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PORTFOLIO INERTIA AND ε-CONTAMINATIONS

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Abstract

This paper analyzes investors' portfolio selection problems in a two-period dynamic model of Knightian uncertainty. We account for the existence of portfolio inertia in this two-period framework. Furthermore, by incorporating investors' updating behavior, we analyze how new observation in the first period will affect investors' behavior. By this analysis, we show that new observation in the first period will expand portfolio inertia in the second period compared with the case in which new observation has not been gained in the first period if the degree of Knightian uncertainty is sufficiently large.

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1. Introduction

In stock markets, it is often observed that *portfolio inertia* exists, that is, a situation where some stocks are not traded or not priced for several minutes. While the existence of portfolio inertia cannot be accounted for[1] under the standard expected utility theory,[2] it can be explained under non-expected utility theories.[3] In this paper, we extend their static frameworks to a two-period dynamic framework in order to incorporate decision maker's updating behavior. Within such a framework, we can consider the following question: Does obtaining new information affect her portfolio inertia, which is derived from her optimization behavior?

As Gilboa and Schmeidler (1993) point out, it is intriguing to cosider whether obtaining new information will shrink portfolio inertia or expand. In order to analyze such a problem, we consider two updating rules, the *Fagin-Halpern rule* and *Dempster-Shafer rule*, which are frequently adopted in economics and statistics.[4] Recently, Epstein and Schneider (2003) and Wang (2003) axiomatize rational decision makers' behaviors incorporating their belief-updatings in the multi-period dynamic framework. In this paper, we consider the condition under which obtaining new information will *expand* portfolio inertia by way of decision makers' belief-updating.

The organization of this paper is as follows. In Section 2 we provide a review of *non-expected utility theories*, in particular, the *Choquet Expected Utility* and the *Maxmin Expected utility*. Section 3 provides the stochastic environment of a two-period model. Section 4 presents the two updating rules, the *Fagin-Halpern* and *Dempster-Shafer rules*. In Section 5, we define the *dilation of Knightian uncertainty*. Section 6 defines the *ε-contamination*, which is a restriction of the set of decision makers' beliefs. Section 7 analyses a portfolio selection problem *à la* Arrow (1965) under the non-expected utility

[1] See Arrow (1965).

[2] We explain the standard expected utility theory including non-expected utility theories in the next section.

[3] See Dow and Werlang (1992) that first account for the existence of portfolio inertia under the Choquet Expected Utility Theory. Also see Asano (2003) that explains the existence of portfolio inertia under the concept of *ambiguity*. For the concept of ambiguity, see Epstein (1999).

[4] See Section 4 in details.

framework. Section 8 provides the main theorem of this paper. Section 9 concludes this paper. Definitions of *Choquet integrals* and *non-additive measures (or capacities)* as well as mathematical results about Choquet integrals are relegated to Appendix.

2. CEU and MMEU

Since Ellsberg (1961) first cast doubt on the validity of the Subjective Expected Utility (henceforth SEU) theory axiomatized by Savage (1954),[5] a number of generalizations have been proposed in order to overcome such a shortcoming and to analyze problems that are not well explained within the framework of SEU.[6] While *risk* is the situation in which decision maker's beliefs are captured by a single probability measure, *uncertainty* is the situation in which decision maker's beliefs are represented by a set of probability measures or a non-additive probability measure. The difference between risk and uncertainty is crucial in analyzing decision maker's behavior under "non-deterministic" environment.

The Maxmin Expected Utility (henceforth MMEU) and the Choquet Expected Utility (henceforth CEU) are two alternatives to SEU that are extensively investigated in economics,[7] finance,[8] game theory,[9] and so forth. MMEU states that if a certain set of axioms is satisfied, then DM's beliefs are captured by *a set of finitely additive measures* and her preferences are represented by *the minimum of expected utilities over the set of these measures*.[10] On the other hand, CEU states that if a certain set of axioms is satisfied, then DM's beliefs are captured by *a non-additive measure* and her preferences are represented by *Choquet integrals*.[11] These two theories are axiomatized by different sets of axioms, respectively. However, it can be shown that CEU with a *convex* non-

[5]SEU states that if a certain set of axioms is satisfied, then DM's beliefs are captured by a unique probability measure and her preferences are represented by the expected utility.

[6]Using an easy-to-understand example, Ellsberg shows that people often violate Savage's crucial axiom, *the Sure Thing Principle*.

[7]For example, see Nishimura and Ozaki (2002b) (job search) or Ghirardato (1994) (agency theory).

[8]For example, see Dow and Werlang (1992), and Epstein and Wang (1994, 1995).

[9]For example, see Dow and Werlang (1994), Lo (1996, 1998, 1999), and Marinacci (2000).

[10]MMEU is axiomatized by Gilboa and Schmeidler (1989) in the Anscombe and Aumunn (henceforth AA) framework and axiomatized by Casadesus-Masanell, Klibanoff and Ozdenoren (2000) in the Savage framework.

[11]CEU is axiomatized by Schmeidler (1989) in the AA framework and axiomatized by Gilboa (1987) in the Savage framework.

additive measure μ is equivalent to MMEU with the set core(μ) as a set of probability measures.[12] In this paper, we further restrict the set of probability measures $\mathcal{P}$ to the *ε-contamination*. The reason of its restriction to the ε-contamination is provided in the following sections.

3. Stochastic Environment

In this section, we provide the formal description of our two-period dynamic model.

Let S be a state space. Let (s_1, s_2) denote a generic element of $S \times S$. Let $\mathscr{F} = \langle \mathscr{F}_t \rangle_{t=0,1,2}$, where $\mathscr{F}_0 \equiv \{\emptyset, S \times S\}$, and $\mathscr{F}_1$ is defined to be the algebra generated by the set of finite partitions of $S \times S$ of the form: $\langle E_i \times S \rangle_i$ for some finite partition $\langle E_i \rangle_{i=1}^m$ of S, and $\mathscr{F}_2$ is defined to be the algebra generated by the set of finite partitions of $S \times S$ of the form: $\langle E_i \times F_j \rangle_{i,j}$ for some finite partitions $\langle F_j \rangle_{j=1}^n$ of S.

