We then provide examples showing that some of the extra complication is needed. Lastly, we characterize sequences of Turing degrees that occur as $(\deg(T \cap \Sigma_n))_{n\in\omega}$, where T is a completion of PA.

§1. Introduction. We use $P(\omega)$ to denote the class of all subsets of ω. Let $\mathcal{L}_{\text{PA}}$ be the usual language of PA: relations $+, \cdot, S$, and $<$; and constants 0 and 1. We abbreviate True Arithmetic, the theory of the standard model of PA, by the initials TA. We use $S^n(0)$ to denote the numeral for n

We continue with some preliminary definitions and results. A B_n formula is a boolean combination of Σ_n formulas. A *complete* B_n *type* is the set of all B_n formulas true of some tuple in some structure. The *open* diagram of a structure $\mathcal{A}$, denoted $D(\mathcal{A})$, is the collection of open sentences, with constants from $\mathcal{A}$, that are true in $\mathcal{A}$. Similarly, the m-diagram of $\mathcal{A}$, denoted $D_m(\mathcal{A})$, is the collection of B_m sentences, with constants from $\mathcal{A}$, that are true in $\mathcal{A}$.

Behind most of what we know about models and completions of PA is the notion of a Scott set:

Definition 1.1. A *Scott set* is a nonempty family of sets $\mathcal{S} \subseteq P(\omega)$ such that

1. if $X \in \mathcal{S}$ and $Y \leq_T X$, then $Y \in \mathcal{S}$,

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2. if $X, Y \in \mathcal{S}$, then $X \oplus Y \in \mathcal{S}$,
3. if $T \subseteq 2^{<\omega}$ is an infinite tree in $\mathcal{S}$, then T has a path in $\mathcal{S}$. Equivalently, if A is a consistent set of sentences in $\mathcal{S}$, then some complete extension of A is in $\mathcal{S}$.

The family of arithmetical sets forms a Scott set. Scott sets are the ω-models of the axiom system WKL_0 as studied in reverse mathematics (and where the model is identified with the power set part of the structure, as in [14]). For a nonstandard model $\mathcal{A} \models$ PA, let $SS(\mathcal{A}) = \{d_a : a \in \mathcal{A}\}$, where

$$d_a = \{n \in \omega : \mathcal{A} \models p_n | a\}$$

where $(p_k)_{k \in \omega}$ is the sequence of primes.

Theorem 1.2 (Scott). *For a nonstandard model $\mathcal{A} \models$ PA, $SS(\mathcal{A})$ is a Scott set.*

We thus call $SS(\mathcal{A})$ the *Scott set* of the model $\mathcal{A}$.

The following well-known lemma is a sort of weak saturation property for bounded types in a Scott set:

Lemma 1.3. *Let $\mathcal{A}$ be a nonstandard model of PA. Let $\Gamma(\overline{u}, x)$ be a complete B_m type, with $\overline{a} \in \mathcal{A}$ a tuple that can be substituted for $\overline{u}$ in Γ. Then $\Gamma(\overline{a}, x)$ is realized by some $c \in \mathcal{A}$ if and only if $\Gamma(\overline{a}, x) \cup D_{m+1}(\mathcal{A})$ is consistent and $\Gamma(\overline{u}, x) \in SS(\mathcal{A})$.*

Scott was originally interested in Scott sets because they are closely tied to the notion of "representability". He wanted to characterize the families of sets representable with respect to completions of PA.

Definition 1.4. For a theory T in the language of PA, a set $X \subseteq \omega$ is *representable* by T if there is a formula φ such that for $n \in X, T \vdash \varphi(S^{(n)}(0))$, and for $n \notin X, T \vdash \neg\varphi(S^{(n)}(0))$.

We denote the collection of sets representable by a theory T by $Rep(T)$. Scott [12] showed the following fact relating Scott sets and $Rep(T)$:

Theorem 1.5 (Scott). *For a countable collection $\mathcal{S} \subseteq P(\omega)$, $\mathcal{S}$ is a Scott set if and only if there exists a completion T of PA such that $Rep(T) = \mathcal{S}$.*

Feferman [3] noted the following fact about nonstandard models of TA:

THEOREM 1.6 (Feferman). *Let $\mathcal{A}$ be a nonstandard model of TA. Then $SS(\mathcal{A})$ contains the arithmetical sets.*

Feferman gave the result only for TA. However, for essentially the same reasons we also get the following result, for any model of PA:

THEOREM 1.7. *Let $\mathcal{A}$ be a nonstandard model of PA. Then $SS(\mathcal{A})$ contains $Rep(T)$. Equivalently, $SS(\mathcal{A})$ contains $T_n = T \cap \Sigma_n$, for all n.*

Theorem 1.7 implies Theorem 1.6, because for $T =$ TA, $T_n \equiv_T \emptyset^{(n)}$ for all n. Theorem 1.7 suggests the following definition:

DEFINITION 1.8. A Scott set $\mathcal{S}$ is *appropriate* for a theory T if $T_n \in \mathcal{S}$ for all n. Equivalently, $\mathcal{S}$ is appropriate for T if $Rep(T) \in \mathcal{S}$.

Using this definition, we can restate Theorem 1.7 as:

THEOREM 1.9. *Let $\mathcal{A}$ be a nonstandard model of PA. Then $SS(\mathcal{A})$ is appropriate for T.*

A notion we shall use in connection with Scott sets is that of an "enumeration".

DEFINITION 1.10. An *enumeration* of a set $\mathcal{S} \subseteq P(\omega)$ is a binary relation R such that $\mathcal{S} = \{R_n : n \in \omega\}$, where

$$R_n = \{k : (n, k) \in R\}.$$

An *R-index* for X is some $k \in \omega$ such that $R_k = X$.

DEFINITION 1.11. For a nonstandard model $\mathcal{A}$ of PA with universe ω,

$$R = \{(a, n) : \mathcal{A} \models p_n | a\}$$

is called the *canonical enumeration* of $SS(\mathcal{A})$.

We have the well-known fact:

PROPOSITION 1.12. *Let $\mathcal{A}$ be any nonstandard model of PA with universe ω and let R be the canonical enumeration of $SS(\mathcal{A})$. Then $R \leq_T D(\mathcal{A})$.*

This follows from the fact that the open diagram $D(\mathcal{A})$ witnesses true instances of the division algorithm. The following corollary follows from the fact that $D(\mathcal{A}) \leq_T D_m(\mathcal{A})$, for $m \geq 0$:

COROLLARY 1.13. *For $\mathcal{A}$ a nonstandard model of PA with universe ω, if R is the canonical enumeration of $SS(\mathcal{A})$, then $R \leq_T D_m(\mathcal{A})$, for $m \geq 0$.*

Solovay defined the notion of an "effective enumeration":

Definition 1.14. For a countable Scott set $\mathcal{S}$, an *effective enumeration* is an enumeration R, with associated functions f, g, and h witnessing that $\mathcal{S}$ is a Scott set. These functions have the following properties:

1. if $\varphi_e^{R_i} = \chi_X$, then $f(i,e)$ is an R-index for X,
2. $g(i,j)$ is an R-index for $R_i \oplus R_j$,
3. if R_i is an infinite tree $T \subseteq 2^{<\omega}$, then $h(i)$ is an R-index for a set X such that χ_X is a path through T.

