LEIBNIZ'S SYNCATEGOREMATIC INFINITESIMALS, SMOOTH INFINITESIMAL ANALYSIS, AND NEWTON'S PROPOSITION 6

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ABSTRACT

In contrast with some recent theories of infinitesimals as non-Archimedean entities, Leibniz's mature interpretation was fully in accord with the Archimedean Axiom: infinitesimals are fictions, whose treatment as entities incomparably smaller than finite quantities is justifiable wholly in terms of variable finite quantities that can be taken as small as desired, i.e. syncategorematically. In this paper I explain this syncategorematic interpretation, and how Leibniz used it to justify the calculus. I then compare it with the approach of Smooth Infinitesimal Analysis (SIA), as propounded by John Bell. Despite many parallels between SIA and Leibniz's approach —the non-punctiform nature of infinitesimals, their acting as parts of the continuum, the dependence on variables (as opposed to the static quantities of both Standard and Non-standard Analysis), the resolution of curves into infinite-sided polygons, and the finessing of a commitment to the existence of infinitesimals— I find some salient differences, especially with regard to higher-order infinitesimals. These differences are illustrated by a consideration of how each approach might be applied to Newton's Proposition 6 of the *Principia*, and the derivation from it of the v^2/r law for the centripetal force on a body orbiting around a centre of force. It is found that while Leibniz's syncategorematic approach is adequate to ground a Leibnizian version of the v^2/r law for the "solicitation" *ddr* experienced by the orbiting body, there is no corresponding possibility for a derivation of the law by nilsquare infinitesimals; and while SIA can allow for second order differentials if nilcube infinitesimals are assumed, difficulties remain concerning the compatibility of nilcube infinitesimals with the principles of SIA, and in any case render the type of infinitesimal analysis adopted dependent on its applicability to the problem at hand.

1. INTRODUCTION

Leibniz's doctrine of the fictional nature of infinitesimals has been much misunderstood. It has been construed as a late defensive parry, an attempt to defend the success of his infinitesimal calculus—understood as implicitly committed to the existence of infinitesimals as actually infinitely small entities—in the face of criticisms of the adequacy of its foundations. It has also been read as being at odds with other defences of the calculus Leibniz gave on explicitly Archimedean foundations. I take the position here (following Ishiguro 1990) that the idea that Leibniz was committed to infinitesimals as actually infinitely small entities is a misreading: his mature interpretation of the calculus was fully in accord with the Archimedean Axiom. Leibniz's interpretation is (to use the medieval term) *syncategorematic*: Infinitesimals are fictions in the sense that the terms designating them can be treated *as if* they refer to entities incomparably smaller than finite quantities, but really stand for variable finite quantities that can be taken as small as desired. As I have argued elsewhere (2005b), this interpretation is no late stratagem, but in place already by 1676.

In section 2 I present this interpretation by tracing its development from Leibniz's early work on infinite series and quadratures to his unpublished attempt in 1701 in the essay "*Cum prodiisset*" to provide a systematic foundation for his calculus. We shall see that as early as 1676 Leibniz had succeeded in providing a rigorous foundation for Riemannian integration, based on the Archimedean Axiom. The appeal to this axiom, generalized by Leibniz into his Law of Continuity, undergirds his interpretation of infinitesimals as fictions that can nevertheless be used in calculations, and forms the basis for his foundation for differentials in "*Cum prodiisset*".

I then turn to a comparison of Leibniz's approach with the recent theory of infinitesimals championed by John Bell, Smooth Infinitesimal Analysis (SIA), of which I give a brief synopsis in section 3. As we shall see, this has many points in common with Leibniz's approach: the non-punctiform nature of infinitesimals, their acting as parts of the continuum, the dependence on variables (as opposed to the static quantities of both Standard and Non-standard Analysis), the resolution of curves into infinite-sided polygons, and the finessing of a commitment to the existence of infinitesimals. Nevertheless, there are also crucial differences. These are brought into relief in section 4, by a consideration of how each approach might be applied to Newton's Proposition 6 of the *Principia*, and the derivation from it of the v^2/r law for the centripetal force on a body orbiting around a centre of force. It is found that while Leibniz's syncategorematic approach is adequate to ground a Leibnizian version of the v^2/r law for the "solicitation" *ddr* experienced by the orbiting body, there is no corresponding possibility for a derivation of the law by nilsquare infinitesimals; and while SIA can allow for second order differentials if nilcube infinitesimals are assumed, difficulties remain concerning the compatibility of nilcube infinitesimals with the principles of SIA, and in any case render the type of infinitesimal analysis adopted dependent on its applicability to the problem at hand.

2. Leibniz's foundation for the infinitesimal calculus

As Leibniz explains in his mature work, the infinite should not be understood to refer to an actual entity that is greater than any finite one of the same kind, i.e. *categorematically*, but rather *syncategorematically*: in certain well-defined circumstances infinite terms may be used *as if* they refer to entities incomparably larger than finite quantities, but really stand for variable finite quantities that can be taken as large as desired. Leibniz's interpretation of infinitesimals as *fictions* is intimately linked with his doctrine of the infinite. Just as the infinite is not an actually existing whole made up of finite parts, so infinitesimals are not actually existing parts which can be composed into a finite whole. As Leibniz explained to Des Bosses at the beginning of their correspondence,

> Philosophically speaking, I hold that there are no more infinitely small magnitudes than infinitely large ones, or that there are no more infinitesimals than infinituples. For I hold both to be fictions of the mind due to an abbreviated manner of speaking, fitting for calculation, as are also imaginary roots in algebra. Meanwhile I have demonstrated that these expressions have a great utility for abbreviating thought and thus for invention, and cannot lead to error, since it suffices to substitute for the infinitely small something as small as one wishes, so that the error is smaller than any given, whence it follows that there can be no error. (March 1706; G I 305)

The roots of this syncategorematic interpretation, like the roots of the calculus itself, can be discerned in Leibniz's earliest work on quadratures and infinite series, specifically in his work on the hyperbola. Already in 1672 in his reading notes on Galileo Galilei's *Two New Sciences (Discorsi*), Leibniz had formulated the basis of his later position. There he notes Galileo's demonstration that there are as many square roots of (natural) numbers as there are natural numbers, and that "therefore there are as many squares as numbers; which is impossible",[1] and then comments:

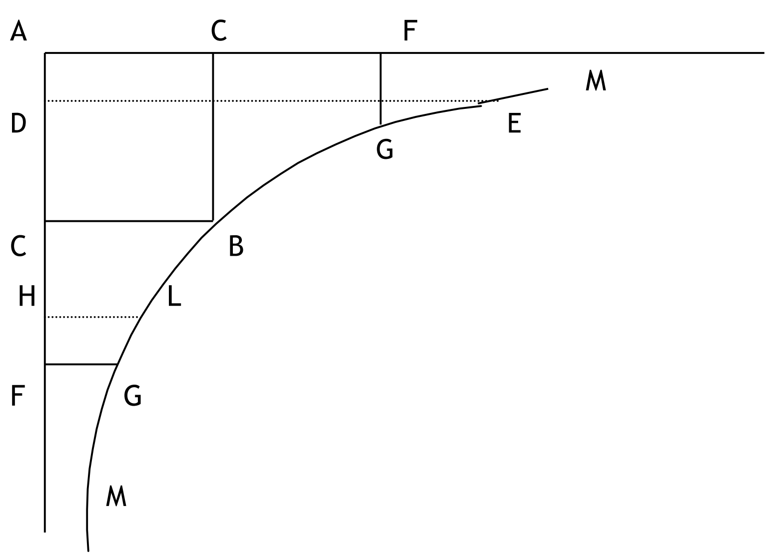
> Hence it follows either that in the infinite the whole is not greater than the part, which is the opinion of Galileo and Gregory of St. Vincent,[2] and which I cannot accept; or that infinity itself is nothing, i.e. that it is not one and not a whole. Or perhaps we should say: when distinguishing between infinities, the most infinite, i.e. all the numbers, is

[1] Relevant extracts from Galileo's *Discorsi* may be found in (Leibniz 2001, pp. 352-357).
[2] At EN 78, Salviati says: "for I believe that these attributes of greatness, smallness and equality do not befit infinities, of which one cannot be said to be greater than, smaller than, or equal to another"; and again at the end of the ensuing proof: "in final conclusion, the attributes of equal, greater, and less have no place in infinities, but only in bounded quantities" (EN 79; Leibniz 2001, pp. 355-56). For Gregory of St Vincent's opinion, the Akademie editors refer us to his *Opus Geometricum*, 1647, lib. 8, pr. 1, theorema, p. 870 ff.

something that implies a contradiction, for if it were a whole it could be understood as made up of all the numbers continuing to infinity, and would be much greater than all the numbers, that is, greater than the greatest number. Or perhaps we should say that one ought not to say anything about the infinite, as a whole, except when there is a demonstration of it. (Fall 1672(?); A VI, iii, 168; *LLC*, 8-9)

Two features of this commentary are seminal for Leibniz's later thought: the upholding of the part-whole axiom for the infinite, with the entailed denial that the infinite is a whole; and the more nuanced claim that one cannot assert a property of the infinite except insofar as one has an independent demonstration of it.

The first claim is graphically illustrated by a calculation Leibniz performs in 1674, in which we are confronted with an infinite whole which has a direct visual representation: the area under a hyperbola between 0 and 1. By this time, Leibniz has made great strides in summing infinite series, and applying the results to calculate "quadratures", the areas under curves. In this example, he uses his knowledge to demonstrate the inapplicability of the part-whole axiom to infinite quantities.



