DENSE NON-REFLECTION FOR STATIONARY COLLECTIONS OF COUNTABLE SETS

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ABSTRACT. We present several forcing posets for adding a non-reflecting stationary subset of $P_{\omega_1}(\lambda)$, where $\lambda \geq \omega_2$. We prove that PFA is consistent with *dense non-reflection* in $P_{\omega_1}(\lambda)$, which means that every stationary subset of $P_{\omega_1}(\lambda)$ contains a stationary subset which does not reflect to any set of size $\aleph_1$. If λ is singular with countable cofinality, then dense non-reflection in $P_{\omega_1}(\lambda)$ follows from the existence of squares.

Introduction

A classical consequence of Jensen's principle $\square_\lambda$ is that reflection of stationary subsets of λ^+ fails densely often; in other words, every stationary subset of λ^+ contains a non-reflecting stationary set. In this paper we focus on the extrapolation of the above conclusion to the context of stationarity in $P_{\omega_1}(\lambda)$: Given a cardinal $\lambda \geq \omega_2$, we say that *dense non-reflection in* $P_{\omega_1}(\lambda)$ holds if for every stationary $S \subseteq P_{\omega_1}(\lambda)$ there is a stationary $T \subseteq S$ which does not reflect to any set of size $\aleph_1$; that is, such that $T \cap P_{\omega_1}(N)$ is non-stationary for every set N of size $\aleph_1$.[1]

One of the purposes of this paper is to contribute to the general project of separating forcing axioms from combinatorial statements, and more specifically to build a model of PFA in which dense non-reflection holds in $P_{\omega_1}(\lambda)$ for many instances of λ.

A well–known fact is that PFA implies the failure of $\square_\lambda$ for all cardinals $\lambda \geq \omega_1$ ([15]). Yet another fact is that if PFA holds, then it is possible to force in such a way that both PFA is preserved and for every regular cardinal $\kappa \geq \omega_2$ there is a non-reflecting stationary subset of κ. This can be seen by combining the forcing construction in [4] in a class forcing as in Section 5 of this paper. On the other hand, the stronger forcing axiom MM implies, for every regular cardinal $\kappa \geq \omega_2$, that every stationary subset of $\kappa \cap cf(\omega)$ reflects ([8]). And, concerning stationarity in $P_{\omega_1}(\lambda)$, MM implies that every stationary subset of $P_{\omega_1}(\lambda)$, for every $\lambda \geq \omega_2$, reflects to a set of size $\aleph_1$ ([8]).[2]

In view of the web of implications given so far, it is natural to ask whether or not there is any connection between PFA and the statement that every stationary

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[1]Dense non-reflection in $P_{\omega_1}(\lambda)$ is basically a global failure of weak reflection in $P_{\omega_1}(\lambda)$.

[2]All implications mentioned so far are known to admit stronger, more general, formulations (for example weak forms of $\square_\lambda$ suffice to yield that reflection of stationary subsets of λ^+ fails densely often, etc.). We do not bother to state the stronger statements here, as we just want to display the general situation we were initially interested in.

subset of $P_{\omega_1}(\lambda)$ reflects to a set of size $\aleph_1$. In this paper we show, in analogy to what happens in the context of stationary sets of ordinals, that this is not the case.

Another purpose of this paper is to show that, when λ is a singular cardinal of countable cofinality, dense non-reflection in $P_{\omega_1}(\lambda)$ follows from fairly general pcf–theoretic assumptions.

The rest of the paper is structured as follows. In Sections 1 and 2 we deal with dense non-reflection in $P_{\omega_1}(\lambda)$ for $\lambda \geq \omega_2$ a regular cardinal. In Section 1 we build, for any such λ with $\lambda^{<\lambda} = \lambda$ and any set $\mathbb{X}$ of size at least λ, a λ–strategically closed poset $\mathbb{P}(\mathbb{X})$ preserving cofinalities and adding a non-reflecting stationary subset of $P_{\omega_1}(\mathbb{X})$ having stationary intersection with every stationary subset of $P_{\omega_1}(\mathbb{X})$ in the ground model. We also show that a suitable product $\mathbb{Q}(\mathbb{X})$ of copies of $\mathbb{P}(\mathbb{X})$ with $< \lambda$–support has the same niceness properties and forces dense non-reflection in $P_{\omega_1}(\mathbb{X})$. In Section 2 we show that the forcing $\mathbb{Q}(\mathbb{X})$ from Section 1 preserves PFA. This shows that PFA is consistent with dense non-reflection.

In Sections 3 and 4 we focus on dense non-reflection in $P_{\omega_1}(\lambda)$ in the case that λ is a singular cardinal of countable cofinality. In Section 3 we describe general situations implying dense non-reflection in $P_{\omega_1}(\lambda)$ for this choice of λ. These situations are phrased in the context of pcf theory. We prove a general result (Theorem 3.2) implying that if λ is a singular cardinal of countable cofinality such that $2^\lambda = \lambda^+$ and $\square^*_\lambda$ holds, then dense non-reflection holds in $P_{\omega_1}(\lambda)$ (Corollary 3.11). By a result of Magidor, it is possible to force over any model of PFA in such a way that PFA is preserved, GCH holds above ω, and $\square_{\kappa,\aleph_2}$ holds for all $\kappa \geq \omega_2$. Hence, in Magidor's model dense non-reflection in $P_{\omega_1}(\lambda)$ holds for all singular cardinals λ of countable cofinality (Corollary 3.13). We finish this section by describing a situation which does not assume $2^\lambda = \lambda^+$ but which nevertheless implies the conclusion that dense non-reflection holds in $P_{\omega_1}(\lambda)$ (Corollary 3.19); this time we assume that there is some $\delta < \lambda$ for which the set $\mathfrak{a}$ of regular cardinals between δ and λ is such that $pcf(\mathfrak{a})$ is countable and such that $\square^*_\kappa$ holds, where $\kappa \geq \lambda$ is such that $2^\lambda = \kappa^+$.[3]

Given a singular cardinal λ of countable cofinality, in Section 4 we introduce a $(\lambda + 1)$–strategically closed forcing $\mathbb{P}$ for adding a non-reflecting stationary subset of $P_{\omega_1}(\lambda)$ having stationary intersection with every stationary subset of $P_{\omega_1}(\lambda)$ in the ground model. A suitable product of copies of $\mathbb{P}$ with $< \lambda^+$–support is also $(\lambda + 1)$–strategically closed and, if $2^\lambda = \lambda^+$, then it has the λ^{++}–c.c. and forces dense non-reflection in $P_{\omega_1}(\lambda)$.

Finally, in Section 5 we present two forcing constructions preserving PFA, while at the same time forcing dense non-reflection in $P_{\omega_1}(\lambda)$ for all cardinals $\lambda \geq \omega_2$ which are either regular or singular of countable cofinality. In both constructions we start by assuming GCH above ω (which can always be forced preserving PFA). In the first construction we build a reverse Easton iteration in which we force at all relevant stages λ with a forcing as in Section 1 or in Section 4 for getting dense non-reflection in $P_{\omega_1}(\lambda)$. The desired forcing is the direct limit of this iteration. In the second construction we start with Magidor's model of PFA, GCH above ω and $\square_{\kappa,\aleph_2}$ for all $\kappa \geq \omega_2$, and build a reverse Easton iteration in which we keep forcing instances of dense non-reflection in $P_{\omega_1}(\lambda)$ only for regular λ.

We should remark that our methods do not seem to work for the case when λ is a singular cardinal of uncountable cofinality. For such a given λ and given any

[3]The existence of such a κ follows already from the existence of a set $\mathfrak{a}$ as above.

$\gamma < \lambda$ with $\gamma^{<\gamma} = \gamma$, it is possible to force dense non-reflection in $P_{\omega_1}(\lambda)$ by a γ–strategically closed forcing. This can be achieved by a special case of the forcing in Section 1. However, this forcing will blow up the power set of all cardinals between γ and λ to at least λ. Hence, iterating this type of forcings in length ORD certainly kills the Power Set Axiom. In fact, we do not know whether dense non-reflection is consistent in $P_{\omega_1}(\lambda)$ for any singular strong limit cardinal of uncountable cofinality.

