Miloš Arsenijević

Truth and the Open Future: The Solution to Aristotle's Sea Battle Challenge with the Principle of Bivalence Retained

Introduction

Although it has been practically used throughout the history of logic, the Principle of Bivalence got its name and its explicit formulation only in Lukasiewicz's *Rector's Speech* in 1922[1], after he mentioned, for the first time, the "three-valued logic", 1918[2], and outlined a "trivalent system of logic", 1920[3]. The reason for such a late recognition of one of the basic logical principles lies probably in the fact that, in traditional logic, the Principle of Bivalence was derivable from the Principle of Contradiction and the Principle of Excluded Middle. Namely, if it holds, for *any* proposition, that the conjunction of it and its negation is always false, whereas the disjunction of it and its negation is always true, then, given the standard way in which conjunction, disjunction and negation are defined, it follows that *every* proposition must have one and only one of the two truth values – truth or falsity – which is exactly what the Principle of Bivalence claims. So, in traditional logic, *being either true or false* becomes a *necessary* condition for *being a proposition*.

The appearance of *intuitionism*, where the axiom that expresses the Principle of Excluded Middle ceased to be a theorem[4], *could have* caused a re-examination of the situation in the way that would consequently lead to an explicit formulation of the Principle of Bivalence, but this didn't happen, probably because, by restricting the universal validity of the Principle of Excluded Middle, nobody questioned the previous "triviality" that there are just two truth-values exclusively ascribable to every proposition. But it is to be noted that in intuitionistic logic, in spite of the fact that in the case where neither p nor $\neg p$ is provable, one is not allowed to claim $p \vee \neg p$ – since this formula is not an instance of a theorem scheme – there supposedly *mustn't be* any case in which one would be allowed to claim that $p \vee \neg p$ is *false,* since this would lead to the denial of the Principle of Contradiction.

So, curiously enough, a clear separation of the Principle of Bivalence from the other two basic principles of traditional logic was made only after the appearance of *trivalent* logic, in which the Principle of Bivalence was *rejected*.

The Principle of Contradiction, the Principle of Bivalence and the Principle of Excluded Middle are three *independent* principles. The Principle of Contradiction holds in any system whose interpretation contains just *truth* and *falsehood*, independently of whether the other two principles hold in it or not. So, the Principle of Contradiction holds *unrestrictedly* in intuitionistic logic, in which the validity of the Principle of Excluded Middle is *restricted*, as well as in the bivalent systems with truth-value gaps. The Principle of Bivalence (in its strong sense) holds only in systems in which *there are no other* truth-values but *truth* and *falsity* and in which *there are no truth-value gaps*, so that each proposition is necessarily *either true or false*.

Now, it can be argued that the fact that the Principle of Bivalence was so late separated from the other two basic principles of traditional logic is precisely that which caused a rather strange confusion present in traditional interpretations of Aristotle's famous Sea Battle example before Lukasiewicz. For, on the one hand, it is Aristotle himself who was the first to formulate the general validity of the Principle of Contradiction as well as of the Principle of Excluded Middle, while, on the other hand, it seems obvious that the Sea Battle example was intended to check *some* basic principle. Which one? It couldn't be the Principle of Excluded Middle, since Aristotle said explicitly that $p \vee \neg p$ should be considered true in the given example (where p is the proposition that the sea battle takes place at a given later time and $\neg p$ its negation). It can hardly be the Principle of Contradiction, because, for Aristotle, this principle is the most general and absolutely valid principle. So, it can only be the Principle of Bivalence that is challenged in the Sea Battle example, but in the absence of its explicit formulation, the interpreters were at pain to cope with the problem without a clear guidance. In particular, the Christian logicians seemed to be in a much worse situation than the ancient Greek philosophers, for, according to Christian theology, God is *assumed* to *know* whether it is p or it is $\neg p$ that is true *already at the time* at which, in the example, it is *supposed* that it is *not yet decided* what the case will be.

In the first section of the paper, I shall start with Aristotle's original formulation of the problem and his solution to it. Then, in the second section, I shall show why Boethius' eternalist view concerning a similar problem is irrelevant for Aristotle' problem and analyze Ockham's solution (or "solution") as a typical failure of the traditional use of modalities to avoid logical determinism without questioning the Principle of Bivalence.

At the same time, however, the Aristotelian concept of *conditional necessity*, which is called *necessity per accidens* in Ockham's analysis, will be of great importance for the solution I am going to offer in the fourth section of the paper.

In the third section, Lukasiewicz will be praised as the first logician in the history who recognized that Aristotle's example is directed against the Principle of Bivalence. Lukasiewicz's solution to the problem of logical determinism, given within a trivalent system of logic, is simple, intuitive and consistent.

The solution that I shall offer in the fourth section is based on the recognition of Lukasiewicz's diagnosis of the source of the problem, and, in a sense, my solution is nothing else but a refinement of Lukasiewicz's solution. The striking difference consists in the fact that the Principle of Bivalence will be retained. But this will appear to be possible only because the whole background in which this principle should hold will have been radically changed. For instance, a modal logic system will be used, along with the possible world semantics, but essentially combined with temporal logic, while a generalization of Tarski's disquotational scheme will turn out to be needed for overcoming the "indeterminateness", which is introduced in trivalent logic as the third truth-value.

Finally, in the fifth section, it will be shown why we need *tenses* if we want to be able to define that which is called the *open future*. From a metaphysical point of view, this means that we need *the flow of time* assumption if we want to speak of *real*, *in-the-world-inherent possibilities*. Time should be conceived as essentially similar to a *line-in-drawing* and not to a *drawn line*.

I. Aristotle's puzzle and his solution: the acceptance of truth-value gaps

The Sea Battle example concerns *future contingent singular propositions*. However, in order to avoid ambiguities I shall take that instead of being *tensed*, p and $\neg p$, as a pair of two contingent singular propositions stating and denying respectively that some sufficiently well specified event occurs at some definite *date* are *considered* at some earlier time. So, p will not mean "the sea battle will take place tomorrow", but "the sea battle takes place at t", while $\neg p$ will be its negation. In this way, the question concerning the truth of p and $\neg p$ becomes the question of whether it is true at some time earlier than t that it is p that will be true at t or it is rather $\neg p$ that will be true at t. This stipulation will do no harm for understanding the problem and its solution suggested by Aristotle.

Now, there is an enormous literature concerning the reading of Aristotle's text, his understanding of the problem and its suggested solution.[5] Instead of analyzing various concepts of necessity and possibility and their different meanings allegedly relevant for the Sea Battle problem as formulated and solved by Aristotle, which characterizes both the medieval and the non-standard modern interpretations,[6] I myself am prone to generalize what William Lane Craig says by discussing one of them: "It seems to me best to take Aristotle at face value […] and not introduce further qualifications". Fortunately, there are no uncertainties about the original wording of the most important passages in which Aristotle summarizes the problem and offers his solution. So, I shall simply start by citing them in full.

As for the way in which Aristotle understands *indeterminism* in the sense relevant for the Sea Battle problem, he says:

"…not everything is or happens of necessity: some things happen as chance has it, and of the affirmation and negation neither is true rather than the other; with other things it is one rather than the other and as a rule, but still it is possible for the other to happen instead".[7]

As for problem of truth and falsity concerning the future contingent propositions, he says:

"What is, necessarily is, when it is; and what is not, necessarily is not, when it is not. But not everything that is, necessarily is; and not everything that is not, necessarily is not. For to say that everything that is, is of necessity, when it is, is not the same as saying unconditionally that it is of necessity. Similarly with what is not. And the same account holds for contradictories: everything necessarily is or is not, and will be or will not be; but one cannot divide and say that one or the other is necessary. I mean, for example: it is necessary for there to be or not to be a sea battle tomorrow; but it is not necessary for a sea battle to take place tomorrow, nor for one not to take place – though it is necessary for one to take place or not to take place. So, since statements are true according to how the actual things are, it is clear that wherever these are such as to allow of contraries as chance has it, the same necessarily holds for the contradictories also. This happens with things that are not always so or are not always not so. With these it is necessary that one part of the contradiction[8] is true or false – not, however, this one or that one, but as chance has it; or for one to be true rather than the other, yet not already true or false.

Clearly, then, it is not necessary that of every affirmation and opposite negation one should be true and the other false. For what holds for things that are does not hold for things that are not but may possibly be or not be; with these it is as we have said."[9]

It is evident that Aristotle speaks of *indeterministic events* in the same way in which we do it today: "some things happen as chance has it". He also allows for the possibility that some events are *more probable than the others* ("with other things it is one rather than the other and as a rule, but still it is possible for the other to happen instead"). However, the difference between the two cases is of no relevance to our problem. We can take events that are *more probable* to happen to be indeterministic as well but characterize them as *not completely indeterministic*, or we can simply ignore them as not relevant to the problem. It is sufficient to consider only *completely indeterministic events*, for which it is supposedly *by no means* determined whether they will happen or not.

What is *logical determinism* according to Aristotle is clearly discernible in what he says in the second quotation. Given the correspondence theory of truth ("statements are true according to how the actual things are"), an indeterministic event, such as a future sea battle supposedly is, *mustn't* also be predetermined by the fact that *this or that particular* (τόδε) of the *two parts of the contradiction* (ἀντίφασις) about it is *true* at the time at which it is not determined in any other way that the battle will happen or that it will not happen: "…it is necessary that *one part* (θάτερον μόριον) of [the two parts of] the contradiction is true or false – not, however, *this one or that one* (τόδε ἢ τόδε), but as chance has it; or for one to be true rather than the other, yet *not* ***already*** *true or false* (οὐ μέντοι **ἤδη** ἀληθῆ ἢ ψευδῆ)."[10] [*italics and bolds added*]).