We say that an effective enumeration is *computable in a set* X if the enumeration and the three functions are all computable in X. Effective enumerations are available to us in light of the following result [7]:

Theorem 1.15 (Marker). *Let $\mathcal{S}$ be a countable Scott set. If $\mathcal{S}$ has an enumeration computable in X, then it also has an effective enumeration computable in X.*

Solovay gave a characterization of the degrees (of open diagrams) of nonstandard models of TA in terms of effective enumerations [15]. Marker simplified Solovay's result by applying Theorem 1.15 [7]. The result is the following characterization:

Theorem 1.16 (Solovay / Marker). *The degrees of nonstandard models of TA are the degrees of enumerations of Scott sets containing the arithmetical sets.*

Solovay also characterized the degrees of (open diagrams of) nonstandard models of other completions of PA. The result is more difficult to state than the result for TA. To see why, let us highlight the difference between TA and arbitrary completions of PA. For a nonstandard model $\mathcal{A}$ of TA, $\mathcal{A}''$ yields the theory (and indices for the Σ_n fragments). For an arbitrary completion of PA this may not be so, as we will illustrate in Section 4.

Solovay found the general relationship between jumps of the model and indices for fragments of the theory. The result is the following characterization:

Theorem 1.17 (Solovay). *Suppose T is a completion of PA. The degrees of nonstandard models of T are the degrees of sets X such that:*

(a) *There is an enumeration* $R \leq_T X$ *of a Scott set* $\mathcal{S}$ *appropriate for* T*; and*
(b) *There are functions* t_n *for* $n \geq 1, \Delta^0_n(X)$ *uniformly in* n*, such that* $\lim_{s\to\infty} t_n(s)$ *is an index for* T_n *and for all* s*,* $t_n(s)$ *is an* R*-index for a subset of* T_n*.*

Solovay did not publish these results we are attributing to him. In [6], Julia Knight has given proofs of Theorems 1.16 and 1.17. Our proofs in Sections 2 and 3 follow those of [6], extending Solovay's results. In Section 2, we extend Solovay and Marker's characterization to include m-diagrams of nonstandard models of TA. In Section 3, we extend Solovay's characterization for arbitrary completions of PA to include m-diagrams. In Section 4, we will develop a class of theories $T(X)$ illustrating why the extra conditions in the more general characterization for arbitrary completions of PA given in Section 3 cannot simply be dropped. As part of doing this, we give a proof of Harrington's result that there exists a nonstandard model $\mathcal{A} \models$ PA such that $\mathcal{A} \leq_T 0'$ and $Th(\mathcal{A})$ is not arithmetical [5]. Lastly, in Section 5, we examine the relationship between sequences of Turing degrees and completions of PA.

§2. True Arithmetic. In this section we characterize the degrees of m-diagrams of nonstandard models of TA as the degrees of enumerations of Scott sets containing the arithmetical sets. We first show that for a nonstandard model of TA, we can find an enumeration below the m-diagram (in terms of Turing reducibility). We then show that for a suitable enumeration, we can find the m-diagram of a nonstandard model of TA below it. The second step requires more work. We use the fact that if R is an enumeration of a Scott set containing the arithmetical sets, then computably in R'' we can compute a sequence $(i_n)_{n\in\omega}$ of indices such that $R_{i_k} =$ TA $\cap \Sigma_k$ for each k. The fact holds because we can use R'' to list $\emptyset'$ and find its index in R; we may then use R'' to list $(\emptyset')'$, find its index in R, and so on. Using this fact, we can construct a nonstandard model $\mathcal{C}$ such that $D_m(\mathcal{C}) \leq_T R''$ and such that the set

$$Q = \{(i, \overline{a}) : R_i \text{ is the complete } B_{m+1} \text{ type of } \overline{a}\}$$

is $\Sigma^0_2(R)$. This is the content of Theorem 2.1. We then use $\Delta^0_1(R)$ to approximate $\mathcal{C}$, building an isomorphic copy $\mathcal{A}$ such that $D_m(\mathcal{A}) \leq_T R$. This is the content of Theorem 2.2. We then combine these results in Theorem 2.3 and apply them in the main result, Theorem 2.6. We now give the results.

THEOREM 2.1. *Let T be a completion of PA and X any set. Suppose $R \leq_T X$ is an enumeration of a Scott set $\mathcal{S}$, and t is a $\Delta^0_3(X)$ function such that for all n, $t(n)$ is an R-index for $T_n = T \cap \Sigma_n$. Then T has a model $\mathcal{A}$ with $SS(\mathcal{A}) = \mathcal{S}$, such that for $m \geq 0$,*

$$Q = \{(i, \overline{a}) : R_i \text{ is the complete } B_{m+1} \text{ type of } \overline{a}\}$$

is $\Sigma^0_2(X)$.

The result for $m = 1$ is Theorem 2.2 in [6]. The proof for arbitrary m is essentially the same, so we omit the details. It is a finite injury priority construction.

Here is the other result we need to establish Theorem 2.3:

THEOREM 2.2. *Let $\mathcal{S}$ be a countable Scott set and let $\mathcal{A}$ be a nonstandard model of PA such that $SS(\mathcal{A}) = \mathcal{S}$. Suppose $\mathcal{S}$ has an enumeration $R \leq_T X$ and $Q = \{(i, \overline{a}) : R_i$ is the complete B_{m+1} type of $\overline{a}\}$ is $\Sigma^0_2(X)$, for $m \geq 0$. Then there exists a nonstandard model $\mathcal{B}$ of PA such that $\mathcal{B} \cong \mathcal{A}$ and $D_m(\mathcal{B}) \leq_T X$.*

The result for $m = 1$ is Theorem 2.1 in [6]. Again, the proof for arbitrary m is essentially the same, so we omit details. Again, it is a finite injury priority construction.

We may combine these two results into the following single result:

THEOREM 2.3. *Let T be a completion of PA and suppose $m \geq 0$. Suppose $R \leq_T X$ is an enumeration of a Scott set $\mathcal{S}$, and $t(n)$ is a $\Delta^0_3(X)$ function such that for all n, $t(n)$ is an R-index for $T_n = T \cap \Sigma_n$. Then T has a model $\mathcal{A}$ with $SS(\mathcal{A}) = \mathcal{S}$ such that $D_m(\mathcal{B}) \leq_T X$.*

We need two more lemmas before we can give our characterization for m-diagrams of nonstandard models of TA. The first lemma is an extension of the well-known fact that the set of degrees of open diagram copies of a nonstandard model of a completion of PA is upward closed (see [8]). The second lemma is a fact about the degrees of enumerations of families of more than one set. We give proofs for each.

LEMMA 2.4. *Let $\mathcal{A}$ be a fixed ordered structure (with universe ω) and let $m \geq 0$. For any $D >_T D_m(\mathcal{A})$, there exists $\mathcal{B} \cong \mathcal{A}$ such that $D_m(\mathcal{B}) \equiv_T D$.*

PROOF. Let $\mathcal{A} = \{a_n : n \in \omega\}$. We indicate how to build $\mathcal{B} = \{b_n : n \in \omega\}$ using D. We need to show that $\mathcal{B} \cong \mathcal{A}$ and $D_m(\mathcal{B}) \equiv_T D$.