For notational abuse, we denote the *first-period measurable space* by $(S, \langle E_i \rangle_i)$, where $\langle E_i \rangle_i$ is the algebra generated by the partition $\langle E_i \rangle$. We denote the set of all the probability measures on it by $\mathscr{M}(S, \langle E_i \rangle)$. Similar notational abuse applies to the *second-period measurable space* denoted by $(S, \langle F_j \rangle_j)$. Furthermore, we denote the set of all the probability measures on $(S \times S, \mathscr{F}_2)$ by $\mathscr{M}(S \times S, \mathscr{F}_2)$, and denote decision maker's set of probability measures by $\mathcal{P} \subseteq \mathscr{M}(S \times S, \mathscr{F}_2)$. In this paper, we consider that decision maker's Knightian uncertainty is represented by the set $\mathcal{P}$.

[12] Let S be a set, 2^S be the set of subsets of S, μ be a convex non-additive measure, $\mathcal{P}$ be the set of finitely additive measures on $(S, 2^S)$, and let $B(S, \mathbb{R})$ denote the space of bounded functions from S into $\mathbb{R}$. Then

$$\int X(s)\mu(ds) = \min \left\{ \int X(s)P(ds) \,|\, P \in \overline{\text{core}}(\mu) \right\},$$

where $X \in B(S, \mathbb{R})$ and the *upper core* of μ is the set of finitely additive measures on $(S, 2^S)$ that dominate μ for all $E \in 2^S$, *i.e.*, $\overline{\text{core}}(\mu) = \left\{ P \in \mathcal{P} | (\forall E \in 2^S)\, P(E) \geq \mu(E) \right\}$, and the integral of the left-hand side is in the sense of Choquet integrals. Note that

$$\int X(s)\mu'(ds) = \max \left\{ \int X(s)P(ds) \,|\, P \in \underline{\text{core}}(\mu') \right\},$$

where $X \in B(S, \mathbb{R})$, μ is a convex capacity on $(S, 2^S)$, μ' is the conjugate of μ, and the set $\underline{\text{core}}(\mu')$ is the *lower core* of μ'. We provide the definitions of Choquet integrals, capacities, convex capacities, and the lower core of capacities in Appendix.

Let $p \in \mathscr{M}(S \times S, \mathscr{F}_2)$. Then we define $p|_1(\cdot) \equiv p(\cdot \times S)$. Formally speaking, $p|_1$ is not a measure on $(S, \langle E_i \rangle)$, but it can be considered as the *first-period marginal probability measure* of p on $(S \times S, \mathscr{F}_2)$. In a similar way, we define the *second-period marginal probability measure*, denoted by $p|_2$, and we consider it as a measure on $(S, \langle F_j \rangle)$.

Let $\mathcal{P} \subseteq \mathscr{M}(S \times S, \mathscr{F}_2)$. Then we define the *first and second period marginal Knightian uncertainty* by

$$\begin{aligned} \mathcal{P}|_1 &= \{p|_1 \,|\, p \in \mathcal{P}\} \\ \text{and} \quad \mathcal{P}|_2 &= \{p|_2 \,|\, p \in \mathcal{P}\}, \end{aligned}$$

respectively.

In order to describe updating rules after observing an event E in the first period, we provide the definition of the *Bayesian updating procedure*. Let $p \in \mathcal{P}$ and let $E \in \langle E_i \rangle$ such that $p(E \times S) > 0$. Then we define the probability measure on $(S \times S, \mathscr{F}_2)$ conditional on the event $E \times S$ by

$$(\forall E \in \langle E_i \rangle)(\forall F \in \langle F_j \rangle)\ p|_2(E \times F|E) \equiv \frac{p(E \times F)}{p(E \times S)}.$$

We define the *Bayesian procedure* by a function $(p, E) \mapsto p|_2(\cdot|E)$, where $p|_2(\cdot|E) \equiv p|_2(E \times \cdot|E)$. That is, it maps a pair of a probability measure p on $(S \times S, \mathscr{F}_2)$ and an event E in $\langle E_i \rangle$ in the first period, to $p|_2(\cdot|E)$, which can be considered as the probability measure on $(S, \langle F_j \rangle)$.

Let $\mathcal{P}$ be decision maker's set of beliefs, and let E be an $\langle E_i \rangle$-measurable set such that $p(E \times S) > 0$ for all $p \in \mathcal{P}$. Then we define an *updating rule* by a function that maps a pair $(\mathcal{P}, E)$ to a set of probability measures on $(S, \langle F_j \rangle)$. In this paper, an updating rule is denoted by ϕ.

4. Updating Rules: The Fagin-Halpern and Dempster-Shafer Rules

In this section, we provide the definitions of two updating rules that are heavily investigated in statistics[13] and economics,[14] the *Fagin-Halpern updating rule* (we abbreviate it

[13]For example, see Dempster (1967, 1968), Shafer (1976), or Fagin and Halpern (1990).

[14]For example, see Gilboa and Schmeidler (1993), Denneberg (1994), or Nishimura and Ozaki (2002a).

as the FH rule) and the *Dempster-Shafer updating rule* (we abbreviate it as the DS rule).

We denote the FH rule as ϕ_{FH} and define it by

$$(\forall \mathcal{P} \subseteq \mathscr{M}(S \times S, \mathscr{F}_2))(\forall E \in \langle E_i \rangle_i)\ \ \phi_{FH}(\mathcal{P}, E) = \{p|_2(\cdot|E) \,|\, p \in \mathcal{P}\}\,. \tag{1}$$

Note that a decision maker who adapts the FH rule updates all the probability measures $p \in \mathcal{P}$ following the Bayesian procedure. Also note that when $E = S$, $\phi_{FH}(\mathcal{P}, S) = \mathcal{P}|_2$ for any $\mathcal{P}$. Thus the second period marginal Knightian uncertainty $\mathcal{P}|_2$ can be derived from the FH rule for $E = S$.

Before we define the DS rule, one more definition is in order. Define a set of first-period probability measures by

$$(\forall E \in \langle E_i \rangle_i)\ \mathcal{P}^*(E) = \operatorname{argmax} \{p|_1(E) \,|\, p \in \mathcal{P}\}\,.$$

Then we are in a position to define the DS rule by

$$(\forall \mathcal{P} \subseteq \mathscr{M}(S \times S, \mathscr{F}_2))(\forall E \in \langle E_i \rangle_i)\ \ \phi_{DS}(\mathcal{P}, E) = \{p|_2(\cdot|E) \,|\, p \in \mathcal{P}^*(E)\}\,. \tag{2}$$

Note that in the first period, a decision maker who follows the DS rule survives probability measures $p \in \mathcal{P}$ that assign the maximum value to an event E, and in the second period, she follows the Bayesian procedure for such a set of probability measures denoted by $\mathcal{P}^*(E)$.