Notice that I translated ἀνάγκη μὲν θάτερον μόριον τῆς ἀντιφάσεως ἀληθὲς εἶναι ἢ ψεῦδος literary as "it is necessary that one part of [the two parts of] the contradiction is true or false" and not as Ackrill did it: "it is necessary for one or the other of the contradictories to be true or false". I did it just in order to make it quite clear what I take to be Aristotle's point, namely, that one of the contradictories must be true or false only if we take it as *a part* of a contradiction and not as a *concrete proposition*. For it is said immediately after this statement that it is *neither this nor that part* (οὐ…τόδε ἢ τόδε) that is true of false, for neither is *already* (**ἤδη**) either true or false. The general lesson is that "it is not necessary that of every affirmation and opposite negation one should be true and the other false." Here "the affirmation" and "the opposite negation" are taken as statements *per se* and not as *parts of a contradiction*, for if we take them as parts of a contradiction, one must be true and another false.

So, Aristotle's solution is rather subtle. The fact that (at some time earlier than *t*) neither *p* nor $\neg p$ is either true of false does not imply that $p \vee \neg p$ is neither true nor false ("everything necessarily is or is not, and will be or will not be"). Obviously, Aristotle thinks that the fact that, *at t*, it is *necessary* that either *p* is true and $\neg p$ false or $\neg p$ true and *p* false, implies that $p \vee \neg p$ is true at *any* time, and so also at *any earlier* time, *independently* of the fact that it could have been that at *some* (earlier) time neither *p* nor $\neg p$ were either true or false. In other words, he holds that in some cases a *complex proposition* can have a *definite truth-value* even if *none* of the *component propositions* has it.

In view of all that has been said, it seems quite obvious that, *in a sense*, the standard modern interpreters[11] are right when arguing – and in fact only following Lukasiewicz – that *what* Aristotle was denying in order to avoid *any* kind of *determinism*, and *logical determinism* in particular, is the Principle of Bivalence. But, however obvious this may be, the *way* in which Aristotle did it might seem tricky. He did not reject the Principle of Bivalence by allowing for the possibility of a third truth-value but by allowing the existence of *truth-value gaps*. And even more curiously, by allowing the existence of truth-value gaps, he *didn't* restrict the *universal validity* of the Principle of Excluded Middle,[12] but allowed for the possibility that, *at some time*, a *complex* proposition can be true in spite of the fact that *no* component proposition has a truth-value. This is the case, according to Aristotle, when *there is nothing yet in reality* that would make either *p* or $\neg p$ either true or false, while $p \vee \neg p$ is *already* true, since it is necessary that, independently of *which* of the two propositions will turn out true, the other one will turn out false. Brilliantly, and quasi-paradoxically, Aristotle *used* here just the Principle of Bivalence to explain why the Principle of Excluded Middle should be said to hold (*tertium non datur*) *even* at the time at which the validity of the *very same* principle – the Principle of Bivalence – is suspended through the introduction of truth-value gaps!

For a neutral reader of Aristotle's text, it can be questionable *whether* Aristotle himself, by allowing truth-value gaps, thought that he was denying the Principle of Bivalence, for what he was denying was not that *if* a proposition has a truth-value it is either *truth* or *falsity*. What he was denying is only a *strong* version of the Principle of Bivalence that claims not only that *if* a proposition has a truth-value it is either *truth* or *falsity* but *also* that *every* proposition *must have* one and only one of the two truth-values. However different from some standard interpretations it may be,[13] I think that only this interpretation of Aristotle's solution fits his own words.

I said in the Introduction that in traditional logic the Principle of Bivalence is derivable from the Principle of Contradiction and the Principle of Excluded Middle. In view of the analysis of Aristotle's solution to the Sea Battle problem, we may now say that this holds only for its *strong* version, namely, under the *assumption* that *there are no truth-value gaps*. For, if there are truth value gaps, it is trivially not true that a proposition must be either true or false.

Now, Aristotle's solution is not only very subtle, so as to be even quasi-paradoxical, but is also *non-standard* from a point of view of traditional logic, however this formulation may seem bizarre, given that he himself is recognized as the philosopher who established traditional logic. But this way or another, traditional logic does not allow the existence of truth-values gaps, however obvious it may be that Aristotle used these gaps to solve the Sea Battle problem. Aristotle's solution is *non-standard* also in view of the fact that, in his solution to the Sea Battle problem, he allowed for the possibility that a complex sentence $p \vee \neg p$ is true even if neither p nor $\neg p$ is either true or false at the time when $p \vee \neg p$ is uttered. In other words, for him, $p \vee \neg p$ is *always* true because *whenever* p is true $\neg p$ is false and *vice versa*, while for his followers $p \vee \neg p$ is *always* true because it is *always* true either that p is true and $\neg p$ false or that $\neg p$ is true and p false.

All in all, I disagree with *any* standard modern interpretation[14] that does not accept *all* of the following five claims: (1) that Aristotle based his solution to the Sea Battle problem on the correspondence theory of truth; (2) that the essence of his solution consists in the acceptance of the truth-value gaps; (3) that he implicitly restricted the Principle of Bivalence to its weak version, but did not assume, even tacitly, a third truth value; (4) that he did not restrict the universal validity of the Principle of Excluded Middle; (5) that the universal validity of the Principle of Excluded Middle is based on the acceptance of the possibility that a complex proposition has a truth-value even if the component propositions do not have any truth-value.

As for the non-standard modern interpretations, it follows that I think that they are all wrong as *interpretations* of Aristotle's solution. But also, practically all logicians before Lukasiewicz, instead of focusing their attention on the *basic logical principles*, tried to solve the Sea Battle problem by concentrating on the analysis of *modalities*. The "solution" of William Ockham's, whose interest for the problem was reinforced by the alleged fact that God has *foreknowledge* about the truth or falsity of p and $\neg p$, can serve as a typical example to show that such attempts do not solve the original problem.

II. The eternalist and the tenseless view: Boethius and Ockham

Boethius, who is said to be the last Roman among Christians and the greatest Christian among Romans, and whose *Consolatio philosophiae* was for centuries the most influential book ever written in Latin, defined *eternity* as "…the perfect and completely simultaneous possession of the endless life" ([*Aeternitas igitur est*] *interminabilis vitae tota simul et perfecta possessio*…)"[15] and used this definition to solve the problem of the alleged incompatibility between human freedom and God's omniscience. This problem seems to be essentially similar to Aristotle's Sea Battle problem. How can God know some future contingent whose truth should depend on what some person *will do* according to his or her own free will? According to Boethius, God can do it only because He transcends time and sees the whole history in one timeless act of awareness. From God's *eternalist* perspective, the endless history is completely finished and simultaneous (*tota simul et perfecta*), so that nothing is "earlier than" or "later than" something else. And then, given this peculiar kind of God's knowledge, the question about the alleged incompatibility between God's omniscience and the existence of the free will *cannot even be raised*.

The problem with Boethius' solution is that it is completely *irrelevant* for Aristotle's question, however the two questions discussed may seem to be similar. Namely, since we mustn't pass from the fact that God knows *sub specie aeternitatis* the truth of a contingent proposition *p* to the statement that He knows it at some time *earlier than* the time the proposition is about, we still may ask whether we should accept that either it is *p* or it is $\neg p$ that is true at that earlier time or we should rather say that both p and $\neg p$ are neither true nor false at that time.

William Ockham didn't separate the two realms that radically as Boethius had suggested. For Ockham, not only that God "knows which part of a contradiction is true and which false (*Deus scit hanc partem contradictionis esse vera vel illam*)"[16], which could be also understood in accordance with Boethius' eternalism, but He "knows with certainty [regarding] *all future contingents* (*omnia futura contingentia*) *which part* (*quae pars*) of the contradiction *will be true* (*erit vera*) and which false…"[17] [*italics added*]. This last statement – as well as the whole context – could hardly be understood in the way in which the contingents were said to be *future* just for the sake of *referring* to them from the *human point of view*. Instead, they should be taken to be future contingents because they are statements about the facts that are *future facts* both for humans and for God. So, in the given case, God's knowledge should be understood as *foreknowledge* (*praescientia*) in a literal sense of the word.

The reason for claiming that Ockham did not just repeat what Boethius meant follows also from the fact that Ockham made a great effort to analyze the concepts of contingency and necessity, and conditional necessity (necessity *per accidens*) in particular, which concern both contingent and necessary truths as well as different kinds of God's knowledge. For this whole analysis would be simply unnecessary if it were supposed from the very beginning that God's knowledge of the whole history has nothing to do with *time* in which the history is situated.

