A well known feature of Leibniz's philosophy is his espousal of the ***actual infinite***, in defiance of the Aristotelian stricture that the infinite can exist only potentially. As he wrote to Foucher in 1692:

> I am so in favor of the actual infinite that instead of admitting that Nature abhors it, as is commonly said, I hold that Nature makes frequent use of it everywhere, in order to show more effectively the perfections of its Author. Thus I believe that there is no part of matter which is not, I do not say divisible, but actually divided; and consequently the least particle ought to be considered as a world full of an infinity of different creatures.[1]

Cantor drew succor from this doctrine of Leibniz, indicating his full agreement that there is an "actual infinity of created individuals in the universe as well as on our earth and, in all probability, even in each ever so small extended part of space."[2] However, no doubt to his disappointment, Leibniz had refused to countenance infinite number, arguing that its supposition was in conflict with the part-whole axiom, since (as Galileo's paradox shows) this would lead to an infinite set (the whole) being equal to an infinite proper subset of its elements (the part). Cantor, on the other hand, had followed Dedekind in taking this equality of an infinite set to its proper subset as the defining property of an infinite set. This development, together with the widespread acceptance today of Cantor's position, has led many commentators to criticize Leibniz's position as unsound: if there are actually infinitely many creatures, there is an infinite number of them.

Here I shall try to show that such a wholesale rejection of Leibniz's position is premature. Basing my constructions on Leibniz's unpublished manuscripts as well as his published work, and drawing on an argument of Antonio Leon, I hope to show that despite his rejection of infinite number, Leibniz could still give a consistent, recursive construal of infinite aggregates that would not involve him in any contradictions.

1. Infinite Aggregates and Actually Infinite Division.

On my construal of Leibniz's position, he advocates an infinite that is ***syncategorematic but actual***. Thus there is, according to him, an actual infinity of parts into which any piece of matter is actually (not merely potentially) divided, but there is no

[1] Letter to Foucher, ***Journal de Sçavans***, March 16, 1693, G I 416.
[2] Georg Cantor, ***Gesammelte Abhandlungen***, ed. E. Zermelo, 1932, p. 399; quoted from Joel Friedman's translation given in Gregory Brown, ***Leibniz Society Review***, 8, 1998, p. 123, from which article I have also drawn the above extract from the letter to Foucher.

totality or collection of all these parts.[3] That is, the infinitude of these parts may legitimately be expressed ***distributively*** as follows:

1. Every part is divided into further parts ("there is no part so small that it is not further divided.") Whatever finite number is proposed, there are more parts than this.

but it may not be expressed ***collectively***:

1. A given part is an infinite collection of parts. The number of these parts is greater than any finite number.

Under the spell of Cantor, many modern commentators cannot envision that the first way of expressing the matter does not automatically entail the second, even though the logical distinction between ***syncategorematic*** and ***categorematic*** infinites was well known in the middle ages. Let us then see whether the difference can be discerned in the present example. Leibniz's position rests on a conception according to which the parts are instituted and individuated by their differing motions. Each body or part of matter would be one moving with a motion in common. But this does not rule out various parts internal to that body having their own common motions, which effectively divide the body within. All that is essential for our purposes here can be captured in the following three premises:

1. Any part of matter (or ***body***) is actually divided into further parts.
2. Each such body is the ***aggregate*** of the parts into which it is divided.
3. Each part of a given body is the result of a division of that body or of a part of that body.

From 1) and 2) it follows that every body is an aggregate of parts. We may note, incidentally, that if one accepts Leibniz's further premise that

4) What is aggregated cannot be a substantial unity.

we already arrive at one of Leibniz's characteristic doctrines, that matter, being an aggregate, cannot be substantial. But I will not pursue that further here.

[3] Here it must be admitted that Leibniz's use of the term "infinite ***aggregate***" may be misleading.

We have the result that every body is an aggregate of parts, each of which is an aggregate of further parts, and so on to infinity. That is, by recursive application of 1),

5) Every body is actually infinitely divided.

And from 1) and 3) it follows that

6) Any part of a given body must be reachable by repeated division of the original body.