From the definitions of FH and DS rules, we can immediately prove that

$$(\forall \mathcal{P} \subseteq \mathscr{M}(S \times S, \mathscr{F}_2))(\forall E \in \langle E_i \rangle_i)\ \ \phi_{DS}(\mathcal{P}, E) \subseteq \phi_{FH}(\mathcal{P}, E).$$

Although this result holds for any set of probability measures $\mathcal{P}$, the converse set inclusion does not necessarily hold for any $\mathcal{P}$. A question is worth considering: Does the converse set inclusion hold by restricting the set of probability measures $\mathcal{P}$? That is the topic of Section 5. Before we investigate this topic, the definition of the dilation of Knightian uncertainty is in order.

5. Dilation of Knightian Uncertainty

In this section, we define the concept of dilation.[15] Let $\mathcal{P} \subseteq \mathscr{M}(S \times S, \mathscr{F}_2)$ be the set of probability measures. We say that the *dilation of Knightian uncertainty* occurs if for any $E \in \langle E_i \rangle$, the next strict set inclusion holds:

$$\phi(\mathcal{P}, E) \supset \mathcal{P}|_2.$$

This means that if a decision maker has observed an event $E \in \langle E_i \rangle$ in the first period and updates her set of beliefs $\mathcal{P}$ by an updating rule ϕ, then the updated set of probability measures $\phi(\mathcal{P}, E)$ expands compared with the case in which she observes no event in the first period, that is, $E = S$. In this paper, a set of probability measures $\mathcal{P}$ is assumed to represent decision maker's Knightian uncertainty. Thus, the occurrence of the dilation of Knightian uncertainty in the sense of this paper forces her to make decisions based on "more ambiguous environment."

6. The ε-Contamination

In this section, we restrict the set of priors $\mathcal{P}$ to the *ε-contamination*. At first, we define the ε-contamination of some probability measure p^0 on $(S \times S, \mathscr{F}_2)$.

[15] Seidenfeld and Wasserman (1993) provide another definition of dilation based on upper and lower probability measures. Let $\mathcal{P} \subseteq \mathscr{M}(S \times S, \mathscr{F}_2)$ and let $B \in \mathscr{F}_2$ such that $(\forall p \in \mathcal{P})\, p(B) > 0$. Define the *upper probability* and the *lower probability* by

$$(\forall A \in \mathscr{F}_2)\ \overline{P}(A) \equiv \sup_{p \in \mathcal{P}} p(A), \quad \text{and}$$
$$(\forall A \in \mathscr{F}_2)\ \underline{P}(A) \equiv \inf_{p \in \mathcal{P}} p(A),$$

respectively, and define the *upper conditional probability* and the *lower conditional probability*

$$(\forall A \in \mathscr{F}_2)\ \overline{P}(A|B) \equiv \sup_{p \in \mathcal{P}} p(A \cap B)/p(B), \quad \text{and}$$
$$(\forall A \in \mathscr{F}_2)\ \underline{P}(A|B) \equiv \inf_{p \in \mathcal{P}} p(A \cap B)/p(B),$$

respectively. They define the concept of dilation as follows: B dilates A if

$$\underline{P}(A|B) < \underline{P}(A) \leq \overline{P}(A) < \overline{P}(A|B).$$

Their necessary and sufficient conditions for the dilation to occur depend on a *particular event* A. On the other hand, in this paper, we capture the concept of dilation by way of *expantions of sets of measures*. Our concept of dilation is more appropriate than their concept when it comes to analyzing economic problmes since our concept corresponds to the notion of Knightian uncertainty.

Let p^0 be a probability measure on $(S \times S, \mathscr{F}_2)$ such that $p^0(E \times S) > 0$, and let $\varepsilon \in (0,1)$. We assume that the decision maker's set of beliefs $\mathcal{P}$ is represented by the *ε-contamination* of p^0 on $(S \times S, \mathscr{F}_2)$ such that

$$\mathcal{P} = \left\{p^0\right\}^{\varepsilon} \equiv \left\{(1-\varepsilon)p^0 + \varepsilon q \,|\, q \in \mathscr{M}(S \times S, \mathscr{F}_2)\right\}.$$

If $\varepsilon = 0$, then $\mathcal{P} = \{p^0\}^{\varepsilon}$ reduces to $\mathcal{P} = \{p^0\}$. This corresponds to the case in which decision maker's belief is captured by a single probability measure. The larger this number ε is, the more uncertain decision makers are about the true probability measure p^0.[16] Thus, the number ε can be considered to be the value that captures the degree of Knightian uncertainty. For further discussions on ε-contaminations, see Berger (1985), Wasserman and Kadane (1990) or Nishimura and Ozaki (2002a, b).

For later use, we define the ε-contamination of $p^0|_2(\cdot|E)$ by

$$(\forall \varepsilon \in (0,1))(\forall E \in \langle E_i \rangle) \ \left\{p^0|_2(\cdot|E)\right\}^{\varepsilon} = \left\{(1-\varepsilon)p^0|_2(\cdot|E) + \varepsilon q_2 \,|\, q_2 \in \mathscr{M}(S, \langle F_j \rangle)\right\}.$$

As we have already pointed out, the converse inclusion $\phi_{DS}(\mathcal{P}, E) \supseteq \phi_{FH}(\mathcal{P}, E)$ is not necessarily true except for the trivial case in which $\mathcal{P}$ is a singleton. However, when we restrict the set of probability measures $\mathcal{P}$ to the ε-contamination, the converse inclusion holds. That is, sets of probability measures updated by FH and DS rules are identical. Moreover, the two updating rules, DS and FH rules can be represented by some ε'-contamination (ε' is defined in the next theorem) of second-period probability measures.

Theorem 1 (Nishimura and Ozaki (2002a)). *Let $\varepsilon \in (0,1)$ and let $E \in \langle E_i \rangle_i$. Then*

$$\phi_{FH}\left(\left\{p^0\right\}^{\varepsilon}, E\right) = \phi_{DS}\left(\left\{p^0\right\}^{\varepsilon}, E\right) = \left\{p^0|_2(\cdot|E)\right\}^{\varepsilon'},$$

where ε' is defined by

$$\varepsilon' = \varepsilon'(\varepsilon, E) \equiv \frac{\varepsilon}{(1-\varepsilon)p^0|_1(E) + \varepsilon}.$$

[16] This observation can be checked by way of expansions of sets of beliefs that are represented by $\mathcal{P} = \left\{p^0\right\}^{\varepsilon}$.

