Quasi-truth and defective knowledge in science: a critical examination

Jonas R. Becker Arenhart*
Décio Krause†

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Abstract

Quasi-truth (a.k.a. pragmatic truth or partial truth) is typically advanced as a framework accounting for incompleteness and uncertainty in the actual practices of science. Also, it is said to be useful for accommodating cases of inconsistency in science without leading to triviality. In this paper, we argue that the given developments do not deliver all that is promised. We examine the most prominent account of quasi-truth available in the literature, advanced by da Costa and collaborators in many places, and argue that it cannot legitimately account for incompleteness in science: we shall claim that it conflates paraconsistency and paracompleteness. It also cannot account for inconsistencies, because no direct contradiction of the form $\alpha \wedge \neg\alpha$ can be quasi-true, according to the framework. Finally, we advance an alternative interpretation of the formalism in terms of dealing with distinct contexts where incompatible information is dealt with. This does not save the original program, but seems to make better sense of the formalism.

Keywords: quasi-truth; pragmatic theories of truth; paraconsistency; incompleteness.

1 Introduction

Scientific theories and scientific knowledge, in general, may be said to be defective in a multitude of ways. Our abilities to generate knowledge are known to be less than perfect, and we are frequently found to be holding mistaken views about the nature of reality. Of course, science itself is self-correcting, but even the optimistic hopes for a final correct and true theory cannot avoid the fact that our situation is less than perfect on what concerns our *actual* theories.

*Department of Philosophy, Federal University of Santa Catarina, Brazil
†Department of Philosophy, Federal University of Santa Catarina, Brazil. Partially supported by CNPq.

One of the senses in which current science is defective concerns the fact that it is not complete in an important sense. There is much that still needs to be investigated, many open questions, and many tentative claims and theories that are not known to be true, or that are certainly false. It seems that a framework able to deal with such situations must accommodate this kind of incompleteness or openness of current knowledge. As da Costa and French [8, pp.13-14] put it, considering the possibility that there is an ideal limit to which scientific knowledge may converge,

> If the final conception is taken to be complete or total, then our conception at any given time prior to the realization of this limit may be said to be *partial*. And because it is, at any given time, partial, it is, at that time, *open* in the sense that it may be completable in a variety of ways.

Philosophically, this incompleteness is thought of as challenging because most of our accounts of scientific theories rely on classical apparatuses that cannot straightforwardly be said to accommodate incomplete knowledge. In a nutshell, the semantic view of scientific theories, for instance, which is taken by many to be the current orthodoxy, regards scientific theories as classes of set theoretical structures (see Krause and Arenhart [14]). These structures, on their turn, are total, not partial, in the sense that relations and properties in them are always defined for every entity in the domain of the structure; no openness or incompleteness allowed. This brings some trouble when it comes to discuss the appropriate epistemic attitude towards such theories, given that they are treated as either completely true or else completely false. It is a matter of all or nothing, it seems.

There is, however, an alternative framework that was advanced precisely in order to deal with these situations: the concept of pragmatic truth or quasi-truth, as introduced by Mikenberg, da Costa, and Chuaqui [15]. The account was further explored by da Costa and French [8], among many others. As da Costa and French [8, p.14] put it, the framework was advanced precisely to model the current situation in science, because "[i]t is precisely this sense of partiality and openness that our account attempts to capture and further explore". Pragmatic truth, then, promises to account for incomplete information in science, with the benefits that are expected to result from such a more realistic characterization of science.

However, there is more than that in pragmatic truth. The view is said to deal with another kind of defective situation in science: the ones involving contradictions. As Bueno and da Costa [5, p.385] claim, an account of scientific rationality must explain how is it possible that scientists entertain inconsistent theories. That inconsistent theories have been frequently entertained is typically argued for by the presentation of examples of theories that were indeed entertained, and were found out to be inconsistent: the early formulation of the calculus, Bohr's atomic model, Frege's original logicist reconstruction of arithmetic, naive set theory, quantum mechanics and general theory of relativity (taken together), to mention a few. Again, the major problem consists in explaining our epistemic attitude towards such theories, and also accounting for their apparent non-triviality. Now, pragmatic truth is said to accommodate also cases of inconsistency in science without entailing triviality (Bueno [4, p.275]; da Costa and French [8, chap.5]; Bueno and da Costa [5, p.392]). Inconsistent theories may be quasi-true in a sense, without logically implying everything. That makes of quasi-truth a nice tool for philosophers to approach current science, no?

In this paper, we shall argue that quasi-truth falls short of delivering what is promised. Basically, our claim is that defective situations in science such as incompleteness and inconsistency cannot be clearly accounted for by the use of the conceptual tools provided by the notion of pragmatic truth. The view fails in providing the appropriate views of incompleteness and also of inconsistency. More than that, it conflates inconsistency and incompleteness, which are clearly distinct phenomena when it comes to knowledge and information. The source of such problems, we argue, comes basically from a much liberal use of the concept of quasi-truth, which in most informal uses of the conceptual framework in hand simply ignore the restrictions to a given structure. Once it is recalled that a sentence may be quasi-true *in a total structure relatively to a given partial structure*, then, a weaker reading of quasi-truth may be provided for. This weaker reading does not do everything that quasi-truth was expected to do by its proponents, but it does a better job in making sense of the formalism.

The paper is structured as follows. In section 2 we recover the formalism of pragmatic truth as it is typically presented by da Costa and collaborators. We then present in section 3 the main problems for the view, and how it is found wanting on its own terms. One of the diagnosis of what has gone wrong is that the formalism has conflated paraconsistency and paracompleteness; it attempts to capture incomplete knowledge with contradictions. In section 4, we argue that the original formalism by da Costa may be open for a distinct, pragmatic interpretation that is better suited for its purposes. The account no longer accounts for contradictions, but it is closer to what the formalism originally introduced actually does. We conclude in section 5.