In the above diagram, which Leibniz has deliberately drawn symmetrically, MGBEM is a hyperbola, and the variable abscissa DE = x (dashed line) is taken to vary between CB (taken as the x-axis) and (the horizontal) ACF... Leibniz calculates the area by "applying" DE to (the vertical) AC (= 1) as a base. He expands DE as $(1 - y)^{-1} = 1 + y + y^2 + y^3 + \ldots$, and gets the result ACBEM = $1 + {}^1/_2 + {}^1/_3 + {}^1/_4 + {}^1/_5 + \ldots$ In modern terms, Leibniz is evaluating the area under the curve $x = (1 - y)^{-1}$ by calculating the definite integral $\int_0^1 x\,dy$ using a series expansion, to obtain

$$\text{ACBEM} = \int_0^1 (1 - y)^{-1}\,dy = [y + y^2/_2 + y^3/_3 + y^4/_4 \ldots]_0^1 = 1 + {}^1/_2 + {}^1/_3 + {}^1/_4 + {}^1/_5 + \ldots$$

Now Leibniz applies the line HL = $(1 + y)^{-1} = 1 - y + y^2 - y^3 + \ldots$ to the line CF (= 1) to obtain the finite area CFGLB. That is,

$$\text{CFGLB} = \int_0^1 (1 + y)^{-1}\,dy = [y - y^2/_2 + y^3/_3 - y^4/_4 \ldots]_0^1 = 1 - {}^1/_2 + {}^1/_3 - {}^1/_4 + {}^1/_5 + \ldots$$

But if we now subtract the finite space CFGLB from the infinite space ACBEM, we obtain

$$\begin{aligned} \text{ACBEM} - \text{CFGLB} &= (1 - 1) + ({}^1/_2 + {}^1/_2) + ({}^1/_3 - {}^1/_3) + ({}^1/_4 + {}^1/_4) + ({}^1/_5 - {}^1/_5) + \ldots \\ &= 1 + {}^1/_2 + {}^1/_3 + {}^1/_4 + {}^1/_5 + \ldots \\ &= \text{ACBEM} \end{aligned}$$

Leibniz comments:

> Which is pretty amazing (*satis mirabilis*), and shows that the sum of the series 1, ${}^1/_2$, ${}^1/_3$ etc. is infinite, and consequently that the area of the space ACBGM remains the same even when the finite space CBGF is taken away from it, i.e. that nothing noticeable is taken away.

By this argument it is concluded that the infinite is not a whole, but only a fiction; for otherwise the part would be equal to the whole. (A VII 3, N. 38_{10}, p.468; October 1674)

Of interest here is the close connection between the visualizable infinite whole —the area under the hyperbola— and its expansion as an infinite series. For the infinite series is treated *as if* it were a whole, has a definite sum, and so forth. But in this case the result leads to a contradiction, given the applicability of the part-whole axiom to the infinite. Even though this establishes the fictional nature of such infinite wholes, however, this does not mean that one cannot calculate with them; only, the viability of the resulting calculation is contingent on the provision of a demonstration.

As an example, we can look at another seminal calculation Leibniz made involving the hyperbolic series, in the process of answering a challenge set him by Huygens soon after his arrival in Paris in 1672. This was to determine the sum of the series of the reciprocal triangular numbers, $^1/_1$, $^1/_3$, $^1/_6$, $^1/_{10}$, $^1/_{15}$, ... Leibniz achieved this by noting that the successive terms are twice those of the differences between successive terms of the hyperbolic series: $1 = 2(1 - {}^1/_2)$; $^1/_3 = 2(^1/_2 - {}^1/_3)$; $^1/_6 = 2(^1/_3 - {}^1/_4)$; and so forth. But the sum of a series of such difference terms will, because of the cancellation of terms, equal the difference between the first term of the hyperbolic series and the last. Leibniz immediately realized the generality of this result, which he enshrined in what I have elsewhere (Arthur 2005a) called his Difference Principle: "the sum of the differences is the difference between the first term and the last" (A VII, 3, p. 95). As is well known, after generalization to infinite series, this result will lead Leibniz to his formulation of the Fundamental Theorem of the Calculus: the sum (integral) of the differentials equals the difference of the sums (the definite integral evaluated between first and last terms), $\int Bdx = [A]_i^f$. Thus the sum of the first *n* terms of the original series,

$$\sum_{i=1}^{n} T_i = {}^1/_1 + {}^1/_3 + {}^1/_6 + {}^1/_{10} + {}^1/_{15} + \ldots + 2/(n^2 + n)$$

will be twice the difference between the $(n + 1)^{th}$ and the first term of the hyperbolic series:

$$\sum_{i=1}^{n+1} H_i = {}^1/_1 + {}^1/_2 + {}^1/_3 + {}^1/_4 + {}^1/_5 + \ldots + 1/(n + 1)$$

That is,

$$\sum_{i=1}^{n} T_i = 2(H_1 - H_{n+1}) = 2[1 - 1/(n + 1)]$$

Applying similar reasoning to infinite series, Leibniz calculates that if the sum of the infinite series $S(H) = \sum_{i=1}^{\infty} H_i = {}^1/_1 + {}^1/_2 + {}^1/_3 + {}^1/_4 + {}^1/_5 + \ldots$, and half the sum of the reciprocal triangular numbers ${}^1/_2S(T) = {}^1/_2\sum_{i=1}^{\infty} T_i = {}^1/_2 + {}^1/_6 + {}^1/_{12} + {}^1/_{20} + {}^1/_{30} \ldots$, then

$$S(H) - {}^1/_2S(T) = {}^1/_2 + {}^1/_3 + {}^1/_4 + {}^1/_5 + \ldots = S(H) - 1$$

giving

$$S(T) = 2.$$

Now one might of course object to this reasoning that, since the hyperbolic series is a diverging infinite series, Leibniz has effectively cancelled infinities in subtracting its sum $S(H) = \sum_{i=1}^{\infty} H_i$ from both sides of the equation above. Clearly this would not be news to Leibniz, given the argument for the fictionality of the infinite treated as a whole given above. And indeed, Leibniz is sensitive to the need for rigour here. He acknowledges that this application of the Difference Principle "ought to be demonstrated to come out in the infinite" (362), and proceeds to show how this can be done. This is crucial, for in the above reasoning Leibniz has treated the infinite series *as if* they are wholes. But this can only be done, according to his commentary on Galileo, if there is a corresponding demonstration. Moreover, treating the infinite series as wholes is equivalent to treating them *as if* they had last terms, since otherwise the Difference Principle would not be applicable. This gives the all-important connection between Leibniz's doctrines of the fictionality of infinite wholes and the fictionality of infinitesimals. A justification of treating the infinite series as wholes is at the same time a justification of treating them as if they had a last, infinitely small term or *terminatio*, $^1/_\infty$.

Leibniz's demonstration proceeds by taking an arbitrarily small y^{th} term as the *terminatio* of the series, where "*y* signifies any number whatever". For the hyperbolic series the *terminatio* of a finite series of terms H(y) will be $1/y$, and for the series of reciprocal triangular numbers ∑T(y), it will be $2/(y^2 + y)$, since the y^{th} triangular number is $(y^2 + y)/2$. Thus when half the series ∑T(y) is subtracted from ∑H(y), the *terminatio* of the resulting series will be $1/y - 1/(y^2 + y)$, or $(y^2 + y - y)/ (y^3 + y^2)$, or $1/(y + 1)$. But this is the *terminatio* for ∑H(y + 1). Leibniz does not complete this demonstration in these terms, preferring to proceed to a geometrical depiction in terms of a triangle, but it entails that for arbitrarily large y, the sum of half the series, $^1/_2$∑T(y), is $1 - 1/(y + 1)$. So the sum of the reciprocal triangular numbers, ∑T(y), approaches 2 arbitrarily closely as y is taken arbitrarily large. Correspondingly, the *terminatio* of either this series or the hyperbolic series is not actually 0, but an arbitrarily small number.

This, then, yields the link between Leibniz's syncategorematic interpretation of the infinite and his interpretation of infinitesimals as fictions. To treat the infinite series as whole is to treat it as if it has an infinitieth term or infinitely small *terminatio*; whereas in fact the number of terms is greater than any number that can be given, and the magnitude of the *terminatio* is correspondingly smaller than any that can be given. This connection is stated explicitly by Leibniz in an annotation he made on Spinoza's Letter on the Infinite in the Spring of 1676:

> Finally those things are *infinite in the lowest degree* whose magnitude is greater than we can expound by an assignable ratio to sensible things, even though there exists

something greater than these things. In just this way, there is the infinite space comprised between Apollonius' Hyperbola and its asymptote, which is one of the most moderate of infinities, to which there somehow corresponds in numbers the sum of this space: $^1/_1 + ^1/_2 + ^1/_3 + ^1/_4 + \ldots$, which is $^1/_0$. Only let us understand this 0, or nought, or rather instead a quantity infinitely or unassignably small, to be greater or smaller according as we have assumed the last denominator of this infinite series of fractions, which is itself also infinite, smaller or greater. For a maximum does not apply in the case of numbers. (A VI 3, 282; *LLC*, 114-115)

For some time Leibniz appears to have hesitated over this interpretation, and as late as February 1676 he was still deliberating about whether the success of the hypothesis of infinities and the infinitely small in geometry spoke to their existence in physical reality too. But by April of the same year the syncategorematic interpretation is firmly in place, and we find Leibniz exploring the implications of the rejection of the infinitely small in a series of mathematico-physical reflections.[3] During the same period he was putting the finishing touches to his treatise on the infinitesimal calculus, *De quadratura arithmetica* (Leibniz, 1993), which he submitted for the Academie Française in the summer of 1676. Here we find his first explicit exposition of the interpretation of infinites and infinitesimals as fictions, and the provision of a theorem which, in Leibniz's words, "serves to lay the foundations of the whole method of indivisibles in the soundest way possible". Indeed, as Eberhard Knobloch has remarked, this theorem is a "model of mathematical rigour" (2002, p. 72).

In the treatise Leibniz promotes his new method of performing quadratures directly "without a *reductio ad absurdum*" (Prop 7, *Scholium*; *De quadratura*, 1999, p. 35), by what we would now call a direct integral. This, he believes, necessarily involves the assumption of "fictitious quantities, namely the infinite and the infinitely small" (35). The traditional

[3] See Arthur (2005b) for a full discussion of the development of Leibniz's views on infinitesimals.