Supposing that Ockham didn't accept Boethius' radical solution, he was confronted with a problem unsolvable also via the theory of truth-value gaps offered by Aristotle. He was quite aware of this when he started his discussion by admitting that "it is impossible to express clearly the way in which God knows future contingents"[18], which is, at the same time, something that "must be held because of the pronouncements of the Saints, who say that God does not know things that are *going to be* (*fienda*) in a way different from that in which [He knows] things that already *are* (*facta*)".[19] [*italics added*]

Let us start with one of Ockham's examples. Given that a person has been and will be doing everything needed for being saved, "God *knows* (*scit*) that this person will be saved" is true, but it is yet possible that "God *will never have known* (*numquam sciverit*) that this person will be saved. And so the proposition is *immutable* and is nevertheless *not necessary* but *contingent*."[20] [*italics added*] What Ockham here means is, obviously, that it is possible that God *will never have known* that the person will be saved because His knowledge about the salvation depends, *inter alia*, on everything that the person will do by using *his* or *her* free will. If he or she has been and will be doing everything needed for being saved, God will have foreknowledge that he or she will be saved, and if he or she hasn't been or will not be doing that, He will have foreknowledge that he or she will not be saved. Doesn't this mean that, in addition to what he or she has been doing, that which he or she will be doing *causes backwardly* God's foreknowledge about his or her salvation or non-salvation?

It is hard to believe that Ockham used, at least tacitly, the concept of backward causation in order to explain God's foreknowledge about future contingents. He rather took that for God all truths are *tenseless* and only wanted to prevent them from being *necessary*. It seems that he believed that for preventing predestination it is sufficient to claim that, and explain why, God's knowledge about salvation is unnecessary. The point should be that God’s knowledge is contingent *just as* what the person will do is something contingent, depending on his or her free will.

Another example of Ockham’s, in which God does not appear, has the same point. If ‘Sortes sits’ is true at some time, ‘Sortes sat’ “will afterwards always be necessary.”[21] But “it is *not necessary* that a *necessary* [proposition] *about the past* corresponds to a proposition necessarily true about the present.”[22] [*italics added*] Evidently, this means that though, if Sortes is sitting now, it will be necessary, at any later time, that he was doing so, it is not true that it was necessary that he would be sitting now, for there was nothing in the past that made Sortes’ sitting now necessary.

But, however brilliant this Ockham’s analysis of the contingency and the only conditional necessity of p as the statement about someone’s salvation as well as God’s knowledge about it may be, it doesn’t even touch directly the problem of truth and falsity of p or $\neg p$ *at some time earlier than* the time at which either p or $\neg p$ is supposedly true. It still remains the question of whether it is p or it is $\neg p$ that is true at this time or there is rather a truth-value gap related to the truth and falsity of p and $\neg p$.

But even worse, since *knowledge implies truth* ($\mathrm{K}p \Rightarrow p$), God’s *foreknowledge* about p, quite *independently* of whether it is necessary or contingent, *implies* that either p is true or $\neg p$ is true at *some* time *earlier* than the time at which it is either p or it is $\neg p$ that is true, which means that, given the truth of p or the truth of $\neg p$ at some time earlier than the time at which it is either p or it is $\neg p$ that is true, it is *logically predetermined* either that a person will be saved or that he or she will not be saved.

All in all, by being sufficiently detached from the human perspective, both Boethius’ *eternalism* and Ockham’s *tenseless view* concerning the truth of all propositions *can* explain God’s omniscience, but they *don’t* solve Aristotle’s puzzle. The former simply has no answer to Aristotle’s question concerning the truth or falsity of some proposition p at some time that is earlier than the time p is about, while the latter cannot even avoid *logical determinism*, because God’s *praescientia* of p that is implied by His *omniscientia* implies also the truth or falsity of p *independently* of the question of modality, which means that what p is about is *predetermined* by the fact that either it is p or it is $\neg p$ that is true already at some time earlier than the time at which either p is true or $\neg p$ is true.

III. Three-valued logic: Lukasiewicz

As already mentioned above, Jan Lukasiewicz was the first philosopher who realized that Aristotle’s Sea Battle problem is not something that could be solved by any analysis, however subtle, of the concepts of contingency, necessity and conditional necessity, because it concerns something much deeper: it represents a challenge for one of the basic principles of traditional logic. This principle is the *Strong Principle of Bivalence*, which states that every proposition is either true of false. Aristotle himself was quite aware of this and that’s why he, as it is shown above, rejected the *Strong Principle of Bivalence* and accepted instead only its *Weak Version*, according to which every proposition is true or false *if* it has a truth-value at all. But, according to him, it must be that there *are* propositions that are neither true nor false *if* logical determinism should be false. And since Aristotle took it for granted that logical determinism is false, he admitted the existence of truth-value gaps.

Now, Lukasiewicz made just one small additional step, but this step was revolutionary: instead of allowing for the existence of truth-value gaps, he introduced a *third truth-value*. He himself compared this change with the introduction of non-Euclidean systems of geometry[23], and I think that he was right in having made that comparison. As he said in his *Rector's Speech*, because "what I call the Principle of Bivalence (sic!)" […] "lies at the very foundations of logic, the principle under discussion cannot be proved. One can only believe it, and he alone who considers it self-evident believes it. To me, personally, the principle of bivalence does not appear to be self-evident. Therefore I am entitled not to recognize it, and to accept the view that besides truth and falsehood there exist other truth-values, including at least one more, the third truth value".[24]

Let us consider a trivalent system of Lukasiewicz's that deviates least from two-valued logic[25] and is, at the same time, sufficient for solving the Sea Battle problem.

The two-valued logic we shall start with contains two logical constants, $\top$ and $\bot$, which denotes truth and falsehood respectively, = as the identity sign, schematic letters a, b, c,… which particular propositions and $\top$ and $\bot$ can be substituted for, and logical constants $\Rightarrow$, $\neg$, $\wedge$ and $\vee$. Now, the principles of two-valued logic are:

1. The principles of the identity of falsehood, identity of truth, and non-identity of truth and falsehood: $(\bot = \bot) = \top$, $(\top = \top) = \top$, $(\bot = \top) = (\top = \bot) = \bot$.
2. The principles of implication: $(\bot \Rightarrow \bot) = (\bot \Rightarrow \top) = (\top \Rightarrow \top) = \top$, $(\top \Rightarrow \bot) = \bot$.
3. The definitions of negation, disjunction and conjunction: $\neg a = (a \Rightarrow \bot)$, $a \vee b =$ $= ((a \Rightarrow b) \Rightarrow b)$, $a \wedge b = \neg(a \vee \neg b)$.

The three-valued system contains all the symbols of the above system of two-valued logic but also ½ as a third logical value. The principles specified above concerning $\bot$ and $\top$, and the definitions of negation, disjunction and conjunction remain the same in the three-valued system, only that ½ may also be substituted for a and b, in which case the additional principles of identity and implication concerning ½ should hold:

1a. The principles of identity: $(\bot = ½) = (½ = \bot) = (\top = ½) = (½ = \top) = ½$, $(½ = ½) = \top$.

2a. The principles of implications: $(\bot \Rightarrow ½) = (½ \Rightarrow \top) = (½ \Rightarrow ½) = \top$, $(½ \Rightarrow \bot) = (\top \Rightarrow ½) = ½$.

Some laws of three-valued logic differ partly from those of two-valued logic. The most important amongst them are the Principle of Contradiction and the Principle of Excluded Middle, since though for $a = \top$ or $a = \bot$, $a \wedge \neg a$ equals $\bot$, for $a = ½$, $a \wedge \neg a$ equals ½ (and not $\bot$), whereas though for $a = \top$ or $a = \bot$, $a \vee \neg a$ equals $\top$, for $a = ½$, $a \vee \neg a$ equals ½ (and not $\top$), since in this case $a = \neg a$ is true.

If we now apply the three-valued system to the Sea Battle problem and interpret the third value not as related to what is true or false but to what is “possible”, we can see directly where Aristotle and Lukasiewicz agree and where they disagree. They agree at the most important point, namely, that neither p nor $\neg p$ (as I defined them above at the beginning of Section I) is either true or false before time t they speak about, whereas, at the same time, it is either p that is true at t or it is $\neg p$ that is true as t. But while Aristotle stopped here, the three-valued system enables Lukasiewicz not only to claim that p and $\neg p$ do have truth-values at some time earlier than t but also, which would sound quite strange to Aristotle, that $p \wedge \neg p$ is *true* before t. Namely, by consulting the above principles of identity of truth-values and the definition of conjunction, it can be seen that, for p = ½, $p = \neg p$, and so, $p \wedge \neg p$ is not false.

One might say that the two solutions, that of Aristotle's and that of Lukasiewicz's, are essentially similar, being based on the same intuitions. A possible advantage of Lukasiewicz's solution may be that it is also based on an independently sketched system of logic, while Aristotle's solution might seem *ad hoc*, since he introduced truth-value gaps at this singe place, just in order to avoid logical determinism. This possible advantage of Lukasiewicz's solution, which is an advantage more from a purely logical than a metaphysical point of view, will become more visible, when we now turn to the implications of the two solutions that concern the Principle of Excluded Middle.

Let us remember that, according to Aristotle, though neither p nor $\neg p$ is either true or false at some time earlier than t, $p \vee \neg p$ is true at *any* time. As already mentioned, this claim of Aristotle could seem a bit strange, since in this case the complex proposition has a truth value in spite of the fact that the component propositions do not. According to the three-valued system, not only that it is not so, but, given that neither p nor $\neg p$ is either true or false at some time, $p \vee \neg p$ is not true at that time, since, as we have seen, if p = ½, $p \vee \neg p$ is not true, because $p = \neg p$.

All in all, if "the indeterministic philosophy […] is the metaphysical substratum of the new logic"[26], as Lukasiewicz put it, the three-valued system of logic shows its force through its natural application to the Sea Battle problem.