To show the former, we specify an isomorphism $F : \mathcal{B} \to \mathcal{A}, F \leq_T D$. We give the isomorphism on two elements of $\mathcal{B}$ at a time. Let $F(b_{2k}) = a_{2k}$ and $F(b_{2k+1}) = a_{2k+1}$ if either $k \in D$ and $\mathcal{A} \models a_{2k} < a_{2k+1}$ or $k \notin D$ and $\mathcal{A} \models a_{2k+1} < a_{2k}$. Let $F(b_{2k}) = a_{2k+1}$ and $F(b_{2k+1}) = a_{2k}$ if either $k \in D$ and $\mathcal{A} \models a_{2k+1} < a_{2k}$ or $k \notin D$ and $\mathcal{A} \models a_{2k} < a_{2k+1}$. Thus the isomorphism F is computable in D. Furthermore, $\mathcal{B} \models b_{2k} < b_{2k+1}$ iff $k \in D$.

Next, we need to show that $D_m(\mathcal{B}) \leq_T D$. Let $\varphi(\overline{x})$ be an arbitrary B_m formula. We indicate how to decide if $\varphi(\bar{b}) \in D_m(\mathcal{B})$ using D. By our isomorphism we have that $\mathcal{B} \models \varphi(\bar{b})$ iff $\mathcal{A} \models \varphi(F(\bar{b}))$. Using oracle D, we compute $F(\bar{b}) = \overline{a}$. Since $D >_T D_m(\mathcal{A})$, we use D to determine whether $\varphi(\overline{a}) \in D_m(\mathcal{A})$.

Finally, we show that $D \leq_t D_m(\mathcal{B})$. We use the fact that $\mathcal{B} \models b_{2k} < b_{2k+1}$ iff $k \in D$. To decide if $k \in D$, we ask $D_m(\mathcal{B})$ if $b_{2k} < b_{2k+1}$. If $b_{2k} < b_{2k+1}$, then $k \in D$; otherwise, $k \notin D$. ⊣

The next lemma is well-known.

LEMMA 2.5. *Let $\mathcal{S}$ be a family of sets containing at least two sets. Let $En(\mathcal{S})$ be the set of all enumerations of $\mathcal{S}$. If $R \in En(\mathcal{S})$ and $R <_T D$, then there exists $R^* \in En(\mathcal{S})$ such that $R^* \equiv_T D$.*

PROOF. Suppose $\mathcal{S} \subseteq P(\omega)$, with $A_0 \neq A_1$ elements of $\mathcal{S}$. Let a_0 be an element witnessing that $A_0 \neq A_1$. Without loss of generality, suppose $a_0 \in A_0 - A_1$.

Given $R \in En(\mathcal{S})$ and $D >_T R$, we indicate how to construct R^*. Let $R^*_{2k} = R_k$ for each $k \in I$. We let $R^*_{2k+1} = A_0$ if $k \in D$ and $R^*_{2k+1} = A_1$ if $k \notin D$.

By our construction we have that $k \in D$ iff $a_0 \in R^*_{2k+1}$. Thus it follows immediately that $R^* \equiv_T D$. ⊣

By Theorem 1.6 (Feferman's result), we know that for any nonstandard $\mathcal{A} \models$ PA and for all n, $T_n \in SS(\mathcal{A})$. For TA, each fragment T_n is Turing equivalent to the arithmetical set $\emptyset^{(n)}$. Thus the only possible Scott sets of nonstandard models of TA are those that contain the arithmetical sets. We may now characterize the degrees of m-diagrams of nonstandard models of TA.

THEOREM 2.6. *For any $m \geq 0$, the degrees of m-diagrams of nonstandard models of TA are the degrees of enumerations of Scott sets containing the arithmetical sets.*

PROOF. By Lemma 2.4, we have that the degrees of m-diagrams of nonstandard models of completions of PA are closed upward. By Lemma 2.5 we have that the set of degrees of enumerations of a given Scott set $\mathcal{S}$ is closed upward.

Suppose $\mathcal{A}$ is a nonstandard model of TA such that $SS(\mathcal{A}) = \mathcal{S}$. Assuming the universe of $\mathcal{A}$ to be ω, we use Corollary 1.13 to see that the canonical enumeration of $SS(\mathcal{A})$ is computable in $D_m(\mathcal{A})$.

Next, suppose $\mathcal{S}$ is a Scott set containing the arithmetical sets and that R is an enumeration of $\mathcal{S}$. We may use Marker's result again and take R to be an effective enumeration. To apply Theorem 2.3 and conclude the proof, we use a $\Delta^0_3(R)$ function $t(n)$ giving an R-index for $T_n = TA \cap \Sigma_n$. Let $t(n)$ be the least R-index of $\mathrm{TA} \cap \Sigma_n$.

We show how to compute $t(n)$ using $\Delta^0_3(R)$. Note first that $\mathrm{TA} \cap \Sigma_n \leq_T TA \cap \Sigma_{n+1}$ and $\mathrm{TA} \cap \Sigma_{n+1} \leq_T (TA \cap \Sigma_n)'$ uniformly in n. Note also that the relation

$$J(i,j) = \{(i,j) : \forall x[x \in R_j \leftrightarrow x \in (R_i)']\}$$

is $\Delta^0_3(R)$. Beginning with $t(r)$, an index for $\mathrm{TA} \cap \Sigma_r$, we use J to get an index for $(TA \cap \Sigma_r)'$. Since $\mathrm{TA} \cap \Sigma_{r+1} \leq_T (TA \cap \Sigma_r)'$, we use our effective enumeration to get an index for $\mathrm{TA} \cap \Sigma_{r+1}$. This index is $t(r+1)$.

We have thus shown $t(n)$ to be $\Delta^0_3(R)$. We may now apply Theorem 2.3 to get a nonstandard model $\mathcal{A}$ of TA such that $SS(\mathcal{A}) = \mathcal{S}$ and $D_m(\mathcal{A}) \leq_T R$. ⊣

As a corollary to the previous result, we have the following:

COROLLARY 2.7. *The degrees of m-diagrams of nonstandard models $\mathcal{A}$ of TA are the same for all $m \geq 0$.*

§3. Other completions of PA. In this section we give a characterization of the m-degrees of nonstandard models of other completions of PA. This new characterization (Theorem 3.4) will be like the characterization for TA (Theorem 2.6) in that it involves enumerations of an appropriate Scott set. It differs from the earlier characterization in that it additionally involves a sequence of approximating functions.

To prove this characterization, we need to use the sequence of oracles $(\Delta^0_i(X))_{i\in\omega}$ to prove a more general version of Theorem 2.1. To prove this result, Theorem 3.1, we use a infinitely nested priority construction. The result for $m = 1$ is Theorem 2.3 in [6]. Again, the proof for arbitrary m is essentially the same, so we omit details and give only a sketch.

As with the TA case, we can break the characterization into two parts. The model-construction part, Theorem 3.2, can itself again be separated into two separate priority constructions. The first priority construction for TA, Theorem 2.1, used $\Delta^0_2(X)$ to approximate a $\Delta^0_3(X)$ function. In the case of arbitrary completions of PA, we need to approximate not a single $\Delta^0_3(X)$ function, but rather a sequence of functions t_{m+n}, $\Delta^0_n(X)$ uniformly in n, approximating $T \cap \Sigma_n$ for each n relative to X. We thus need to prove a more general version of Theorem 2.1. Here we use an infinitely nested priority construction.