The paper should be understandable to a reader familiar with forcing, iterated forcing, proper forcing, and generalized stationarity (see [2] and [3]). If X is a set of size at least $\aleph_1$, a set $S \subseteq P_{\omega_1}(X) = \{a \subseteq X : |a| < \omega_1\}$ is *stationary* if for any function $F : [X]^{<\omega} \to X$, there is a set b in S which is closed under F. A forcing poset $\mathbb{P}$ is *α-strategically closed*, where α is an ordinal, if Player II has a winning strategy in the following game. Players I and II take turns to build a descending sequence of conditions in $\mathbb{P}$, $\langle p_i : 1 \leq i < \alpha \rangle$, where Player I plays p_i for odd i, and Player II plays p_j for even j. If at each even stage less than α, Player II is able to play some condition, then Player II wins. The *Proper Forcing Axiom*, or *PFA*, is the statement that for any proper forcing poset $\mathbb{P}$, if $\{D_i : i < \omega_1\}$ is a family of dense subsets of $\mathbb{P}$, then there is a filter G on $\mathbb{P}$ which meets D_i for all $i < \omega_1$.

1. Adding Non-Reflecting Stationary Sets

Let $\lambda \geq \omega_2$ be a regular cardinal. Let $\mathbb{X}$ be a set of size at least λ. We define a forcing poset $\mathbb{P}(\mathbb{X})$ which adds a non-reflecting stationary subset of $P_{\omega_1}(\mathbb{X})$. A condition in $\mathbb{P}(\mathbb{X})$ is a set X such that:

- $X \subseteq P_{\omega_1}(\mathbb{X})$,
- $|X| < \lambda$,
- for any set N in $[\mathbb{X}]^{\aleph_1}$, $P_{\omega_1}(N) \cap X$ is non-stationary in $P_{\omega_1}(N)$.

We let $Y \leq X$ if:

- $X \subseteq Y$,
- for all y in $Y \setminus X$, y is not a subset of $\bigcup X$.

We will prove the following main properties of $\mathbb{P}(\mathbb{X})$.

(1) $\mathbb{P}(\mathbb{X})$ is λ-strategically closed.
(2) If $\lambda^{<\lambda} = \lambda$, then $\mathbb{P}(\mathbb{X})$ is λ^+-c.c., and therefore preserves all cardinals and cofinalities.
(3) The union of a generic filter for $\mathbb{P}(\mathbb{X})$ is a stationary subset of $P_{\omega_1}(\mathbb{X})$ which does not reflect to any set of size $\aleph_1$.
(4) For any stationary set $S \subseteq P_{\omega_1}(\mathbb{X})$ in the ground model, the union of a generic filter has stationary intersection with S.

After establishing these facts, we will show that a suitable product forcing of $\mathbb{P}(\mathbb{X})$ will force dense non-reflection in $P_{\omega_1}(\mathbb{X})$.

Proposition 1.1. *Let $\langle X_i : i < \delta \rangle$ be a descending sequence of conditions in $\mathbb{P}(\mathbb{X})$, where δ is a limit ordinal less than λ, such that for every limit ordinal $\nu < \delta$, $X_\nu = \bigcup_{i<\nu} X_i$. Then $\bigcup_{i<\delta} X_i$ is a condition in $\mathbb{P}(\mathbb{X})$ which is below X_i for all $i < \delta$.*

Proof. Let $Y = \bigcup_{i<\delta} X_i$. Clearly Y is a subset of $P_{\omega_1}(\mathbb{X})$ of size less than λ, and for all $i < \delta$, $X_i \subseteq Y$. Moreover, if $y \in Y \setminus X_i$, then there is $i < j < \delta$ such that y is in $X_j \setminus X_i$, and therefore y is not a subset of $\bigcup X_i$. So if the statement of the proposition fails, then there exists a set N in $[\mathbb{X}]^{\aleph_1}$ such that $P_{\omega_1}(N) \cap Y$ is stationary in $P_{\omega_1}(N)$.

Fix N in $[\mathbb{X}]^{\aleph_1}$ and assume for a contradiction $P_{\omega_1}(N) \cap Y$ is stationary in $P_{\omega_1}(N)$. Let $\langle a_i : i < \omega_1 \rangle$ be an increasing and continuous sequence of countable sets with union equal to N. Then there is a stationary set $A \subseteq \omega_1$ such that $\{a_i : i \in A\}$ is a subset of Y.

Claim 1.2. *The cofinality of δ is ω_1.*

Proof. If $\mathrm{cf}(\delta) > \omega_1$, then there is $\gamma < \delta$ such that $\{a_i : i \in A\} \subseteq X_\gamma$. Thus $P_{\omega_1}(N) \cap X_\gamma$ is stationary, which contradicts that X_γ is a condition. Suppose $\mathrm{cf}(\delta) = \omega$. Fix a cofinal function $f : \omega \to \delta$. Then $P_{\omega_1}(N) \cap Y = \bigcup_{n<\omega}(P_{\omega_1}(N) \cap X_{f(n)})$. Since the club filter on $P_{\omega_1}(N)$ is countably complete, there is $n < \omega$ such that $P_{\omega_1}(N) \cap X_{f(n)}$ is stationary. This contradicts that $X_{f(n)}$ is a condition. □

Let $\langle \beta_i : i < \omega_1 \rangle$ be an increasing and continuous sequence cofinal in δ. Then $X_\delta = \bigcup\{X_{\beta_i} : i < \omega_1\}$. Define $g : A \to \omega_1$ by letting $g(i)$ be the least ordinal such that a_i is in $X_{\beta_{g(i)}}$. Note that $g(i)$ is always a successor ordinal, since the map $j \mapsto \beta_j$ is normal and $X_\nu = \bigcup_{j<\nu} X_j$ for every limit ordinal $\nu < \delta$.

Claim 1.3. *There exists a club $C \subseteq \omega_1$ such that for all $i < j$ in C, if i is in A then $i < g(i) < j$.*

Proof. Let C_1 be the club set of limit ordinals α in ω_1 such that for all i in $\alpha \cap A$, $g(i) < \alpha$. Suppose for a contradiction there does not exist a club set $C \subseteq C_1$ such that for all i in $C \cap A$, $i < g(i)$. Since $g(i) = i$ is impossible when i is a limit ordinal, there is a stationary set $A' \subseteq A$ such that for all i in A', $g(i) < i$. By Fodor's Lemma, there is a stationary set $A'' \subseteq A'$ and $\gamma < \omega_1$ such that for all i in A'', $g(i) = \gamma$. Then for all i in A'', $a_i \in X_{\beta_\gamma}$. So $\{a_i : i \in A''\}$ is a subset of X_{β_γ}, and therefore $P_{\omega_1}(N) \cap X_{\beta_\gamma}$ is stationary, contradicting that X_{β_γ} is a condition. □

Fix a club C as in Claim 1.3. Since A is stationary in ω_1, it has non-empty intersection with the limit points of $A \cap C$. So we can choose a closed set x contained in $A \cap C$ with order type $\omega + 1$. Let $\nu = \max(x)$. Since $\langle a_i : i < \omega_1 \rangle$ is increasing and continuous, $a_\nu = \bigcup\{a_i : i \in x \cap \nu\}$. Consider i in $x \cap \nu$. Then $g(i) < \nu$, so $a_i \in X_{\beta_{g(i)}} \subseteq X_{\beta_\nu}$. It follows that $a_i \subseteq \bigcup X_{\beta_\nu}$. Since this is true for all i in $x \cap \nu$, $a_\nu \subseteq \bigcup X_{\beta_\nu}$. On the other hand, $\nu < g(\nu)$, so a_ν is in $X_{\beta_{g(\nu)}} \setminus X_{\beta_\nu}$. By the definition of the ordering on $\mathbb{P}(\mathbb{X})$, a_ν is not a subset of $\bigcup X_{\beta_\nu}$, which is a contradiction. □

Corollary 1.4. *The forcing poset $\mathbb{P}(\mathbb{X})$ is λ-strategically closed.*

