Visual Thinking in Mathematics
By MARCUS GIAQUINTO
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Our visual experience seems to suggest that no continuous curve can cover every point of the unit square, yet in the late nineteenth century Giuseppe Peano proved that such a curve exists. Examples like this, particularly in analysis (in the sense of the infinitesimal calculus) received much attention in the nineteenth century. They helped instigate what Hans Hahn called a "crisis of intuition", wherein visual reasoning in mathematics came to be thought to be epistemically problematic. Hahn described this "crisis" as follows:

> Mathematicians had for a long time made use of supposedly geometric evidence as a means of proof in much too naive and much too uncritical a way, till the unclarities and mistakes that arose as a result forced a turnabout. Geometrical intuition was now declared to be inadmissible as a means of proof... (p. 67)

Avoiding geometrical evidence, Hahn continued, mathematicians aware of this crisis pursued what he called "logicization", "when the discipline requires nothing but purely logical fundamental concepts and propositions for its development." On this view, an epistemically ideal mathematics would minimize, or avoid altogether, appeals to visual representations. This would be a radical reformation of past practice, necessary, according to its advocates, for avoiding "unclarities and mistakes" like the one exposed by Peano.

Many mathematicians have eagerly pursued this reformed mathematics, and philosophers of mathematics have rightly attended to it. Nevertheless, visual representations like diagrams have continued to be used to good purposes across mathematics. A philosophy of mathematics that solely treats "logicized" mathematics ignores or distorts an important component of mathematical practice.

This book is an attempt to take visual representations in mathematics seriously. While it leaves us with more questions than definitive answers, that is perfectly good for a work intended to stimulate further research. Giaquinto's primary aim is to show that visual representations have epistemic value in mathematics, particularly in the *discovery* of new results in geometry and arithmetic.

Giaquinto begins with visual thinking in geometry in Chapters 2–5. He claims to have answered Kant's question of how synthetic *a priori* judgments are possible in geometry. I will (roughly) describe his approach, because it grounds much of what follows in the book. Firstly, he gives an account of shape perception, focused on the single example of squares. Drawing on recent psychological work on visual perception, he arrives at what he calls a "visual category specification" for squares, that is, the set of features whose detection by the visual system is sufficient for perceiving a figure as a square. Secondly, he gives an account of "perceptual concepts" of shapes, which are specified by the feature descriptions comprising that shape's category specification (up to some unspecified degree of accuracy). To possess a perceptual concept of a shape is to be disposed to judge something to be that shape when it appears to be that shape, and the appearing seems reliable. He then isolates a class of perceptual concepts he calls "geometric concepts"; to possess a geometric concept of a shape is to be disposed to judge something to be that shape when it appears to be *a perfect instance* of that shape, and the appearing seems reliable. Finally, Giaquinto explains that visual experiences can "trigger" these belief-forming dispositions, and when such a belief-forming disposition is reliable, the resulting beliefs are knowledge.

There's a lot to worry about here. For instance: are beliefs formed this way really *a priori*, in light of the key role played by perception? How is reliability to be measured for mathematical truths? Are any or all these geometric belief-forming dispositions latent, and if so, which ones? Should the identity conditions of geometric concepts depend so much on empirical matters, particularly those of an area as fluid as psychology? Giaquinto has responses to these questions, but I hope that others will engage these arguments critically so that the arguments may be improved.

In Chapters 6–8, Giaquinto turns to arithmetic, arguing that visual thinking plays an essential epistemic role in arithmetic. He considers reasoning with number lines; computations using both finger counting for single digit numbers and place value for multiple digit numbers; and the discovery of general numerical formulas like those used in combinatorics. He again draws on contemporary work in cognitive psychology, most notably regarding whether humans have an innate "number sense" for judging which of two numbers is larger.

Recognizing that mathematics today stretches far beyond elementary geometry and arithmetic, in Chapters 9 and 11 Giaquinto applies the framework of earlier chapters to visual thinking in analysis, and to the type of visual reasoning concerning structures common in abstract algebra and logic today. The work on analysis is particularly valuable, for as explained above, it was one of the main sources of the "crisis of intuition" that devalued visual reasoning in the nineteenth century. In dwelling on examples (particularly Rolle's theorem) rather than a systematic treatment of visual reasoning in analysis, the chapter is more a prospectus for future work than other chapters. In Chapter 10 Giaquinto considers algebraic symbol manipulation as an instance of visual reasoning. This is a great idea, and quite different from the usual formal language treatments of algebraic reasoning by philosophers. He extends this idea by arguing in Chapter 12 that algebraic and geometric reasoning differ less than traditional accounts would have it, in light of the essential role of visual thinking in both.

I recommend this book highly. It is rich with insight, stimulating, and worth careful thought. Furthermore, its concise presentation of recent psychological work is invaluable.

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