Infinitely nested priority constructions are difficult to do in general. However, there is a metatheorem giving conditions under which some may be done. Solovay's theorem and our generalization follow from the metatheorem 4.1 in [6].

As with TA, our plan is to build a nonstandard model $\mathcal{B}$ such that $D_m(\mathcal{B}) \leq_T X''$ and such that the set

$$Q = \{(i, \overline{a}) : R_i \text{ is the complete } B_{m+1} \text{ type of } \overline{a}\}$$

is $\Sigma^0_2(X)$. The metatheorem shows that under certain conditions such a construction can be effected.

The result of this construction is the following:

Theorem 3.1. *Let T be a completion of PA and let $m \geq 0$. Suppose $R \leq_T X$ is an enumeration of a Scott set $\mathcal{S}$, with functions t_{m+n} for $n \geq 2, \Delta^0_n(X)$ uniformly in n, such that $\lim_{s\to\infty} t_{m+n}(s)$ is an index for T_{m+n} and for all s, $t_{m+n}(s)$ is an index for a subset of T_{m+n}. Then T has a model $\mathcal{A}$ such that $SS(\mathcal{A}) = \mathcal{S}$ and $Q = \{(i, \overline{a}) : R_i$ is the complete B_{m+1} type of $\overline{a}\}$ is $\Sigma^0_2(X)$.*

We may now reuse Theorem 2.2, using $\Delta^0_1(X)$ to approximate $\mathcal{B}$, building an isomorphic copy $\mathcal{A}$ such that $D_m(\mathcal{A}) \leq_T X$. These constructions can then be combined into one result:

Theorem 3.2. *Let T be a complete theory and let $m \geq 0$. Suppose $R \leq_T X$ is an enumeration of a Scott set $\mathcal{S}$, with functions t_{m+n} for $n \geq 2, \Delta^0_n(X)$ uniformly in n, such that $\lim_{s\to\infty} t_{m+n}(s)$ is an index for T_{m+n} and for all s, $t_{m+n}(s)$ is an index for a subset of T_{m+n}. Then T has a model $\mathcal{A}$ with $SS(\mathcal{A}) = \mathcal{S}$ such that $D_m(\mathcal{A}) \leq_T X$.*

To show the enumeration half of the main theorem, we need a modified version of Solovay's Approximation Lemma for m-diagrams.

The original version for $m = 1$ appears in [6] along with a proof. We omit the details here, as the proof for arbitrary m is essentially the same.

LEMMA 3.3. *Let $\mathcal{A}$ be a nonstandard model of PA with universe ω, and let R be the canonical enumeration of $SS(\mathcal{A})$. Then for any $m \geq 0$, there are functions $t_{m+n}, \Delta^0_n(D_m(\mathcal{A}))$ uniformly in n, such that $\lim_{s\to\infty} t_{m+n}(s)$ is an R-index for $T_{m+n}(\mathcal{A})$. Furthermore, for $r < s, R_{t_n(r)} \subseteq R_{t_n(s)}$.*

We can now give the main result giving the characterization for an arbitrary completion of PA:

THEOREM 3.4. *Suppose T is a completion of PA. For any $m \geq 0$, the degrees of m-diagrams of nonstandard models of T are the degrees of sets X such that:*

(a) *There is an enumeration $R \leq_T X$ of a Scott set $\mathcal{S}$ appropriate for T; and*
(b) *There are functions t_{m+n} for $n \geq 1, \Delta^0_n(X)$ uniformly in n, such that $\lim_{s\to\infty} t_{m+n}(s)$ is an index for T_{m+n} and for all s, $t_{m+n}(s)$ is an R-index for a subset of T_{m+n}.*

PROOF. Suppose first that $R \leq_T X$ is an enumeration $\mathcal{S}$ satisfying condition (2) above. Using Theorem 3.2, we get a model $\mathcal{A} \models T$ with $SS(\mathcal{A}) = \mathcal{S}$ such that $D_m(\mathcal{B}) \leq_T R$. Next, suppose we start with $\mathcal{A} \models T$ with $SS(\mathcal{A}) = \mathcal{S}$ such that $D_m(\mathcal{B}) \leq_T X$. Using the canonical enumeration R of $SS(\mathcal{A})$, we get that $R \leq_T D_m(\mathcal{A})$. Then by Lemma 3.3, functions satisfying (b) exist as needed. ⊣

§4. Examples. In this section, we present examples illustrating aspects of Solovay's results. First, we give a theory T with enumeration R of $Rep(T)$ such that there is no model of T computable in R. Next, we present Harrington's result that there is a model $\mathcal{A}$ of PA that is computable in $0'$, but $Th(\mathcal{A})$ is not arithmetical. Hence, $Th(\mathcal{A}) \not\leq_T \mathcal{A}^{(n)}$ for any n. Thus, Solovay's results in general require an infinite sequence of approximating functions. In this sense especially, arbitrary completions of PA differ from TA.

We provide a general procedure for constructing the theories we use in these examples in Theorem 4.4. The construction uses the Gödel-Rosser Incompleteness Theorem, as well as Scott's modification of this theorem. We will review the Gödel-Rosser and Scott results before giving our results.

Independence was first explored by Gödel in his landmark 1931 paper [4]. Rosser tightened the result by modifying the sentence shown to be independent [11]. We state a variant of the Gödel-Rosser Theorem that we will make use of later:

LEMMA 4.1 (Gödel-Rosser). *There is a computable sequence of sentences* $(\varphi_n)_{n\in\omega}$ *such that* φ_n *is* Π_{n+1} *and for any set* Γ *of* B_n *sentences consistent with PA,* φ_n *is independent over* $\mathrm{PA}\cup\Gamma$.

Note that we may also extend the axioms of PA by any computable set and preserve the result.

We continue with Scott's results. In arriving at his results regarding Scott sets, Scott investigated the notion of independence for formulas.

DEFINITION 4.2. For a set of sentences Γ and a formula $\varphi(x)$, $\varphi(x)$ is *independent* over Γ if for all $X\subseteq\omega$, the set

$$\Gamma\cup\{\varphi(S^{(n)}(0)) : n\in X\}\cup\{\neg\varphi(S^{(n)}(0)) : n\notin X\}$$

is consistent.

By varying the Gödel-Rosser independent sentence, Scott was able to show the following result [12]:

LEMMA 4.3 (Scott). *There is a computable sequence of formulas* $(\varphi_n(x))_{n\in\omega}$ *such that* φ_n *is* Π_{n+2} *and if* Γ *is a set of* B_n *sentences such that* $\mathrm{PA}\cup\Gamma$ *is consistent, then* φ_n *is independent over* $\mathrm{PA}\cup\Gamma$.