Proof. By Proposition 1.1, the following strategy works: Player II plays anything at successor stages, and plays the union of the previous plays at limit stages. □

Corollary 1.5. *The forcing poset $\mathbb{P}(\mathbb{X})$ is ω_1-closed. In fact, if $\langle X_n : n < \omega \rangle$ is a descending sequence of conditions, then $\bigcup_{n<\omega} X_n$ is a condition which is below X_n for all $n < \omega$.*

Proof. For a descending sequence of order type ω, the hypotheses of Proposition 1.1 hold trivially. □

Lemma 1.6. *Suppose X is in $\mathbb{P}(\mathbb{X})$ and E is a subset of $\mathbb{X}$ of size less than λ such that $\bigcup X \subseteq E$. Then there is $Y \leq X$ such that $\bigcup Y = E$.*

Proof. Let $\{E_i : i \in I\}$ be a partition of $E \setminus \bigcup X$ into countable sets, where $|I| < \lambda$. Let $Y = X \cup \{E_i : i \in I\}$. If N is in $[\mathbb{X}]^{\aleph_1}$ and $P_{\omega_1}(N) \cap Y$ is stationary, then by the countable completeness of the club filter on $P_{\omega_1}(N)$, either $P_{\omega_1}(N) \cap X$ is stationary or $P_{\omega_1}(N) \cap \{E_i : i \in I\}$ is stationary. The former is impossible since X is a condition, and the latter is impossible since otherwise there are distinct i and j in I such that $E_i \subseteq E_j$. So Y is a condition, and clearly $\bigcup Y = E$. It is easy to see that $Y \leq X$. □

Corollary 1.7. *Let $\dot{T}$ be a $\mathbb{P}(\mathbb{X})$-name for the union of the generic filter. Then $\mathbb{P}(\mathbb{X})$ forces that $P_{\omega_1}(N) \cap \dot{T}$ is non-stationary in $P_{\omega_1}(N)$ for every set N in $[\mathbb{X}]^{\aleph_1}$.*

Proof. Suppose X forces $\dot{N}$ is in $[\mathbb{X}]^{\aleph_1}$. Since $\mathbb{P}(\mathbb{X})$ is λ-strategically closed, there is $Y \leq X$ and N in $[\mathbb{X}]^{\aleph_1}$ such that Y forces $\dot{N} = \check{N}$. By Lemma 1.6 fix $Z \leq Y$ such that $\bigcup Z = \bigcup Y \cup N$. Then Z forces $P_{\omega_1}(\dot{N}) \cap \dot{T} = P_{\omega_1}(N) \cap Z$. For any condition which is compatible with Z does not contain any subsets of $\bigcup Z$ which are not already in Z. Since $P_{\omega_1}(N) \cap Z$ is non-stationary, Z forces $P_{\omega_1}(\dot{N}) \cap \dot{T}$ is non-stationary. □

Lemma 1.8. *If $\lambda^{<\lambda} = \lambda$, then $\mathbb{P}(\mathbb{X})$ is λ^+-c.c.*

Proof. Let $\langle X_i : i < \lambda^+ \rangle$ be a sequence of conditions in $\mathbb{P}(\mathbb{X})$. Then $\langle \bigcup X_i : i < \lambda^+ \rangle$ is a sequence of sets of size less than λ. Since $\lambda^{<\lambda} = \lambda$, by the Δ-System Lemma there is an unbounded set $A \subseteq \lambda^+$ and a set $a \subseteq \mathbb{X}$ such that for all $i < j$ in A, $\bigcup X_i \cap \bigcup X_j = a$. Now $\mathcal{P}(a)$ has size at most λ, so there are at most $\lambda^{<\lambda} = \lambda$ many possibilities for $\mathcal{P}(a) \cap X_i$, for i in A. Fix $i < j$ in A such that $\mathcal{P}(a) \cap X_i = \mathcal{P}(a) \cap X_j$.

We claim that X_i and X_j are compatible. Clearly $X_i \cup X_j$ is a condition. Suppose for a contradiction $X_i \cup X_j$ is not a common refinement of X_i and X_j. Then without loss of generality, there is y in $(X_i \cup X_j) \setminus X_i$ such that y is a subset of $\bigcup X_i$. Since y is not in X_i, y is in X_j. So $y \subseteq \bigcup X_i \cap \bigcup X_j = a$. Therefore y is in $\mathcal{P}(a) \cap X_j$. But $\mathcal{P}(a) \cap X_j = \mathcal{P}(a) \cap X_i$. So y is in X_i, which is a contradiction. □

Proposition 1.9. *The forcing poset $\mathbb{P}(\mathbb{X})$ forces that $\dot{T} = \bigcup \dot{G}$ is stationary in $P_{\omega_1}(\mathbb{X})$. In fact, for every stationary set $S \subseteq P_{\omega_1}(\mathbb{X})$, $\mathbb{P}(\mathbb{X})$ forces that $\dot{T} \cap \check{S}$ is stationary.*

Proof. Let S be a stationary subset of $P_{\omega_1}(\mathbb{X})$. Let X be a condition in $\mathbb{P}(\mathbb{X})$ and suppose X forces $\dot{F} : [\mathbb{X}]^{<\omega} \to \mathbb{X}$ is a function. Fix a regular cardinal θ such that $\mathbb{X}$, $\mathbb{P}(\mathbb{X})$, and $\dot{F}$ are in $H(\theta)$. Since S is stationary, we can choose a countable set N such that:

- $N \prec H(\theta)$,
- $\{\mathbb{X}, \mathbb{P}(\mathbb{X}), X, \dot{F}\} \subseteq N$,
- $N \cap \mathbb{X} \in S$.

Define a descending sequence $\langle X_n : n < \omega \rangle$ of conditions in $N \cap \mathbb{P}(\mathbb{X})$ such that $X_0 = X$ and for any set D in N which is dense open in $\mathbb{P}(\mathbb{X})$, there is $n < \omega$ such that $X_n \in D$. This is possible since N is countable.

By Corollary 1.5, the set $\bigcup_{n<\omega} X_n$ is a condition in $\mathbb{P}(\mathbb{X})$. In particular, $\bigcup_{n<\omega} X_n$ does not reflect to any set of size $\aleph_1$. Let $Y = \bigcup_{n<\omega} X_n \cup \{N \cap \mathbb{X}\}$. Then Y does not reflect to any set of size $\aleph_1$. We claim $Y \leq X_n$ for all $n < \omega$. Clearly $X_n \subseteq Y$. Suppose y is in $Y \setminus X_n$. If y is in $\bigcup_{n<\omega} X_n$, then there is $m > n$ such that $y \in X_m \setminus X_n$. Then y is not a subset of $\bigcup X_n$, since $X_m \leq X_n$. Otherwise

$y = N \cap \mathbb{X}$. Since X_n is in N, by elementarity $(N \cap \mathbb{X}) \setminus \bigcup X_n$ is non-empty. Therefore $y = N \cap \mathbb{X}$ is not a subset of $\bigcup X_n$.

Clearly Y forces that $N \cap \mathbb{X}$ is in $\dot{T} \cap \check{S}$. So it suffices to show that Y forces $N \cap \mathbb{X}$ is closed under $\dot{F}$. Let $a_1, \ldots, a_n$ be in $N \cap \mathbb{X}$. Let D be the dense open set of conditions in $\mathbb{P}(\mathbb{X})$ which decide the value of $\dot{F}(a_1, \ldots, a_n)$. By elementarity, D is in N. Fix n such that X_n is in D, and let $b \in \mathbb{X}$ be such that X_n forces $\dot{F}(a_1, \ldots, a_n) = b$. By elementarity, b is in N. Since $Y \leq X_n$, Y forces $\dot{F}(a_1, \ldots, a_n) = b$ is in $N \cap \mathbb{X}$. □

We now consider a product forcing of the poset $\mathbb{P}(\mathbb{X})$ with $<\lambda$-support. Let κ be the cardinality of $P_{\omega_1}(\mathbb{X})$. Define $\mathbb{Q}(\mathbb{X})$ as the set of partial functions $p : \kappa^+ \to \mathbb{P}(\mathbb{X})$ with domain of size less than λ, and let $q \leq p$ if $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$ and for all i in $\mathrm{dom}(p)$, $q(i) \leq p(i)$ in $\mathbb{P}(\mathbb{X})$.