Archimedean method of demonstration was by a double *reductio ad absurdum*: it would be shown that a contradiction could be derived on the assumption that the quantity in question was smaller than a given value, and another contradiction on the assumption that the quantity in question was greater than that value, thus proving that it equalled it. Leibniz's method is instead to proceed by an application of the Archimedean Axiom. This axiom (actually due to Eudoxus) asserts that for any two geometric quantities x and y (with $y > x$), a natural number n can be found such that $nx > y$. This also entails that no matter how small a geometric quantity is given, a smaller can be found, and it is this property to which Leibniz appeals. Thus he prefers a justification "which simply shows that the difference between two quantities is nothing, so that they are then equal (whereas it is otherwise usually proved by a double reductio that one is neither greater nor smaller than the other)" (35). That is, he applies the Archimedean axiom to demonstrate that the error involved in calculations with infinitely small differences can be reduced to a quantity less than any given quantity by taking a difference sufficiently small, rendering it effectively null.

Moreover, this justification does not have to be effected in every case: it is enough to show that it can be done in a general case. This Leibniz does in a case that is surprisingly general, given the usual accusations about the parlous lack of justification he and Newton are alleged to have provided for their methods. For the key theorem that Leibniz successfully demonstrates in *De quadratura arithmetica* using this Archimedean method is what is now known as Riemannian Integration, as Eberhard Knobloch has shown in detail (2002).[4] This is Proposition 6; (Leibniz provides a similar justification for his Theorems 7 and 8). In his demonstration of Proposition 6 (1999, pp. 30-33), Leibniz first identifies and then relates the sum of the "elementary rectangles [*rectangula elementales*]" (that is the Riemannian sum of

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[4] The exposition I give is indebted to Knobloch's (2002), but in the symbolization I have followed Leibniz's own from the *De quadratura* (Leibniz 1999).

unequal rectangles by which the curve is being approximated, which we may denote Q), and that of the mixtilinear figure [*spatium gradiformis*] that is the area under the curve between two ordinates $_1L$ and $_4D$, which we may denote A. Then he demonstrates that the difference between the area and the sum of the elementary rectangles, A – Q, can be no greater than the area of a certain rectangle whose height is the maximum height □$_4$D of any of the elementary rectangles, and whose width is the distance between the two ordinates $_1L$ and $_4D$. Thus A – Q □ $_1L_4D$ □ □$_4$D. But because the curve is assumed continuous, Archimedes' Axiom applies. Thus the height □$_4$D, even though it is greater than the heights of all the other elementary rectangles, "can be assumed smaller than any assigned quantity, for however small it is assumed to be, still smaller heights could be taken." Therefore the area of the rectangle $_1L_4D$□ "can also be made smaller than any given surface". Thus the difference A – Q too "can also be made smaller than any given quantity. QED." (pp. 30-33) There is therefore no error involved in calculating the quadrature as the sum of an infinity of infinitesimal areas, provided this is understood to mean that there are more little finite areas than can be assigned, and that their magnitude is smaller than any that can be assigned.

The point here is not that Leibniz has two methods, one committed to the existence of infinitesimals and the other Archimedean;[5] nor is it the case that he simply uses the infinitesimal calculus and then airily refers to the fact that one could *instead* have used an Archimedean method. It is that, as examples like this demonstrate, the Archimedean Axiom justifies proceeding as if there are infinitesimals, and at the same time demonstrates that what they really stand for are finite quantities which can be taken as small as desired. Once this is demonstrated in a suitably general case, it also justifies the use of these fictions in other analogous cases. As Leibniz himself writes, "Nor is it necessary always to use inscribed or circumscribed figures, and to infer by *reductio ad absurdum*, and to show that the error is

[5] Cf. Bos (1974/75) on "Leibniz's two different approaches to the foundation of the calculus" (p. 55).

smaller than any assignable; although what we have said in *Props. 6, 7 & 8* establishes that it can easily be done by those means." (Scholium to *Prop. 23*, Leibniz 1993, 69)

In effect, the application of the Archimedean Axiom enables a kind of Arithmetic of the Infinite. In his article, Knobloch identifies a number of rules which are tacitly applied by Leibniz in *De quadratura*, "without demonstrating them, only relying on the 'law of continuity'" (2002, 67). Examples are "1. Finite + infinite = infinite", "2.1 Finite ± infinitely small = finite", "2.2 x = (y + infinitely small) □ $x - y$ □ 0 (is unassignable)". 2.2 can be demonstrated by Leibniz's method as follows. Suppose $x = y + dx$, where $dx > 0$, and suppose dx is actually infinitely small, i.e. smaller than any given difference, yet is a geometric quantity. It will then have an assignable ratio to another geometric quantity z such that, by the Archimedean Axiom, a number $n > 0$ can be chosen to make $nz = dx$. Therefore z will be dx/n and thus smaller than dx, contrary to supposition. Therefore $x - y = dx$ is unassignable; the difference between x and y is smaller than any quantity assignable; it is incomparable with (has no finite ratio to) any finite quantity. What this means is that there can be no assignable error in equating $x - y$ with 0. Similar demonstrations can be given for the other rules.

The full "Law of Continuity" that Leibniz will publicize for the first time in 1688 is an attempt to codify the method that we have just been examining:

> *When the difference between two instances in a given series, or in whatever is presupposed, can be diminished until it becomes smaller than any given quantity whatever, the corresponding difference in what is sought, or what results, must of*

necessity also be diminished or become less than any given quantity whatever. (A VI 4, 371, 2032)[6]

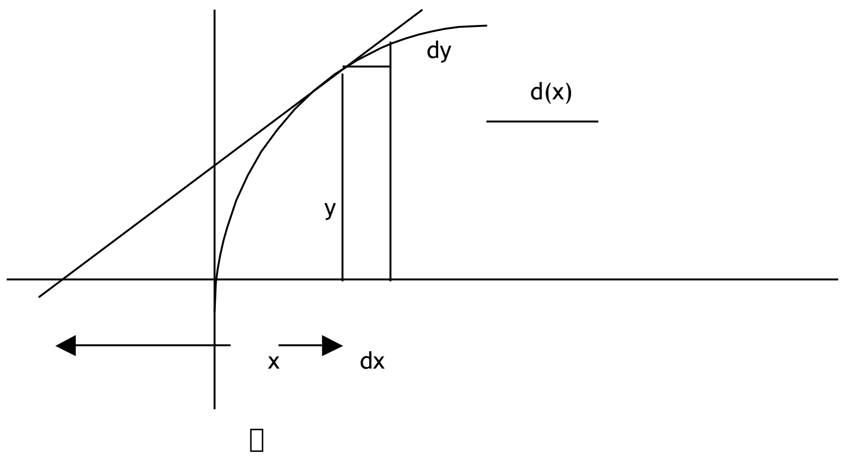
But in a second formulation it also appears to be a generalization of the approach to infinite series we examined above, where the infinite limiting term or *terminatio* is included as if it were an infinitieth term in the series:

> If any continuous transition is proposed that finishes in a certain limiting case (*terminus*), then it is possible to formulate a general reasoning which includes that final limiting case.[7]

It is the Law of Continuity in this form that forms the basis of Leibniz's later attempts to ground the rules of the calculus. His first attempt, "*Nova Methodus pro Maximis et Minimis*" (1684), was not clear to his contemporaries, and was in any case vitiated by a number of errors. He made another attempt in a paper drafted around 1701, "*Cum prodiisset*", which was first published by Gerhardt in 1846. The contents of this paper have been lucidly explained by Henk Bos in his classic article on Leibniz's calculus (1974/75). In it the differentials dx and dy are finite, arbitrarily small, and variable: they are neither fixed quantities, nor infinitely small ones. Leibniz proceeds by letting $(d)x$ be a fixed finite line segment, and then defining another segment $(d)y$, for all finite dx and $dy > 0$, by the proportion

$$(d)y:(d)x = dy:dx \qquad (2.1)$$

[6] Leibniz adds: "Or, to put it more commonly, *when the cases (or given quantities) continually approach one another, so that one finally passes over into the other, the consequences or events (or what is sought) must do so too.*" (A VI 4, 371, 2032; translations mine.)
[7] Translated from the Latin quotation from (Leibniz, 1701, 40) given by Bos (1974/75, 56).



But the same *(d)y* can also be given an interpretation in the limit when the variable *dx* = 0, namely through the proportion

(d)y:(d)x = *y*: (2.2)

where is the subtangent to the curve.[8]

Now, since the resulting formula is still interpretable even in the case where *dx* = 0, the Law of Continuity asserts that this limiting case may also be included in the general reasoning: *dy*:*dx* can be substituted for *(d)y:(d)x* in the resulting formulas even for the case where *dx* = 0, with *dy* and *dx* in this case interpreted as fictions. If a third variable *v* is involved, which varies with *x*, *(d)v:(d)x* can be defined in an entirely analogous way.

That this foundation suffices for first-order differentials and the rules of the calculus is best shown by an example. In *Cum prodiisset*… Leibniz offers the following proof of the rule for the differentiation of a product *d(xv)* = *xdv* + *vdx*. He lets *ay* = *xv* (here the purpose of the constant *a* is to conserve the homogeneity of the equation), and then allows *x*, *y*, and *v* all to increase infinitesimally. Following Bos, I quote Leibniz's own demonstration:

[8] As Bos reports (1974-75, 62-63), this idea at the foundation of Leibniz's justification of the calculus in *Cum prodiisset*", defining the ratio of *dy* and *dx* at the beginning of the interval by means of a proportion to finite line segments, is also found in his first publication on the calculus (1684).