This is the crux of the matter—each part of a body must be reachable by repeated division of the original body. It is either a part, a part of a part, or a part of a part of a part, and so on. (We are not talking about "reaching" a part by a division as if this reaching and dividing were processes occurring ***in time***[4]; indeed, each motion causing these divisions within matter is conceived by Leibniz as an instantaneous motion. It is just that in order for a piece of matter to be a part of the original body, it must be obtainable by division of the body.) Our question is, can this state of affairs be interpreted in accordance with the Cantorian notion of infinity? Let us see.

Suppose that a given body has an actual infinity of parts in Cantor's sense, i.e. suppose that there is a number of divisions ω, greater than any finite number $\boldsymbol{r}$ of divisions. In particular, according to Cantor's definition ω is the first ordinal number > $\boldsymbol{r}$ for all finite numbers $\boldsymbol{r}$. Call the part $P_{\boldsymbol{r}}$ any part reachable by $\boldsymbol{r}$ divisions of the original body. Thus by 3) $P_{\boldsymbol{r}}$ must have resulted from the division of a part $P_{\boldsymbol{r}-1}$. Likewise, P_ω must have resulted from the division of a part $P_{\omega-1}$, a part resulting from $\omega - 1$ divisions. But $\omega - 1$ cannot be infinite, otherwise ω is not the first ordinal > $\boldsymbol{r}$ for all $\boldsymbol{r}$, contrary to our Cantorian supposition. But neither can it be finite, since one cannot obtain an actually infinite division by one more division of a finite aggregate of parts. Therefore there cannot be an infinite number ω of divisions.[5]

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[4] Cf. Cantor's stricture: "I have to declare that in my opinion reliance on the concept of time or the intuition of time in the much more basic and general concept of the continuum is quite wrong." (1932, pp. 191-2, quoted from the translation given by Michael Hallett in his thorough study of Cantor's philosophy of mathematics, ***Cantorian set theory and the limitation of size***, Clarendon Press: Oxford 1984; p. 15).

[5] I owe this form of argument to Antonio Leon, who used it in his article of his published on the internet which he generously brought to my attention: "On the infiniteness of the set of natural numbers", (http://www.terra.es/personal3/eubulides/infinit.htm). He interpreted it to show that the natural numbers cannot be infinite in Cantor's sense.

This does not, of course, prove anything inconsistent in Cantor's notion of an infinite cardinal. But it does show that it cannot automatically be assumed that any actual infinite will automatically have a corresponding infinite number, an infinite cardinality. That is, it raises questions about the ***applicability*** of Cantorian transfinite mathematics. In any case where there is a requirement of a ***recursive connection*** between any pair of the things numbered, the Cantorian conception of the infinite will not be valid. This is because the set ***N*** of natural numbers ordered by the relation > (is 'greater than') is recursively connected if and only if every number is finite. If limit ordinals (Cantor's ω, ω^2 etc.) are included, recursive connectedness fails.[6]

2. Infinite Numbers

Questions of applicability, of course, lead us to some other salient issues. One of these concerns the applicability of the mathematics of the real line. If one rejects the (Cantorian) actual infinite, Cantor claimed, then one must also reject irrationals:

> The transfinite numbers are in a certain sense ***new irrationalities***, and in my view the best method of defining the ***finite*** irrational numbers is quite similar to, and I might even say in principle the same as, my method of introducing transfinite numbers. One can say unconditionally: the transfinite numbers ***stand or fall*** with the finite irrational numbers: they are alike in their innermost nature, since both kinds are definitely delimited forms or modifications of the actual infinite.[7]

Here Cantor alludes to the fact that just as irrationals can be conceived as limits of infinite sequences of rational numbers, so transfinite numbers can be conceived as limits of infinite sequences of natural numbers, in each case added in immediately after the sequence they limit. If one rejects transfinites, what right has one to allow the extension of the number system to include irrationals? A reluctance to jettison the theory of the real line thus explains the widespread acceptance among modern mathematicians of the Cantorian theory of the infinite.