Proposition 1.10. *The forcing poset $\mathbb{Q}(\mathbb{X})$ is ω_1-closed and λ-strategically closed.*

This proposition follows by an easy argument using the corresponding properties of $\mathbb{P}(\mathbb{X})$. So by the next proposition, if $\lambda^{<\lambda} = \lambda$ then $\mathbb{Q}(\mathbb{X})$ preserves all cardinals and cofinalities.

Proposition 1.11. *If $\lambda^{<\lambda} = \lambda$, then $\mathbb{Q}(\mathbb{X})$ is λ^+-c.c.*

Proof. Let $\langle p_i : i < \lambda^+ \rangle$ be a sequence of conditions in $\mathbb{Q}(\mathbb{X})$. Consider $i < \lambda^+$. Let $E_i = \bigcup\{\bigcup(p_i(\alpha)) : \alpha \in \mathrm{dom}(p_i)\}$. Then E_i is a subset of $\mathbb{X}$ of size less than λ, and for each α in $\mathrm{dom}(p_i)$, $\bigcup(p_i(\alpha)) \subseteq E_i$. Let $q_i \leq p_i$ be a condition such that $\mathrm{dom}(q_i) = \mathrm{dom}(p_i)$ and for all α in $\mathrm{dom}(q_i)$, $\bigcup(q_i(\alpha)) = E_i$. This is possible by Lemma 1.6.

It suffices to find $i < j$ such that q_i and q_j are compatible. Since $\lambda^{<\lambda} = \lambda$, apply the Δ-System Lemma to find an unbounded set $A \subseteq \lambda^+$ and a set $a \subseteq \kappa^+$ such that for all $i < j$ in A, $\mathrm{dom}(q_i) \cap \mathrm{dom}(q_j) = a$. Applying the Δ-System Lemma again to the collection $\{E_i : i \in A\}$, find an unbounded set $B \subseteq A$ and a set $b \subseteq \mathbb{X}$ such that for all $i < j$ in B, $E_i \cap E_j = b$.

The set $\mathcal{P}(b)$ has size at most λ. So there are at most $\lambda^{<\lambda} = \lambda$ many possibilities for a sequence $\langle \mathcal{P}(b) \cap q_i(\alpha) : \alpha \in a \rangle$, for i in B. Fix $i < j$ in B which have the same such sequence.

To show q_i and q_j are compatible, it suffices to show that if α is in $\mathrm{dom}(q_i) \cap \mathrm{dom}(q_j) = a$, then $q_i(\alpha)$ and $q_j(\alpha)$ are compatible in $\mathbb{P}(\mathbb{X})$. If not, then without loss of generality there is α in a and a set x in $(q_i(\alpha) \cup q_j(\alpha)) \setminus q_i(\alpha)$ such that x is a subset of $\bigcup(q_i(\alpha)) = E_i$. Then x is in $q_j(\alpha)$, so x is a subset of $\bigcup(q_j(\alpha)) = E_j$. Hence x is a subset of $E_i \cap E_j = b$. Therefore x is in $\mathcal{P}(b) \cap q_j(\alpha)$. But $\mathcal{P}(b) \cap q_j(\alpha) = \mathcal{P}(b) \cap q_i(\alpha)$. So x is in $q_i(\alpha)$, which is a contradiction. □

Let α be an ordinal less than κ^+. For p in $\mathbb{Q}(\mathbb{X})$, let $p_\alpha = p \restriction \alpha$ and $p^\alpha = p \restriction [\alpha, \kappa^+)$. Define $\mathbb{Q}_\alpha(\mathbb{X}) = \{p_\alpha : p \in \mathbb{Q}(\mathbb{X})\}$ and $\mathbb{Q}^\alpha(\mathbb{X}) = \{p^\alpha : p \in \mathbb{Q}(\mathbb{X})\}$, with the obvious orderings. Then $\mathbb{Q}(\mathbb{X})$ is isomorphic to $\mathbb{Q}_\alpha(\mathbb{X}) \times \mathbb{Q}^\alpha(\mathbb{X})$ by the map $p \mapsto \langle p_\alpha, p^\alpha \rangle$. Note that $\mathbb{Q}(\mathbb{X}) = \bigcup_{\alpha<\kappa^+} \mathbb{Q}_\alpha(\mathbb{X})$.

The following easy lemma is similar to Proposition 1.10.

Lemma 1.12. *For all $\alpha < \kappa^+$, $\mathbb{Q}_\alpha(\mathbb{X})$ and $\mathbb{Q}^\alpha(\mathbb{X})$ are both ω_1-closed and λ-strategically closed.*

Lemma 1.13. *Assume $\lambda^{<\lambda} = \lambda$. Let $\dot{S}$ be a $\mathbb{Q}(\mathbb{X})$-name for a subset of $P_{\omega_1}(\mathbb{X})$. Then there is $\alpha < \kappa^+$ such that $\mathbb{Q}(\mathbb{X})$ forces $\dot{S}$ is in $V^{\mathbb{Q}_\alpha(\mathbb{X})}$.*

Proof. Since $\mathbb{Q}(\mathbb{X})$ does not add any new countable subsets of $\mathbb{X}$, $\mathbb{Q}(\mathbb{X})$ forces that $\dot{S}$ is a subset of the ground model. Let $\dot{F}$ be a $\mathbb{Q}(\mathbb{X})$-name for a surjection of κ onto $\dot{S}$. For each $i < \kappa$ let A_i be a maximal antichain of $\mathbb{Q}(\mathbb{X})$ which is contained in the dense open set of conditions which decide the value of $\dot{F}(i)$. Since $\mathbb{Q}(\mathbb{X})$ is λ^+-c.c., each A_i has size at most λ. As $\lambda \le \kappa$, $\bigcup_{i<\kappa} A_i$ has size at most κ. Since $\mathbb{Q}(\mathbb{X}) = \bigcup_{\alpha<\kappa^+} \mathbb{Q}_\alpha(\mathbb{X})$, there is $\alpha < \kappa^+$ such that $\bigcup_{i<\kappa} A_i$ is a subset of $\mathbb{Q}_\alpha(\mathbb{X})$.

Let G be a generic filter for $\mathbb{Q}(\mathbb{X})$ over V. Let $S = \dot{S}^G$ and $F = \dot{F}^G$. Then S is equal to the set of a in $P_{\omega_1}(\mathbb{X})$ such that there is $i < \kappa$ and a condition X in $A_i \cap G$ such that X forces $\dot{F}(i) = a$. Let $G_\alpha = \{p_\alpha : p \in G\}$. Then G_α is a generic filter for $\mathbb{Q}_\alpha(\mathbb{X})$ over V. By the choice of α, $p_\alpha = p$ for all p in $\bigcup_{i<\kappa} A_i$. So $A_i \cap G = A_i \cap G_\alpha$ for all $i < \kappa$. Therefore S is equal to the set of a in $P_{\omega_1}(\mathbb{X})$ such that there is $i < \kappa$ and a condition X in $A_i \cap G_\alpha$ which forces $\dot{F}(i) = a$. So S is in $V[G_\alpha]$. □

Theorem 1.14. *Assume $\lambda^{<\lambda} = \lambda$. Then $\mathbb{Q}(\mathbb{X})$ preserves all cardinals and cofinalities, and forces that every stationary subset of $P_{\omega_1}(\mathbb{X})$ contains a stationary subset which does not reflect to any set of size $\aleph_1$.*

Proof. Let G be a generic filter for $\mathbb{Q}(\mathbb{X})$ over V. Let S be a stationary subset of $P_{\omega_1}(\mathbb{X})$ in $V[G]$. By the last lemma, there is $\alpha < \kappa^+$ such that S is in $V[G_\alpha]$, where $G_\alpha = \{p_\alpha : p \in G\}$.