Proof: $ay + dy = (x + dx)(v + dv)$

$$= xv + xdv + vdx + dxdv,$$

and, subtracting from each side the equals *ay* and *xv*, this gives

$$ady = xdv + vdx + dxdv$$

or $$ady/dx = xdv/dx + v + dv$$

and transposing the case as far as possible to lines that never vanish, this gives

$$a(d)y/(d)x = x(d)v/(d)x + v + dv$$

so that the only remaining term which can vanish is *dv*, and in the case of vanishing differences, since $dv = 0$, this gives

$$a(d)y = x(d)v + v(d)x$$

as was asserted. … Whence also, because $(d)y:(d)x$ always = $dy:dx$ one may assume this in the case of vanishing *dy, dx* and put

$$ady = xdv + vdx.$$ [9]

As a second example of this approach, it will be instructive to examine Leibniz's criticisms of Newton's proof of his Lemma 9 in the *Principia* and the demonstration he offers in its stead. For, as Leibniz correctly recognized, Lemmas 9-11 are crucial for Newton's proofs of the inverse square law; Lemma 10 in particular, is a corollary of Lemma 9, and is appealed to by Newton (in the first edition of the *Principia*) in the proof of Proposition 6. Each of Newton's lemmas is an instance of his Method of First and Last Ratios, and establishes a ratio between quantities as they are on the very point of being generated or vanishing. The figures, accordingly, are "ultimate" ones, depicting what Whiteside has aptly termed "limit

[9] Leibniz, "*Cum prodiisset*", pp. 46-47; quoted from Bos (1974-75, 58). The ellipsis omits an obvious error in calculation not important for the argument.

motions",[10] the motions a body would undergo during a "moment" if it continued with the velocity it had at the beginning of that moment. Leibniz had no objection to this procedure; but he did object to what he found in Proposition 6, namely the treatment of a curved trajectory in such an ultimate moment as compounded from two motions, one a rectilinear inertial motion along the tangent, and the other an acceleration towards the centre. This was an objection of his of long standing, and is closely related to his syncategorematic understanding of moments and of infinite division. In a piece written at the beginning of April 1676, he reasoned that if there were such a thing as a perfect fluid —by which he meant matter divided all the way down into points, each individuated by a differing motion— then "a new endeavour [would] be impressed at any moment whatever" on a body moving in the fluid. But this would be to compose a curved line form points and time from instants, and would also entail an "impossible" composition at every single instant:

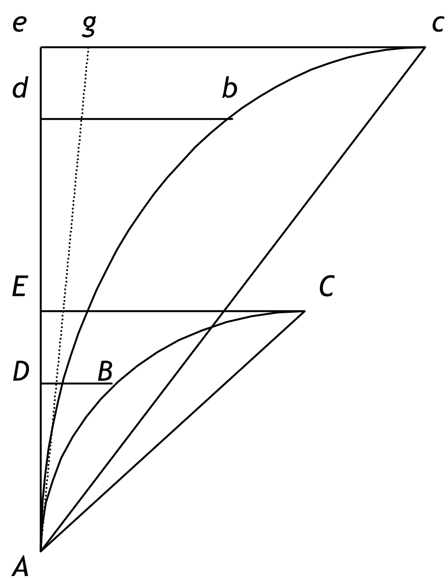
> But if this is conceded, time will actually be divided into instants, which is not possible. So there will be no uniformly accelerated motion anywhere, and so the parabola will not be describable in this way. And so it is quite credible that circles and parabolas and other things of that kind are all fictitious.... For supposing a point moves in a parabolic line, it will certainly be true of it that at any instant it is moving with a uniform motion in one direction, and with a uniformly accelerated motion in another, which is impossible. (A VI 3, 492; *LLC*, 74-77)[11]

When he first confronted Newton's *Principia* some 12 years later, Leibniz's initial reaction was therefore to believe that there was a mistake in Newton's composition of motions in deriving Proposition 6. On his understanding, in an ultimate moment only straight line motions

[10] Whiteside 1967, p. 154 and *passim*.
[11] Here Leibniz appears to be applying a principle he had just formulated: "If a given motion can be resolved into two motions, one of them possible and the other impossible, the given motion will be impossible." (A VI 3, 492; *LLC*, 72-73)

(and equivalently the lines they would traverse in such a moment) can be compounded, since ultimately the curve is resolved into an infinite-sided polygon with fictional straight sides, each one representing a geometric "indivisible" or ultimate difference between successive values. Seeing the dependence of Proposition 6 on Lemmas 9 and 10, therefore, he concentrated his attention on them.

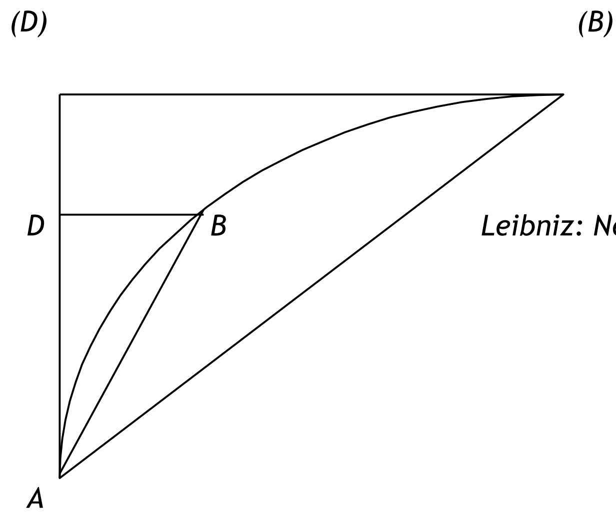
In Lemma 9, Newton has a curved trajectory depicted in an ultimate moment, and a claim is made that "the areas of the [curvilinear] triangles *ADB* and *AEC* will ultimately be to each other in the squared ratio of the sides", i.e. $\Delta ADB:\Delta AEC = AD^2:AE^2$ (1999, 437). Newton performs his proof by using his Method Of Finite Surrogates: here these surrogates are the lines *Ae* and *Ad*, which are finite, with *Ae* remaining fixed, and in proportion to *AD* and *AE*, which are supposed variable; likewise *Abc* is a fixed portion of curve similar to *ABC*, with the elements of the curve *AB* and *AC* supposed variable, gradually shrinking toward *A* until *B* and *C* "come together" with it. The line *AGg* is tangent to both curves at *A*. Newton's proof then proceeds by noting that in the limit *A*, *B* and *C* coincide, $\angle cAg$ vanishes, and "the curvilinear areas *Abd* and *Ace* will coincide with the rectilinear areas *Afd* and *Age*, and thus (by Lemma 5) will be in the squared ratio of these sides *Ad* and *Ae*" (437). Therefore, given the assumed proportionality between these finite surrogates and the infinitesimals, "the areas ABD and ACE also are ultimately in the squared ratio of the sides *AD* and *AE*. QED" (437).



Newton's Lemma 9

Leibniz's reaction to Newton is fraught with multiple ironies. One of them is his not noticing that Newton's method of finite surrogates is to all intents and purposes identical to his own way of justifying the calculus, as he would later adumbrate it in "*Cum prodiisset*", explained above.[12] For in his notes he dispenses with the finite surrogates completely, so as to concentrate on the curvilinear figures that Newton has depicted in his ultimate moment. (This irony is compounded by the fact that Leibniz rather arbitrarily replaces Newton's letters *E* and *C* by (*D*) and (*B*), the same bracket notation he usually uses for finite lines.) Here is his diagram, statement and criticism of Newton's Lemma 9, following the description given by Domenico Bertoloni Meli:

[12] One's first thought would be that perhaps Leibniz had cribbed the method from his reading of Newton; in his excellent discussion Guicciardini (1999, 161) draws attention to some other wording of Leibniz's later justifications that show Newton's influence. But the dates undermine this more cynical view about Leibniz's "finite surrogates": in his first publication on the calculus (1684), Leibniz had promoted his differentials as finite variable quantities bearing a proportion to fixed lines (albeit in a form that was not easy for his contemporaries to understand). See Bos (1974/75, 57, 63).



Leibniz: Newton's Lemma 9

> If *AB*, *A(B*)are unassignable, △*ADB* and △*A(D)(B)* will be in the squared ratio of the sides (+ namely, it must be understood that the △s are similar, and so angle *BA(B)* has a ratio to angle *DAB* that is infinitely small, but this does not seem to be true, since in △*AB(B)* the side *B(B)* has an assignable ratio to *AB* or *A(B*), and so to *AD* and *DB*.+)

That is, Leibniz objects that in the limit △s *ADB* and *A(D)(B)* will not be similar, since Newton's diagram *already* represents the ultimate moment, and in it angle *BA(B)* has a finite rather than infinitesimal ratio to the sides of the triangles. In an effort to show that Newton has made an error in calculating with infinitesimals, Leibniz proceeds with his own calculation, in which the infinitesimal order of the angle *BA(B)* with respect to *AB* and *AD* is rendered explicit by setting *AD* = *x*, *DB* = *y*, *A(D)* = *x* + *dx*, and *D(B)* = *y* + *dy*. Leibniz does not succeed in demonstrating any error on Newton's part; nor in fact is there an error, as Meli observes, since as *x* ends to zero so does *dx*, so that ultimately the triangles become similar. Nevertheless, even though he fails to show anything wrong in Newton's procedure, Leibniz's workings provide us with a nice illustration of his approach to the differential calculus. First he expresses *y* as a series expansion in *x*: $y = a + bx + cx^2 + ex^3 + ...$ Now since *x* = *0* when *y* = *0*, we have *a* = *0* —a point Leibniz first realized, and then subsequently forgot, rendering his calculation inconclusive. But as Meli shows, the lemma follows fairly easily from these premises. For

$$xy = bx^2 + cx^3 + ex^4 + \ldots \qquad (2.3)$$

so that since (assuming Lemma 5) area[ADB] = $AD \bullet DB/2$,

$$\text{area}[ADB] = xy/2 = {}^1/_2(bx^2 + cx^3 + ex^4 + \ldots) \qquad (2.4)$$

Meli remarks that "If x becomes infinitesimal, the terms in x^3, x^4, ...are negligible, and the area of ADB (= $xy/2$) is indeed proportional to AD^2 (= x^2)" (242). But in fact we can be more rigorous. For although Leibniz apparently thought Newton's finite surrogates could be dispensed with, they are completely in accord with his way of proceeding. Thus let $DB:AD$ = $db:Ad$, etc., as Newton had assumed, with db and Ad finite, and with this ratio remaining in a finite and non-vanishing proportion to ec and Ae (with Ae fixed) even as $x \to 0$. Then

$$DB/AD = AD \bullet DB/AD^2 = b + cx + ex^2 + \ldots \qquad (2.5)$$

But since this equals $db:Ad$, which is finite and non-vanishing as $x \to 0$, it follows that

$$2 \times \text{area}[ADB)]/AD^2 = AD \bullet DB/AD^2 = b \text{ (const.)} \qquad (2.6)$$

The same reasoning will apply for $\Delta A(D)(B)$, giving

$$\text{area}[ADB)]:\text{area}[A(D)(B)] = AD^2:A(D)^2 \qquad (2.7)$$

Newton's Lemma is therefore sound even by Leibniz's lights, as indeed he came to realize despite not having successfully completed this calculation.[13] Much the same applies to Lemma