2 Quasi-truth: the basic definitions

Let us begin by reviewing the basic concepts of the first kind of definition of quasi-truth or pragmatic truth. Basically, this kind of approach requires a detour through the Tarskian definition of truth, so that this notion plays a central role (and this will be important for our discussion).

When investigating a domain of knowledge Δ, we formulate a conceptual scheme in order to deal with the entities in this domain and their relations. This requires setting a set D, the domain of entities, which may have as elements both concrete entities, such as tracks in a cloud chamber, as well as non-directly observable posits of our theories, such as quarks and strings. Accounting for the behavior of these entities also involves devising a family K of relations and properties that hold for these entities, on which we are interested in. This will give rise to a set theoretical structure $\mathfrak{A} = \langle D, R_k \rangle_{k \in K}$ (we use standard Zermelo-Fraenkel set theory with the axiom of choice in the metalanguage).

In classical structures (that is, structures in classical model theory), we typically define each R_k as being a *total* relation, that is, if R_k is an n-ary relation, for each n-tuple of elements of D, either R_k holds of this n-tuple, or else it doesn't; no further options available. In science, however, there are situations in which our knowledge about the entities in D and the relations R_k is incomplete, so that it may not be completely clear whether some n-tuple holds of R_k or not. That is, the relation R_k is only *partially defined*, because for some n-tuples it is *left open* whether they are related by R_k or not. The concept of *partial relation* was devised precisely to accommodate such cases of *incomplete information*.

In formal terms, a partial n-ary relation R over D is defined as a triple $R = \langle R_1, R_2, R_3 \rangle$ of sets of n-tuples of elements of D such that these three sets are mutually disjoint and their union is D^n (that is, the sets are mutually exclusive and jointly exhaustive of D^n). The explanations for the division are typically framed in terms of our lack of knowledge or lack of definition (as we did in the previous paragraph): R_1 is the set of n-tuples of elements of D for which *we know* that R holds, R_2 is the set of n-tuples of which *we know not to hold* of R, and R_3 is the set of n-tuples for which we do not know whether they are R-related or not (see Bueno [4, p.279]). Other times, the triple division in a partial relation is explained not in epistemic terms, but in terms of 'definition', so that "R_1 is the set of n-tuples that belong to R, R_2 is the set of n-tuples that do not belong to R, and R_3 is the set of n-tuples for which it is not defined whether they belong or not to R" (Bueno [4, p.279], see also da Costa and French [8, p.18]). Sometimes, the idea is further explained as follows (for binary relations, where A is used as the domain set):

> R_1 is the set of ordered pairs which are satisfied by those sentences expressing the relationship between the entities concerned, R_2 is the set of ordered pairs not satisfied by these sentences, and R_3 is the set of ordered pairs for which it is left open whether they are satisfied . . . It is precisely this which is meant when we say that R_k is "not necessarily defined for all n_k-tuples of elements of A" (da Costa and French [8, p.19])

That is, a relation 'not being defined' for some n-tuples is thought to be 'precisely' explained in terms of a relation being 'left open' for those entities. Our point is that such characterizations are not clearly equivalent, given that they vary from epistemic to semantic. The intuitive interpretation that is expected to be captured by the formalism is important, however, because it reveals the intended meaning conferred to the third component of a partial relation, and because it is thought that it is precisely this component which accounts for the incompleteness of scientific knowledge.

A *partial structure* is a structure $\mathfrak{A} = \langle D, R_k \rangle_{k \in K}$, where D is the domain of the structure and R_k is *a family of partial relations* over D. But that is not enough. We also need to enrich partial structures with a set P of sentences of the language to be interpreted in the partial structure (a language of the same similarity type as that of the structure), sentences to be accepted as true in the Tarskian sense. Which sentences are to be found in P? It all depends on which sentences of the theory one is wiling to countenance as true in the classical, Tarskian sense. Empiricists may be willing to accept as true only empirically decidable sentences, describing the results of experiments; scientific realists may go a step further and include in P statements or laws about unobservable entities. The resulting structure $\langle D, R_k, \mathsf{P}\rangle_{k\in K}$ is called *a simple pragmatic structure*.

The rationale behind the introduction of P is simple. Given a partial structure $\mathfrak{A}$, and a partial n-ary relation R, for any n-tuple $\vec{x} = \langle x_1, \ldots x_n\rangle$ of elements of D, if $\vec{x}$ is in R_3, we may extend R in two directions: according to one extension, $\vec{x}$ is in R_1, or, according to another extension, $\vec{x}$ is in R_2. That provides for too many extensions of a structure, given that this process may be repeated for each n-tuple and for each relation in the structure. In pragmatic structures, then, the role of P is to limit the number of extensions: "P introduces constraints on the ways that a partial structure can be extended" (Bueno and da Costa [5, p.388]). The idea is that an extension makes for a completion of the relations leading to a complete structure, and that these extensions are allowed only in the cases where they are consistent with the sentences in P.

Now, we are almost ready to define quasi-truth. In order to do that, we need the concept of a $\mathfrak{A}$-normal structure, which is complete or total structure associated with a partial structure $\mathfrak{A}$, in which the notion of truth is just the classical, Tarskian notion.