Let $H = \{p(\alpha) : p \in G\}$. Then H is a generic filter for $\mathbb{P}(\mathbb{X})$ over $V[G_\alpha]$. Let $T = \bigcup H$. By Corollary 1.7 and Proposition 1.9, T does not reflect to any set of size $\aleph_1$ and $T \cap S$ is stationary. Clearly then $T \cap S$ does not reflect to any set of size $\aleph_1$, and since $\mathbb{Q}(\mathbb{X})$ does not add any subsets of $\mathbb{X}$ of size $\aleph_1$, this remains true in $V[G]$. Since $\mathbb{Q}^{\alpha+1}(\mathbb{X})$ is ω_1-closed in V and $\mathbb{Q}_{\alpha+1}(\mathbb{X})$ does not add any new countable subsets of V, $\mathbb{Q}^{\alpha+1}(\mathbb{X})$ is ω_1-closed in $V[G_\alpha * H]$. Therefore $\mathbb{Q}^{\alpha+1}(\mathbb{X})$ is proper in $V[G_\alpha * H]$. It follows that $T \cap S$ remains stationary in $V[G]$. □

2. Dense Non-Reflection and PFA

Let $\lambda \ge \omega_2$ be a regular cardinal such that $\lambda^{<\lambda} = \lambda$ and $\mathbb{X}$ a set of size at least λ. Let κ be the cardinality of $P_{\omega_1}(\mathbb{X})$. Using the notation of Section 1, let $\mathbb{P}(\mathbb{X})$ denote the forcing poset which adds a non-reflecting stationary subset of $P_{\omega_1}(\mathbb{X})$ with conditions of size less than λ, and let $\mathbb{Q}(\mathbb{X})$ denote the $<\lambda$-support product of κ^+ many copies of $\mathbb{P}(\mathbb{X})$. We will abbreviate $\mathbb{Q}(\mathbb{X})$ as $\mathbb{Q}$ in what follows. Recall that $\mathbb{Q}$ is λ-strategically closed and λ^+-c.c., and so preserves all cardinals and cofinalities.

The goal of this section is to prove:

Theorem 2.1. *Assume* PFA*. Then $\mathbb{Q}$ forces* PFA *and that every stationary subset of $P_{\omega_1}(\mathbb{X})$ contains a stationary subset which does not reflect to any set of size $\aleph_1$.*

By Theorem 1.14, $\mathbb{Q}$ forces that every stationary subset of $P_{\omega_1}(\mathbb{X})$ contains a stationary subset which does not reflect to any set of size $\aleph_1$. So it suffices to show that $\mathbb{Q}$ forces PFA.

If A and B are subsets of $\mathbb{Q}$, we say that A and B are *cofinally interleaved* if for all p in A there is q in B such that $q \le p$, and for all q in B there is r in A such that $r \le q$. When we write $q(i)$ for a condition q in $\mathbb{Q}$, this will denote the empty set if i is not in $\mathrm{dom}(q)$, although strictly speaking $q(i)$ is undefined.

Let $\dot{\mathbb{R}}$ be a $\mathbb{Q}$-name for a proper forcing poset. We will prove that $\mathbb{Q}$ forces that PFA holds with respect to $\dot{\mathbb{R}}$. Here is a rough outline of the proof. Suppose q^* is a condition in $\mathbb{Q}$ which forces $\{\dot{D}_i : i < \omega_1\}$ is a family of dense open subsets of $\dot{\mathbb{R}}$. We will define a $\mathbb{Q} * \dot{\mathbb{R}}$-name $\dot{\mathbb{S}}$ for a forcing poset such that the iteration $\mathbb{Q} * \dot{\mathbb{R}} * \dot{\mathbb{S}}$ is proper. An application of PFA to this iteration will enable us to obtain a condition in $\mathbb{Q}$ below q^* which forces there is a filter on $\mathbb{R}$ which intersects $\dot{D}_i$ for all $i < \omega_1$.

To define $\dot{\mathbb{S}}$, consider a generic filter $G * H$ for $\mathbb{Q} * \dot{\mathbb{R}}$ over V. Working in $V[G * H]$, define $\mathbb{S}$ as the set of sequences $s = \langle s^i : i \leq \gamma \rangle$ satisfying:

- $\gamma < \omega_1$,
- s^i is in G for all $i \leq \gamma$,
- $s^j \leq s^i$ for $i \leq j \leq \gamma$,
- for every limit ordinal $\nu \leq \gamma$, $\mathrm{dom}(s^\nu) = \bigcup_{i<\nu} \mathrm{dom}(s^i)$, and for every ξ in $\mathrm{dom}(s^\nu)$, $s^\nu(\xi) = \bigcup_{i<\nu} s^i(\xi)$.
- for every limit ordinal $\nu \leq \gamma$, there is a countable set $A \subseteq \mathbb{Q}$ in V such that $\{s^i : i < \nu\}$ and A are cofinally interleaved.

The ordering on $\mathbb{S}$ is by extension of sequences.

Note that for every q in G, there is a dense set of conditions $t = \langle t^i : i \leq \delta \rangle$ in $\mathbb{S}$ such that $t^i \leq q$ for some $i \leq \delta$. Indeed, let $s = \langle s^i : i \leq \gamma \rangle$ in $\mathbb{S}$ be given. Since s^γ and q are both in G, we can choose $s^{\gamma+1}$ in G which is below s^γ and q. Then $t = \langle s^i : i \leq \gamma + 1 \rangle$ is as desired. Also note that for every ordinal $\xi < \omega_1$, there is a dense set of conditions $t = \langle t^i : i \leq \delta \rangle$ in $\mathbb{S}$ such that $\delta \geq \xi$. For example, given a condition $s = \langle s^i : i \leq \gamma \rangle$ in $\mathbb{S}$, where $\gamma < \xi$, extend s to $t = \langle s^i : i \leq \xi \rangle$ by letting $s^i = s^\gamma$ for $\gamma \leq i \leq \xi$.

Let $\dot{\mathbb{S}}$ be a $\mathbb{Q} * \dot{\mathbb{R}}$-name for the forcing poset $\mathbb{S}$ described above.

Proposition 2.2. *The iteration $\mathbb{Q} * \dot{\mathbb{R}} * \dot{\mathbb{S}}$ is proper.*

Proof. Let θ be a regular cardinal larger than $2^{|\mathbb{Q} * \dot{\mathbb{R}} * \dot{\mathbb{S}}|}$ such that $\mathbb{Q} * \dot{\mathbb{R}} * \dot{\mathbb{S}}$ is in $H(\theta)$. Let N be a countable elementary substructure of $H(\theta)$ with $\mathbb{Q} * \dot{\mathbb{R}} * \dot{\mathbb{S}}$ in N. Consider a condition $q * \dot{r} * \dot{s}$ in N. We will find a condition $q' * \dot{r}' * \dot{s}'$ below $q * \dot{r} * \dot{s}$ which is N-generic. First, since $\mathbb{Q}$ forces that $\dot{\mathbb{R}}$ is proper, choose a $\mathbb{Q}$-name $\dot{r}'$ for a condition in $\dot{\mathbb{R}}$ which is below $\dot{r}$ and is $N[\dot{G}]$-generic.

To define q', choose a descending sequence $\langle q_n : n < \omega \rangle$ of conditions in $N \cap \mathbb{Q}$ such that $q_0 = q$ and for every D in N which is a dense open subset of $\mathbb{Q}$, there is $n < \omega$ such that q_n is in D. This is possible since N is countable. Define q' so that the domain of q' is equal to $\bigcup_{n<\omega} \mathrm{dom}(q_n)$, and for α in the domain of q', $q'(\alpha) = \bigcup_{n<\omega} q_n(\alpha)$. By Corollary 1.5, for all α in $\mathrm{dom}(q')$, $q'(\alpha)$ is a condition in $\mathbb{P}(\mathbb{X})$ which is below $q_n(\alpha)$ for all $n < \omega$. So clearly q' is a condition in $\mathbb{Q}$ and $q' \leq q_n$ for all $n < \omega$.

In order to define $\dot{s}'$, consider a generic filter $G * H$ for $\mathbb{Q} * \dot{\mathbb{R}}$ over V which contains $q' * \dot{r}'$. Let $M = N[G * H]$. Then M is an elementary substructure of $H(\theta)$ in $V[G * H]$. Since $q' * \dot{r}'$ is N-generic, $M \cap V = N$. Let $s = \dot{s}^{G*H}$.