Let's consider briefly the construction of these independent formulas. Fix n. We sketch the construction of the formula $\varphi_n(x)$ in two steps. The first step is to define a sequence of Π_{n+1} sentences $(\psi_\sigma)_{\sigma\in 2^{<\omega}}$, which we think of as being on a binary-branching tree τ. We describe the first few levels of τ. At level 0 of τ, let the root be a variant of the Gödel-Rosser sentence that says "for any proof of me from PA and true B_n sentences, there is a smaller proof of my negation from the same axioms"; call this sentence $\psi_{<\emptyset>}$. The root $\psi_{<\emptyset>}$ branches left to a sentence $\psi_{<0>}$ that says, "for any proof of me from PA, true B_n sentences, and $\psi_{<\emptyset>}$, there is a smaller proof of my negation from the same axioms". Similarly, $\psi_{<\emptyset>}$ branches right to $\psi_{<1>}$, which says, "for any proof of me from PA, true B_n sentences, and $\neg\psi_{<\emptyset>}$, there is a smaller proof of my negation from the same axioms". Both $\psi_{<0>}$ and $\psi_{<1>}$ are at level 1 of τ. We may continue and define the level 2 sentences of τ similarly: $\psi_{<0>}$ branches to the left to a sentence $\psi_{<00>}$ that says "for any proof of me from PA, true B_n sentences, $\psi_{<\emptyset>}$, and $\psi_{<0>}$, there is a smaller proof of my negation from the same axioms", while $\psi_{<0>}$ branches to the right to a sentence $\psi_{<01>}$ that says "for any proof of me from PA, true B_n sentences, $\psi_{<\emptyset>}$, and $\neg\psi_{<0>}$, there is a smaller proof of my negation from the same axioms". Accordingly, $\psi_{<1>}$ branches to sentences $\psi_{<10>}$ and $\psi_{<11>}$. For each $\sigma \in 2^{<\omega}$, the sentence ψ_σ is defined as above, using σ to determine which axioms ψ_σ mentions. Each sentence ψ_σ is independent over PA, Γ, and the axioms $\pm\psi_\varsigma$ that ψ_σ mentions.

Using this sequence $(\psi_\sigma)_{\sigma\in 2^{<\omega}}$ of Π_{n+1} sentences, we specify another sequence of sentences $(\mu_n)_{n\in\omega}$. Each sentence μ_i expresses the disjunction of all paths of length $i+1$ through τ that branch to the left at level i. We illustrate this by giving the first three sentences of this sequence. First, let

$$\mu_0 = \psi_{<\emptyset>}.$$

Next, let

$$\mu_1 = (\psi_{<\emptyset>} \wedge \psi_{<0>}) \vee (\neg\psi_{<\emptyset>} \wedge \psi_{<1>}).$$

Continuing, let

$$\mu_2 = (\psi_{<\emptyset>} \wedge \psi_{<0>} \wedge \psi_{<00>}) \vee (\psi_{<\emptyset>} \wedge \neg\psi_{<0>} \wedge \psi_{<01>}) \vee$$
$$(\neg\psi_{<\emptyset>} \wedge \psi_{<1>} \wedge \psi_{<10>}) \vee (\neg\psi_{<\emptyset>} \wedge \neg\psi_{<1>} \wedge \psi_{<11>}).$$

Continue this way for all levels i. Since these sentences μ_n are boolean combinations of Π_{n+1} sentences, each μ_n may be taken to be B_{n+1}.

We are now finally ready to describe the formula $\varphi_n(x)$ described in the lemma. Let $\varphi_n(x) = \mathrm{Sat}_{B_{n+1}}(\mu_x)$. We may take $\mathrm{Sat}_{B_{n+1}}(x)$ to be both Π_{n+2} and Σ_{n+2}.

We will use Lemmas 4.1 and 4.3 for our examples, by way of the following construction. We remark that Marker proved essentially the same result in his Ph.D. thesis [9], using essentially the same proof. The result appears there as Theorem 1.27.

THEOREM 4.4. *Let R be an enumeration of a Scott set $\mathcal{S}$. For any set X, there is a completion $T(X,R)$ of PA with $Rep(T(X,R)) = \mathcal{S}$ and $T(X,R) \cap B_{3n} \leq_T (X \cap n) \oplus R$, uniformly in n.*

PROOF. We may suppose R is an effective enumeration, by Marker's Theorem 1.15. We construct the appropriate theory $T(X)$. We start with a computable sequence $(\varphi_n(x))_{n\in\omega}$ of independent formulas as in Lemma 4.3, where $\varphi_n(x)$ is Π_{n+2}. We also start with a computable sequence $(\varphi_n^*)_{n\in\omega}$ of independent sentences as in Lemma 4.1, where φ_n^* is Π_{n+1}. Let T be any completion of PA. We build $T(X,R)$ using the following list of requirements:

Code$_0$: Take the Π_1 sentence φ_0^* from the sequence given by Lemma 4.1, where φ_0^* is independent over PA.

If $0 \in X$, let $T_1^* =$ a completion of PA $\cup \{\varphi_0^*\}$ in $\mathcal{S}$.

If $0 \notin X$, let $T_1^* =$ a completion of PA $\cup \{\neg\varphi_0^*\}$ in $\mathcal{S}$.

We may do this because φ_0^* and $\neg\varphi_0^*$ are both consistent with PA $\cup (T \cap B_0)$. In either case we can effectively find the index i_1^* of the completion.

Let $T_1 = T_1^* \cap B_1$. We can find its index i_1 effectively as well. Informally, T_1 'codes' whether or not $0 \in X$.

Code$_1$: Take the Π_3 formula $\varphi_1(x)$ from the sequence given by Scott's Lemma 4.3, where $\varphi_1(x)$ is independent over PA $\cup T_1$. For $k \in R_0$, put $\varphi_1(S^{(k)}(0))$ into T_3^*. For $k \notin R_0$, put $\neg\varphi_1(S^{(k)}(0))$ into T_3^*.

Next, we find the index for a completion of PA $\cup T_1 \cup \{\varphi_1(S^{(k)}(0)) : k \in R_0\} \cup \{\neg\varphi_1(S^{(k)}(0)) : k \notin R_0\}$. Then let T_3 be the B_3 part of this completion, again finding its index i_3. Informally, T_3 codes that R_0 is in $Rep(T)$.

Code$_{2n}$: Take the Π_{3n+1} sentence φ_{3n}^*, where φ_{3n}^* is independent over PA $\cup T_{3n}$.

If $n \in X$, let $T_{3n+1}^* =$ a completion of PA $\cup (T_{3n} \cap B_{3n}) \cup \{\varphi_{3n}^*\}$ in $\mathcal{S}$.

If $n \notin X$, let $T_{3n+1}^* =$ a completion of PA $\cup (T_{3n} \cap B_{3n}) \cup \{\neg\varphi_n^*\}$ in $\mathcal{S}$.

Once again, we can effectively find the index i_{3n+1}^* of T_{3n+1}^*. Let $T_{3n+1} = T_{3n+1}^* \cap B_{3n+1}$. We can find its index i_{3n+1} effectively as well.

Code$_{2n+1}$: Take the Π_{3n+3} formula $\varphi_{3n+1}(x)$ of our sequence, where $\varphi_{3n+1}(x)$ is independent over PA $\cup T_{3n+1}$. For $k \in R_n$, put $\varphi_{3n+1}(S^{(k)}(0))$ into T_{3n+3}^*. For $k \notin R_n$, put $\neg\varphi_{3n+1}(S^{(k)}(0))$ into T_{3n+3}^*.

Next, we find an index for a completion of

$$\mathrm{PA} \cup T_{3n+1} \cup \{\varphi_{3n+1}(S^{(k)}(0)) : k \in R_n\} \cup \{\neg\varphi_{3n+1}(S^{(k)}(0)) : k \notin R_n\}.$$

Then let T_{3n+3} be the B_{3n+3} part of this completion, finding its index i_{3n+3}.