At this point it is reasonable to ask, to what extent could Leibniz have been aware of these kinds of issues, writing as he was prior to the modern theories of the transfinite and of irrationals? Some unpublished papers of his written in Paris in the Spring of 1676

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[6] This is the basis of Leon's argument referred to in the previous footnote.
[7] Cantor, 1887-8,; quoted from Hallett, ***op. cit.,*** p. 80. See also p. 26.

show that he was more alive to some of the necessary distinctions than might have been supposed. In one of these, “On the Secrets of the Sublime, or on the Supreme Being”, written 11 February 1676, he offers the following proof that there is no such thing as infinite number:

> It seems that all that needs to be proved is that the number of finite numbers cannot be infinite. If numbers can be assumed as continually exceeding each other by one, the number of such finite numbers cannot be infinite, since in that case the number of numbers is equal to the greatest number, which is supposed to be finite. (A VI iii 477)

but then retracts it, granting that “It must be responded that there is no such thing as the greatest number.” This would be music to the ears of Cantor, who objected to proofs that depended on a conception of the infinite as an “unincreasable actual infinite”. By contrast, for him the transfinite shares with the finite the property of “increasability in just that sense that 3 can be increased to larger numbers by the addition of new unities” (Hallett, p. 41). After some further elaboration of a similar proof that “the last number will always be greater than the number of all numbers. Whence it follows that the number of numbers is not infinite”, Leibniz then objects in a footnote “No, N.B. this only proves that such a series is unbounded” (A VI iii 477). Thus like Cantor he realizes that one cannot regard the series of all numbers as completed by a greatest or last number. The same point is given an extended treatment in a paper Leibniz penned about two months later, “Infinite Numbers”. There, anticipating the essence of the modern conception of a converging infinite series in terms of a sequence of partial sums, he writes:

> Whenever it is said that a certain infinite series of numbers has a sum, I am of the opinion that all that is being said is that any finite series with the same rule has a sum, and that the error always diminishes as the series increases, so that it becomes as small as we would like. For numbers do not ***in themselves*** go absolutely to infinity, since then there would be a greatest number. But they do go to infinity when applied to a certain space or unbounded line divided into parts. Now here there is a new difficulty. Is the last number of a series of this kind the last one that would be ascribed to the divisions of an unbounded line? It is not, otherwise there would also be a last number in the unbounded series. Yet there does seem to be, because the number of terms of the series will be the last number. Suppose to the point of division we ascribe a number always greater by unity than the preceding one, then of course the number of terms will be the last number of the series. But in fact there is no last number of the series, since it is unbounded; especially if the series is unbounded at both ends. (A VI iii 503)

Leibniz concludes these reflections with the remarks:

> Thus if you say that in an unbounded series there exists no last finite number that can be written in, although there can exist an infinite one: I reply, not even this can exist if there is no last number. The only other thing I would consider replying to this reasoning is that the number of terms is not always the last number of the series. That is, it is clear that even if finite numbers are increased to infinity, they never —unless eternity is finite, i.e. never— reach infinity. This consideration is extremely subtle. (A VI iii 504)

As Sam Levey has pointed out in his insightful commentary on this passage,[8] Leibniz is close here to Cantor's distinction between ***cardinal*** number and ***ordinal*** number. At least, he has distinguished the possibility that "the number of terms of the series is not always the last number of the series." In Cantor's theory, it will be remembered, the cardinality of the natural numbers is $\aleph_0$, but whereas the finite natural numbers are regarded as succeeded by ω_0, the first infinite ordinal number, this by no means represents the greatest ordinal number. But of course Leibniz does not accept that there is an ordinal that is the least number greater than all finite ordinals. Thus on his view infinite cardinals and infinite ordinals are alike impossible.

But what then of Cantor's "stand or fall" objection? Surely Leibniz's position would rule out irrationals too? I think there is evidence that Leibniz would accept this inference, but would be unfazed by it. For he did not accept the correspondence between real numbers and the real line we take for granted. In "Infinite Numbers" he reasons that irrationals such as "the ratio of the diagonal to the line" do require infinite numbers, which, like infinitesimals, are "nothings" or "fictions"; but argues that one might be able still to give a finitist account of irrational ratios of lines analogous to the treatment of converging infinite series quoted above. (A VI iii 503). Whatever is made of this, we must leave the implications of this issue for another time.