[13] As Meli reports (Bertoloni 1993, 103), Leibniz's attitude seems to have undergone a sea change on moving on from Section 1 of the *Principia* on First and Last Ratios to section 2, on the determination of central forces. Accordingly, in the first set of *Excerpts* Leibniz took from the *Principia* the following

11, stating that the curvature of a trajectory at a given point is the same as the curvature of the osculating circle at that point, a result that will be of importance to us below. In the first stage of composition of his *Marginalia* identified by Meli, Leibniz objects to Newton's Corollary 7 to Proposition 4, in which the proportionality of centripetal forces to v^2/r, proved in Proposition 4 for bodies rotating uniformly in circular orbits, is extended to non-circular orbits for limit-motions: "Since I do not yet accept the generality of Lemma 11, I also doubt the generality of this corollary 7". Somewhat later, according to Meli, he corrected himself: "On the contrary, this is true, because the considerations on the secant of the angle made by the radius from the centre to the curve, and that of the radius of the osculating circle to the curve, vanish on account of the similarity of the figures."[14] That is, according to Meli's analysis, "since Proposition 4 is stated in the form of a proportion between the homologous elements of two similar figures, their similarity cancels out the dependence of paracentric conatus [—the endeavour toward the centre—] on the secant of the angle and on the osculating radius" (Meli, p. 107). What all this means, in effect, is that in calculations of centripetal force, in the limit the force can be treated as if directed towards the centre of the osculating circle and the radius (or distance from the centre of force to the curve) may be set equal to the radius of that circle. (We shall exploit this result in proving Proposition 6 below.)

year, "lemma 9 is transcribed without commentary, and seems to be accepted without difficulty" (242).

[14] *Marginalia*, M 42 A: translated from the Latin quoted by Meli, p. 107.

3. Smooth infinitesimal analysis

Smooth Infinitesimal Analysis has many features in common with Leibniz's approach. It begins, like Leibniz, by eschewing the composition of the continuum from an infinity of points. In contrast to the account of the continuum established by Cantor based on Point Set Theory, according to which a continuous line is an infinite (indeed, nondenumerably infinite) set of points, SIA is rooted in Category Theory, and mappings rather than points are taken as basic. The category of smooth manifolds, ***Man***, is embedded in an enlarged category ***C*** which contains "infinitesimal" objects, and a topos $\boldsymbol{Set}^{\boldsymbol{C}}$ is then formed of sets varying over ***C***. "Each smooth topos ***E*** is then identified as a certain subcategory of $\boldsymbol{Set}^{\boldsymbol{C}}$. Any of these toposes has the property that its objects are undergoing a form of smooth variation, and each may be taken as a smooth world." (Bell 1998, 14). With this foundation secured, a great many results of differential calculus may be obtained "—with full rigour— using straightforward calculations with infinitesimals in place of the limit concept" (Bell, 1998, 4), as if infinitesimals exist.

Here I say "as if infinitesimals exist" advisedly. For another feature that SIA has in common with Leibniz's approach is that, whilst it licenses certain infinitesimal techniques, it is not committed to the existence of infinitesimals in the continuum. That is, as in Leibniz's theory, infinitesimals are *fictions* in a precisely defined sense. The sense in which they are fictions in SIA, however, is that although it is denied that an infinitesimal neighbourhood of a given point, such as 0, reduces to zero, it cannot be inferred from this that there exists any point in the infinitesimal neighbourhood distinct from 0. Thus the Law of Excluded Middle, and with it the Law of Double Negation, both fail in smooth worlds. Bell explains this as follows. Define two points *a* and *b* on the real line (as represented in a smooth world □) as *distinguishable* iff they are not identical, i.e. iff □$a = b$, where '=' denotes identity. Now define the *infinitesimal neighbourhood □(0)* of a given point 0 as the set of all those points indistinguishable from 0. That is, define □(0) as follows:

$$\square(0) =_{def} \{ x \mid \square\square x = 0] \qquad (3.1)$$

Now if the Law of Double Negation (or, equivalently, the Law of Excluded Middle) held in □, we could infer that *x* = 0 for each *x* in □(0), so that the infinitesimal neighbourhood of 0 □(0) would reduce to {0}. But we know that this neighbourhood does not reduce to {0} in □. So we cannot infer the identity of points from their indistinguishability. Again, suppose there is a point *a* in □ that is distinguishable from 0, i.e. suppose there is a point *a* □ □ such that □*a* = 0. But since *a* □ □ □□*a* = 0 by definition. But this is a contradiction. Therefore it is not the case that there exists a point *a* in □ that is distinguishable from 0. That is, the logic of smooth worlds is intuitionistic logic: the Law of Noncontradiction holds, as we have just seen; but the Laws of Excluded Middle, Double Negation, and one form of Quantifier Negation do not hold in smooth worlds. From □(□*x* □ □)*x* = 0 (it is not the case that all members of □ are identical with 0), we can not infer that (□*x* □ □)□*x* = 0 (there is a member of □ that is distinguishable from 0).[15] This makes precise the older conception of an infinitesimal difference as a difference smaller than any assignable, but not zero. It does so, in effect, by denying that an unassignable difference reduces to zero, but not allowing the inference from this that there exists an unassignable difference different from zero. it is in this sense that the infinitesimal intervals of SIA are fictional.

On this foundation Bell erects the theory of SIA. First he defines □ as consisting in those points *x* in R such that $x^2 = 0$. The letter □ then denotes a variable ranging over □ (21). The fundamental assumption is then that every curve is microstraight (pp. 9, 22). That is,

[15] More precisely, the logic of smooth toposes is *free first-order intuitionistic* or *constructive logic*. See Bell 1998, 101-102.

arbitrary functions f: R □ R are assumed to behave locally like polynomials, so that with $f(x) = a_0 + a_1x + a_2x^2 + \ldots a_nx^n$ and □^{2} = 0, we have

$$f(□) = a_0 + a_1□ \text{ for any } □ \text{ in } □. \qquad (3.2)$$

This is the *Principle of Microstraightness*, and it implies that (□,f(□) lies on the tangent to the curve at the point (0,a_0). It also entails (if one considers only the restriction of g of f to □) that g is affine on □. That consideration motivates the *Principle of Microaffineness* (p. 23):

For any map g: □ □ R, there exists a unique b in R such that, for all □ in □, we have

$$g(□) = g(0) + □b. \qquad (3.3)$$

This principle allows one to define the *derivative* of an arbitrary function f: R □ R as follows (p. 26). Define the function g_x(□) = $f(x$ + □). By Microaffineness it follows that there is a unique b_x such that for all □ in □,

$$f(x + □) = g_x(□) = g_x(0) + □b_x = f(x) + □b_x \qquad (3.4)$$

If we allow x to vary, the values b_x will constitute a new function, the derivative $f'(x)$:

$$f(x + □) = f(x) + □f'(x) \qquad (3.5)$$

If the function f is a function of time, and □ is an infinitesimal time, we have as a direct consequence of this the *Principle of Microuniformity* (of natural processes): any such process may be considered as taking place at a constant rate over the "timelet" □ (p. 9). Another important consequence of Microaffineness is the *Principle of Microcancellation*: *for any a,b in R, if □a = □b for all □ in □, then a = b* (p. 24). I quote Bell's proof:

Suppose that, for all $\square$ in $\square$, all $\square a = \square b$ and consider the function g: $\square$ $\square$ R defined by $g(\square) = \square a$. The assumption then implies that g has both slope a and slope b: the uniqueness clause in Microaffineness yields $a = b$. (p. 24)

I will now illustrate the neatness and simplicity of SIA by means of two examples: (i) the proof of the product rule for derivatives, and (ii) the proof of Newton's Proposition 1 of the *Principia*, the Kepler Area Law. For both of these illustrations I shall follow Bell's exposition.

(i) *The Product Rule for Derivatives*

Let y(x) = f(x)g(x). We wish to prove that $y' = f'g + fg'$. By the definition of the derivative,

$$f(x + \square) = f(x) + \square f'(x), \tag{3.6}$$

and likewise for g and y. Thus since

$$y(x + \square) = f(x + \square) \bullet g(x + \square), \tag{3.7}$$

we have

$$y(x) + \square y'(x) = [f(x) + \square f'(x)] \bullet [g(x) + \square g'(x)] \tag{3.8}$$

$$= y(x) + \square [f'(x)g(x) + g'(x)f(x)] + \square^2 f'(x)g'(x) \tag{3.9}$$

Now subtracting y from both sides and assuming $\square^2 = 0$, we have

$$\square y'(x) = \square [f'(x)g(x) + g'(x)f(x)], \text{ and thus} \tag{3.10}$$

$$y'(x) = f'(x)g(x) + g'(x)f(x) \tag{3.11}$$

The inference from (3.10) to (3.11) is guaranteed by the Principle of Microcancellation: if $\square a = \square b$ for all $\square$ in $\square$, then $a = b$. As we saw, this in turn depends on the nilsquare property $\square^2 = 0$. So in the proof, this property is invoked three times: indirectly in the definition of the derivative (3.6), in the inference from (3.9) to (3.10) directly, and again indirectly in proving the Microcancellation property involved in the inference from (3.10) to (3.11).

(ii) *Newton's Proposition 1, the Kepler Area Law*

In his proof of this law, Bell assumes that the area A, the radius r and the angle θ are all functions of t, which increases by a nilsquare infinitesimal ε "with ε in Δ". "Then by Microstraightness the sector *OPQ* is a triangle of base $r(t + \varepsilon) = r + \varepsilon r'$ and height

$$r \sin[\theta(t + \varepsilon) - \theta(t)] = r \sin \varepsilon\theta' = r\varepsilon\theta' \text{" (Bell , 69)} \qquad (3.12)$$

Here the nilsquare property is invoked in Microstraightness, in the definitions of the derivatives r' and θ', and again in the equating of $\sin\varepsilon\theta'$ with $\varepsilon\theta'$. Since the area is half the base times the height, this gives for the area of sector *OPQ*

$$OPQ = {}^1/_2(r + \varepsilon r')r\varepsilon\theta' = {}^1/_2\varepsilon r^2\theta' \qquad (3.13)$$

if the term ${}^1/_2\varepsilon^2 rr'\theta'$ is dropped, again invoking the nilsquare property. But the area *OPQ* is the increment in area produced by the motion of the radius vector,

$$OPQ = A(t + \varepsilon) - A(t) = \varepsilon A'(t) \qquad (3.14)$$

so that (3.8) and (3.9) give, by Microcancellation,

$$A'(t) = {}^1/_2 r^2\theta' \qquad (3.15)$$

From this Bell then proves that $A''(t) = 0$, so that, assuming $A(0) = 0$, we have the Area Law

$$A(t) = kt, \qquad (3.16)$$

where k is a constant.