Proof. See Nishimura and Ozaki (2002a). □

It can be proved that the next equality holds:

$$\overline{\mathrm{core}}(\theta_1) = \left\{p^0|_2(\cdot|E)\right\}^{\varepsilon'} \equiv \left\{(1-\varepsilon')p^0|_2(\cdot|E) + \varepsilon' q \,|\, q \in \mathscr{M}(S, \langle F_j\rangle_j)\right\},$$

where $\theta_1 : \langle F_j\rangle_j \to [0,1]$ is defined by

$$(\forall A \in \langle F_j\rangle_j)\ \ \theta_1(A) = \begin{cases} (1-\varepsilon')p^0|_2(A|E) & \text{if} \quad A \neq S \\ 1 & \text{if} \quad A = S \end{cases} \tag{3}$$

From Fact 4, it can be shown that θ_1 is a convex capacity. Thus it follows that

$$\begin{aligned}(\forall s_1 \in E)\ \int_S I(s_1, s_2)\theta_1(ds_2) &= \min\left\{\int_S I(s_1, s_2)P(ds_2) \mid P \in \overline{\mathrm{core}}(\theta_1)\right\} \\ &= \min\left\{\int_S I(s_1, s_2)P(ds_2) \mid P \in \left\{p^0|_2(\cdot|E)\right\}^{\varepsilon'}\right\},\end{aligned}$$

where the integral of the left-hand side is in the sense of the *Choquet integral*.

Next, we investigate the relation between the set of the first-period marginals of the ε-contamination, $\mathcal{P}|_1$, and the ε-contamination of the first-period marginal probability measures, $\{p^0|_1\}^{\varepsilon}$.

Lemma 1. *Let $\varepsilon \in (0,1)$. Let $\mathcal{P} \equiv \{p^0\}^{\varepsilon} = \{(1-\varepsilon)p^0 + \varepsilon q \,|\, q \in \mathscr{M}(S \times S, \mathscr{F}_2)\}$, and let $\mathcal{P}|_1 \equiv \left\{p|_1 \,|\, p \in \{p^0\}^{\varepsilon}\right\}$. Then*

$$\mathcal{P}|_1 = \left\{p^0|_1\right\}^{\varepsilon} = \left\{(1-\varepsilon)p^0|_1 + \varepsilon q \,|\, q \in \mathscr{M}(S, \langle E_i\rangle)\right\}.$$

Proof. At first, we show that $\{p^0\}^{\varepsilon}|_1 \subseteq \{p^0|_1\}^{\varepsilon}$. Let $p_1 \in \{p^0\}^{\varepsilon}|_1$. Then there exists $p \in \{p^0\}^{\varepsilon}$ such that $p_1 = p(\cdot \times S)$. Since $p \in \{p^0\}^{\varepsilon}$, there exists $q \in \mathscr{M}(S \times S, \mathscr{F}_2)$ such that $p = (1-\varepsilon)p^0 + \varepsilon q$. Thus, $p_1 = p(\cdot \times S) = (1-\varepsilon)p^0|_1(\cdot) + \varepsilon q|_1(\cdot)$, which implies that $p_1 \in \{p^0|_1\}^{\varepsilon}$.

Next, we show that $\{p^0|_1\}^{\varepsilon} \subseteq \{p^0\}^{\varepsilon}|_1$. Let $p_1 \in \{p^0|_1\}^{\varepsilon}$. Then, there exists $q_1 \in \mathscr{M}(S, \langle E_i\rangle_i)$ such that $p_1 = (1-\varepsilon)p^0|_1 + \varepsilon q_1$. Let $q_2 \in \mathscr{M}(S, \langle F_j\rangle_j)$ and let $p = (1-\varepsilon)p^0 + \varepsilon(q_1 \times q_2)$. Then, $p \in \{p^0\}^{\varepsilon}$ and $p|_1 = (1-\varepsilon)p^0|_1 + \varepsilon q_1 = p_1$, which implies that $p_1 \in \{p^0\}^{\varepsilon}|_1$. □

By this lemma, we can show that

$$\mathcal{P}|_1 = \left\{p^0|_1\right\}^\varepsilon = \overline{\text{core}}(\theta_0),$$

where $\theta_0 : \langle E_i \rangle_i \to [0,1]$ is defined by

$$(\forall E \in \langle E_i \rangle_i)\ \ \theta_0(E) = \begin{cases} (1-\varepsilon)p^0|_1(E) & \text{if} \quad E \neq S \\ 1 & \text{if} \quad E = S \end{cases} \tag{4}$$

From Fact 4, it can be shown that θ_0 is a convex capacity. Thus it follows that

$$\begin{aligned} (\forall I)\ \int_S I(s_1,s_2)\theta_0(ds_1) &= \min\left\{\int_S I(s_1,s_2)P(ds_1) \mid P \in \overline{\text{core}}(\theta_0)\right\} \\ &= \min\left\{\int_S I(s_1,s_2)P(ds_1) \mid P \in \left\{p^0|_1\right\}^\varepsilon\right\}. \end{aligned}$$

Again, the integral of the left-hand side is in the sense of the Choquet integral.

Similarly, we can characterize the set of second-period marginals of the ε-contamination, $\mathcal{P}|_2$, by the ε-contamination of the second-period marginal probability measures, $\{p^0|_2\}^\varepsilon$.

Lemma 2. *Let $\varepsilon \in (0,1)$. Let $\mathcal{P}|_2 \equiv \left\{p|_2 \mid p \in \{p^0\}^\varepsilon\right\}$. Then,*

$$\mathcal{P}|_2 = \left\{p^0|_2\right\}^\varepsilon = \left\{(1-\varepsilon)p^0|_2 + \varepsilon q \mid q \in \mathscr{M}(S, \langle F_j \rangle)\right\}.$$

By this lemma, it follows that

$$\mathcal{P}|_2 = \left\{p^0|_2\right\}^\varepsilon = \overline{\text{core}}(\theta_2),$$

where $\theta_2 : \langle F_j \rangle_j \to [0,1]$ is defined by

$$(\forall F \in \langle F_j \rangle_j)\ \ \theta_2(F) = \begin{cases} (1-\varepsilon)p^0|_2(F) & \text{if} \quad F \neq S \\ 1 & \text{if} \quad F = S. \end{cases} \tag{5}$$

From Fact 4, we can show that θ_2 is a convex capacity.