IV. A solution to the Sea Battle Problem within temporal-modal logic

It might be said that the solution to the Sea Battle problem that I am now going to offer relates to that of Lukasiewicz's in a way rather similar to that in which Lukasiewicz's solution relates to that of Aristotle's. Namely, I completely agree with Lukasiewicz just as he agrees with Aristotle that if we accept that it is *universally true* that "if *A* is *b* at instant *t*, then it is true at every instant earlier than *t* that *A* is *b* at instant *t*"[27], where *A* stands for a contingent proposition and *b* for one of the only two truth values, logical determinism is unavoidable, since either truth or falsity of a contingent future proposition *at every time earlier than t predetermines* what is to happen at *t* in at least the same (if not a stronger) sense in which this is so according to any other type of determinism. But just as Lukasiewicz supplemented Aristotle's solution, which assumes truth-value gaps, by introducing "possibility"[28] as a third truth-value, so I want to analyze "possibility" a bit further.

The most important thing to be noticed is that, though the above quotation of Lukasiewicz's implies that if (in accordance with the above notation) $\boldsymbol{A}e(\mathrm{t}_n)$ is supposed to mean that it is asserted (**A)** that the sea-battle (*e*) happens at t_n that denotes the time-interval that is exactly *tomorrow* in relation to the interval at which $\boldsymbol{A}e(\mathrm{t}_n)$ is assereted (say t_m), the truth of $\boldsymbol{A}e(\mathrm{t}_n)$ implies logical determinism, it need *not* be so in relation to the truth of $\neg\boldsymbol{A}e(\mathrm{t}_n)$, because $\neg\boldsymbol{A}e(\mathrm{t}_n)$ can be understood as stating straightforwardly and only that, *today* (at t_m), there is no such thing as a *tomorrow's-sea-battle*, without the implication that there will be no sea-battle tomorrow. This open space for claiming that *there will be a sea-battle tomorrow* and *there will be not a sea-battle tomorrow* are both *false today*, whereas, when *tomorrow* becomes *today*, one of the two contradictory statements will be *true*. The Principle of Excluded Middle holds *today* and will hold *tomorrow*, though for different reasons. Everything depends on whether it is only t_m on which a world is actualized or there is a world actualized on t_n as well. Moreover, once a world on t_n is actualized — but only then — it becomes possible to say that if $\boldsymbol{A}e(\mathrm{t}_n)$ is true, it was true on t_m that it would be true on t_n, because everything that has happened actually belongs to one and the same reality.

From both metalogical and metaphysical points of view, more is to be said about the difference between the reality and the actual and the possible worlds as well as about the relation between the real world and the actual and possible worlds.

Though I don't want to claim that there cannot be more real worlds, the whole system will be sketched under the assumption that there is just one real world. In any case, the modal realism of David Lewis'[29] will not be presupposed. Moreover, it will be assumed that there is an essential difference between worlds that are actual and those that are possible but not actual, where actual worlds will be taken to be time segments of the real world, i.e., parts of the real world that have some temporal extension.[30]

The axioms defining the structure of time that will be used in what follows are cited in the Appendix.[31] Intuitively, time will be taken to consist of time stretches (intervals or periods) which time variables $t_1, t_2, \ldots, t_n, \ldots$ range over, while particular stretches will be denoted by constants $\mathrm{t}_1, \mathrm{t}_2, \ldots, \mathrm{t}_n, \ldots$. Relations in which time intervals can stand are the identity relation, the precedence relation, the abutment relation, the overlapping relation and the inclusion relation, denoted by $=$, $\prec$, $\{$, $\cap$ and $\subset$ respectively.

As for the possible worlds, I shall take, of course, that all actual worlds are possible (because *actuality* implies *possibility*), whereas *merely possible worlds* will be taken to be worlds that are *not actual*, be they accessible or not. *Accessible possible* worlds will be taken to be possible worlds that are either already *actual* or *merely possible but also actualizable starting from a real world segment* (this will be formally defined below). *Actualizable (but not actual) possible worlds* will be worlds that are accessible from a real world segment according to the *accessibility relation* (that will be also formally introduced below).

From a metalogical point of view, we need to adjust Tarski's *disquotational scheme* so that it becomes applicable not only to the real world but also to worlds that are possible but not actual (i.e., at least not yet real).[32] Using Tarski's example that concerns the truth of the proposition "snow is white", we may generalize the *disquotational scheme* in the following way:

”snow is white” *is true* in world *w*, be it actual or possible, if and only if snow is white in *w*.

But though the bi-conditional should hold in any case, we shall take that

if *w* is a time segment of the *real* world, ”snow is white” *is true because*, at the corresponding time and in the corresponding actual world as a segment of the real world, snow is *de facto* white,

while,

if *w* is *not actual* but *possible*, snow is white in *w because* "snow is white" is supposedly one of the *true* propositions from a set of propositions through which *w* is described.

More is to be said about the meaning of "a set of propositions through which a possible world *w* is described". Let us remember our central example and take that $\boldsymbol{A}e(\mathrm{t}_n)$ is the assertion $\boldsymbol{A}$ that the sea battle *e* happens at t_n, and $\neg\boldsymbol{A}e(\mathrm{t}_n)$ its negation. It is clear that we want to say that $\boldsymbol{A}e(\mathrm{t}_n)$ is true in *any* possible world that contains $\boldsymbol{A}e(\mathrm{t}_n)$ as one of the propositions that describe it consistently, while $\neg\boldsymbol{A}e(\mathrm{t}_n)$ is true in *any* possible world that contains $\neg\boldsymbol{A}e(\mathrm{t}_n)$ as one of the propositions that describe it consistently. So, we are rather interested in the equivalence classes of the worlds in which $\boldsymbol{A}e(\mathrm{t}_n)$ and $\neg\boldsymbol{A}e(\mathrm{t}_n)$ are true respectively than in just two particular worlds in which $\boldsymbol{A}e(\mathrm{t}_n)$ and $\neg\boldsymbol{A}e(\mathrm{t}_n)$ are true respectively.

If we consider another pair of assertions, $\boldsymbol{A}e'(\mathrm{t}_n)$ and $\neg\boldsymbol{A}e'(\mathrm{t}_n)$, and take that $\boldsymbol{A}e'(\mathrm{t}_n)$ says that it rains at t_n and $\neg\boldsymbol{A}e'(\mathrm{t}_n)$ that it doesn't, it is clear that both $\boldsymbol{A}e(\mathrm{t}_n)$ and $\neg\boldsymbol{A}e(\mathrm{t}_n)$ are compatible with each of $\boldsymbol{A}e'(\mathrm{t}_n)$ and $\neg\boldsymbol{A}e'(\mathrm{t}_n)$. This means that $\boldsymbol{A}e(\mathrm{t}_n)$ and $\neg\boldsymbol{A}e(\mathrm{t}_n)$ divide the whole class of not actual but actualizable worlds into two exclusive equivalence subclasses $[w_1]$ and $[w_2]$, whereas $\boldsymbol{A}e'(\mathrm{t}_n)$ and $\neg\boldsymbol{A}e'(\mathrm{t}_n)$ divide it into two other exclusive equivalence subclasses $[w_3]$ and $[w_4]$. If so, $[w_1]$, $[w_2]$, $[w_3]$ and $[w_4]$ are all not identical among each other, but also none of the four cuts $[w_1] \cap [w_3]$, $[w_1] \cap [w_4]$, $[w_2] \cap [w_3]$ and $[w_2] \cap [w_4]$ is empty. If we proceed in this way, by introducing always new and new pairs of assertions that divide the class of all the actualizable worlds in a new way, that is $\boldsymbol{A}e''(\mathrm{t}_n)$ and $\neg\boldsymbol{A}e''(\mathrm{t}_n)$, $\boldsymbol{A}e'''(\mathrm{t}_n)$ and $\neg\boldsymbol{A}e'''(\mathrm{t}_n)$, …, etc., the number of worlds described by a cut of equivalence classes will become smaller and smaller. The *ideal limit* would be reached by such a partition of the class of all the actualizable worlds according to which every subset were a singleton. At the end, any singleton, containing just one *completely individuated possible world* would be described by a *maximally consistent set of contingent propositions*, so that the addition of any independent contingent proposition that preserves the consistency of the description were not possible any longer.

The concept of the *maximally consistent set of contingent propositions* may seem tricky for various reasons. But independently of this, our temporal modal logic is temporal logic of *events*, and I don't want to extend the concept of event so that every contingent proposition is about an event. So, no description obtained through a set of our elementary formulae concerning events could ever be the maximally consistent set of contingent propositions, since it would always be possible to add consistently some contingent proposition that is not about an event but about something else. But actually, we don't need a set of propositions that describes a possible world completely. We should only be aware of the fact that whenever we talk about "a world *w* described by a set of propositions" we talk *ipso facto* about the equivalence class [*w*] in which the set of propositions describing *w* is true.