This ends our inductive definition of $T(X,R)$. By our construction, it is clear that $Rep(T(X,R)) = S$.

⊣

Note that our construction also gives that $X \leq_T T(X,R)$. To determine if $n \in X$, we may ask $T(X,R)$ which of $\pm\varphi^*_{3n} \in T(X,R)$. If $\varphi^*_{3n} \in T(X,R)$, then $n \in X$; if $\neg\varphi^*_{3n} \in T(X,R)$, then $n \notin X$.

We can use this construction to build the following theory, demonstrating that the extra conditions requiring approximating functions for the fragments of the theory in Theorems 1.17 and 3.4 cannot be dropped:

Corollary 4.5. *For any enumeration R of a Scott set $\mathcal{S}$, there is a completion T of PA such that $Rep(T) = \mathcal{S}$ and there is no model $\mathcal{A} \models T$ such that $\mathcal{A} \leq_T R$.*

Proof. Let X be a set such that $X \not\leq_T R^{(\omega)}$. Let T be a completion given by the construction of Theorem 4.4. We show that if $\mathcal{A} \models T$, then $\mathcal{A} \not\leq_T R$. If $\mathcal{A} \models T$ and $\mathcal{A} \leq_T R$, then we have $X \leq_T T \leq_T \mathcal{A}^{(\omega)} \leq_T R^{(\omega)}$, contradicting the fact that $X \not\leq_T R^{(\omega)}$. ⊣

We can use Theorems 1.17 and 4.4 to prove the following related theorem of Harrington [5]:

Theorem 4.6 (Harrington). *There exists a nonstandard model $\mathcal{A} \models$ PA such that $\mathcal{A} \leq_T 0'$ and $Th(\mathcal{A})$ is not arithmetical.*

Proof. We show how to use Solovay's Theorem 1.17 and Theorem 4.4 to prove Harrington's Theorem. Choose $X \equiv_T \mathrm{TA} \equiv_T \emptyset^{(\omega)}$ as follows:

$$n = < n_0, n_1 > \in X \Leftrightarrow n_1 \in \emptyset^{(n_0)}.$$

Let $R \leq_T \emptyset'$ be an enumeration of a Scott set $\mathcal{S}$. By Marker's result, we may take R to be an effective enumeration. Use Theorem 4.4 to obtain a completion $T(X,R)$ of PA.

We claim that there is a model $\mathcal{A}$ with $SS(\mathcal{A}) = S$ such that $\mathcal{A} \leq_T R$. In order to use Solovay's Theorem to get $\mathcal{A}$, we need to specify the functions t_n for $n \geq 1, \Delta^0_n(X)$ uniformly in n, such that $\lim_{s\to\infty} t_n(s)$ is an index for $T(X,R) \cap \Sigma_n$ and for all s, $t_n(s)$ is an index for a subset of $T(X,R) \cap \Sigma_n$. We begin by letting $t_1(0) = R$-index for $T(X,R) \cap \Sigma_1$. We show how to use $\Delta^0_n(R)$ to find R-indices for $T(X,R) \cap \Sigma_1$. These functions t_n will be constant for all s. Thus the requirement that for all s, $t_n(s)$ is an index for a subset of $T(X,R) \cap \Sigma_n$ will be satisfied.

Fix n. We define $t_n(s)$ as follows. Using $\Delta^0_n(R)$, we proceed through the first $2n$ steps of the construction in the proof of Theorem

4.4, finding an R-index for $T(X, R) \cap B_{3n}$. Since $T(X, R) \cap \Sigma_n \leq_T T(X, R) \cap B_{3n}$, we may use our effective enumeration R to obtain an R-index for $T(X, R) \cap \Sigma_n$. Let $t_n(s)$ equal this R-index, for all s.

We may now apply Solovay's Theorem 1.17. We obtain a model $\mathcal{A} \models T(X, R)$ such that $SS(\mathcal{S}) = S$ and $\mathcal{A} \leq_T R$. Since $R \leq_T \emptyset'$, we get that $\mathcal{A} \leq_T \emptyset'$, as required. Since $X \leq_T T(X, R)$, we get that $Th(\mathcal{A})$ is not arithmetical.

⊣

As a consequence of Harrington's Theorem we get that the following holds:

COROLLARY 4.7. *There is a completion T of PA with a nonstandard model $\mathcal{A}$ such that $T \not\leq_T \mathcal{A}^{(n)}$ for any n.*

The examples we have given here show that we cannot simply drop condition (2) of Solovay's Theorem 1.17. In a related paper [1], we have shown that Solovay's Theorem 1.17 cannot be simplified as follows. We cannot simplify the result by restricting the approximating functions to being only (i) constant functions, (ii) functions that change values only k many times for some fixed k, or (iii) functions that change values $f(n)$ many times for each n, for some computable function f.

§5. Sequences of degrees and completions of PA. In this last section we leave Solovay's results behind and consider the possible sequences of degrees $deg(T \cap \Sigma_n)$, where T is a completion of PA. We make use of the following notion: for sets A and B, $A << B$ iff there is a completion T such that $T \leq_T B$ and $A \in Rep(T)$. This notion extends naturally from sets to degrees. For more results regarding this notion, see [13].

Our main result in this section is:

THEOREM 5.1. *For any sequence $(\mathbf{d_n})_{n\in\omega}$ of Turing degrees, the following are equivalent:*

(1) *There exists a completion T of PA such that for all n, $\mathbf{d_n} = deg(T_n)$.*
(2) $\mathbf{0} = \mathbf{d_0} << \mathbf{d_1} << \mathbf{d_2} << \dots$.

In Theorem 5.1 and in the rest of the section, T_n denotes $T \cap \Sigma_n$, when T is a completion of PA.

In what follows we use the fact, due to Gödel [4] and Matijasevich [10], that we may bound quantifiers by a primitive recursive function without increasing the complexity of formulas in which we use these quantifiers. Thus, we may take the primitive recursion function $p(s) = \prod_{i<s} p_i$, as a bound on existential quantifiers without increasing the complexity of formulas. For perspicuity, we will abuse notation and identify this function with its represented counterpart.

We prove Theorem 5.1 by breaking it into two pieces, Theorems 5.2 and 5.5. Here is the first direction:

THEOREM 5.2. *Let T be any completion of PA. Then $T_n << T_{n+1}$, for each $n \geq 0$.*

For any completion T of PA and any n, let $\widetilde{T_n} = T_n + \{\neg\varphi : \varphi \text{ is } \Sigma_n, \varphi \notin T_n\}$. To prove Theorem 5.2, we use the following two lemmas:

LEMMA 5.3. *Let T be a completion of PA and let n be given. If $\tau \subseteq 2^{<\omega}$ is computable in T_n, then there are formulas, Π_{n+1} and Σ_{n+1} respectively, such that each formula represents τ in $\mathrm{PA} + \widetilde{T_n}$.*

Note that these formulas represent τ in any completion T^* such that $T^* \cap \Sigma_n = T_n$.

LEMMA 5.4. *Let T be a completion of PA and let n be given. If $\tau \subseteq 2^{<\omega}$ is representable by both a Π_{n+1} and a Σ_{n+1} formula in T, then τ has a path computable in $T \cap \Sigma_{n+1}$.*

Using Lemmas 5.3 and 5.4, here is how we prove Theorem 5.2:

PROOF. Fix T. We show that for every fragment T_n of T, there is another completion $T^* \leq_T T_{n+1}$, with $T_n \in Rep(T^*)$. We find $T^* \leq_T T_{n+1}$ by using T_{n+1} to compute a path through a tree of completions of $\mathrm{PA} + \widetilde{T_n}$. This path is our T^*.