Given a first-order language $\pounds$ of the same similarity type as of that of a simple pragmatic structure $\mathfrak{A} = \langle D, R_k, \mathsf{P}\rangle_{k\in K}$, in which $\pounds$ is interpreted, and a simple pragmatic structure $\mathfrak{B} = \langle D', R'_k, \mathsf{P}\rangle_{k\in K}$, we say $\mathfrak{B}$ is an $\mathfrak{A}$-*normal structure* if (notice that the set P is the same in both structures):

i) $D = D'$,

ii) every constant in the language is interpreted in the same object in both $\mathfrak{A}$ and $\mathfrak{B}$,

iii) R'_k extends the corresponding R_k for a total relation,

iv) $\mathfrak{B}$ is a model of P, in the sense of Tarski.

A sentence S is *quasi-true in* $\mathfrak{A}$ according to $\mathfrak{B}$ iff

i) $\mathfrak{A} = \langle D, R_k, \mathsf{P}\rangle_{k\in K}$ is a simple pragmatic structure,

ii) $\mathfrak{B} = \langle D', R'_k, \mathsf{P}\rangle$ is an $\mathfrak{A}$-normal structure,

iii) S is true in $\mathfrak{B}$ in the Tarskian sense.

If S is not quasi-true in $\mathfrak{A}$ according to $\mathfrak{B}$, then S is *quasi-false* in $\mathfrak{A}$ according to $\mathfrak{B}$.

Also, S is quasi-true in $\mathfrak{A}$ if there is an $\mathfrak{A}$-normal structure $\mathfrak{B}$ such that S is quasi-true in $\mathfrak{A}$ according to $\mathfrak{B}$. Otherwise, S is quasi-false in $\mathfrak{A}$.

Notice that there are two related concepts of quasi-truth being defined:

(I) a sentence S is quasi-true in a structure $\mathfrak{A}$ according to $\mathfrak{B}$ (and quasi-false in $\mathfrak{A}$ according to $\mathfrak{B}$), and

(II) a sentence S is quasi-true in $\mathfrak{A}$ (no direct mention of which $\mathfrak{B}$ is being considered).

Bueno introduces a further definition

(III) "we say that a sentence S is *quasi-true* if there is a partial structure $\mathfrak{A}$ and a corresponding $\mathfrak{A}$-normal structure $\mathfrak{B}$ such that S is true in $\mathfrak{B}$ (according to Tarski's account). Otherwise, S is *quasi-false*" (Bueno [4, p.280], with notation adapted for the sake of uniformity).

Notice the differences between the three cases. In (I), a sentence S may be quasi-true in $\mathfrak{A}$ according to $\mathfrak{B}$, but $\neg S$ cannot be quasi-true in $\mathfrak{A}$ according to $\mathfrak{B}$, because $\mathfrak{B}$ is a Tarskian classical structure. In (II), S and $\neg S$ *may be quasi-true according to* $\mathfrak{A}$, it just requires that S is quasi-true according to a given total structure $\mathfrak{B}$, and $\neg S$ is quasi-true according to a *distinct* total structure $\mathfrak{C}$. This seems to allow for a contradiction (more on this soon). The definition (III) introduces a predicate of quasi-truth *simpliciter*, not relative to any specific structure. As a result, notice, a sentence S may be quasi-true, with its negation $\neg S$ also quasi-true: S may be quasi-true because it is quasi-true in a given partial structure $\mathfrak{A}$, and $\neg S$ may be quasi-true because it is quasi-true in *another* partial structure $\mathfrak{B}$.

As a result, we have two notions of quasi-truth in a model (I and II), and a notion of quasi-truth which does not mention any model (III). The fact that there are three concepts invites confusion when it comes to discuss the idea that partial structures capture incompleteness and contradictions in science. One must always keep in mind which of these three definitions of quasi-truth one is talking about in discussing the prospects of quasi-truth and whether it achieves its goals. Let us do that now.

3 Discussion

3.1 Partiality and incompleteness

Once these definitions are in order, let us check how that apparatus is supposed to deal with incompleteness and inconsistency. We begin with incompleteness and partiality, but this is directly related with inconsistency, as we shall see. According to da Costa and French [8, p.19] "the incomplete and imperfect nature of the majority of our representations of the world is, we claim, represented by the simple pragmatic structures just provided". That is, the hope is that pragmatic structures somehow accommodate incomplete knowledge by the fact that some relations are partial, which lead to quasi-truth, and not the whole truth. However, that the partial structures approach does indeed capture such incompleteness is something that must be argued for, and not just claimed. Let us try to enlighten this issue.

Let us begin by following how Bueno and da Costa [5, p.388] describe the workings of the partial truth conceptual machinery in order to capture partiality and incompleteness. Intuitively, they claim, the idea is that a quasi-true sentence does not describe the whole domain to which it refers to, but only an aspect of it, the aspect modelled by the relevant partial structure. The explanation for this: there are different ways a partial structure may be extended, and in some of them the target sentence may be true, in others false. Incompleteness means, then, that completion of the partial relations may be performed in distinct, incompatible, ways.

To begin with, notice that there seems to be two (somehow incompatible) claims about incompleteness here: *first*, that a quasi-true sentence is model of a partial part of the domain, not of the whole of it. *Second*, that incompleteness is accommodated in the account by the possibility of distinct incompatible extensions of the partial relations. Let us check these claims.

The *first claim* is easier to deal with. The idea seems to be that a quasi-true sentence accommodates partiality because it models only the aspects of the domain that are known or defined, not the undefined or unknown parts. That is, given a partial structure $\mathfrak{A}$, a sentence involving a relation symbol R interpreted in $\langle R_1, R_2, R_3 \rangle$ describes only the behavior of those entities for which it is defined that the entities are in the corresponding R_1 or in R_2. The behavior of the entities in R_3 is not accounted for by the sentence, because their status is unknown or undefined. If this is what is being claimed, then, notice, that this is far from incomplete or uncertain. That is, the aspects being said to be modelled by the sentence are precisely the well-known, well-defined aspects of the structure. The sentence is a complete description of the part that cannot change. So, it cannot be claimed that partial truth is partial because of *this* aspect of the model.