Yet another thing that should be clarified concerns what I call Prior's Axiom (due to a suggestion he made in Oberwolfach shortly before he died).[33] We say normally of a green board that it is completely green in spite of the fact that, according to the current theory of colour, it should not be said to be green if we took it to consist of very tiny regions to which the concept of colour were not applicable. We do say that the board *is* completely green because we take greenness to be a *holistic* property, so that *if* the board is green, *every* part of it is green. Analogously, I shall take that *if* $\boldsymbol{A}e(t_n)$ is true, $\boldsymbol{A}e(t_m)$ is also true for any t_m included in t_n, that is, I shall take that

$$(t_n)(\boldsymbol{A}e(t_n) \Rightarrow (t_m)(t_m \subset t_n \Rightarrow \boldsymbol{A}e(t_m)))$$

is axiomatically true, and call it *Prior's Axiom*. Intuitively, it is correct to say that the Second World War lasted *uninterruptedly* in spite of the fact that, taken per se, the concept *war* is inapplicable in a case in which the time interval is, say, just a billionth part of a second. But, of course, if $t_n \subset t_m$, i.e., if it is not t_m that is included in t_n but conversely, t_n is included in t_m, $\boldsymbol{A}e(t_m)$ can be only contingently true if $\boldsymbol{A}e(t_n)$ is true.

After all these preliminaries, let me introduce the central components of the apparatus. As for modal logic, we need a standard model for modal logics.[34] Purely formally, a standard model is a structure

$M = \langle W, R, P \rangle$, where

(1) W is a set;
(2) R is a binary relation on W (i.e. $R \subseteq W \times W$);
(3) P is a mapping from natural numbers to subsets of W (i.e. $P_n \subseteq W$, for each natural number n).

When interpreted quite generally, W is a set of possible worlds, R a relation between them and P an assignment of sets of possible worlds to atomic sentences. In our case, (in addition to elementary formulae of the time system defined in the Appendix) the elementary formula will be $\boldsymbol{A}e(t_n)$ as it is understood above as well as any formula obtainable by substituting e', e'', e''',… for e and/or t_1, t_2,…, t_n,… or t_1, t_2,…, t_m,… for t_n. If ordered, the atomic sentences will be denoted by p_1, p_2,..., p_n,… . In accordance with what is said above about the relation between atomic sentences and the possible worlds which they describe and the way in which each atomic sentence is associated with an equivalence class of (accessible) possible worlds in which it is true, P, by mapping natural numbers to subsets of W, is to be understood as mapping each of p_1, p_2,..., p_n,… to exactly those possible worlds in which it is true.

The interpretation of the relation R in a standard model can vary significantly.[35] In view of our purposes, it should be understood as the *accessibility* relation whose domain is the set of all the worlds that are actual and whose anti-domain is the set of all the possible worlds that are actual and actualizable. This means that in $\alpha R \beta$ α is a real world time segment and β either a real world time segment or a possible world that can become actual but is either not yet actual or not yet completely actualized at least.

It should be noted that, according to the interpretation of *R* just given, its domain changes by the flow of time, because the set of all actual worlds will differ tomorrow, when some not yet actual worlds become actualized, from the set of actual worlds today.

As for the anti-domain of *R*, it is clear that it will also change by the flow of time because it consists partly of all the actual worlds, but it is less clear which worlds among those possible worlds that are not actual are to be treated as actualizable. The first guess could be that an actualizable world is any possible (not yet actual) world that is a member of an equivalence class of all the worlds described by a consistent set of future contingents. However, such a definition may seem odd if we look at a simple example. For the sea battle between Athenians and Spartans would hardly be said to be possible to happen tomorrow if Athenians and Spartans actually don't have any ships and arms today, any reason for fighting, and so on. The suggested definition allows too much. It seems necessary to take into account the state of the actual world today and restrict the set of actualizable worlds to those which seem to be *really* possible.[36] It seems also necessary to take into account natural laws and different levels of reality. For instance, after tossing a coin, it seems reasonable to say that there are two possible outcomes concerning how the coin will fall down. But, from the point of view of classical physics, it can be said that these possibilities are based on our ignorance, while one of the two outcomes is actually predetermined.[37] It can be also said that in some cases what is really possible depends on a specific physical theory or on its interpretation. So, for instance, some events from the domain of quantum mechanics are really indeterministic according to the mainstream interpretation but not according to some others.

Yet, however strange this may be, I shall follow the first guess and endorse, in this paper, the broadest possible definition of actualizable worlds. For, however unlikely it may be that, in the given example, the sea battle between Athenians and Spartans will happen tomorrow, it is still *logically possible* that it happens. We can easily imagine that, during the night, God supplies both parties with ships and arms and inspires them to fight. The broadest possible definition will be useful for our purposes (however useless it may be for analyzing ordinary cases and cases we come across in sciences), and then, once the main logico-metaphysical lesson has been learnt, one can restrict the anti-domain of the accessibility relation at will.

Now, since in accordance with the intended interpretation we want to speak of truths in actual and possible worlds that are accessible from a real world segment, W in $M = \langle W, R, P\rangle$ will consist of an infinite number of actual worlds, where some possible worlds are supposed to be accessible from some actual worlds but not from others. So, we need a temporal (not tense!) operator that can be prefixed to any given formula — let us denote it by $\{t_m\}$ (where another subscript specifying some other real world segment may be substituted for m) — which would tell us *at* which world the given formula is said to be true.

For any formula $\boldsymbol{F}$, $\{t_m\}\boldsymbol{F}$ states that

at the world actual at t_m, $\vDash \boldsymbol{F}$ ($\boldsymbol{F}$ is true).

If there were no world actualized at t_m, $\boldsymbol{F}$ would not be assertable. We may write this as $\neg\{t_m\}\boldsymbol{F}$ and stipulate it to mean:

not at a world actual at t_m, $\vDash \boldsymbol{F}$.

It is important to note that $\{t_m\}$ does not commute with negation, since $\{t_m\}\neg\, \boldsymbol{F}$ means:

at the world actual at t_m, **not** $\vDash \boldsymbol{F}$.

So, if there is no actual world at t_m, $\{t_m\}\boldsymbol{F}$ is false even if $\boldsymbol{F}$ is a logical truth, of course not because $\boldsymbol{F}$ itself is false but because there is no world at t_m at which it would hold.

The last fact renders possible to state exactly that which we need in the case of the tomorrow's sea battle. Given that $\boldsymbol{A}e(t_n)$ asserts ($\boldsymbol{A}$) that the sea battle (e) happens at t_n, then, if t_n is *tomorrow*, both $\{t_n\}\boldsymbol{A}e(t_n)$ and $\{t_m\}\neg\boldsymbol{A}e(t_n)$ are false, because both $\boldsymbol{A}e(t_n)$ and $\neg\boldsymbol{A}e(t_n)$ are not assertable at t_n. And if we allow the iteration of temporal operators — what we of course should do — and if t_m denotes *today*, it becomes possible to say that $\{t_m\}\{t_n\}\boldsymbol{A}e(t_n)$ and $\{t_m\}\{t_n\}\neg\boldsymbol{A}e(t_n)$ are both false. At the actual world at t_m (today) it is *both false* that the sea battle will as well as that it will not happen at t_n (tomorrow). But, at the same time, $\{t_m\}\neg\boldsymbol{A}e(t_n)$ is *true*, because, today, there is no such a thing in the real world as the sea battle at t_n.

In any standard model $M = \langle W, R, P\rangle$, at some world α, the necessity and the possibility of a proposition $\boldsymbol{A}$ in view of all the accessible worlds β, denoted by $\Box\boldsymbol{A}$ and $\Diamond\boldsymbol{A}$ respectively, are defined via the truth of $\boldsymbol{A}$ in the worlds β accessible by R from α:

$\vDash \Box\boldsymbol{A}$ iff for *every* β such that $\alpha R\beta$, $\vDash\boldsymbol{A}$,

$\vDash \Diamond\boldsymbol{A}$ iff for *some* β such that $\alpha R\beta$, $\vDash\boldsymbol{A}$.

Since the truth of $\Box\boldsymbol{A}$ and $\Diamond\boldsymbol{A}$ depends on α, which may be actual, actualizable or non-actualizable, for any $\boldsymbol{A}$, $\Box\boldsymbol{A}$ and $\Diamond\boldsymbol{A}$ are *not closed* (just as $f(x)$ in the predicate calculus) before being prefixed by a temporal operator.

Now, given that we have assumed that there is just one real world, then, according to the concept of *conditional necessity,* which the medieval logicians called *necessity per accidens* and which we may, in the given context, rather call *inaccessibility per accidens*,[38] the fact that some world β is actual at t_n precludes the possibility that some other world, different from β, is actual at t_n. This means that

$$(\forall t_n)(\neg t_m \prec t_n \wedge \neg t_m \cap t_n \Rightarrow (\{t_m\}\boldsymbol{A}e(t_n) \Rightarrow \{t_m\}\Box\boldsymbol{A}e(t_n)))$$

for, in any such case, $\boldsymbol{A}e(t_n)$ is a proposition about a world that is already actualized. Notice that if there is no world that is actual at t_m, the formula is trivially true because the second antecedent is false.

If $t_m \prec t_n$, the modality of $\boldsymbol{A}e(t_n)$ depends on whether the world at t_n is actual, partly actualized, or accessible but not actualized at all. If it is actual,

$$\{t_m\}\boldsymbol{A}e(t_n) \Rightarrow \{t_m\}\Box\boldsymbol{A}e(t_n)$$

holds as well. If it is not actualized at all, it holds only

$$\{t_m\}\boldsymbol{A}e(t_n) \Rightarrow \{t_m\}\Diamond\boldsymbol{A}e(t_n)$$

but not

$$\{t_m\}\boldsymbol{A}e(t_n) \Rightarrow \{t_m\}\Box\boldsymbol{A}e(t_n),$$

since in this case

$$\{t_m\}\Diamond\neg\boldsymbol{A}e(t_n)$$

also holds.