Here is how we define this tree of completions. Let $(\varphi_k)_{k\in\omega}$ be a computable list of all sentences in $\mathcal{L}_{\mathrm{PA}}$. We construct the tree of completions of $\mathrm{PA} + \widetilde{T_n}$, denoted τ_n, as an infinite binary-branching tree as follows. A node $\sigma \in 2^{<\omega}$ is in τ_n iff there is no proof of a contradiction of length less than $len(\sigma)$ from the set

$$\mathrm{PA} \cup \widetilde{T_n} \cup \{\varphi_k : \sigma(k) = 1\} \cup \{\neg\varphi_k : \sigma(k) = 0\}.$$

Each path through τ_n corresponds to a completion of $\mathrm{PA} + \widetilde{T_n}$, since paths decide every sentence from $(\varphi_k)_{k\in\omega}$ consistently.

Fix $n \geq 0$. Note that $\tau_n \leq_T T_n$. By Lemmas 5.3 and 5.4, there is a path T^* through τ_n computable in T_{n+1}. Note that we have that

$T_n = T_n^*$ and $T_n^* \leq_T T^*$. Then, since $Rep(T^*)$ is a Scott set, we have that $T_n^* \in Rep(T^*)$. Hence $T_n \in Rep(T^*)$.

⊣

Next, we prove Lemmas 5.3 and 5.4. First, we prove Lemma 5.3:

PROOF. Fix n. Since $\tau \leq_T T_n$, there is some e such that $\chi_\tau(x) = \varphi_e^{T_n}(x)$. It is well-known that there are Π_1 and Σ_1 formulas $\psi_\Pi(e,x,y,z,\sigma)$ and $\psi_\Sigma(e,x,y,z,\sigma)$ representing that y is a computation of φ_e on input x using oracle σ with output y. To represent the oracle T_n, we make use of the formula $\mathrm{Sat}_n(x)$, defining truth for Σ_n sentences. It is well-known that $\mathrm{Sat}_n(x)$ is Σ_n. Using $\mathrm{Sat}_n(x)$, we give a Σ_{n+1} formula for representing $\varphi_e^{T_n}(x)$:

$$\delta_\Sigma(x) = \exists y \exists \sigma[\psi_\Sigma(e,x,y,1,\sigma) \wedge \forall t < len(\sigma)[(\sigma(t)=1 \rightarrow \mathrm{Sat}_n(t)) \wedge (\sigma(t)=0 \rightarrow \neg\mathrm{Sat}_n(t))]].$$

We also give a Π_{n+1} representing formula:

$$\delta_\Pi = \forall y \forall \sigma[\psi_\Sigma(e,x,y,0,\sigma) \rightarrow \exists t < len(\sigma)[(\sigma(t)=1 \wedge \neg\mathrm{Sat}_n(t)) \vee (\sigma(t)=0 \wedge \mathrm{Sat}_n(t))]].$$

⊣

We now prove Lemma 5.4:

PROOF. There are two cases to consider. In Case 1, we assume that T proves that τ has an infinite path. In Case 2, we assume that T proves that τ does not have an infinite path. In both cases, we show how to find ζ, a path through τ.

Case 1: T proves that τ has an infinite path.

We first present a Π_{n+1} formula infinite-left(τ) that holds iff a node $\sigma \in \tau$ has an infinite extension in τ to its left:

$$\text{infinite-left}(\sigma) := \forall s > len(\sigma) \exists \gamma \leq p(s+1)[(len(\gamma) = s+1) \wedge ((\sigma \wedge 0) \subseteq \gamma) \wedge \delta_\Pi(\gamma)].$$

Suppose we have determined an initial segment σ_i of our path ζ through τ, where σ_i has length i. Here is how we decide whether to branch to the left or right at the i^{th} level in our path. We update our path to $\sigma_i \wedge 0$ if infinite-left(σ_i) $\in T_{n+1}$. We update our path to $\sigma_i \wedge 1$ if $\neg$infinite-left(σ_i) $\in T_{n+1}$. We do this for every $i \geq 0$. Let

$$\zeta = \bigcup_{i \in \omega} \sigma_i.$$

Since we use T_{n+1} as an oracle, we get that $\zeta \leq_T T_{n+1}$.

Case 2: T does not prove that τ has an infinite path.

In this case, T proves that there is some level past which no node in the tree can be consistently extended. We extend an initial segment to this maximum level, and take this initial segment to be our ζ. Since we will use T_{n+1} as an oracle, we will have $\zeta \leq_T T_{n+1}$.

Suppose we have determined an initial segment σ_i of our path ζ through τ, where σ_i has length i. Here is how we decide whether to branch to the left or right at the i^{th} level in our path. Since we are in Case 2, T witnesses that τ is finite. Thus T proves that there is some first level s_0 to the left of σ_i and some first level s_1 to the right of σ_i, beyond which no path can be consistently extended. We extend to $\sigma_i \wedge 0$ if $s_0 > s_1$, while we extend to $\sigma_i \wedge 1$ if $s_1 \geq s_0$.

To decide whether $s_0 > s_1$ or $s_1 \geq s_0$, we use two Σ_{n+1} formulas, $\psi_0(\pi)$ and $\psi_1(\pi)$. The formula $\psi_0(\pi)$ holds if there is a level s such that there is some node extending the node π to the left that is contained in τ; while at the same time, there is no node at level s extending π to the right that is contained in τ. The formula ψ_1 is similar but considers extensions to the right. Here are the formulas:

$$\psi_0(\pi) := \exists s \exists \sigma \leq p(s+1)(len(\sigma) = s+1) \wedge ((\pi \wedge 0) \subseteq \sigma) \wedge \delta_\Sigma(\sigma) \wedge$$

$$\forall \lambda \leq p(s+1)[(len(\lambda) = s+1) \wedge ((\pi \wedge 1) \subseteq \lambda) \rightarrow \neg \delta_\Pi(\lambda)]$$

and

$$\psi_1(\pi) := \exists s \exists \sigma \leq p(s+1)[[(len(\sigma) = s+1) \wedge ((\pi \wedge 1) \subseteq \sigma) \wedge \delta_\Sigma(\sigma)] \wedge$$

$$\forall \lambda \leq p(s+1)[(len(\lambda) = s+1) \wedge ((\pi \wedge 0) \subseteq \lambda) \rightarrow \neg \delta_\Pi(\lambda)]].$$

If $\psi_0(\tau_t) \in T_{n+1}$, then $s_0 > s_1$. If $\psi_1(\tau_t) \in T_{n+1}$, then $s_1 \geq s_0$. If neither is in T_{n+1} for some level i^*, then we have reached the maximum extendible level of τ, according to T_{n+1}. Let

$$T^* = \bigcup_{i<i^*} \sigma_i.$$

Since we use T_{n+1} as an oracle, we get that $T^* \leq_T T_{n+1}$.