As a historical note, we may remark that Bell's procedure is closely related to that pioneered by the 17th C Dutch mathematician Bernard Nieuwentijt.[16] In his treatment (which was independent of Newton's and Leibniz's) Nieuwentijt laid down a number of axioms. The

[16] See the succinct account of Nieuwentijt's theory in Mancosu (1996, p. 158ff).

first states (in Mancosu's rendering) that "anything that when multiplied, however many times, cannot equal another given [finite] quantity, however small, cannot be considered a quantity —geometrically, it is a mere nothing." (159) Axiom 2 states that any arbitrary finite quantity can be divided into arbitrarily many equal or unequal parts less than any given quantity, so that the division of a finite quantity b by an infinite number m yields an infinitesimal quantity. (This is in accord with Axiom 1, since b/m may be multiplied by the product of the infinite number m and the finite number c/b so that it does equal any other finite quantity c.) But it now follows (lemma 10) that if two infinitesimal quantities b/m and c/m are multiplied together, their product bc/mm is zero. For when multiplied by the largest possible number m, the product bc/m is still infinitesimal, and therefore cannot be made equal to any other finite quantity; by Axiom 1, the product of any two infinitesimal quantities is therefore zero. Nieuwentijt's infinitesimals are nil-square infinitesimals.

There are, of course, many profound differences between SIA and Nieuwentijt's approach. For SIA, like Leibniz's approach, is based on smoothly varying geometric quantities, whereas Nieuwentijt's infinitesimals are defined through division by an infinite number m, where m is the largest number. SIA presupposes intuitionistic logic, whereas Nieuwentijt's logic is classical. Nieuwentijt's theory does not admit higher order differentials, whereas, as we shall see, this is possible in SIA. But the key points it has in common with Nieuwentijt's approach are these: the Principle of Microaffineness, which "entails that all curves are 'locally straight'" (110); and the Nilsquare Property, which guarantees that all squares and higher powers of infinitesimals are intrinsically zero, and not just by comparison with other quantities.[17]

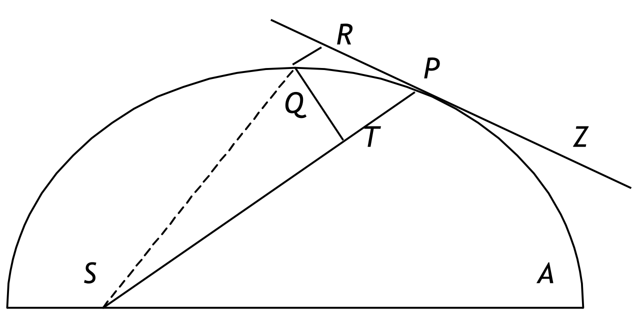
[17] Mancosu has very aptly summarized the difference between Nieuwentijt's approach and Leibniz's on p. 160. One of these differences is that on Nieuwentijt's theory, the infinitesimal b/m cannot be eliminated from computations; but Bell's Principle of Microcancellation shows how to circumvent this problem. Bell has provided a useful comparison of the difference between SIA and Nonstandard Analysis in his book on p. 110.

4. PROPOSITION 6 OF NEWTON'S *PRINCIPIA*

The comparison between Leibniz's syncategorematic approach and that of SIA can be set in sharp relief by applying both to the same example. Because of its historical importance, and also because it gave both Leibniz and Varignon some measure of grief[18] and therefore presents itself as an excellent touchstone, I shall take Newton's Proposition 6 of Book 1 of the *Principia*, the theorem that is the basis of his derivation of the inverse square law of attraction due to gravity, together with its application to find an expression for the centripetal force in terms of the instantaneous velocity v and radius r. I shall first present Newton's statement and his own proof of the proposition (from the first 1687 edition of the *Principia*) with some commentary. Then I will proceed to a proof of the proposition using the infinitesimal methods of Leibniz, following the method outlined by Nico Bertoloni Meli in his proof of Proposition, and showing how it is justifiable on the syncategorematic interpretation of infinitesimals. Then I will proceed to a consideration of how a proof might be effected using SIA. We will see that this is not as straightforward as might be supposed from Bell's presentation of SIA and his proof of Proposition 1 above; and this in turn will lead to some reflections on the relationship between SIA and Leibniz's approach.

Newton states Proposition 6 as follows:

Figure 2



[18] On Varignon and Leibniz on central force see Bertoloni Meli (1993, 81-83, 201ff.).

> If a body *P*, revolving about a centre *S*, describes any curved line *APQ*, while the straight line *ZPR* touches that curve at any point *P*; and to the tangent from any other point *Q* of the curve, *QR* is drawn parallel to the distance *SP*, and the perpendicular *QT* is dropped onto the distance *SP*: I say that the centripetal force is as the reciprocal of the solid $SP^2 \bullet QT^2/QR$, provided that the quantity of this solid is always taken as that which is made when the points *P* and *Q* come together.

In the first edition[19] Newton gives the following proof:

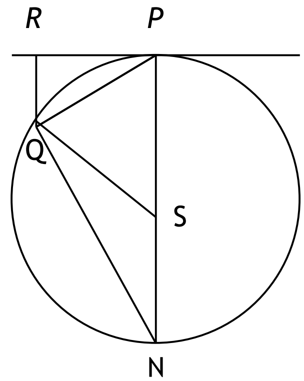
> For in the indefinitely small figure *QRPT* the nascent linelet *QR*, if the time be given, is as the centripetal force (by law 2), and if the force be given, as the square of the time (by lemma 10), and thus, if neither be given, as the centripetal force and the square of the time jointly, and thus the centripetal force is as the linelet *QR* directly and the square of the time inversely. But the time is as the area *SPQ*, or its double *SP* □ *QT*, that is, as *SP* and *QT* jointly, and thus the centripetal force is as *QR* directly and SP^2 □ QT^2 inversely, that is, as $SP^2 \bullet QT^2/QR$ inversely. QED.

We may now determine an expression for the centripetal force in an infinitesimal timelet *dt*, or moment *o*, in terms of *v* and *r* as follows. In the limit, the curvature of the arc *QP* is represented by the curvature of the corresponding osculating circle: this means that *SP* can be taken as the radius of this circle.[20] As is clear from the figure below, if *NP* = *2SP is* the diameter of he circle, then □*NQP* is a right angle, so that *QR*:*PQ* = *PQ:NP* = *PQ:2SP*, giving

$$QR = PQ^2/2SP \qquad (4.1)$$

[19] As Cohen and Whitman explain, Proposition 6 of the 1st edition becomes Corollary 1 of Proposition 6 in the 2nd and 3rd editions (453-54).

[20] See discussion above; see also Newton's Corollary 7 (8 in the 2nd and 3rd editions) to Proposition 4: "And the application is made by substituting the uniform description of areas for uniform motion, and by using the distances of bodies from the centres for the radii." (Newton, *Principia*, Cohen and Whitman, p. 451.)



But in the moment o (i.e. dt) the velocity v is as PQ and $SP = r$, so the centripetal force F is as

$$QR/o^2 = (PQ^2/o^2)/2SP = {}^1/_2(v^2/r) \qquad (4.2)$$

In fact, the factor of proportionality $^1/_2$ cancels, since in modern terms $QR = {}^1/_2 a\, dt^2$. Thus

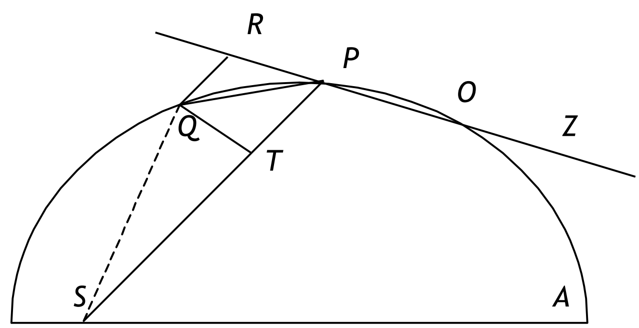
$$F/m = v^2/r \qquad (4.3)$$

As explained above, Leibniz could not accept the validity of Newton's demonstration of Proposition 6, involving as it did a composition of a uniform motion with a uniformly accelerated motion. Subsequently he came to see that Newton's way of applying his Lemmas to limit-motions involved no mathematical error, but still preferred a composition of two uniform motions, rejecting the physicality of Newton's instantaneous acceleration.[21] The precise development of Leibniz's views from his initial reactions to Newton's *Principia* to the development of his own rival dynamics of celestial motion has been analysed in a brilliant and careful study of the surviving documents by Bertoloni Meli (1993). As Meli has shown with respect to Newton's Proposition 4, the two ways of analyzing the composition of motion, Newton's composition of a rectilinear uniform motion along the tangent with a uniform acceleration towards the centre, and Leibniz's composition of a rectilinear uniform motion along the chord with a rectilinear uniform motion towards the centre, each lead to the same mathematical result (pp. 78-84). Following his

[21] "La voye est plus simple," Leibniz wrote to Varignon in October 1706, "qui ne met pas l'acceleration dans les elemens, lorsqu'on n'en a point besoin. Je m'en suis servi depuis de 30 ans." (GM IV 150-151; Bertoloni Meli, 1993, 81).

lead, I shall sketch here a slightly modernized account of a Leibnizian proof of Proposition 6 using Leibniz's own preferred composition (which Newton had also followed in Proposition 1), where the body undergoes an inertial motion between successive impulses so as to arrive at successive points on the curve, with the inertial and impulse motion composed to form the new resultant motion in each case.