7. Portfolio Selection Problem

In this section, we consider a portfolio selection problem *à la* Arrow (1965) in a two period framework. At first, the setup is in order. After that, we analyze the portfolio selection problem in two different setting. The first setting is the one in which a decision

maker observes an event E in the first period and updates her set of probability measures $\mathcal{P} = \{p^0\}^\varepsilon$ by the FH rule. The second is the one in which she observes no event in the first period and updates $\mathcal{P}$ by the FH rule, which is equal to the second-period marginal of $\mathcal{P}$, $\mathcal{P}|_2$, as we have already pointed out.

Let $W \in \mathbb{R}_+$ be the wealth at $t = 0$, $N \in \mathbb{R}$ be the amount of money invested at $t = 0$, $q > 0$ be the price of a risky asset at $t = 0$, and X be the random payoff of the asset at $t = 2$, where X is $\mathscr{F}_2$-measurable. Moreover, let $u : \mathbb{R} \to \mathbb{R}$ be a monotonic increasing concave function. We assume that an investor obtains new information at $t = 1$, but she does not gain any payment at $t = 1$. We are in a position to define her objective function.

The objective function is

$$
\begin{aligned}
& V(X(s_1, s_2)) \\
= \; & \min_{p \in \mathcal{P}} E^p \left[\min_{p \in \mathcal{P}} E^p \left[X \middle| \mathscr{F}_2 \right] \middle| \mathscr{F}_1 \right] \qquad (6) \\
= \; & \min_{p \in \mathcal{P}} \int \left[\min_{p \in \mathcal{P}} E^p \left[X \middle| \mathscr{F}_2 \right] \right] p|_1(ds_1) \\
= \; & \min_{p \in \mathcal{P}} \int \left[\min_{q \in \phi_{FH}(\mathcal{P}, E)} \int u\left(W - N + (N/q) X(s_1, s_2)\right) q(ds_2) \right] p|_1(ds_1) \\
= \; & \min_{p \in \mathcal{P}|_1} \int \left[\min_{q \in \phi_{FH}(\mathcal{P}, E)} \int u\left(W - N + (N/q) X(s_1, s_2)\right) q(ds_2) \right] p(ds_1) \\
= \; & \min_{p \in \{p^0|_1\}^\varepsilon} \int \left[\min_{q \in \{p^0|_2(\cdot|E)\}^{\varepsilon'}} \int u\left(W - N + (N/q) X(s_1, s_2)\right) q(ds_2) \right] p(ds_1) \\
= \; & \int_S \left[\int_S u\left(W - N + (N/q) X(s_1, s_2)\right) \theta_1(ds_2) \right] \theta_0(ds_1), \qquad (7)
\end{aligned}
$$

where $\theta_0 : \langle E_i \rangle_i \to [0, 1]$ is defined by

$$
(\forall E \in \langle E_i \rangle_i) \;\; \theta_0(E) = \begin{cases} (1 - \varepsilon) p^0|_1(E) & \text{if} \quad E \neq S \\ 1 & \text{if} \quad E = S, \end{cases}
$$

$\theta_1 : \langle F_j \rangle_j \to [0, 1]$ is defined by

$$
(\forall A \in \langle F_j \rangle_j) \;\; \theta_1(A) = \begin{cases} (1 - \varepsilon') p^0|_2(A|E) & \text{if} \quad A \neq S \\ 1 & \text{if} \quad A = S, \end{cases}
$$

and ε' is defined in Theorem 1. Note that DM's set of beliefs $\mathcal{P}$ is captured by the ε-contamination of p^0, and her updating rule $\phi(\mathcal{P}, E)$ is characterized by the FH rule

$\phi_{FH}(\mathcal{P}, E)$. Also note that the penultimate equality holds by Lemma 1 and Theorem 1. For axiomatizations of this preference (6), see Epstein and Schneider (2001) or Wang (2003).

An investor is supposed to invest in two assets, a riskless asset and a risky asset. The problem is to choose the amount of money N so as to maximize

$$\int_S \left[\int_S u\left(W - N + (N/q)X(s_1, s_2)\right) \theta_1(ds_2) \right] \theta_0(ds_1).$$

That is, she is supposed to choose the amount of money N to invest the risky asset so as to maximize her non-expected utility of the terminal payoff.

Theorem 2. *The investor will neither buy nor sell the risky asset, if q satisfies:*

$$\int_S \int_S X(s_1, s_2)\theta_1(ds_2)\theta_0(ds_1) < q < \int_S \int_S X(s_1, s_2)\theta_1'(ds_2)\theta_0'(ds_1),$$

where θ_0' and θ_1' are the conjugate of θ_0 defined by Equation (4) *and the conjugate of θ_1 defined by Equation* (3)*, respectively.* [17]

This theorem states that there exists portfolio inertia in a two-period setting, in which DM's beliefs are represented by the ε-contamination $\mathcal{P} = \{p^0\}^{\varepsilon}$, and her set of probabilities $\mathcal{P}$ is updated by the FH rule (DS rule).

Proof. Assume that

$$q > \int_S \int_S X(s_1, s_2)\theta_1(ds_2)\theta_0(ds_1). \tag{8}$$

[17] Observe that

$$\begin{aligned}
& \int_S \int_S X(s_1, s_2)\theta_1'(ds_2)\theta_0'(ds_1) \\
= & \max_{p \in \underline{\mathrm{core}}(\theta_0')} \int \left[\max_{q \in \underline{\mathrm{core}}(\theta_1')} \int X(s_1, s_2) q(ds_2) \right] p(ds_1) \\
= & \max_{p \in \overline{\mathrm{core}}(\theta_0)} \int \left[\max_{q \in \overline{\mathrm{core}}(\theta_1)} \int X(s_1, s_2) q(ds_2) \right] p(ds_1) \\
= & \max_{p \in \{p^0|_1\}^{\varepsilon}} \int \left[\max_{q \in \{p^0|_2(\cdot|E)\}^{\varepsilon'}} \int X(s_1, s_2) q(ds_2) \right] p(ds_1).
\end{aligned}$$

Then, for all $N \geq 0$,

$$\begin{aligned}
& \int_S \left[\int_S u\left(W - N + (N/q)X(s_1, s_2)\right) \theta_1(ds_2) \right] \theta_0(ds_1) \\
\leq\ & \int_S u\left(\int_S \left(W - N + (N/q)X(s_1, s_2)\right) \theta_1(ds_2) \right) \theta_0(ds_1) \\
=\ & \int_S u\left(W - N + (N/q) \int_S X(s_1, s_2)\theta_1(ds_2) \right) \theta_0(ds_1) \\
\leq\ & u\left(\int \left(W - N + (N/q) \int X(s_1, s_2)\theta_1(ds_2) \right) \theta_0(ds_1) \right) \\
=\ & u\left(W - N + (N/q) \int_S \int_S X(s_1, s_2)\theta_1(ds_2)\theta_0(ds_1) \right) \\
<\ & u(W - N + (N/q)q) \\
=\ & u(W),
\end{aligned}$$

where the first and second inequalities follow from Jensen's inequality, the first and second equalities follow from Fact 2, and the strict inequality follows from Equation (8).