$\dashv$

This completes the proof of the (1) $\Rightarrow$ (2) direction of Theorem 5.1. Next, we give the (2) $\Rightarrow$ (1) direction, with proof:

THEOREM 5.5. *Suppose* $(\mathbf{d_n})_{n\in\omega}$ *is a sequence of Turing degrees such that*

$$\mathbf{0} = \mathbf{d_0} << \mathbf{d_1} << \mathbf{d_2} << \dots .$$

Then there exists a completion T *of PA such that for all* n, $\mathbf{d_n} = deg(T_n)$.

PROOF. We build the completion T inductively by determining each of its fragments T_i. Let $T_0 = \text{PA} \cap \Sigma_0$. For our inductive step, suppose we have specified T_{n-1}. We build T_n so that $T_n \equiv_T D_n$, for D_n a fixed representative from $\mathbf{d_n}$. After we show how to construct T_n, we will show that $T_n \equiv_T D_n$, by how we have constructed T_n.

By assumption, there is a completion T^* such that $T^* \leq_T D_n$ and $T_{n-1} \in Rep(T^*)$. Let φ_k be a computable list of the Σ_n sentences of $\mathcal{L}_{\text{PA}}$. We break our construction into attempts to meet the following requirements, for $k \geq 0$:

R_{2k}: Put one of φ_k or $\neg\varphi_k$ into T_n

R_{2k+1}: Code whether $k \in D_n$ into T_n.

To meet these requirements, we define sets A_i such that

$$\bigcup_{i\in\omega} A_i = T_n.$$

First, let $A_0 = \text{PA} \cup T_{n-1}$. Suppose we have already defined A_j. There are two cases to consider:

Case 1: j is even

Then $j = 2k$, for some $k \geq 0$. We define A_{2k+1}, in an attempt to meet requirement R_{2k}. To decide whether to add φ_k or $\neg\varphi_k$ to T_n, we use the following notion. We say that $A_{2k} \cup \{\varphi_k\}$ is *more inconsistent* than $A_{2k} \cup \{\neg\varphi_k\}$, according to T^*, iff T^* proves that there is a smaller proof of an inconsistency from $A_{2k} \cup \{\varphi_k\}$ than there is from $A_{2k} \cup \{\neg\varphi_k\}$. Let γ_k be the sentence in $\mathcal{L}_{\text{PA}}$ expressing that $A_{2k} \cup \{\varphi_k\}$ is more inconsistent than $A_{2k} \cup \{\neg\varphi_k\}$.

Claim 1: If $\gamma_k \in T^*$, then $A_{2k} \cup \{\neg\varphi_k\}$ is consistent.

If we are in this case, then we put $\neg\varphi_k$ into T_n. Let $A_{2k+1} := A_{2k} \cup \{\neg\varphi_k\}$.

Claim 2: If $\gamma_k \notin T^*$, then $A_{2k} \cup \{\varphi_k\}$ is consistent.

If we are in this case, then we put φ_k into T_n. Let $A_{2k+1} := A_{2k} \cup \{\varphi_k\}$.

To finish describing how to meet the even requirements, we need to prove Claims 1 and 2. We leave those proofs until the end.

Case 2: j is odd
Then $j = 2k+1$, for some $k \geq 0$. We build A_{2k+2}, in an attempt to meet requirement R_{2k+1} Use Lemma 4.1, the variant of the Gödel – Rosser Theorem, to get a Π_n sentence ψ_k, independent over A_{2k+1}.
If $k \in D_n$, put $\neg\psi_k$ into T_n. Let $A_{2k+2} = A_{2k+1} \cup \{\neg\psi_k\}$.
If $k \notin D_n$, put ψ_k into T_n. Let $A_{2k+2} = A_{2k+1} \cup \{\psi_k\}$.

This ends our description of the construction. No injury ever threatens these requirements, so in the limit they will all be met. Let $\bigcup_{i\in\omega} A_i = T_n$.

To show that $T_n \equiv_T D_n$, we must show both that $D_n \leq_T T_n$ and $T_n \leq_T D_n$. First, we show that $D_n \leq_T T_n$. To do this, we need to decode if $k \in D_n$, computably in T_n, by following the construction through requirement R_{2k+1}. We reuse the computable list $(\varphi_k)_{k\in\omega}$ of Σ_n sentences of $\mathcal{L}_{\text{PA}}$. To begin decoding, ask T_n if $\varphi_0 \in T_n$. Using the answer, update the set A_1. Using A_1, we may use Lemma 4.1 to compute the Π_n sentence ψ_0 that is independent over A_1. Using T_n, we check whether $\pm\psi_0 \in T_n$. If $\psi_0 \in T_n$, then we know that $0 \notin D_n$, by construction. If $\neg\psi_0 \in T_n$, then we know $0 \in D_n$. In either case, we have decoded whether or not $0 \in D_n$. Use this answer to update A_2.

At step $2k$, ask T_n if $\pm\varphi_k$ is in T_n, as described above for step 0. Update A_{2k+1}. Do step $2k+1$, deciding if $\pm\psi_k$ in T_n as above for step 1, and hence decoding whether $k \in D_n$.

Next, we show that $T_n \leq_T D_n$. For a B_n sentence α, we want to determine whether $\alpha \in T_n$. By assumption, we have that there is a completion T^* such that $T^* \leq_T D_n$. We may follow through the steps of the construction given above, computably in D_n. At each even step $2k$, building A_{2k+1}, we check if $\alpha = \pm\varphi_k$. If it is, by following through the steps in Case 1, we determine whether or not we put $\alpha \in A_{2k+1}$. If it is not, we follow through the steps of Case 1 and Case 2, reaching the next even step. Since our computable list $(\varphi_i)_{i\in\omega}$ contains every Σ_k sentence in $\mathcal{L}_{\text{PA}}$, we will eventually reach an even step $2k$ where $\alpha = \pm\varphi_k$.

Finally, we give the proofs of Claims 1 and 2.

Proof of Claim 1: Suppose $\gamma_k \in T^*$. Let p witness γ_k in T^*. Then p is a proof of $\perp$ from $A_{2k} \cup \{\varphi_k\}$ in T^*, and for all $q < p$, q is not a proof of $\perp$ from $A_{2k} \cup \{\neg\varphi_k\}$ in T^*. If p is standard, then there is no standard proof of $\perp$ from $A_{2k} \cup \{\neg\varphi_k\}$, so $A_{2k} \cup \{\neg\varphi_k\}$ is consistent. If p is nonstandard, then since there is no smaller proof of $\perp$ from $A_{2k} \cup \{\neg\varphi_k\}$, in particular there can be no *standard* proof of $\perp$ from $A_{2k} \cup \{\neg\varphi_k\}$. Again, $A_{2k} \cup \{\neg\varphi_k\}$ is consistent.

Proof of Claim 2: Suppose $\gamma_k \notin T^*$. If there is no proof of $\perp$ from $A_{2k} \cup \{\varphi_k\}$, then we are finished. Suppose p is a proof of $\perp$ from $A_{2k} \cup \{\varphi_k\}$. If p is standard, then there is a proof $q < p$ of $\perp$ from $A_{2k} \cup \{\neg\varphi_k\}$. We then have that $A_{2k} \vdash \neg(\varphi_k \vee \neg\varphi_k)$, or equivalently, $A_{2k} \vdash \varphi_k \wedge \neg\varphi_k$. This contradicts the fact that we have built A_{2k} to be consistent. Thus p cannot be standard, so $A_{2k} \cup \{\varphi_k\}$ is consistent.

$\dashv$

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