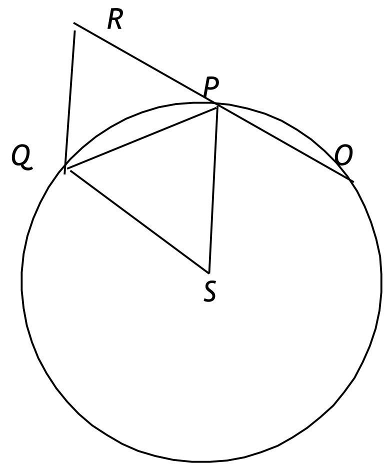
Figure 3



As in Proposition 1, we suppose the deflections to occur at equal time intervals, *dt*, so that if the uniform velocity along *PQ* is *v*, *PQ* = *vdt*. Likewise, if the deflection due to the impulse is *QR*, and the velocity along *QR* parallel to *SP* is *w*, *QR* = *wdt*. Now we need to find a relation between *QR*, *PQ*, and the radius *SP*. This can be achieved by noting that the curvature of the arc *QP* can be taken in the limit to be represented by the curvature of the osculating circle. This is the substance of Newton's Lemma 11, mentioned above. Leibniz had eventually concluded that Lemma 11 was correct, so that, as stated above, the force can be treated as directed towards the centre of the osculating circle with *SP* taken as its radius.[22] As is clear from the figure below, with *QR* and *SP* parallel and equal, *QR*/*PQ* = *PQ*/*SP*, so that

[22] Cf. Leibniz, *De conatu*: "Let E be the centre of the osculating circle to the curve at CC; it is clear that the angle $_2CE_3C$ is equal to the angle of deflection K_2C_3C. But the elements of the curve are in the compound ratio of the osculating radii EC, and of the angles of deflection. Therefore the angles of deflection are in the compound ratio of the elements directly and the osculating radii reciprocally. Therefore finally *the paracentric endeavours are in the squared ratio of the velocities compounded directly with the simple ratios of the secants which the angles of the radii from the centre of*

$$QR = PQ^2/SP. \qquad (4.4)$$



But SP is finite and PQ is first order infinitesimal; therefore QR must be second order infinitesimal. Thus the velocity w, although constant because it represents a uniform motion, must be a first order differential du, so that

$$QR = w \bullet dt = du \bullet dt = d(dr/dt) \bullet dt = ddr \qquad (4.5)$$

Meanwhile

$$PQ^2/SP = v^2dt^2/r \qquad (4.6)$$

Combining (4.4), (4.5) and (4.6) we have

$$ddr = v^2dt^2/r \qquad (4.7)$$

As Leibniz would express this, the solicitation in equal infinitesimal times dt is as v^2 and inversely as r. Leibniz's notion of solicitation, as Meli explains very clearly, is not to be confused with Newton's acceleration: solicitation is as an element of velocity (here du), velocities are as u = □du, and forces as □du or u^2 (p. 88). We shall return to this point later.

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endeavour have to the curve, taken with the osculating radii reciprocally." (See Meli, 1993, 255, 260.) The result is repeated by Leibniz in "*Inventum a me est*" (265).

This addresses the mathematical physics. But a crucial question remains: is Leibniz entitled to his second order differentials, given his understanding of first order differentials as fictions? It is known that he had some confusions about second order differentials, mistakenly believing that they can be taken as elements of first order infinitesimals in the same way that first order infinitesimals can be taken as elements of finite quantities, and not appreciating that relations involving them in general depend on a specification of the progression of the variables (or, equivalently, on the choice of an independent variable). Given this, is it possible to allow terms like *ddr* above? Can it be given a syncategorematic interpretation analogous to the one we saw for first order differentials above? The answer to these questions has already been provided by Henk Bos in his penetrating study of the Leibnizian calculus (1974/75). If one assumes that *ddr* is absolute —i.e. not dependent on which variable is taken as independent variable— as Leibniz did in his attempt to derive a formula for second order differentials in "*Cum prodiisset*", then one will run into trouble. But this is not necessary, and as Bos has shown, a derivation of a formula for second order differentials consistent with the syncategorematic interpretation may successfully be achieved along the lines Leibniz had tried, provided an independent variable is specified.

That is the case here. We are tacitly assuming, as do Leibniz and Newton, that time may be taken as independent variable, so that *dt* is constant, and $ddt = 0$. Under this assumption one may derive legitimate formulas involving second order differentials such as *ddr*, such as the following. Assume $r = wt$, where *w* is a velocity. We will show that

$$ddr = t \bullet ddw + 2\, dw \bullet dt \qquad (4.8)$$

For this to be justified syncategorematically, let *(d)r* and *(d)(d)r* be finite lines and *(d)w* and *(d)(d)w* finite velocities such that in the fixed finite time *(d)t* the following proportions always hold for all finite $dt \neq 0$:

$$(d)r:(d)t = dr:dt \tag{4.9}$$

$$(d)w:(d)t = dw:dt \tag{4.10}$$

$$(d)(d)r:(d)t^2 = ddr:dt^2 \tag{4.11}$$

$$(d)(d)w:(d)t^2 = ddw:dt^2 \tag{4.12}$$

Assume further that these ratios all have an interpretation when $dt = 0$. Then each can be substituted for its counterpart involving *dt* in any formula, as we assumed with Leibniz above. Now from $r = wt$ the following relation is derivable by elementary algebra:

$$ddr = t \bullet ddw + w \bullet ddt + 2\, dw \bullet dt + 2dw \bullet ddt + 2dt \bullet ddw + ddt \bullet ddw \tag{4.13}$$

Since *dt* is constant, and $ddt = 0$, we may drop the terms in *ddt*. Dividing by dt^2 gives

$$ddr:dt^2 = t \bullet ddw:dt^2 + 2\, dw:dt + 2dt \bullet ddw:dt^2 \tag{4.14}$$

Now "transposing the case as far as possible to quantities that never vanish", i.e. by substituting the equivalences (1)-(4), we obtain

$$(d)(d)r:(d)t^2 = t \bullet (d)(d)w:(d)t^2 + 2(d)w:(d)t + 2dt \bullet (d)(d)w:(d)t^2 \tag{4.15}$$

By hypothesis, this formula remains interpretable when $dt = 0$, when the last term vanishes. So the Law of Continuity asserts that this limiting case may also be included in the general reasoning: *dr*:*dt* can be substituted for *(d)r*:*(d)t* etc. in the resulting formulas even for the case where $dt = 0$, with *dr*, *ddr*, etc. in this case interpreted as fictions. Therefore

$$ddr:dt^2 = t \bullet ddw:dt^2 + 2\, dw:dt \tag{4.16}$$

or

$$ddr = t \bullet ddw + 2\, dw \bullet dt \qquad \text{QED.} \tag{4.17}$$

Now let us turn to the task of deriving this result using Smooth Infinitesimal Analysis. Here we are immediately confronted with the difficulty that the basic principles of SIA as expounded so far, as embodied in the Principle of Microuniformity, will not countenance time variation of geometric quantities across an infinitesimal interval. As Whiteside (1966) has shown, if we want the infinitesimal elements of the curve to be rectilinear, as they are by the Principle of Microstraightness, then they must be second order infinitesimals, in contradiction to Microuniformity. Relatedly, if we try to duplicate the Leibnizian calculation above using nilsquare infinitesimals, this means that any second order differential, such as Leibniz's *ddr* representing solicitation, is identically zero. This can be seen as follows. If the moment dt is taken to be a nilsquare infinitesimal, then $PQ^2 = (vdt)^2 = v^2dt^2 = 0$. Thus

$$QR = ddr = PQ^2/SP = 0$$

That is, to use Leibniz's terms, there can be no solicitation along QR. In Newton's terms, if the force at the very beginning of the interval is "given as the square of the time inversely", i.e. as $1/(SP\bullet QT)^2$ with $SP\bullet QT$ proportional to dt, then the acceleration in the moment dt is undefined. For the nascent triangle SPQ has an area $dA = {}^1/_2 SP\bullet QT = \mathrm{k}dt$, so $dA^2 = \mathrm{k}^2dt^2 = 0$.

This does not entail any inconsistency within SIA, which is perfectly able to derive results analogous to the Leibnizian relation (1) above for second order *derivatives*, as opposed to second order differentials. Thus supposing $r = wt$, and that t is the independent variable as before, with respect to which all derivatives are taken. Then, by the iterative definition of the derivative in SIA (Bell, 1998, 27), $\varepsilon r' = [r]^{t+\varepsilon}{}_t$ and $\varepsilon r'' = [r']^{t+\varepsilon}{}_t$. Therefore

$$\varepsilon r' = [r]^{t+\varepsilon}{}_t = [wt]^{t+\varepsilon}{}_t = (t+\varepsilon)\bullet w(t+\varepsilon) - t\bullet w(t) \qquad (4.18)$$

$$= t\bullet[w(t+\varepsilon) - w(t)] + \varepsilon\bullet w(t+\varepsilon)$$

$$= \varepsilon t w' + \varepsilon\bullet w(t+\varepsilon)$$

$$= \square t w\square + \square \bullet [w(t) + \square w\square(t)]$$

$$= \square t w\square + \square w + \square^2 w\square \qquad (4.19)$$

Thus, by the principle of microcancellation,

$$r\square = tw\square + w + \square w\square \qquad (4.20)$$

From this it is easy to derive

$$\square r\square\square\square = [r\square]^{t+\square}_{t} = [tw\square + w + \square w\square]^{t+\square}_{t}$$

$$= \square t w\square\square\square + 2\square w\square + 2\square^2 w\square\square\square \qquad (4.21)$$

Again, by microcancellation and applying $\square^2 = 0$,

$$r\square\square\square = tw\square\square\square + 2w\square \qquad (4.22)$$

The problem, however, is that with $\square^2 = 0$, we are unable to apply SIA to Proposition 6.