In the case that the world at t_n is only partly actualized, that is, actualized only on some t_k, $t_k \subset t_n$, the modality of $\boldsymbol{A}e(t_n)$ at t_m depends on *how* the world is actualized at t_k. If the actualization at t_k is such that $\boldsymbol{A}e(t_k)$ would be true at t_k if $\boldsymbol{A}e(t_n)$ were true at t_n, then, since $\boldsymbol{A}e(t_k)$ should be true if $\boldsymbol{A}e(t_n)$ were true according to Prior's Axiom introduced above, $\boldsymbol{A}e(\mathrm{t}_n)$ is possible at t_m, i.e., $\{\mathrm{t}_m\}\Diamond\boldsymbol{A}e(\mathrm{t}_n)$, while, if the actualization is such that $\neg\boldsymbol{A}e(\mathrm{t}_k)$ is true at t_k, $\boldsymbol{A}e(\mathrm{t}_n)$ is impossible at t_m, i.e., $\{\mathrm{t}_m\}\neg\Diamond\boldsymbol{A}e(\mathrm{t}_n)$, and so $\{\mathrm{t}_m\}\Box\neg\boldsymbol{A}e(\mathrm{t}_n)$.

The same holds if $\mathrm{t}_m \cap \mathrm{t}_n$. Namely, if t_k is the overlapping interval of t_m and t_n, $\{\mathrm{t}_m\}\neg\Diamond\boldsymbol{A}e(\mathrm{t}_n)$ holds if and only if $\boldsymbol{A}e(\mathrm{t}_k)$ is false already at t_k. This overlapping case is especially interesting because it shows that, in spite of the fact that the world at t_k is *actualized*, $\boldsymbol{A}e(\mathrm{t}_k)$ may be only possible and yet not necessary at t_k! But this is so only because, according to Prior's Axiom, the truth of $\boldsymbol{A}e(\mathrm{t}_k)$ depends also on what will happen *after* t_k. So, for instance, if it is actually snowing in the morning, this doesn't mean that it will be snowing the whole day. The proposition about the *whole day snowing* is only *possibly true* in the morning. But if it is not snowing in the morning, the same proposition is *necessarily false* already in the morning.

As we see, all modalities concerning propositions that are not logical truths are *conditional modalities*, even if we define the anti-domain of R in the broadest possible way, as I did it. So, as in the case of three-valued logic, the Sea Battle case is just one particular case to which the "new logic" is applicable. It is easy to see that if $\boldsymbol{A}e(\mathrm{t}_n)$ is the proposition stating that the sea battle e happens at t_n and if t_n is a time interval at which no world is yet actualized at all, both $\{\mathrm{t}_m\}\Diamond\boldsymbol{A}e(\mathrm{t}_n)$ and $\{\mathrm{t}_m\}\Diamond\neg\boldsymbol{A}e(\mathrm{t}_n)$ are true at a world that is actual at some time t_m that is earlier than t_n. But this is so neither because, at t_m, $\boldsymbol{A}e(\mathrm{t}_n)$ and $\neg\boldsymbol{A}e(\mathrm{t}_n)$ don't have truth-values, as it should be according to Aristotle, nor because, at t_m, their truth-value is indeterminate, as it should be according to Lukasiewicz, but because $\{\mathrm{t}_m\}\neg\boldsymbol{A}e(\mathrm{t}_n)$, $\{\mathrm{t}_m\}\neg\{\mathrm{t}_n\}\boldsymbol{A}e(\mathrm{t}_n)$, $\{\mathrm{t}_m\}\neg\{\mathrm{t}_n\}\neg\boldsymbol{A}e(\mathrm{t}_n)$, $\{\mathrm{t}_m\}\Diamond\boldsymbol{A}e(\mathrm{t}_n)$ and $\{\mathrm{t}_m\}\Diamond\neg\boldsymbol{A}e(\mathrm{t}_n)$ are all consistently true.

Let me summarize the point of the offered solution to the Sea Battle Puzzle by citing the six consistent propositions whose truth depends on whether, for $t_m \prec t_n$, it is only t_m on which a world is actualized or it is also t_n. In addition to putting them formally, I shall also express them in idiomatic English tense language (where *e* will denote *the sea battle* and t_m and t_n *today* and *tomorrow* respectively) and also, since the puzzle was originally introduced in Aristotle's *De Interpretatione*, in Ancient Greek.[39]

1. If there is a world actualized on t_m but not on t_n, then $\vDash\{t_m\}\neg \boldsymbol{A}e(t_n)$, i.e., It is true today that the tomorrow's sea battle has not happened; Σήμερον ἀληθὲς εἰπεῖν ὅτι ἡ αὔριον ναυμαχία οὐκ ἐγένετο.

2. If there are worlds actualized both on t_m and on t_n, then, if $\vDash\{t_n\}\boldsymbol{A}e(t_n)$, then $\vDash\{t_n\}\{t_m\}\boldsymbol{A}e(t_n)$, i.e., If the sea battle really happens tomorrow, it will be true tomorrow that it was true the day before that the sea battle would happen the day after; Ἐὰν γένηται αὔριον ἡ ναυμαχία, ἔσται αὔριον ἀληθὲς εἰπεῖν ὅτι ἦν τῇ προτεραίᾳ ἀληθὲς εἰπεῖν ἔσεσθαι τῇ ὑστεραίᾳ τὴν ναυμαχίαν.

3. If there is a world actualized on t_m but not on t_n, then $\vDash\neg\{t_m\}\{t_n\}\ \boldsymbol{A}e\ (t_n)$, i.e., It is not true today that it will be true tomorrow that the sea battle has happened that day (independently of whether it happens tomorrow or not); Σήμερον οὐκ ἀληθὲς εἰπεῖν ἔσεσθαι αὔριον ἀληθὲς εἰπεῖν ὅτι ἐγένετο ἡ ναυμαχία ταύτῃ τῇ ἡμέρᾳ.

4. If there is a world actualized on t_m but not on t_n, then $\vDash\neg\{t_m\}\{t_n\}\neg\ \boldsymbol{A}e\ (t_n)$, i.e., It is not true today that it will be true tomorrow that the sea battle has not happened that day (independently of whether it happens tomorrow or not); Σήμερον οὐκ

ἀληθὲς εἰπεῖν ἔσεσθαι αὔριον ἀληθὲς εἰπεῖν ὅτι οὐκ ἐγένετο ἡ ναυμαχία ταύτῃ τῇ ἡμέρᾳ.

5. If there are worlds actualized both on t_m and on t_n, then, if $\vDash\{t_n\}\boldsymbol{A}e(t_n)$, then $\vDash\{t_n\}\{t_m\}\{t_n\}\boldsymbol{A}e(t_n)$, i.e., If the sea battle really happens tomorrow, it will be true tomorrow that it was true the day before that it would be true the day after that the sea battle had happened that day; Ἐὰν γένηται αὔριον ἡ ναυμαχία, ἔσται αὔριον ἀληθὲς εἰπεῖν ὅτι ἦν τῇ προτεραίᾳ ἀληθὲς εἰπεῖν ἔσεσθαι τῇ ὑστεραίᾳ ἀληθὲς εἰπεῖν ὅτι ἐγένετο ἡ ναυμαχία ταύτῃ τῇ ἡμέρᾳ.
6. If there are worlds actualized both on t_m and on t_n, then, if $\vDash\{t_n\}\neg\boldsymbol{A}e(t_n)$, then $\vDash\{t_n\}\{t_m\}\{t_n\}\neg\boldsymbol{A}e(t_n)$, i.e., If the sea battle does not happen tomorrow, it will be true tomorrow that it was true the day before that it would be true the day after that the sea battle had not happened that day; Ἐὰν μὴ γένηται αὔριον ἡ ναυμαχία, ἔσται αὔριον ἀληθὲς εἰπεῖν ὅτι ἦν τῇ προτεραίᾳ ἀληθὲς εἰπεῖν ἔσεσθαι τῇ ὑστεραίᾳ ἀληθὲς εἰπεῖν ὅτι οὐκ ἐγένετο ἡ ναυμαχία ταύτῃ τῇ ἡμέρᾳ.

The possible advantage of this solution to the Sea Battle problem can be assessed only indirectly, since it is clear that all the three solutions are essentially similar. I suggested above that a possible advantage of Lukasiewicz's solution over that of Aristotle's may be: (1) that it is based on an independently sketched system of logic, while Aristotle's solution might seem *ad hoc*; and (2) that it does not violate the principle according to which the complex proposition has a truth-value only if the component propositions have truth values as well. A possible advantage of the third solution over that of Lukasiewicz's might be said to consist in the fact that *indeterminateness* does not look as a good candidate for being a truth-value. However, it could still be said that the solution within temporal-modal logic represents just a further analytical refinement of what Lukasiewicz had in mind when he suggested *possibility* as that to which the third truth-value refers. Namely, even if we are reluctant to accept that *possibility* as such, whatever it may be otherwise, is a truth-value, we may agree that truth and falsehood are not *directly* applicable to those propositions which are *only to become true* or *false*. That's why modal logic, naturally combined with temporal logic, seems to be the best choice that can do the job, as the system outlined above may show.