3. Inconsistent Multiplicities and the Universe as a Whole

How then are we to regard this opposition between Leibniz and Cantor? In an oft-quoted passage from a letter he wrote to Dedekind, Cantor distinguished between two kinds of multiplicities:

> [O]n the one hand a multiplicity can be such that the assumption that ***all*** of its elements 'are together' leads to a contradiction, so that it is impossible to conceive

[8] Samuel Levey, "Leibniz on Mathematics and the Actually Infinite Division of Matter", ***The Philosophical Review***, 107, No. 1 (January 1998).

> of the multiplicity as a unity, as 'one finished thing'. Such multiplicities I call ***absolutely infinite*** or ***inconsistent multiplicities***.
>
> As one easily sees, the 'totality of everything thinkable', for example, is such a multiplicity; later still other examples will present themselves.
>
> When, on the other hand, the totality of elements of a multiplicity can be thought without contradiction as 'being together', so that their collection into '***one*** thing' is possible, I call it a ***consistent multiplicity*** or a ***set***.[9]

These definitions, I believe, would have been perfectly acceptable to Leibniz. The problem is, according to what premises are we to judge the consistency? A case in point is the very doctrine on which Cantor thought he was following Leibniz, concerning the actual infinity of substances in the universe. Following Hobbes, Leibniz argued that the world or universe, conceived as "an accumulation of an infinite number of substances, is not a whole any more than infinite number itself."[10] Allow me to offer a proof that Leibniz might have found congenial, based on his part-whole axiom:

A1. A whole is the aggregate of all its parts.

A2. A whole is greater than any of its parts.

Now suppose a whole is part of itself. Then it will not be greater than itself, in contradiction to A2. From this it follows that

A3. No whole is part of itself.

Now let U be the aggregate of all wholes. Suppose U is a whole. Then it will be included in the aggregate of all wholes, so by A1 it will be a part of itself. But this contradicts A3. Therefore, from A1 and A2 it follows that U, the aggregate of all wholes, is not a whole. (As will be evident, this is an example of what has come to be called the Burali-Forte paradox.) Now this argument seems to justify Leibniz's claim that the universe is "no more a whole than infinite number itself", since one can construct a proof based on analogous reasoning for Leibniz's rejection of infinite number as follows:

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[9] Letter to Dedekind, 28 July 1899, quoted from the translation of Michael Hallett, ***Cantorian Set Theory***, p. 166.
[10] G. W. Leibniz, ***Theodicy,*** §195, p. 249 in E. M. Huggard, trans. (Open Court: La Salle, 1985)**.**

N1. A number is an aggregate of unities.[11]

N2. A number is greater than any proper subset of its constituent unities.

This obviously implies

N3. No number is equal to any proper subset of its constituent unities.

Now using one-one correspondence as the criterion for equality, it will follow from Galileo's Paradox that there is no number of natural numbers.

Thus on Leibniz's account, both the notion of an infinite collection such as U, the aggregate of all wholes, and that of the number of all numbers, are alike inconsistent multiplicities.

Now of course everything crucially depends on what premises are allowed. Cantor explicitly rejects the classical definition of number N1 on which Leibniz depends, as well as the contended part-whole axiom N2. But this does not prove Leibniz wrong, since Cantor's postulation of limit ordinals is simply that, a postulation, albeit one that is justified by the enormous success of the theory based upon it. But as we have seen with the interesting example of Leibnizian actual division, it does not follow from the success of that theory either that it is the only consistent conception of the actual infinite, or even that it is adequate to all cases. Besides one may note, as was objected already by Hessenberg,[12] that Cantor has not managed to lay down a theory that will determine in advance which multiplicities are to be rejected as inconsistent: this may depend on future developments.

In conclusion, then, it would seem that Leibniz has distinguished a perfectly coherent middle ground between Cantor's actual infinite and Aristotle's potential infinite. Claims that it is some sort of unsound halfway house between the two are in error: it is quite coherent to say that there are actually infinitely many things without accepting Cantor's theory of the transfinite.

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[11] A characteristic definition is quoted by Levey in his paper (***op. cit.***, p. 77): "I define number as one and one and one etc. or as unities" (A VI ii 441).

[12] G. Hessenberg, "Grundbegriffe der Mengenlehre", 1906; quotations are given in translation in Hallett, ***op. cit.***, pp. 167-8, where the vagueness of Cantor's conception of set is also discussed.