Nevertheless, it is possible to define second- and higher-order differentials within SIA, as Bell explains in his book. In particular, following Bell's prescription, we may define a *second-order differential* as one for which $dx^3 = 0$. In other words, instead of nilsquare infinitesimals we adopt nilcube ones, ones such that $\square^3 = 0$.[23] For such a differential in general dx and dx^2 (i.e. $\square$ and $\square^2$) are not equal to zero. Thus with $dt = \square$, $ddt = 0$ and $dw/dt = w\square$ etc., if we multiply both sides of (4.21) by $\square$, we may obtain an expression for *ddr*:

$$ddr = \square^2 r\square\square\square = \square^2 t \bullet w\square\square\square + 2\square^2 w\square + 2\square^3 \bullet w\square\square\square \qquad (4.23)$$

in agreement with the Leibnizian formula (4.17), since $\square^3 = 0$.

Such an expedient, however, seems to me to fail on two counts. On the one hand, the solution is *ad hoc*: the introduction of nilcube infinitesimals is motivated by nothing other than the inadequacy to the problem of an infinitesimal analysis based on nilsquare

[23] This was suggested to me by John Bell in conversation as a way of tackling Newton's Proposition 6.

infinitesimals. On the other hand, all the principles of SIA that we have depended upon above depend critically on the nilsquare property. As can be seen by reviewing the SIA proofs given above, the proof of the Product Rule depends on three applications of the nilsquare property, and the proof of Proposition 1 on six such applications. Even the formula for *ddr* given above relies on multiple applications of the definition of the derivative, and of the Principle of Microcancellation. But both of these depend on the nilsquare property. These difficulties are compounded when it is considered that Newton's Proposition 6 appeals to the Area Law that was demonstrated in Proposition 1, so that the moments *dt* have to be the same in each case, and cannot be represented by different kinds of infinitesimals.

Even assuming these difficulties could be overcome, however, the first objection concerning the adhocness of assuming nilcube infinitesimals in place of nilsquare ones to prove Proposition 6 is acute. For if we have no *independent* means to determine whether nilsquare infinitesimals are adequate to a given problem, then the kind of analysis we should apply is left to depend only on the specifics of the problem.

How does this situation bear on Leibniz's foundation for infinitesimals? For, as Bell observes, the procedure in SIA is very close to Leibniz's. Leibniz's polygonal representation of curves is closely related to Bell's Principle of Microstraightness: in each case the curve is analyzed as compounded of infinitesimal straight segments that are in a certain sense fictional parts. But whereas in SIA the "area deficit" is simply stipulated to be zero (p. 8) —that is, it is shown to be of the order of dx^2, where x is the independent variable, and thus rigorously equal to zero— in Leibniz's justification of Riemannian integration the Area Deficit is shown to be zero in the limit by an application of the Archimedean Axiom without any assumption about the nilpotency of infinitesimals. In fact, using Leibniz's method it is possible —as we saw in the above proofs of Lemma 9 and Proposition 6— to have the second order differential *ddr* proportional to the square of a first order infinitesimal. Because $QR = QP^2/SP$, with *QP* first order infinitesimal and *SP* finite, *QR* must be second order infinitesimal to preserve orders of infinity. This is impossible with nilsquare infinitesimals.

As we saw, Leibniz was able to derive the result (4.7),

$$du = ddr \quad v^2/r \tag{4.24}$$

for the solicitation in an arbitrary infinitesimal interval of time *dt*. Here *du* is an element of velocity, and a simple integration of it gives a velocity, i.e., on Leibniz's understanding, increases its order of infinity without changing its dimension. (In his calculations he tends to leave the dependence on the time interval tacit, as here with the dependence on dt^2.)[24] Similarly, two integrations of *ddr* raise it two orders of infinity, and give a finite line *r*. We would read Leibniz's expression, however, as giving us a second derivative with respect to time,

$$d^2r/dt^2 = v^2/r = du/dt \tag{4.25}$$

namely an acceleration, from which the velocity *u* is obtainable by an integration with respect to time, and the radius *r* by two such integrations. So from a modern perspective a non-zero solicitation in a moment or "timelet" *dt* is equivalent to an acceleration, contrary to Leibniz's understanding. Leibniz's conception of his differentials as independent of a specification of the variables is able to survive so long as he remains within first order infinitesimals. But it gets him into trouble with second order differentials, where relations involving them have to specify an independent variable, as Bos has explained with admirable clarity. Indeed, Bos argues that Leibniz's approach to the foundations of the calculus leads naturally to the introduction of differential quotients and even derivatives, and from there to

[24] See Bos (1974/75, 5-10, 12-35) and Meli (1993, 66-73) for clear expositions of the difference between Leibniz's understanding of differentiation and integration and the modern conception.

the concept of a function (1974/75, 59-66). But when Leibniz's second order differentials are recognized to be functions of an independent variable (here, time), the corresponding adjustment to the understanding of differentials strips him of his grounds for resisting the composition of a uniform motion from an inertial motion and an accelerated one: in motion under a central force, Microstraightness implies the failure of Microuniformity.

Thus Leibniz is able to uphold the Principle of Microuniformity because of a different understanding of how physical quantities are integrated in the calculus; but a successful grounding of the calculus through the syncategorematic interpretation leads to an abandonment of this position, and with it, Principle of Microuniformity. SIA with nilsquare infinitesimals is able to uphold Microuniformity, à la Nieuwentijt, only by equating the squares of all first-order infinitesimals to zero, and therewith rejecting the entire apparatus underlying Newton's calculation of central forces, from Lemma 9 through to Proposition 6 and its corollaries and applications —all of which involve time variation of quantities across an infinitesimal time, and quantities depending on the squares of nascent or evanescent areas which are, as in the Nieuwentijtian calculus, necessarily equal to zero.

5. Conclusion

Despite many points in common, we have seen that Leibniz's syncategorematic approach to infinitesimals and the theory of Smooth Infinitesimal Analysis are by no means equivalent. Indeed, the conceptions of infinitesimal quantities at the heart of each approach are radically diverse. According to John Bell, "The property of being a nilsquare infinitesimal is an *intrinsic* property, in no way dependent on comparisons with other magnitudes or numbers." (Bell, 1998, 2). Therein, it would appear, lies the source of the difficulties identified above. Nilsquareness is intrinsic to the nilsquare infinitesimal, but the nilsquare infinitesimal is not intrinsic to the problem. Thus the type of infinitesimal assumed (nilsquare, nilcube, etc.) has to be determined by its applicability to the problem at hand, i.e. by extrinsic criteria. There is no such problem in principle with Leibniz's syncategorematic infinitesimals. On this approach, the vanishing of infinitesimal quantities is always a comparative affair, and is grounded on a strictly Archimedean geometry. It remains to be seen, however, whether the Leibnizian syncategorematic approach can be set on a foundation adequate to modern mathematical standards of rigour.

Acknowledgements

[to be added in the proofs].

References

Arthur R. T. W. (2005a). "The remarkable fecundity of Leibniz's work on infinite series", a review article of A VII 3 and A II 5, forthcoming in *Annals of Science*.

Arthur R. T. W. (2005b). "From Actuals to Fictions: Four Phases in Leibniz's Early Thought on Infinitesimals" forthcoming in *Studia Leibnitiana*, ed. Mogens Laerke and Mark Kulstad.

Bell, J. L. (1998). *A Primer of Infinitesimal Analysis*. Cambridge: Cambridge University Press.

Bertoloni Meli, Domenico (1993). *Equivalence and Priority: Newton versus* Leibniz. Oxford, Clarendon Press.

Bos, H. J. M. (1974-75). "Differentials, Higher-Order Differential and the Derivative in the Leibnizian Calculus", *Archive for the History of the Exact sciences*, **14**, 1-90.

Gerhardt, C. I. ed. (1849-63): *Leibnizens Mathematische Schriften* (Berlin and Halle: Asher and Schmidt; reprint ed. Hildesheim: Olms, 1971), 7 vols; cited by volume and page, e.g. GM.II.316.

Gerhardt, C. I. ed. (1875-90): *Der Philosophische Schriften von Gottfried Wilhelm Leibniz* (Berlin: Weidmann; reprint ed. Hildesheim: Olms, 1960), 7 vols; cited by volume and page, e.g. G.II.268.

Guicciardini, Niccolò. *Reading the Principia*. Cambridge: Cambridge University Press.

Ishiguro, Hidé (1990). *Leibniz's Philosophy of Logic and Language* (2nd ed.) Cambridge: Cambridge University Press.

Knobloch, Eberhard (2002). "Leibniz's Rigorous Foundation of Infinitesimal Geometry by means of Riemannian Sums." *Synthese* 133: 59-73.

Leibniz, G. W. (1684). *"Nova methodus pro maximis et minimis, itemque tangentibus, quae nec fractas, nec irrationales quantitates moratur, et singulare pro illis calculi genus."* *Acta Eruditorum*, October 467-473. GM v 220-226.

Leibniz, G. W. (1701). *"Cum prodiisset atque increbuisset Analysis mea infinitesimalis …"* in Leibniz (1846), pp. 39-50.

Leibniz, G. W. (1846). *Historia et Origo calculi differentialis a G. G. Leibnitzio conscripta* (ed. C. I. Gerhardt). Hannover.

Leibniz, G. W. (1923-): *Sämtliche Schriften und Briefe*, ed. Akademie der Wissenschaften der DDR (Darmstadt and Berlin: Akademie-Verlag; cited by series, volume and page, e.g. A VI 2, 229.

Leibniz, G. W. (1993). *De quadratura arithmetica circuli ellipseos et hyperbolae cujus corollarium est trigonometria sine tabulis*. Ed. and commentary by Eberhard Knobloch. Göttingen: Vandenhoek & Ruprecht.

Leibniz, G. W. (2001). *The Labyrinth of the Continuum: Writings on the Continuum Problem, 1672-1686*. Ed., sel. & transl. R. T. W. Arthur. New Haven: Yale University Press; abbreviated *LLC* with page number.

Mancosu, Paolo (1996). *Philosophy of Mathematics and Mathematical Practice in the Seventeenth century*. Oxford: Oxford University Press.

Newton, Isaac (1999). *The Principia*: *Mathematical Principles of Natural Philosophy*. Trans. I. Bernard Cohen and Anne Whitman. Berkeley: University of California Press.

Whiteside, D. T. (1966). "Newtonian Dynamics". *History of Science* 5, 104-117.

Whiteside, D. T. (ed.) (1967). *The Mathematical Papers of Isaac Newton*, Volume 1, 1664-1666. Cambridge: Cambridge University Press.