Choose a descending sequence $\langle s_n : n < \omega \rangle$ of conditions in $M \cap \mathbb{S}$ such that $s_0 = s$ and for any D in M which is dense open in $\mathbb{S}$, there is $n < \omega$ such that s_n is in D. This is possible since M is countable. For each n let $s_n = \langle s^i : i \leq \gamma_n \rangle$. Let $\gamma = \sup\{\gamma_n : n < \omega\}$. Note that for each n, s_n is countable, and therefore s_n is a subset of M. So s^i is in M for all $i < \gamma$.

Claim 2.3. *The set $\{s^i : i < \gamma\}$ is cofinally interleaved with $\{q_n : n < \omega\}$.*

Proof. Consider q_n. Since q' is in G and $q' \leq q_n$, q_n is in G. Let D be the dense open set of conditions $\langle t_i : i \leq \delta \rangle$ in $\mathbb{S}$ such that $t_i \leq q_n$ for some $i \leq \delta$. Then D is in M by elementarity. So there is m such that $s_m = \langle s^i : i \leq \gamma_m \rangle$ is in D. Then for some $i \leq \gamma_m$, $s^i \leq q_n$.

On the other hand, consider s^i. Since s^i is in $M \cap \mathbb{Q}$ and $M \cap V = N$, s^i is in $N \cap \mathbb{Q}$. Let E be the dense open set of conditions in $\mathbb{Q}$ which are either incompatible with s^i or below s^i. By elementarity, E is in N. Fix n such that q_n is in E. Since q' is in G and $q' \leq q_n$, q_n is in G. Also s^i is in G. So q_n and s^i are compatible. Therefore $q_n \leq s^i$. □

Let us recall the choice of q'. The domain of q' is equal to $\bigcup_{n<\omega} \mathrm{dom}(q_n)$, which by Claim 2.3 is equal to $\bigcup_{i<\gamma} \mathrm{dom}(s^i)$. For each ξ in the domain of q', $q'(\xi) = \bigcup_{n<\omega} q_n(\xi)$, which by Claim 2.3 is equal to $\bigcup_{i<\gamma} s^i(\xi)$. So clearly $q' \leq s^i$ for all $i < \gamma$. Also q' is in G. Let $s^\gamma = q'$ and define $s' = \langle s^i : i \leq \gamma \rangle$. Since $\{q_n : n < \omega\}$ is in V, s' is in $\mathbb{S}$ and $s' \leq s_n$ for all $n < \omega$.

Let $\dot{s}'$ be a $\mathbb{Q} * \dot{\mathbb{R}}$-name for s'. Then $q' * \dot{r}' * \dot{s}'$ is a condition in $\mathbb{Q} * \dot{\mathbb{R}} * \dot{\mathbb{S}}$ which is below $q * \dot{r} * \dot{s}$ and is N-generic. □

It follows that $\mathbb{Q} * \dot{\mathbb{R}} * \dot{\mathbb{S}}$ preserves ω_1.

Fix a sequence $\langle \dot{q}^i : i < \omega_1 \rangle$ of $\mathbb{Q} * \dot{\mathbb{R}} * \dot{\mathbb{S}}$-names such that $\mathbb{Q} * \dot{\mathbb{R}} * \dot{\mathbb{S}}$ forces:

- $\dot{q}^i$ is in $\dot{G}$ for all $i < \omega_1$,
- for every q in $\dot{G}$, there is $i < \omega_1$ such that $\dot{q}^i \leq q$,
- $\dot{q}^j \leq \dot{q}^i$ for $i \leq j < \omega_1$,
- for every limit ordinal $\nu < \omega_1$, $\mathrm{dom}(\dot{q}^\nu) = \bigcup_{i<\nu} \mathrm{dom}(\dot{q}^i)$, and for all ξ in $\mathrm{dom}(\dot{q}^\nu)$, $\dot{q}^\nu(\xi) = \bigcup_{i<\nu} \dot{q}^i(\xi)$.
- for every limit ordinal $\nu < \omega_1$, there is a countable set $A \subseteq \mathbb{Q}$ in V such that $\{\dot{q}^i : i < \nu\}$ and A are cofinally interleaved.

We will use the following standard fact (for a proof see Theorem 2.53 of [17]).

Proposition 2.4. *Assume PFA and let $\mathbb{P}$ be a proper forcing poset. Let θ be a regular cardinal such that $\mathbb{P}$ is in $H(\theta)$. Then there are stationarily many N in $[H(\theta)]^{\aleph_1}$ such that $\omega_1 \subseteq N$, $N \prec H(\theta)$, $\mathbb{P} \in N$, and there exists a filter on $\mathbb{P}$ which is N-generic.*

Suppose q^* is in $\mathbb{Q}$, $\{\dot{D}_i : i < \omega_1\}$ is a family of $\mathbb{Q}$-names, and q^* forces $\dot{D}_i$ is a dense open subset of $\dot{\mathbb{R}}$ for all $i < \omega_1$.

Fix a regular cardinal θ such that $\mathbb{Q} * \dot{\mathbb{R}} * \dot{\mathbb{S}}$, $\langle \dot{q}^i : i < \omega_1 \rangle$, and $\{\dot{D}_i : i < \omega_1\}$ are in $H(\theta)$. Applying Proposition 2.4, fix N, K, and K_1 satisfying:

- N is in $[H(\theta)]^{\aleph_1}$ and $\omega_1 \subseteq N$,
- $N \prec H(\theta)$,
- $\mathbb{Q} * \dot{\mathbb{R}} * \dot{\mathbb{S}}$, $\langle \dot{q}^i : i < \omega_1 \rangle$, q^*, and $\{\dot{D}_i : i < \omega_1\}$ are in N,
- K is a filter on $\mathbb{Q} * \dot{\mathbb{R}} * \dot{\mathbb{S}}$ which is N-generic,
- K_1 is the set of q for which there is $q * \dot{r} * \dot{s}$ in $N \cap K$,
- q^* is in K_1.

The last statement can be arranged since $(\mathbb{Q}/q^*) * \dot{\mathbb{R}} * \dot{\mathbb{S}}$ is proper.

Using the fact that K is a filter and is N-generic, it is easy to show that K_1 is a filter on $N \cap \mathbb{Q}$.

For each $i < \omega_1$ there is a dense set of conditions in $\mathbb{Q} * \dot{\mathbb{R}} * \dot{\mathbb{S}}$ which decide the value of $\dot{q}^i$. By elementarity, this dense set is in N. Since K is N-generic, let q^i be the unique condition in $\mathbb{Q}$ such that there is a condition in $N \cap K$ which forces $\dot{q}^i = q^i$. By elementarity, q^i is in N.

Lemma 2.5. *The sequence* $\langle q^i : i < \omega_1 \rangle$ *satisfies:*

(1) q^i *is in* K_1 *for all* $i < \omega_1$,
(2) *for all* q *in* $N \cap K_1$, *there is* $i < \omega_1$ *such that* $q^i \le q$,
(3) $q^j \le q^i$ *for all* $i \le j < \omega_1$,
(4) *for every limit ordinal* $\nu < \omega_1$, $\mathrm{dom}(q^\nu) = \bigcup_{i<\nu} \mathrm{dom}(q^i)$, *and for all* ξ *in* $\mathrm{dom}(q^\nu)$, $q^\nu(\xi) = \bigcup_{i<\nu} q^i(\xi)$.

Proof. (1) Let D be the dense open set of conditions in $\mathbb{Q}$ which are either incompatible with q^i or below q^i. Let E be the dense open set of conditions in $\mathbb{Q} * \dot{\mathbb{R}} * \dot{\mathbb{S}}$ of the form $q * \dot{r} * \dot{s}$ such that q is in D. By elementarity, D and E are in N. Since K is N-generic, fix $q * \dot{r} * \dot{s}$ in $N \cap K \cap E$. Then q is in $K_1 \cap D$.