Suppose that

$$q < \int_S \int_S X(s_1, s_2)\theta_1'(ds_2)\theta_0'(ds_1). \tag{9}$$

Then, for all $N \leq 0$,

$$\begin{aligned}
& \int_S \left[\int_S u\left(W - N + (N/q)X(s_1, s_2)\right) \theta_1(ds_2) \right] \theta_0(ds_1) \\
\leq\ & \int_S u\left(\int_S \left(W - N + (N/q)X(s_1, s_2)\right) \theta_1(ds_2) \right) \theta_0(ds_1) \\
=\ & \int_S u\left(W - N - (N/q) \int_S -X(s_1, s_2)\theta_1(ds_2) \right) \theta_0(ds_1) \\
=\ & \int_S u\left(W - N - (N/q) \left(- \int_S X(s_1, s_2)\theta_1'(ds_2) \right) \right) \theta_0(ds_1) \\
\leq\ & u\left(\int_S \left(W - N + (N/q) \int_S X(s_1, s_2)\theta_1'(ds_2) \right) \theta_0(ds_1) \right) \\
=\ & u\left(W - N - (N/q) \int_S - \left[\int_S X(s_1, s_2)\theta_1'(ds_2) \right] \theta_0(ds_1) \right) \\
=\ & u\left(W - N - (N/q) \left(- \int_S \int_S X(s_1, s_2)\theta_1'(ds_2)\theta_0'(ds_1) \right) \right) \\
<\ & u\left(W - N - (N/q)(-q)\right) \\
=\ & u(W),
\end{aligned}$$

where the first and second inequalities follow from Jensen's inequality, the first and third equalities follow from Fact 2, the second and fourth equalities follow from Fact 3, and the strict inequality follows from Equation (9). □

Next, we consider the case in which she does not update her set of belifes $\mathcal{P}$, which is still assumed to be characterized by the ε-contamination of p^0. That is, it is assumed that her set of second-period beliefs is represented $\mathcal{P}|_2$, which is the set of her second-period marginal probability measures defined by

$$\mathcal{P}|_2 = \{p|_2 \,|\, p \in \mathcal{P}\}\,.$$

As we have already pointed out, $\mathcal{P}|_2$ is equal to the FH rule, $\phi_{FH}(\mathcal{P}, E)$ for $E = S$. In this case, her objective function turns out to be

$$\begin{aligned}
& V(X(s_1,s_2)) \\
= & \min_{p\in\mathcal{P}} \int \left[\min_{q\in\phi_{FH}(\mathcal{P},S)} \int u\left(W - N + (N/q)X(s_1,s_2)\right) q(ds_2) \right] p|_1(ds_1) \\
= & \min_{p\in\mathcal{P}|_1} \int \left[\min_{q\in\phi_{FH}(\mathcal{P},S)} \int u\left(W - N + (N/q)X(s_1,s_2)\right) q(ds_2) \right] p(ds_1) \\
= & \min_{p\in\mathcal{P}|_1} \int \left[\min_{q\in\mathcal{P}|_2} \int u\left(W - N + (N/q)X(s_1,s_2)\right) q(ds_2) \right] p(ds_1) \\
= & \min_{p\in\{p^0|_1\}^\varepsilon} \int \left[\min_{q\in\{p^0|_2\}^\varepsilon} \int u\left(W - N + (N/q)X(s_1,s_2)\right) q(ds_2) \right] p(ds_1) \\
= & \int_S \left[\int_S u\left(W - N + (N/q)X(s_1,s_2)\right) \theta_2(ds_2) \right] \theta_0(ds_1), \qquad (10)
\end{aligned}$$

where $\theta_0 : \langle E_i \rangle_i \to [0,1]$ is defined by

$$(\forall E \in \langle E_i \rangle_i)\ \ \theta_0(E) = \begin{cases} (1-\varepsilon)p^0|_1(E) & \text{if} \quad E \neq S \\ 1 & \text{if} \quad E = S, \end{cases}$$

and $\theta_2 : \langle F_j \rangle_j \to [0,1]$ is defined by

$$(\forall F \in \langle F_j \rangle_j)\ \ \theta_2(F) = \begin{cases} (1-\varepsilon)p^0|_2(F) & \text{if} \quad F \neq S \\ 1 & \text{if} \quad F = S. \end{cases}$$

Note that the fourth equality holds by Lemmas 1 and 2.

The problem is to choose the amount of money N so as to maximize

$$\int_S \left[\int_S u\left(W - N + (N/q)X(s_1,s_2)\right) \theta_2(ds_2) \right] \theta_0(ds_1).$$

Similarly, we can prove the following theorem.

Theorem 3. *The investor will neither buy nor sell the risky asset, if q satisfies:*

$$\int_S \int_S X(s_1, s_2)\theta_2(ds_2)\theta_0(ds_1) < q < \int_S \int_S X(s_1, s_2)\theta_2'(ds_2)\theta_0'(ds_1),$$

where θ_0' and θ_2' are the conjugate of θ_0 defined by Equation (4) *and the conjugate of θ_2 defined by Equation* (5)*, respectively.*

Proof. The proof is omitted. □

This theorem states that there also exists portfolio inertia in a two-period model, in which DM's beliefs are represented by the ε-contamination $\mathcal{P} = \{p^0\}^\varepsilon$, and she does not obtain any information at $t = 1$, that is, $\mathcal{P}$ is restricted to the second period, $\mathcal{P}|_2$.

Now, a question is worth raising: whether portfolio inertia expands or shrinks when she obtains new information in the first period compared with the case in which she has not gain any information in the first period? In the next section, we provide an answer to this question.