V. Concluding considerations: Time as a line-in-drawing

One might say that, curiously enough, the third solution is *conservative* when compared with the solutions of Aristotle's and Lukasiewicz's, since, though it uses the tools of modern modal and temporal logics, it neither assumes the existence of truth-value gaps nor rejects the Principle of Bivalence. It is *conservative* in another respect as well, namely, by being compatible only with the tensed and not with the so-called new theory of time, which is tenseless.[40]

The contemporary tenseless theory of time has nothing to do with Boethius' eternalism or with McTaggart's theory of the unreality of time.[41] The contemporary *detensers* accept the reality of time insofar as they accept the reality of the *earlier than* relation that, in addition to the *simultaneity* relation, can hold between two events. If the structure of time were represented within a period-based system, as it is done in the

Appendix, detensers would certainly also accept the reality of the abutment, the overlapping and the inclusion relation. What detensers deny is the reality of *tenses*, i.e., the reality of pastness, presentness and futurity (or any other tenses whatsoever). As Einstein once put it: “For us faithful physicists, the separation between past, present and future has only the meaning of an illusion, although a persisting one.”[42]

Now, it may be unclear *prima facie* why my solution requires the reality of tenses, since it is given without introducing the properties of pastness, presentness and futurity,[43] without introducing tense operators (e.g. as Prior does[44]) and without using temporal indexicals.[45] Remarkably, it is formulated in a *tenseless language*!

The reason why the tenses still occur after all follows only from the way in which different modalities are defined and not from the fact that we always look at the world from the present perspective.[46] The reality of tenses has nothing to do with the existence of an observer. So, the difference between pastness and futurity, as I am now going to show, may be defined *via* what is true about modalities that are inherent to the real world as such.[47] This will turn out to be so because all modalities concerning propositions that are not logical truths are *conditional*, depending solely on the *relations* between the times at which the propositions should supposedly hold and the times they speak about, which becomes most clearly visible if the anti-domain of the accessibility relation is defined in the broadest possible way, just as I defined it above.

Let us suppose that

$$(\forall t_n)(\mathrm{t}_m \,\lbrace\, t_n \Rightarrow \{\mathrm{t}_m\}(\Diamond \boldsymbol{A}E(t_n) \wedge \Diamond\neg\boldsymbol{A}E(t_n)))$$

is true independently of which particular event (*e, e’, e’’*,…) is substituted for *E* (remember that $\lbrace$ is the abutment relation, which is definable *via* $\prec$ relation – see

Appendix). In this case, we can say that t_m is *one of the present intervals*, while any time interval that is later than t_m belongs to the future.[48] The reason that t_m is just *one* of the present intervals follows from the fact that we have implicitly, by introducing the axioms cited in the Appendix, defined time as a *non-quantized* one-dimensional continuum, so that t_m can be a second, a minute, an hour, a day, a year, and so on, *ad libitum*. But whatever it is, it is *present* because any interval that abuts it lies in the future.

It should be noted that if we want to speak about *absolute presentness*, we can easily have it. We should only define *instant* as the abutment place of two equivalence classes of abutting intervals. The *present instant* will be then the abutment place of the equivalence class of present intervals and the equivalence class of intervals that abut them. So, after all, the absolute presentness will be the abutment place of some t_m and the intervals abutting it such that

$$(\forall t_n)(\mathrm{t}_m \lmoustache t_n \Rightarrow \{\mathrm{t}_m\}(\Diamond \boldsymbol{A}E(t_n) \wedge \Diamond\neg\boldsymbol{A}E(t_n)))$$

is true.[49]

Let us suppose now that

$$(\exists t_n)(\mathrm{t}_m \prec t_n \Rightarrow \{\mathrm{t}_m\}(\Box \boldsymbol{A}e(t_n) \vee \Box\neg\boldsymbol{A}E(t_n)))$$

is true independently of which particular event (*e*, *e'*, *e''*,…) is substituted for *E*. In this case, t_m lies in the *past*, since a proposition about an event that happens at some time later than t_m can be necessary only if the world at that later time is already actualized.

And finally, let us suppose that

$$(\forall t_n)(\ \mathrm{t}_m \lmoustache t_n \Rightarrow \{\mathrm{t}_m\}(\Diamond \boldsymbol{A}E(t_n) \wedge \Diamond\neg\boldsymbol{A}E(t_n)))$$

is false independently of which particular event (*e*, *e'*, *e''*,…) is substituted for *E*. Since this cannot be because at the world that is actual at t_m neither $\Diamond \boldsymbol{A}E(t_n)$ nor $\Diamond\neg\boldsymbol{A}E(t_n)$ is true, it can be so only because there is no actual world at t_m at which they would be assertable. And this means, consequently, that t_m lies in the future.

In relation to the last fact, it is a good place to notice that the accessibility relation is unconditionally *reflexive*, but only conditionally *symmetric* and *transitive*. It is symmetric if and only if the worlds between it would be stated to hold were already actualized, and it is transitive if and only if all the three worlds between it would be stated to hold were already actualized.

Not accidentally has it turned out that, although formulated in a tenseless language, the solution to the Sea Battle problem within temporal modal logic implies the reality of tenses. *The flow of time* assumption is also implicitly present in the solutions of Aristotle's and Lukasiewicz's, since the rejection of the tenseless view is equivalent with the rejection of the principle that, as Lukasiewicz put it, "if *A* is *b* at instant *t*, then it is true at every instant earlier than *t* that *A* is *b* at instant *t*", where *A* stands for a contingent proposition and *b* for one of the only two truth-values.[50]

As a general lesson, we may conclude that though time can be represented as an endless one-dimensional continuous line, this line should be conceived not as a drawn line but as a *line-in-drawing*. This is the only way to avoid logical determinism or, actually, any sort of complete determinism. The contemporary tenseless theory of time implies complete determinism, whereas, if there are indeterministic events, the tensed theory of time is true. Thus if there is libertarian free will or an indeterminism implied by quantum mechanics, the tensed theory of time is true. In spite of "the pronouncements of the Saints, who say that God does not know things that are going to be in a way different from that in which [He knows] things that are"[51], the Ockhamists cannot stick to these pronouncements and claim, at the same time, that whether a person will be saved or not depends on his or her free will.

Appendix

The interval-based structure of time axiomatized in the infinitary language $L\omega_1\omega_1$

Let, in the intended model of the interval-based system of time S_I, the individual variables $t_1, t_2, \ldots, t_i, \ldots, t', t'', t''', \ldots$ range over one-dimensional time stretches, and let the relation symbols $=, \prec, \succ, \lmoustache, \rmoustache, \cap$ and $\subset$ be interpreted as the identity, precedence, succession, abutment, overlapping and inclusion relations respectively. Let the elementary formulae be $t_m = t_n$, $t_m \prec t_n$, $t_m \succ t_n$, $t_m \lmoustache t_n$, $t_m \rmoustache t_n$, $t_m \cap t_n$ and $a_m \subset a_n$, as well as any formula obtainable by substituting other variables and/or individual constants $\mathrm{t}_1, \mathrm{t}_2, \ldots, \mathrm{t}_i, \ldots$ for t_m and/or t_n, where

$t_m \succ t_n \Leftrightarrow_{\text{def.}} t_n \prec t_m$ and $t_m \rmoustache t_n \Leftrightarrow_{\text{def.}} t_n \lmoustache t_m$,
$t_m \lmoustache t_n \Leftrightarrow_{\text{def.}} t_m \prec t_n \wedge \neg(\exists t_l)(t_m \prec t_l \wedge t_l \prec t_n)$,
$t_m \cap t_n \Leftrightarrow_{\text{def.}} (\exists t_l)(\exists t_k)(t_l \prec t_n \wedge \neg t_l \prec t_m \wedge t_m \prec t_k \wedge \neg t_n \prec t_k)$,
$t_m \subset t_n \Leftrightarrow_{\text{def.}} \neg t_m = t_n \wedge (t_l)(t_l \cap t_m \Rightarrow t_l \cap t_n)$.

The Axiom Schemes of S_I

1. $(t_n)\neg t_n \prec t_n$
2. $(t_k)(t_l)(t_m)(t_n)(t_k \prec t_m \wedge t_l \prec t_n \Rightarrow t_k \prec t_n \vee t_l \prec t_m)$
3. $(t_m)(t_n)(t_m \prec t_n \Rightarrow t_m \lmoustache t_n \vee (\exists t_l)(t_m \lmoustache t_l \wedge t_l \lmoustache t_n))$
4. $(t_k)(t_l)(t_m)(t_n)(t_k \lmoustache t_m \wedge t_k \lmoustache t_n \wedge t_l \lmoustache t_m \Rightarrow t_l \lmoustache t_n)$
5. $(t_k)(t_l)(t_m)(t_n)(t_k \lmoustache t_l \wedge t_l \lmoustache t_n \wedge t_k \lmoustache t_m \wedge t_m \lmoustache t_n \Rightarrow t_l = t_m)$
6. $(t_m)(\exists t_n)t_m \prec t_n$
7. $(t_m)(\exists t_n)t_n \prec t_m$
8. $(t_m)(\exists t_n)\, t_n \subset t_m$
9. $(t_1)(t_2)\ldots(t_i)\ldots((\exists t')(\wedge_{1\leq i<\omega}\ t_i \prec t') \Rightarrow$
   $\Rightarrow (\exists t'')\ (\wedge_{1\leq i<\omega} t_i \prec t'' \wedge \neg(\exists t''')(\wedge_{1\leq i<\omega} t_i \prec t''' \wedge t''' \prec t'')))$
10. $(t_1)(t_2)\ldots(t_i)\ldots((\exists t')(\wedge_{1\leq i<\omega}\ t_i \succ t') \Rightarrow$
    $\Rightarrow (\exists t'')(\wedge_{1\leq i<\omega} t_i \succ t'' \wedge \neg(\exists t''')(\wedge_{1\leq i<\omega} t_i \succ t''' \wedge t''' \succ t'')))$

NOTES

[1] Lukasiewicz 1922, p. 126.
[2] Lukasiewicz 1918, p. 86.
[3] Lukasiewicz 1920, p. 87.
[4] Brouwer 1908, pp. 109ff.