We claim that q is compatible with q^i, and hence $q \le q^i$ since q is in D. Fix a condition $q_0 * \dot{r}_0 * \dot{s}_0$ in $N \cap K$ which forces $\dot{q}^i = q^i$. Then q_0 is in K_1. Since K_1 is a filter, let q_1 be in K_1 which is below q and q_0. Then $q_1 * \dot{s}_0 * \dot{r}_0$ forces that q and q^i are both in $\dot{G}$, and hence are compatible. So indeed q and q^i are compatible.

Now q is in K_1, $q \le q^i$, and q^i is in $N \cap \mathbb{Q}$. Since K_1 is a filter on $N \cap \mathbb{Q}$, q^i is in K_1.

(2) Let q be in K_1. Fix $q * \dot{r} * \dot{s}$ in $N \cap K$. Since $q * \dot{r} * \dot{s}$ forces that q is in $\dot{G}$, it forces that there is $i < \omega_1$ such that $\dot{q}^i$ is below q. Let E be the dense open set of conditions in $\mathbb{Q} * \dot{\mathbb{R}} * \dot{\mathbb{S}}$ which are either incompatible with $q * \dot{r} * \dot{s}$, or force for some $i < \omega_1$ that $\dot{q}^i \le q$, and in addition decide the value of $\dot{q}^i$. By elementarity, E is in N. Since K is N-generic, fix $q_0 * \dot{r}_0 * \dot{s}_0$ in $N \cap K \cap E$. Fix $i < \omega_1$ such that $q_0 * \dot{r}_0 * \dot{s}_0$ forces $\dot{q}^i \le q$ and $\dot{q}^i = q^i$. Then $q_0 * \dot{r}_0 * \dot{s}_0$ forces $q^i \le q$. So in fact $q^i \le q$.

(3) Choose conditions $q_0 * \dot{r}_0 * \dot{s}_0$ and $q_1 * \dot{r}_1 * \dot{s}_1$ in $N \cap K$ which decide $\dot{q}^i$ and $\dot{q}^j$ respectively as q^i and q^j. Since K is a filter, let $q * \dot{r} * \dot{s}$ be a refinement of these two conditions. Since $\mathbb{Q} * \dot{\mathbb{R}} * \dot{\mathbb{S}}$ forces $\dot{q}^j \le \dot{q}^i$, $q * \dot{r} * \dot{s}$ forces $q^j \le q^i$. So indeed $q^j \le q^i$.

(4) Consider a limit ordinal $\nu < \omega_1$. Then $\mathbb{Q} * \dot{\mathbb{R}} * \dot{\mathbb{S}}$ forces $\mathrm{dom}(\dot{q}^\nu) = \bigcup_{i<\nu} \mathrm{dom}(\dot{q}^i)$. Since $q^\nu \le q^i$ for all $i < \nu$, $\bigcup_{i<\nu} \mathrm{dom}(q^i) \subseteq \mathrm{dom}(q^\nu)$. Let E be the dense open set of conditions in $\mathbb{Q} * \dot{\mathbb{R}} * \dot{\mathbb{S}}$ which decide $\dot{q}^\nu$ and also decide for some countable set $A \subseteq \mathbb{Q}$ in V that $\{\dot{q}^i : i < \nu\}$ and A are cofinally interleaved. By elementarity, E is in N. Since K is N-generic, fix $q * \dot{r} * \dot{s}$ in $N \cap K \cap E$. Let A be a countable subset of $\mathbb{Q}$ which $q * \dot{r} * \dot{s}$ forces is cofinally interleaved with $\{\dot{q}^i : i < \nu\}$. Then A is in N, and $q * \dot{r} * \dot{s}$ forces that $\mathrm{dom}(\dot{q}^\nu) = \bigcup \{\mathrm{dom}(s) : s \in A\}$. Note that since $\{\dot{q}^i : i < \nu\}$ is forced to be a subset of $\dot{G}$, $q * \dot{r} * \dot{s}$ forces $A \subseteq \dot{G}$.

Consider ξ in $\mathrm{dom}(q^\nu)$. Let E_1 be the dense open set of conditions which are either incompatible with $q * \dot{r} * \dot{s}$, or are below it and decide for some s in A that ξ is in $\mathrm{dom}(s)$. Since $N \cap K \cap E_1$ is non-empty, clearly there is s in A such that ξ is in $\mathrm{dom}(s)$. Let E_2 be the dense open set of conditions which are either incompatible with $q * \dot{r} * \dot{s}$, or are below it and decide for some $i < \nu$ that $\dot{q}^i \le s$, and moreover decide $\dot{q}^i$. Then there is $i < \nu$ such that $q^i \le s$, and therefore ξ is in $\mathrm{dom}(q^i)$. This proves the first part of (4). The proof of the second part of (4) is almost the same argument. □

Define q' in $\mathbb{Q}$ as follows. The domain of q' is equal to $\bigcup_{i<\omega_1} \mathrm{dom}(q^i)$. For each ξ in $\mathrm{dom}(q')$, let $q'(\xi) = \bigcup_{i<\omega_1} q^i(\xi)$. By Proposition 1.1 and Lemma 2.5, for all ξ in $\mathrm{dom}(q')$, $q'(\xi)$ is in $\mathbb{P}(\mathbb{X})$ and $q'(\xi) \le q^i(\xi)$ for all $i < \omega_1$. Therefore q' is in

$\mathbb{Q}$ and $q' \leq q^i$ for all $i < \omega_1$. By Lemma 2.5(2), $q' \leq q$ for every q in $N \cap K_1$. In particular, $q' \leq q^*$.

Let G be a generic filter for $\mathbb{Q}$ over V which contains q'. Let $\mathbb{R} = \dot{\mathbb{R}}^G$, and for $i < \omega_1$ let $D_i = \dot{D}_i^G$. In $V[G]$, define H as the set of conditions r in $\mathbb{R}$ such that there is $q * \dot{r} * \dot{s}$ in $N \cap K$ such that $r = \dot{r}^G$.

We claim that H generates a filter on $\mathbb{R}$. Consider r_1 and r_2 in H. Then there are $q_i * \dot{r}_i * \dot{s}_i$ in $N \cap K$ such that $r_i = \dot{r}_i^G$ for $i < 2$. Let E be the dense open set of conditions in $\mathbb{Q} * \dot{\mathbb{R}} * \dot{\mathbb{S}}$ which are either below $q_i * \dot{r}_i * \dot{s}_i$ for $i < 2$, or incompatible with at least one of them. By elementarity, E is in N. Since K is N-generic, fix $q * \dot{r} * \dot{s}$ in $N \cap K \cap E$. Clearly $q * \dot{r} * \dot{s}$ is below $q_i * \dot{r}_i * \dot{s}_i$ for $i < 2$. Let $r = \dot{r}^G$. Then r is in H. Now q is in $N \cap K_1$, so $q' \leq q$. Therefore q is in G. But q forces $\dot{r} \leq \dot{r}_1, \dot{r}_2$. So $r \leq r_1, r_2$.

To complete the proof, we show that $H \cap D_i$ is non-empty for all $i < \omega_1$. Let E be the set of conditions $q * \dot{r} * \dot{s}$ in $\mathbb{Q} * \dot{\mathbb{R}} * \dot{\mathbb{S}}$ such that either q is incompatible with q^*, or q forces $\dot{r}$ is in $\dot{D}_i$. Then E is dense open, because q^* forces $\dot{D}_i$ is dense open in $\dot{\mathbb{R}}$. By elementarity, E is in N. So let $q * \dot{r} * \dot{s}$ be in $N \cap K \cap E$. Then q is in K_1 and q forces $\dot{r}$ is in $\dot{D}_i$. Since $q' \leq q$, q is in G. Therefore $r = \dot{r}^G$ is in $H \cap D_i$. This completes the proof of Theorem 2.1

3. Dense Non-Reflection at a Singular Cardinal

We describe some natural situations in which dense non-reflection holds in $P_{\omega_1}(\lambda)$, where λ is a singular cardinal with countable cofinality, and show that such dense non-reflection is consistent with PFA.