8. Expansion of Portfolio Inertia

In this section, we consider the problem posed in the last section: whether obtaining new information will shrink or expand portfolio inertia?

In a two-period setting, we consider the following two cases. The first case is the one in which a decision maker observes an event $E \in \langle E_i \rangle_i$ in the first period, and she evaluates a second-period event by the probability measures updated by the Bayesian procedure. The second case is the one in which she observes no event in the first period, that is, $E = S$, and she evaluates the second-period event similar to the first case.

In order to analyze this problem, we define the "informational value" according to Nishimura and Ozaki (2002a). Let $E \in \langle E_i \rangle_i$ and let $\delta(E)$ be defined by

$$\delta(E) = \max_{j=1,\cdots,n} \left| p^0|_2(F_j|E) - p^0|_2(F_j) \right|,$$

where $\delta(E) \in [0,1]$. This number $\delta(E)$ is one of the measures to capture the informational value of knowing that an event E has occurred in the first period. Note that $\delta(E) = 0$ implies that observing an event E in the first period will not affect decision maker's evaluation of an event F in the second period.

Some comments are in order before we provide the main result of this paper. In the following theorem, we assume two additional conditions. The first is the condition that $m \geq 2$. This condition together with the assumption that $p^0(E \times S) > 0$ imply that $0 < p^0|_1(E) < 1$. The second is the condition that $F \in \langle F_j \rangle_{j=1}^n$ is such that $p^0|_2(F) > 0$. These two conditions imply that the first and the fourth inequalities in the following theorem hold with strict inequality.

Now we are in a position to present the main result of this paper. If ε, which is the degree of contamination of p^0, and which can be considered as the degree of Knightian uncertainty, is sufficiently large with respect to the informational value of the observation E, $\delta(E)$, then obtaining new information *expands* portfolio inertia compared with the case in which an investor does not obtain new information.

Theorem 4. *An investor neither buys nor sells the risky asset if q satisfies:*

$$\int_S \int_S X(s_1, s_2)\theta_2(ds_2)\theta_0(ds_1) < q < \int_S \int_S X(s_1, s_2)\theta_2'(ds_2)\theta_0'(ds_1),$$

where θ_0' and θ_2' are the conjugate of θ_0 defined by (4) *and the conjugate of θ_2 defined by* (5)*, respectively. Let $\mathcal{P} = \{p^0\}^\varepsilon$ and let $m \geq 2$. Furthermore, let $F \in \langle F_j \rangle_{j=1}^n$ such that $p^0|_2(F) > 0$. Suppose that the following inequality holds:*

$$\varepsilon > \frac{p^0|_1(E)}{(1 - p^0|_1(E)) \min_j p^0|_2(F_j)} \delta(E). \tag{11}$$

Then, for such an asset price, the following inequalities hold:

$$\begin{aligned} &\int_S \int_S X(s_1, s_2)\theta_1(ds_2)\theta_0(ds_1) < \int_S \int_S X(s_1, s_2)\theta_2(ds_2)\theta_0(ds_1) < \\ q \;\; < \;\; &\int_S \int_S X(s_1, s_2)\theta_2'(ds_2)\theta_0'(ds_1) < \int_S \int_S X(s_1, s_2)\theta_1'(ds_2)\theta_0'(ds_1), \end{aligned}$$

where θ_0', θ_1' and θ_2' are the conjugate of θ_0 defined by (4)*, the conjugate of θ_1 defined by* (3) *and the conjugate of θ_2 defined by* (5)*, respectively.*

This theorem states that obtaining new information expands portfolio inertia compared with the case in which she does not obtain new information as long as ε is sufficiently large.

Proof. In order to prove the statement, it suffices to show that

$$\theta_1(F) < \theta_2(F),$$

for any $F \in \langle F_j \rangle_j$ such that $p^0|_2(F) > 0$.

When $\delta(E) = 0$, it follows that

$$\begin{aligned} & (1-\varepsilon)p^0|_2(F) - (1-\varepsilon')p^0|_2(F|E) \\ = \; & (1-\varepsilon)\left(p^0|_2(F) - p^0|_2(F|E)\right) - (\varepsilon - \varepsilon')p^0|_2(F|E) \\ = \; & (\varepsilon' - \varepsilon)p^0|_2(F|E) > 0, \end{aligned}$$

where the second equality holds since $\delta(E) = \max_j |p^0|_2(F|E) - p^0|_2(F_j)| = 0$ and $\varepsilon' > \varepsilon$, and the inequality holds since $\varepsilon' > \varepsilon$ and $p^0|_2(F|E) > 0$.

When $\delta(E) > 0$, it follows that

$$\begin{aligned} & (1-\varepsilon)p^0|_2(F) - (1-\varepsilon')p^0|_2(F|E) \\ = \; & \varepsilon' \left(\frac{1-\varepsilon}{\varepsilon'} p^0|_2(F) - \frac{1-\varepsilon'}{\varepsilon'} p^0|_2(F|E) \right) \\ = \; & (1-\varepsilon)\varepsilon' \left[\left(\frac{1-\varepsilon}{\varepsilon} p^0|_1(E) + 1 \right) p^0|_2(F) - \frac{1}{\varepsilon} p^0|_1(E) p^0|_2(F|E) \right] \\ \geq \; & (1-\varepsilon)\varepsilon' \left[\left(\frac{1-\varepsilon}{\varepsilon} p^0|_1(E) + 1 \right) p^0|_2(F) - \frac{1}{\varepsilon} p^0|_1(E) \left(p^0|_2(F) + \delta(E) \right) \right] \\ = \; & (1-\varepsilon)\varepsilon' \left[\left(1 - p^0|_1(E) \right) p^0|_2(F) - \frac{\delta(E)}{\varepsilon} p^0|_1(E) \right] \\ \geq \; & (1-\varepsilon)\varepsilon' \left[\left(1 - p^0|_1(E) \right) \min_j p^0|_2(F_j) - \frac{\delta(E)}{\varepsilon} p^0|_1(E) \right] \\ > \; & (1-\varepsilon)\varepsilon' \left[\left(1 - p^0|_1(E) \right) \min_j p^0|_2(F_j) - \delta(E) p^0|_1(E) \left(\frac{p^0|_1(E)}{(1 - p^0|_1(E)) \min_j p^0|_2(F_j)} \delta(E) \right)^{-1} \right] \\ = \; & 0, \end{aligned}$$

where the second equality follows from the definition of ε', the first inequality follows from the definition of $\delta(E)$, and the strict inequality follows from the assumption. □

9. Conclusion

In this paper, we investigate a portfolio selection problem *à la* Arrow (1965) in a two period dynamic model of Knightian uncertainty. We explain the existence of portfolio inertia in the two-period framework in which decision maker's preference is represented by MMEU and her beliefs are captured by the ε-contamination. Moreover, by incorporating her updating behavior (the FH rule or the DS rule), we show that new observation in the first period will expand portfolio inertia in the second period compared with the case in which new observation has not been gained in the first period if the degree of Knightian uncertainty is sufficiently large.