[5] For the condensed but clear survey of the main differences between modern interpretations see Craig 1988, Ch. I, and pp. 2ff. and 281-283 in particular; see also Gasking 1995.
[6] See Anscombe 1956, Strang 1960, Rescher 1968, Hintikka 1964 and 1973.
[7] Aristoteles 1831, 19a18ff. (Ackrill's translation; cf. Aristotle 2002).
[8] In this place, the translation differs from that of Ackrill for the reason given below.
[9] Aristoteles 1831, 19a23.
[10] Ibid. 19a35-38.
[11] Cf. Ross 1953, Kneale 1967, Ackrill 2002, Frede 1970.
[12] For me, this is quite obvious; cf. also Craig 1988, pp. 10ff.
[13] See, for instance, Strang 1960, pp. 460ff.
[14] Cf. Craig 1988, p. 2.
[15] Boethius 1984, V, 6.
[16] Ockham 1945, Q. I, Supp. VI.
[17] Loc. cit.
[18] Loc. cit.
[19] Loc. cit.
[20] Ibid. Q. I, Dub. V.
[21] Ibid. Q. I, Supp. III.
[22] Ibid. Q. I, Supp. IV.
[23] Lukasiewics 1922, p. 126.
[24] Loc. cit.
[25] Cf. Lukasiewicz 1920, p. 87.
[26] Ibid., p. 88.
[27] Lukasiewicz 1922, p. 127.
[28] Lukasiewicz 1918, p. 86 and Lukasiewicz 1922, p. 126.
[29] Lewis 1973 and 1986.
[30] Cf. Arsenijević 2002, pp. 134ff. and Arsenijević 2003b, pp. 342ff.
[31] Cf. Arsenijević and Kapetanović 2008a, pp. 82ff., and Arsenijević and Kapetanović 2008b, pp. 42ff.
[32] In this paper, I am going, for the first time, to use this generalization.
[33] Cf. Arseniojević 2002, pp. 126ff.
[34] Cf. Chellas 1980, pp. 67ff.
[35] Loc. cit.
[36] Cf. Deutsch 1990, and as directly relevant for my purposes, the quotation from Belnap in note 46 below.
[37] For Le Poidevin, "the future cannot be *ontologically*" but only "*epistemologically indeterminate*" (Le Poidevin 1991, p. 130).
[38] Arsenijević 2003b, p. 345.
[39] Thanks to Vojin Nedeljković.
[40] Cf. Oaklander and Smith 1994 and Mellor 1998.
[41] Cf. Section II above and McTaggart 1908.
[42] Einstein 1949, p. 537.
[43] This is how tenses are defined in the *tense system* of events that I outlined in Arsenijević 2003, pp. 328ff.
[44] Prior 1967.
[45] Ludlow 1999, pp. 124ff.
[46] Cf. Belnap, Perloff, and Xu 2001 and Belnap 2007.
[47] "If a certain possibility is real, then if it has any relevance at all for us, it must be part and parcel of *Our World*. Conclusion: The brilliantly conceived doctrine of Lewis 1986 (and elsewhere) ought to be rejected" (Belnap 2007, p. 87, note 1).
[48] This is how the difference between the pastness and the futurity is to be defined in the period-based system of time as it is axiomatized in the Appendix.
[49] This is how the difference between the pastness and the futurity is to be defined in the instant-based system of time that is presupposed in Jokić 2003. However, the period-based and the instant-based systems are both syntactically and semantically only trivially different amongst themselves (cf. Arsenijević 2003a, and Arsenijević and Kapetanović 2008a).
[50] Lukasiewicz 1922, p. 127.
[51] Ockham 1945, Q. I, Supp. VI.

REFERENCES

Anscombe, G. E. M. 1956: “Aristotle and the Sea Battle”, *Mind* 65 (pp. 1-15).
Aristoteles 1831: Περ± □ρμφνε°αϖ, in: *Aristoteles graece ex recensione Immanuelis Bekkeri*, edidit Academia Regia Borussica, vol. i, Berolini, apud Georgium Reimerum (pp. 16-24).
Aristotle 2002: *Categories and De interpretatione*, translated with notes by J. L. Ackrill; Clarendon Press.
Arsenijević, M. 2002: “Determinism, Indeterminism and the Flow of Time”, *Erkenntnis* 56/2 (pp. 123-150).
Arsenijević, M. 2003a: “Generalized Concepts of Syntactically and Semantically Trivial Differences and Instant-Based and Period-Based Time Ontologies”, *Journal of Applied Logic* 1 (pp. 1-12).
Arsenijević, M. 2003b: “Real Tenses”, in: *Time, Tense, and Reference* (A. Jokić and Q. Smith eds.); MIT Press (pp. 325-354).
Arsenijević and Kapetanović 2008a: “The ‘Great Struggle’ between Cantorians and Neo-Aristotelians: Much Ado about Nothing”, *Grazer Philosophische Studien* 76 (pp. 79-90).
Arsenijević and Kapetanović 2008b: “An $L_{\omega_1\omega_1}$ Axiomatization of the Linear Archimedean Continua as Merely Relational Structures”, *WSEAS Transactions on Mathematics* 7/2 (pp. 39-47).
Belnap, N., Perloff, M., and Xu. M. 2001: *Facing the Future: Agents and Choices in Our Indeterminisic World*; Oxford University Press.
Belnap, N. 2007: “An Indeterministic View of the Parameters of Truth”, in: *Philosophie der Zeit* (Th. Müller ed.); Vittorio Klostermann (pp. 87-113).
Boethius 1984: *Consolatio philosophiae* (O’Donnell and James J. eds.); Thomas Library, Bryn Mawr College.
Brouwer, J. E. L. 1908: “The Unreliability of the Logical Principles”, in: *Collected Works I* (Heyting, A.); North-Holland 1975.
Craig, W. L. 1988: *The Problem of Divine Foreknowledge and Future Contingents from Aristotle to Suarez*; E.J. Brill, Leiden, New York, Kobenhavn, Köln.
Chellas, B. F. 1980: *Modal Logic*; Cambridge University Press.
Deutsch, H. 1990: “Real Possibility”, *Nous* 24 (pp. 751-755).
Einstein, A. 1949: *Einstein-Besso Correspondence*; Herman, Paris.
Frede, D. 1970: *Aristoteles und die ‘Seeschlacht’*, Hypomnenata 27, Göttingen.
Gasking, D. 1995: *The Sea Battle and the Master Argument*; Walter de Gruyter, Berlin.
Hintikka, J. 1964: “The Once and Future Sea Fight: Aristotle’s Discussion of Future Contingents in *De interpretatione* 9”, *Philosophical Review* 73 (pp. 461-492).
Hintikka, J. 1973: *Time and Necessity Studies in Aristotle’s Theory of Modality*, Clarendon Press, Oxford.
Jokić, A. 2003: “Tensed and Tenseless Existence of Nature “, *Filo* 6/3 (pp. 194-204); the updated and expanded version in: Jokić, A. 2009: “Ontological Status of Temporal Properties”, *Filozofske Studije* 30 (pp. 9-27).
Kneale, W. and Kneale, M. 1967: *The Development of Logic*; Clarendon Press, Oxford.
Le Poidevin, R. 1991: *Chance, Cause and Contradiction: A Defense of the Tenseless Theory of Time*; St. Martin’s Press, New York.
Lewis, D. 1973: *Counterfactuals*; Harvard University Press.
Lewis, D. 1986: *On the Plurality of Worlds*; Blackwell, Oxford.
Lukasiewicz, J. 1918: “Farewell Lecture by Professor Jan Lukasiewicz, delivered in the Warsaw University Lecture Hall on March 7, 1918”, in *Jan Lukasiewicz – Selected Works*; North-Holland, 1970 (pp. 84-86).
Lukasiewicz, J. 1920: “On Three Valued-Logic”, in: *Jan Lukasiewicz – Selected Works*; North-Holland, 1970 (pp. 87-88).
Lukasiewicz, J. 1922: “On Determinism”, in: *Jan Lukasiewicz – Selected Works*; North-Holland, 1970 (pp. 110-128).
Ludlow, P. 1999: *Semantics, Tense, and Time*; MIT Press.

Mellor, D. H. 1998: *Real Time II*; Routledge, London.
McTaggart, J. M. E. 1908: “The Unreality of time”, *Mind* 17 (pp. 457-474).
Ockham, W. 1945: *Tractatus de praedestinatione et de praescientia dei et futuris contingentibus*; The Franciscan Institute St. Bonaventure College, New York.
Oaklander N. and Smith Q. (eds.) 1994: *The New Theory of Time*; Yale University Press.
Prior, A. 1967: *Past, Present, Future*; Clarendon Press, Oxford.
Rescher, N. 1968: “Truth and Necessity in Temporal Perspective”, in: *The Philosophy of Time* (R. Gale ed.), Macmillan, London.
Ross, W. D. 1953: *Aristotle*; Methuen, London.
Strang, C. 1960: “Aristotle and the Sea Battle”, *Mind* 69 (pp. 447-465).