This kind of certainty also spreads to some of the entities that are in the R_3 part of an interpretation of a relation symbol R, in some cases. Notice that it may well be the case that, once the set P of sentences is added to the partial structure, then, the sentences in P constrain the $\mathfrak{A}$-normal structures in such a way that, for some partial relations R, the elements of R_3 may not be allowed to be extended consistently both to R_1 in some of them, and to R_2 in others. In that case, there is only one kind of extension for R consistent with P, and this is no longer a legitimate case incomplete knowledge. That is, there are some situations in which P adds constraints to the structure so that part of the incompleteness goes away.

So, the specific partiality and incompleteness that quasi-truth and partial structures are said to capture cannot concern the well-defined, well-known aspects of the relations and properties. There is nothing uncertain or partial *there*. In other words: in cases where we are dealing with a sentence that is not concerned with entities in the R_3 part of the model, our knowledge is not partial or incomplete. Suppose we are interested in a relation R in a partial structure $\mathfrak{A}$, and we speak about an n-tuple which lies in R_1 or in R_2. Then, the corresponding sentence is true or false in any $\mathfrak{A}$-normal structure extending $\mathfrak{A}$. No uncertainty or incompleteness is involved here.

Let us focus on the *second claim* above, *viz.* the one to the effect that partiality is accounted for in those cases where it is left open that, for some $\mathfrak{A}$-normal structures, an n-tuple in R_3 is extended to R_1, and in another $\mathfrak{A}$-normal structure, to R_2. Is it a good *representation* of a case of incomplete knowledge? Is incompleteness accounted for in the formal apparatus provided for?

To make our discussion simpler, and without losing generality, let us restrict ourselves to the case of a predicate symbol P, interpreted in a pragmatic structure $\mathfrak{A}$ as $P^{\mathfrak{A}} = \langle P_1, P_2, P_3 \rangle$, and a single element x in the corresponding P_3, which is the denotation of an individua constant a. It seems that the claim that our knowledge of whether the corresponding sentence Pa is the case is uncertain is due to the fact that one may have an $\mathfrak{A}$-normal structure where the corresponding sentence Pa is quasi-true, and another, where $\neg Pa$ is quasi-true. We just don't know which one is the structure describing reality correctly. So, we have incomplete information, it is said.

But now, let us consider whether that really represents a case of incomplete knowledge or lack of information in the formal apparatus just presented. We begin by considering the definition of quasi-truth given in (I). The plan seems to be as follows: the incompleteness is represented in terms of the lack of possibility or knowledge to determine which, among at least two incompatible $\mathfrak{A}$-normal structures, are to be chosen. We know that Pa is quasi-true in an $\mathfrak{A}$-normal structure $\mathfrak{B}$, and we also know that $\neg Pa$ is quasi-true in an $\mathfrak{A}$-normal structure $\mathfrak{C}$, but we don't know which one is to be taken as a representation of reality; this reflects our epistemic situation of uncertainty and incomplete information, and this is represented in the framework by the fact that we have statements "Pa is quasi-true in $\mathfrak{A}$ according to $\mathfrak{B}$" and "$\neg Pa$ is quasi-true in $\mathfrak{A}$ according to $\mathfrak{C}$".

However, if that is what is being taken to be a representation of incomplete or partial knowledge, then it is not easy to see that putting the problem in terms of quasi-truth has any advantage over classical logic. The classical logician could put the same situation in the same terms: we know that the sentence Pa is true (in the Tarskian sense) in a total structure $\mathfrak{B}$, and that $\neg Pa$ is true (idem) in a total structure $\mathfrak{C}$. We just don't know which is the case, *i.e.* which of the structures should be chosen or adopted. Then, the situations of uncertainty or incompleteness as described by quasi-truth (according to definition (I)) and classical logic are completely parallel. There is nothing essentially different in the classical case and in the quasi-true case. Then, it could be claimed, classical logic also models our incomplete information!

But that would be going too far. It is preferable to claim that quasi-truth fails to account for incomplete knowledge under definition (I). Certainly, the problem is that the uncertainty or incompleteness is not represented *inside* the model or conceptual apparatus of quasi-truth, when the definition of quasi-truth is stated in terms of definition (I), because that definition leads us to the problem of choosing a classical, Tarskian, structure, which is a problem the classical logician also faces. The diagnosis for that may be put as follows: typically, lack of knowledge is represented inside a framework by a failure of the law of excluded middle (LEM), either in its syntactical formulation, *viz.* $\alpha \vee \neg\alpha$, or else in a semantical formulation, stating that a sentence and its negation cannot both be false. One could hope that, if our knowledge of whether P holds for a in $\mathfrak{A}$ is uncertain, then, neither Pa nor $\neg Pa$ is the case *in the model*. That is, LEM, in some version of it, should fail in some cases. That is typically how lack of information is accounted for. However, $Pa \vee \neg Pa$ is quasi-true in every pragmatic structure $\mathfrak{A}$ in relation to any $\mathfrak{A}$-normal structure (quasi-true according to definition (I)). Also, $Pa \vee \neg Pa$ is quasi-true in any pragmatic structure $\mathfrak{A}$ (definition II). Finally, this sentence is also quasi-true in the third sense (III), defined by Bueno, above. Other versions of the LEM are also valid: it is the case that for any $\mathfrak{A}$-normal structure, Pa is quasi-true, or else $\neg Pa$ is quasi-true. In semantic terms: for each $\mathfrak{A}$-normal structure, Pa is quasi-true or quasi-false. How can that accommodate incompleteness? The incompleteness is only accounted for outside of the model, in the terms we have already explained: we don't know which $\mathfrak{A}$-normal structure should be chosen, the one in which Pa is true, or the one in which $\neg Pa$ is true. But that problem is also available for the classical logician, and if that kind of problem represents incompleteness, then, the classical logician can also 'represent' such situations.