In this paper, we provide the results in which decision maker's beliefs are captured by the ε-contamination. However, this set of beliefs is more restrictive than the set core(μ), which is the set of probability measures dominating a convex non-additive measure μ, and which is extensively investigated in the literature on non-expected utility theories. Thus, one of the interesting topics for future research is to extend the results based on ε-contaminations to the ones based on sets of probability measures, core.

Appendix

A.1 Choquet Integrals

Let S be a set and let 2^S be the power set of S. A set function $\mu : 2^S \to [0,1]$ is a *non-additive measure (or capacity)* if (a) $\mu(\emptyset) = 0$ and (b) $E, F \in 2^S$ and $E \subset F$ imply $\mu(E) \leq \mu(F)$, where $\emptyset$ denotes the empty set. The *conjugate* of a capacity μ is defined by $\mu'(E) = 1 - \mu(E^c)$ for all $E \in 2^S$. A non-additive measure μ is *convex* if $\mu(E \cup F) + \mu(E \cap F) \geq \mu(E) + \mu(F)$ for all $E, F \in 2^S$. The *upper core* and *lower core* of a capacity μ, denoted by $\overline{\text{core}}(\mu)$ and $\underline{\text{core}}(\mu)$, respectively, are defined by

$$\begin{aligned} \overline{\text{core}}(\mu) &= \left\{p \in \mathcal{P} |\, (\forall E \in 2^S)\, p(E) \geq \mu(E)\right\}, \\ \underline{\text{core}}(\mu) &= \left\{p \in \mathcal{P} |\, (\forall E \in 2^S)\, p(E) \leq \mu(E)\right\}, \end{aligned}$$

where $\mathcal{P}$ is the set of probability measures on $(S, 2^S)$. If a capacity μ is convex (concave), then $\overline{\text{core}}(\mu)$ ($\underline{\text{core}}(\mu)$) is non-empty. See Fact 6.

Let $B(S, \mathbb{R})$ denote the space of bounded functions from S into $\mathbb{R}$ and let $X \in B(S, \mathbb{R})$. The integral of X with respect to a non-additive measure μ is called *the Choquet integral*, and is defined by

$$\begin{aligned} &\int X(s)\, \mu(ds) \\ = &\int_0^\infty \mu\left(\{s \in S |\, X(s) \geq \alpha\}\right) d\alpha + \int_{-\infty}^0 \left[\mu\left(\{s \in S |\, X(s) \geq \alpha\}\right) - 1\right] d\alpha, \end{aligned}$$

where integrals on the right hand side are in the sense of Riemann integrals.

A.2 Mathematical Results

Fact 1.

$$(\forall X,\, Y \in B(S, \mathbb{R}))\; X \geq Y \Rightarrow \int X(s)\, \mu\,(ds) \geq \int Y(s)\, \mu\,(ds).$$

Fact 2.

$$(\forall X \in B(S, \mathbb{R}))(\forall a \in \mathbb{R})(\forall b \in \mathbb{R}_+) \int (a + bX(s))\mu\,(ds) = a + b \int X(s)\mu\,(ds).$$

Fact 3.

$$(\forall X \in B(S,\mathbb{R})) \quad \int X(s)\,\mu(ds) = -\int -X(s)\,\mu'(ds),$$

where μ' is the conjugate of μ.

Fact 4. *Let P be a probability measure, and let $f : [0,1] \to [0,1]$ be a monotonic increasing function satisfying $f(0) = 0$ and $f(1) = 1$. Furthermore, define a real valued function $f \circ P : 2^S \to [0,1]$ such that*

$$(\forall A \in 2^S) \quad f \circ P(A) = f(P(A)).$$

Then $f \circ P$ is a non-additive measure. If f is a concave (convex) function, then $f \circ P$ is concave (convex). In particular, any real valued function $P^\alpha : 2^S \to [0,1]$, defined by

$$(\forall A \in 2^S) \quad P^\alpha(A) = (P(A))^\alpha,$$

is a concave non-additive measure when $\alpha \in (0,1)$ and is a convex non-additive measure when $\alpha \in (1,\infty)$.

Fact 5. *Let μ and ν be probability capacities on $(S, 2^S)$. If $\mu(E) \leq \nu(E)$ for all $E \in 2^S$, then*

$$(\forall X \in B(S,\mathbb{R})) \quad \int X(s)\mu(ds) \leq \int X(s)\nu(ds).$$

Fact 6. *If μ is a convex capacity, then $\overline{\text{core}}(\mu)$ is non-empty.*

Proof. See Kelley (1959). □

Fact 7. *If μ is a capacity, then*

$$\overline{\text{core}}(\mu) = \underline{\text{core}}(\mu'),$$

where μ' is the conjugate of μ.

Theorem 5. *If μ is a convex capacity, then the following equality holds:*

$$(\forall X \in B(S,\mathbb{R})) \quad \int X(s)\mu(ds) = \min\left\{ \int X(s)P(ds) \,\middle|\, P \in \overline{\text{core}}(\mu) \right\}.$$

Proof. See Schmeidler (1989). □

Corollary 1. *If μ is a concave capacity, then the following equality holds:*

$$(\forall X \in B(S,\mathbb{R})) \quad \int X(s)\mu(ds) = \max\left\{ \int X(s)P(ds) \middle| P \in \underline{\text{core}}(\mu) \right\}.$$

Theorem 6 (Jensen's Inequality). *Let $(S, 2^S, \mu)$ be a measure space with $\mu(S) = 1$, where μ is a non-additive measure. Let $X : S \to (a, b)$ be in $B(S,\mathbb{R})$, and u be a concave, increasing and real-valued function on (a, b). Then for all non-additive measures μ,*

$$\int u(X(s))\,\mu(ds) \leq u\left(\int X(s)\,\mu(ds) \right).$$

Proof. See Asano (2003). □

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