Perhaps the idea that one may represent incomplete information is better represented by definitions (II) and (III) of partial truth. It could be claimed that they represent incomplete information inside the model by allowing inconsistent sentences to be both quasi-true. For the definition (II) (quasi-truth in a pragmatic structure $\mathfrak{A}$), it is possible to say that, in cases of elements x of P_3,[1] both Pa and $\neg Pa$ quasi-true in a pragmatic structure $\mathfrak{A}$. For (III), one ends up being able to say that both Pa and $\neg Pa$ are quasi-true *tout court*. Then, for every case where information is claimed to be lacking, one ends up discovering that both a sentence and it negation are quasi-true, either in the same partial structure (definition II), or else just quasi-true in distinct partial structures (definition III). This, it could be claimed, represents the incompleteness of information.

It seems da Costa and French are really willing to buy into that ambiguity between incompleteness and inconsistency. Regarding the Bohr model of the atom, which delivered inconsistencies by selectively applying fragments of classical mechanics and Planck's formula, they claim:

> Structurally, these were both inconsistent and incomplete in that with the inconsistency we do not know whether the relevant properties and relations hold in the domain or not ... this openness or lack of closure ... is representable model theoretically by a partial structure. (da Costa and French [8, p.105])

That is, incompleteness is present because there is inconsistency. However,

[1] We are still restricting our discussion to the case of a single predicate letter P.

that is again a bad idea. According to this explanation, every incomplete relation generates a kind of contradictory information, in the sense that we have in fact *overabundant* information, not lack of information. If this is the representation inside the partial structures approach of incomplete and uncertain information, then, it seems, it misses the target by providing a representation of cases where we have in fact a lot of information. It confuses cases where we have gaps (lack of truth value) with cases where we have gluts (abundance of truth values). These must be distinguished, because they give rise to distinct kinds of treatments. To wit, systems of logic such as FDE[2] may be seen as having a four-valued functional semantics, with gaps and gluts clearly distinguished. Even those who prefer to deal with inconsistency in more epistemic terms, not in terms of truth values, as Carnielli and Rodrigues [6], hold that incompleteness of information and inconsistent information are distinct phenomena. Incompleteness must be accounted for by paracompleteness, not by paraconsistency. Incompatible evidence generates some kind of paraconsistency. Violation of a version of the law of non-contradiction (LNC) is the wrong way to represent lack of information; in these cases, it is LEM that should be violated.

Notice also that if such inconsistency as saying that Pa is quasi-true and that $\neg Pa$ is also quasi-true represents incomplete information, then, according to the definition (III) above, the classical logician also has the resources to express this kind of incompleteness. Clearly, for most sentences α (except for logical validities and logical falsities), there is a classical structure which makes α true, and another structure which makes $\neg\alpha$ true. Then, classical logic would also be able to express the same situation. But that simply shows that the account, following definition (III), is inadequate.

In order to avoid such direct conflation between inconsistency and incompleteness, one could claim that definition (II) really leads us back to definition (I), given that it is presented in terms of (I), and making the definitions explicit we see that there is no real contradiction, but a kind of uncertainty over which normal structure to choose. This avoids confusing inconsistency and incompleteness, but then the claim that we have incomplete knowledge just comes back to the claim that we are uncertain about which $\mathfrak{A}$-normal structure to choose, and that is just the same situation as in classical logic, as we have already argued.

So, our preliminary conclusions may be stated as follows: in the case of definition (I) of quasi-truth, there is no representation of incomplete knowledge inside the framework, and there is no progress over classical logic; in the case of definitions (II) and (III), the representation is inadequate, because it models overabundant information by contradictory sentences being quasi-true; (III) also has the disadvantage of being excessively general, so that if this kind of definition is allowed to account for incomplete knowledge, then classical logic may also represent incomplete knowledge.

[2] First-degree entailment; see Priest [17, chap.8].

3.2 Inconsistency

This leads us to the treatment of inconsistency by the apparatus of pragmatic truth. We have already seen that quasi-truth leads to some kinds of contradictions, with both a sentence and its negation being quasi-true in some structures. Perhaps it is this that da Costa and French [8, p.85], have in mind when claiming that "we offer a model theoretic account in which regarding theories in terms of partial structures offers a straightforward and natural way of accommodating inconsistency".

In order for us to check whether this claim holds good of the pragmatic truth approach, we must first make clear what kind of contradiction or inconsistency is being dealt with here. As Priest [16, p.144] has remarked, there are at least three types of inconsistency in empirical science:

i) inconsistency between theory and observation,

ii) inconsistency between theories,

iii) inconsistency internal to the theory.

The case i), inconsistency between theory and observation, is probably not the case one is aiming at with quasi-truth. No one really wants to accommodate such cases. In these circumstances, when observation contradicts the previsions of a theory, one typically revise the theory, or else provides for an explanation that accounts for the incompatibility, showing that what was once perceived as a conflict between theory and observation is not really so. For instance, the orbit of Uranus was not in complete agreement with Newtonian theory; however, the discrepancy was later explained with the discovery of Neptune. What matters for us here is that quasi-truth adds no special ingredient on the relation between theory and observation. Sentences of the theory are quasi-true or quasi-false, and their relation to observation is accounted for empirically, not by the apparatus of the framework.

Case ii) is also not a case to be dealt with by quasi-truth, although some examples of this kind of inconsistency have been used to motivate quasi-truth (as the case of the incompatibility between quantum mechanics and general relativity, one of the most mentioned examples of inconsistency in science). In this situation, distinct theories are considered, not the same theory. The contradiction comes from distinct sources, and the problem is not a matter of extending the same structure in distinct ways, but of unification of theories. Then, quasi-truth is of no help here.

The only case that quasi-truth could really account for is case iii), internal contradictions. However, it seems that quasi-truth is not the correct tool for that too. As Bueno and da Costa [5, p.390] remark (with notation modified for the same of uniformity with the one employed in this paper):

> An important feature to note here is that a sentence *and* its negation can be both quasi-true. Of course, inconsistent sentences are not quasi-true in the *same* $\mathfrak{A}$-normal structure, but they can still be

> both quasi-true — as long as they are true in some $\mathfrak{A}$-normal structure. In other words, as defined above, if a theory is quasi-true, it is consistent (given that it is true in some full $\mathfrak{A}$-normal structure).

This remark is confusing on what concerns the distinct notions of quasi-truth involved. Remember: when it comes to definition (I), a sentence α cannot be quasi-true in a pragmatic structure $\mathfrak{A}$ relative to a $\mathfrak{A}$-normal structure $\mathfrak{B}$ with its negation be quasi-true *in the same* $\mathfrak{A}$-normal structure $\mathfrak{B}$. In relation to definition (II), however, α and $\neg\alpha$ may be both quasi-true in the same pragmatic structure (because they are true in *distinct* $\mathfrak{A}$-normal structures). In relation to (III), α and $\neg\alpha$ may be both quasi-true, each in its own structure, and with distinct structures for each sentence, of course (not being inconsistent or logical validities, there are structures modeling them).

However, despite the lack of specification, the above quote seems to put quasi-true contradictions in the correct perspective. As Bueno [4, p.281] indicates, if we use Q as a predicate of sentences indicating quasi-truth, we may have $Q(\alpha) \wedge Q(\neg\alpha)$ (in definitions II and III), however, we do not have $Q(\alpha \wedge \neg\alpha)$ (and this holds for the three definitions of quasi-truth). The first indicates merely that a pragmatic structure $\mathfrak{A}$ may be extended in incompatible ways, so that in a $\mathfrak{A}$-normal structure $\mathfrak{B}$ it may be the case that α is quasi-true, while in *another* $\mathfrak{A}$-normal structure $\mathfrak{C}$, $\neg\alpha$ is the case. A quasi-true sentence is consistent, in the classical sense. The idea that contradictions can be accommodated in the quasi-truth apparatus comes from ignoring that a sentence and its negation are quasi-true *in distinct* $\mathfrak{A}$-normal structures. However, $Q(\alpha \wedge \neg\alpha)$ cannot be the case, because no $\mathfrak{A}$-normal structure models a contradiction; they are Tarskian, classical structures. Then, if a theory provides for a sentence $\alpha \wedge \neg\alpha$, there is no hope of accommodating it in the apparatus of pragmatic truth.

But that means that the quasi-truth framework cannot really accommodate inconsistencies. On the one hand, a contradiction of the form $\alpha \wedge \neg\alpha$ is never quasi-true, and this is precisely the kind of inconsistency one finds in cases of theories internally inconsistent. On the other hand, it is an exaggeration to claim that $Q(\alpha)$ and $Q(\neg\alpha)$ is a contradiction, or lead to a contradiction. Both are true according to *different* $\mathfrak{A}$-normal structures.[3] It is easy to introduce parameters to account for the apparent incompatibility: α is true according to an $\mathfrak{A}$-normal structure $\mathfrak{B}$, and $\neg\alpha$ is true according to an $\mathfrak{A}$-normal structure $\mathfrak{C}$. A legitimate contradiction would require $Q(\alpha \wedge \neg\alpha)$ or $Q(S) \wedge \neg Q(\alpha)$. The first case is not allowed for, as we have already commented. $Q(\alpha) \wedge \neg Q(\alpha)$ is also not possible. Given that the metalanguage is classical, $\neg Q(\alpha)$ means that α is not quasi-true, that is, α is quasi-false, in a given $\mathfrak{A}$-normal structure. However, no sentence can be quasi-true and quasi-false in the same $\mathfrak{A}$-normal structure. That is, the apparatus of quasi-truth fails in accommodating inconsistencies.

[3] As we shall discuss in section 4, this may be interesting indeed for it seems to accommodate typical "Bohrian" complementary situations, where we need to have both situations in order to fully understand a phenomenon (say the particle and the wave behaviour of matter), but they cannot be taken together in a same situation. See also [9].

However, Bueno and da Costa seem to acknowledge that limitation, and attempt to overcome it. As they claim [5, p.390]: "But in some contexts, we may need to assert that an *inconsistent* theory is quasi-true. How can we do that?" That is, it is recognized that one cannot have a contradiction being quasi-true, but, anyway, some inconsistent theories (in the sense of internal inconsistency) should be said to be quasi-true. Bueno and da Costa present a solution to the problem that, curiously, does not require that we admit a contradiction $\alpha \wedge \neg\alpha$ as quasi-true. Their solution to the problem, however, is presented in a very informal way (again, adapting the notation):

> If a theory T is inconsistent, we say that T is quasi-true in a partial structure $\mathfrak{A}$ if there are "strong" subsets of $T's$ theorems that are true in some $\mathfrak{A}$-normal structure. (Bueno and da Costa [5, p.390])

The plan is explained as follows: one just selects consistent "strong" subsets of $T's$ theorems and check for some $\mathfrak{A}$-normal (that is, complete, Tarskian) structure that satisfies them. The idea of "strong" subsets of theorems is to be understood pragmatically, but that only adds to the mystery, of course. As an instance ([5, p.391]), they discuss the case of naive set theory, which clearly derives a contradiction as one of its theorems (Russell's paradox, let us say). Select a consistent set P of restrictions (the unproblematic postulates, union axiom, power set axiom, and so on), and let membership be the only partial relation of the structure. This membership relation, it is said, may be extended in distinct ways, provided that they are consistent with P. They claim then, for instance, that the membership relation may be extended to obtain ZFC, or to obtain Quine's NF or ML, or to obtain von Neumann-Bernays-Gödel set theory (NBG) ([5, p.391]). But that example is very implausible. These theories are formulated in quite distinct languages,[4] and they cannot be seen as literally extending a common core of the membership relation. Obviously, the theories must have something in common, but to claim that they are precisely the same in some rather undetermined part is unreasonable.

More than that: notice that the apparatus of quasi-truth is performing no essential job in this strategy. A classical logician may also restrict her interest to a class of consistent theorems of the naive set theory and reconstruct the theory in a consistent way. That throws no light on the inconsistent original theory's truth or quasi-truth. Nothing is made of the Russell contradiction in the original theory. Why is Cantorian set theory said to be quasi-true, given that it is trivial? No explanation is given. The classical logician seems even to have an advantage. There is an explanation for why such re-constructions are pursued: because they are admittedly consistent. Cantor's theory is trivial, and must be fixed. No need for quasi-truth. Something similar may be said

[4] As it is known, NF and ML require a stratification procedure in the language (a kind of typified language), where formulas must obey some constraints in order to be well-formed. This does not happen in ZFC or NBG. NBG, on its turn, allows for a distinction between classes and sets, which ZFC and NF do not. Many other differences in language and in the theories themselves could be pointed ou. The reader is referred to Fraenkel, Bar-Hillel and Levy for a very useful introductory account [12]).

of Frege's arithmetic. The development of Frege's theorem indicates that the inconsistency may be eliminated, and arithmetic developed in a second order logic with Hume's Principle (Zalta [18]). No need to say that the original theory is quasi-true.

From these discussion, it results that the claim by Bueno and da Costa [5, p.391] seems unjustified:

> As a result, the partial structures approach provides the right sort of framework to examine issues regarding inconsistency in science. In terms of the approach, it's possible to represent, without triviality, inconsistent theories as being quasi-true.

As we noticed, the quasi-truth approach does not represent contradictions, and the strategy employed to deal with inconsistent theories does not appeal to quasi-truth.

4 Interpreting quasi-truth

So far, we have argued that the traditional definition of quasi-truth does not deliver what was promised, that is, an account of incomplete knowledge, which may also account for inconsistencies in science.

What can we do with the formalism of quasi-truth? Here, we shall argue that the quasi-truth approach provides for a model of a very restricted class of situations.

What we have in mind is the contextual approach to inconsistency defended by Brown in [2], and critically discussed by da Costa and French [8, chap.5]. In a nutshell, according to Brown, some inconsistent theories are not to be accepted as true when taken as a whole. We accept the theory on some contextual limits, avoiding to bring into each context incompatible claims. As da Costa and French [8, p.88] comment, the plan consists in breaking the inconsistent theory into sub-contexts where the incompatible principles are not brought into play together.[5] So, if a theory somehow endorses both a proposition α and its negation $\neg\alpha$, we break it into a context where α is accepted, and in another context, where $\neg\alpha$ is accepted. We never apply the theory using both α and $\neg\alpha$ (this interpretation of quasi-truth, as a case of a paraconsistent treatment of contradictions, was also advanced in Arenhart [1]).

This approach is best illustrated by the example of the Bohr 'model' of the atom (as discussed by Brown [2]). It is widely accepted that the model involved some tensions between classical and quantum principles, and the theory is frequently cited as a case of inconsistent theory. However, as Brown argues, the inconsistent principles are never applied together. On Bohr's model, an electron in a Hydrogen atom is always in some discrete orbit, which form its so-called stationary states. In these cases, classical mechanics is employed to account for the dynamics of the electron in the stationary state it finds itself

[5] Later, these consistent fragments of the theory where termed "chunks" — see below.

in. When, however, one wishes to account for the transition of the electron between the distinct discrete stationary states, quantum principles are called for, Planck's formula giving the relation between the amount of energy and frequency of radiation emitted.

As a result of such a confinement, no real contradiction obtains. Classical principles hold in one context; quantum principles in another.[6] Although da Costa and French [8, p.89] are rather critical of Brown's approach,[7] we believe that the formal apparatus of quasi-truth is a model of precisely this kind of situation, where distinct contexts are applied in order to accommodate incompatible sentences. In order to see why, consider the informal discussion of a modal logic of quasi-truth, as discussed by da Costa [7, pp.135-136]. Given a partial structure $\mathfrak{A}$, the $\mathfrak{A}$-normal structures extending $\mathfrak{A}$ may be seen as possible worlds in a Kripke semantics for S5. A sentence is quasi-true in $\mathfrak{A}$ if there is a world where it is true. A sentence is strictly valid if it is true in every world. Then, obviously, each world (*i.e.*, $\mathfrak{A}$-normal structure) operates as a context, completely classical, where no contradiction is admitted. Of course, a sentence may be quasi-true, and its negation too, but in distinct possible worlds. This captures the idea of a confining of consistent principles in a context.[8]

The main problem with this approach, as da Costa and French [8, p.89] see it in their criticism of Brown, is that it has rather limited application, given that it is not clear that every inconsistent theory will allow for such a division into consistent contexts. However, that criticism applies to the quasi-truth approach also, due to its relation to the Kripke semantics of S5, as briefly discussed above, and, which is equivalent to the fact already discussed that a sentence in the form of $\alpha \wedge \neg\alpha$ is never quasi-true, so that such a contradiction must be broken in two contexts, one verifying α, another one verifying $\neg\alpha$.

In this interpretation, then, quasi-truth is not about incompleteness and inconsistency, but rather, more pragmatically, about assuming incompatible sets of commitments according to our needs. Distinct total structures represent the commitments one temporarily has assumed to account for a given context. This has a much less ambitious aim than the one originally proposed by da Costa and French [8], but it is, we believe, closer to what the formalism really presents us with.

As a further minor remark, this interpretation also points to another inade-

[6] This is precisely what we meant above in mentioning complementarity. There is a "rationale" for such a procedure by means of *paraclassical logic*; these are non-adjunctive logics where we can have two propositions α (say, A) and β (say, $\neg A$) but not their conjunction $\alpha \wedge \beta$ (a contradiction). Furthermore, due to its typical concept of deduction, from two contradictory theses we cannot derive any proposition, that is, the logic is not trivial; see da Costa and Krause [9], [10].

[7] Later, Brown has improved his ideas in several papers, until the 2004 paper with Priest [3] introducing the "chunk and permeate" technique which, roughly speaking, consists in separating logically consistent "chunks" in a theory and study the kinds of information can "permeate" from one chunk to another without conducing to trivialization. For a general explanation and references, see M. Friend and M.del R. Martínez-Ordaz [13] and, of course, [3].

[8] The discussion here could go to deeper levels. In [11], da Costa, Bueno and French discussed "the logic of pragmatic truth", arguing that is consist of a kind of modal non-adjunctive logic, called Jaśkowski's discussive logic, which is grounded on S5. But our sketch takes the minimal points.

quacy of the typical rendering of the formalism of the pragmatic truth: the idea that scientific knowledge progresses or improves by choosing this or that filling of gaps in our knowledge (that is, by a choice of this or that total structure). However, the real problem, as it happened in the case of the Bohr model, is not that we are ignorant of total structure should we choose (classical mechanics or quantum principles), but rather, we are asked to provide for a new, unifying framework which accounts for both situations under the same set of principles (which the 'new' quantum theory of Heisenberg and Schrödinger did). Then, while this temporary use of incompatible information in distinct contexts may be accommodated by our interpretation of quasi-truth, the interpretation also allows us to remark that *unification* of incompatible models is what is typically sought.

5 Concluding remarks

Quasi-truth, as a framework advanced to accommodate more realistically the actual vicissitudes of science, was part of a highly ambitious plan on the philosophy of science. We have argued that on what concerns accommodation of incompleteness and inconsistency in science, the approach falls short of delivering the promised goods. Let us review shortly what was achieved.

On what concerns the definition of quasi-truth advanced directly by da Costa and collaborators, it is implausible to claim that incompleteness is accommodated by the framework. If one considers that a sentence is quasi-true in a structure $\mathfrak{A}$ with relation to a $\mathfrak{A}$-normal structure $\mathfrak{B}$, then, incompleteness reduces to the possibility of choosing between distinct complete models ($\mathfrak{A}$-normal structures) where a sentence may be false or true in the Tarskian sense. This is just the same situation as in the classical case, and no real gain is obtained by this approach. The incompleteness is not codified in the language of the framework, it is an extra-systematic issue. However, when one adopts the other two definitions of quasi-true (definitions II and III), then, it seems, the mark of incompleteness in the approach consists in the fact that a sentence may be quasi-true, while its negation may also be quasi-true in some pragmatic structure. However, we have argued, this is the wrong path to incompleteness, because a contradiction (true or quasi-true) may be better understood as representing excess of information, not lack of information. We have contrasted this paraconsistent approach with the paracomplete approaches, which we believe better capture lack of information.

When it comes to deal with contradictions and inconsistency, quasi-truth seems to fail again. The fact is that even though a sentence and its negation may be both quasi-true, they are quasi-true in distinct normal structures. A parametrization strategy accounts for the apparent contradiction. As we have discussed, this is not an account of contradictions (see also Arenhart [1]). Also, a contradiction of the form $\alpha \wedge \neg\alpha$ cannot be quasi-true; however, this is precisely what one needs in order to account for most cases of internally inconsistent theories.

Finally, we have advanced a more modest reading of the formal apparatus of quasi-truth as a kind of contextualization process. Quasi-truth models the workings of sentences in distinct contexts, with a common core of sentences. Perhaps this idea can be associated with the Chunk and Permeate technique, something to be pursued further, although we still are not able to fully understand the criteria the proponents use to get the chunks and which kind of "information" can pass from one chunk to another. The suggestion by da Costa [7] according to which one may interpret quasi-truth in possible world semantics for S5 gave further evidence to the plausibility of this interpretation (paraclassical logic could be used here as an alternative). In this (ours) interpretation, there are no contradictions or incompleteness accommodated, but it is possible to make sense of the formalism in realistic situations, in which distinct sets of assumptions are held in distinct contexts, perhaps for distinct purposes. If this does not carry forward the program initiated by da Costa and collaborators, at least it provides for a much more plausible understanding of the formalism, it seems to us.

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