First we review some basic pcf theoretic definitions. Let $\mathfrak{a}$ be a set of regular cardinals with $|\mathfrak{a}| < \min(\mathfrak{a})$. We write $\prod \mathfrak{a}$ for the set of functions f with domain $\mathfrak{a}$ such that $f(\kappa) \in \kappa$ for all κ in $\mathfrak{a}$. Suppose D is a filter on $\mathfrak{a}$. For functions f and g in $\prod \mathfrak{a}$, let $f \leq_D g$ if the set $\{\kappa \in \mathfrak{a} : f(\kappa) \leq g(\kappa)\}$ is in D, and similarly with $f <_D g$. For a regular cardinal μ, a sequence of functions $\vec{f} = \langle f_i : i < \mu \rangle$ in $\prod \mathfrak{a}$ is a *scale in* $\prod \mathfrak{a}/D$ if $f_i \leq_D f_j$ for $i < j < \mu$, and for any function h in $\prod \mathfrak{a}$, $h \leq_D f_i$ for some $i < \mu$. The *cobounded filter* on $\mathfrak{a}$ is the filter generated by the complements of the proper initial segments of $\mathfrak{a}$.

In Theorem 1.5 of Chapter 2 of [14], Shelah proved:

Theorem 3.1. *Let λ be a singular cardinal. Then there is a set $\mathfrak{a}$ of regular cardinals cofinal in λ with order type* $\mathrm{cf}(\lambda)$ *and* $\mathrm{cf}(\lambda) < \min(\mathfrak{a})$ *such that there exists a scale $\vec{f} = \langle f_i : i < \lambda^+ \rangle$ in $\prod \mathfrak{a}/D$, where D is the cobounded filter on $\mathfrak{a}$.*

A scale $\vec{f} = \langle f_i : i < \mu \rangle$ in $\prod \mathfrak{a}/D$ is said to be an ω_1-*better scale* if for every limit ordinal $\alpha < \mu$ with cofinality ω_1, there is a club set $c \subseteq \alpha$ with order type ω_1 such that for any β in c, there is a set A in D such that for all γ in $c \cap \beta$ and κ in A, $f_\gamma(\kappa) < f_\beta(\kappa)$. (See [6] for more information on better scales.)

Let F be a filter on a set X and κ a cardinal. We say that F is κ-*generated* if there is a family $\{A_i : i < \xi\}$ of fewer than κ many sets in F such that for any set $A \subseteq X$, A is in F iff $A_i \subseteq A$ for some $i < \xi$. We say that F is *countably generated* if F is ω_1-generated. For any infinite cardinal λ, the club filter on $P_{\omega_1}(\lambda)$ is $(2^\lambda)^+$-generated, since for every club $C \subseteq P_{\omega_1}(\lambda)$ there is a function $F : [\lambda]^{<\omega} \to \lambda$ such that C contains the club set of a in $P_{\omega_1}(\lambda)$ which are closed under F.

The main theorem of this section is:

Theorem 3.2. *Let λ be a singular cardinal with countable cofinality. Let $\mathfrak{a}$ be a countable set of regular uncountable cardinals cofinal in λ, and μ a regular cardinal larger than λ. Suppose that:*

1. *The club filter on $P_{\omega_1}(\lambda)$ is μ^+-generated,*
2. *There is a countably generated filter D on $\mathfrak{a}$ and an ω_1-better scale $\langle f_i : i < \mu\rangle$ in $\prod \mathfrak{a}/D$.*

Then every stationary subset of $P_{\omega_1}(\lambda)$ contains a stationary subset which does not reflect to any set of size $\aleph_1$.

Proof. For any countable set $b \subseteq \lambda$, let χ_b denote the function in $\prod \mathfrak{a}$ defined by letting $\chi_b(\kappa) = \sup(b \cap \kappa)$ for all κ in $\mathfrak{a}$. Note that for any function h in $\prod \mathfrak{a}$, there are club many b in $P_{\omega_1}(\lambda)$ such that $h <_D \chi_b$.

Fix a stationary set $S \subseteq P_{\omega_1}(\lambda)$. We will find a stationary set $T \subseteq S$ which does not reflect to any set of size $\aleph_1$. Fix a family $\{C_i : i < \mu\}$ of club subsets of $P_{\omega_1}(\lambda)$ such that for every club $C \subseteq P_{\omega_1}(\lambda)$, there is $i < \mu$ such that $C_i \subseteq C$. Clearly any set $T \subseteq P_{\omega_1}(\lambda)$ is stationary if it has non-empty intersection with C_i for all $i < \mu$.

To construct T, we define by induction a sequence $\langle b_i : i < \mu\rangle$ and a normal function $G : \mu \to \mu$ such that for all $i < \mu$:

(I) b_i is in $S \cap C_i$,
(II) $f_{G(i)} <_D \chi_{b_i} <_D f_{G(i+1)}$.

Suppose $i < \mu$ and $\langle b_j : j < i\rangle$ and $G \restriction i$ are defined as required. If $i = 0$ let $G(i) = 0$. If i is a limit ordinal, let $G(i) = \sup\{G(j) : j < i\}$. If $i = j + 1$, choose $G(i)$ larger than $G(j)$ so that $\chi_{b_j} <_D f_{G(i)}$. This is possible since $\vec{f}$ is a scale. Now applying the stationarity of S, choose b_i in $S \cap C_i$ satisfying $f_{G(i)} <_D \chi_{b_i}$. This completes the construction.

Let $T = \{b_i : i < \mu\}$. By property (I), T is a stationary subset of S. Suppose for a contradiction that N is a subset of λ of size $\aleph_1$ and $P_{\omega_1}(N) \cap T$ is stationary in $P_{\omega_1}(N)$.

Let β be the least ordinal such that $P_{\omega_1}(N) \cap \{b_i : i < \beta\}$ is stationary in $P_{\omega_1}(N)$. Choose an increasing and continuous sequence $\langle a_i : i < \omega_1\rangle$ of countable sets with union equal to N. Then $\{a_i : i < \omega_1\} \cap \{b_i : i < \beta\}$ is stationary in $P_{\omega_1}(N)$. So there is a stationary set $A \subseteq \omega_1$ such that $\{a_i : i \in A\} \subseteq \{b_i : i < \beta\}$.

Claim 3.3. *The cofinality of β is equal to ω_1.*

Proof. If $\mathrm{cf}(\beta) > \omega_1$, then there is $\gamma < \beta$ such that $\{a_i : i \in A\} \subseteq \{b_i : i < \gamma\}$, which contradicts the minimality of β. Suppose that $\mathrm{cf}(\beta) = \omega$, and fix a cofinal function $f : \omega \to \beta$. Then $\{b_i : i < \beta\} \cap P_{\omega_1}(N)$ is equal to $\bigcup_{n<\omega}(\{b_i : i < f(n)\} \cap P_{\omega_1}(N))$. Since the club filter on $P_{\omega_1}(N)$ is countably complete, there is $n < \omega$ such that $\{b_i : i < f(n)\} \cap P_{\omega_1}(N)$ is stationary in $P_{\omega_1}(N)$. Again this contradicts the minimality of β. □

Fix a collection of sets $\{X_n : n < \omega\}$ which generates D. Since $G : \mu \to \mu$ is a normal function, the cofinality of $G(\beta)$ is equal to ω_1. Applying the fact that $\vec{f}$ is an ω_1-better scale, let $c \subseteq G(\beta)$ be a club subset of $G(\beta)$ with order type ω_1 satisfying: for all α in c, there is $n < \omega$ such that for all γ in $c \cap \alpha$ and κ in X_n, $f_\gamma(\kappa) < f_\alpha(\kappa)$. By intersecting c with the club $G[\beta]$, we may assume without loss of generality that c is contained in the range of G.

By the normality of G, $G^{-1}[c]$ is a club subset of β with order type ω_1. Enumerate $G^{-1}[c]$ in increasing order as $\langle \beta_i : i < \omega_1 \rangle$. Then the function $i \mapsto \beta_i$ is a normal cofinal function from ω_1 to β.

For $i < \omega_1$ let $h_i = f_{G(\beta_i)}$. Then by the choice of c,

$$\forall i < \omega_1 \; \exists n < \omega \; \forall j < i \; \forall \kappa \in X_n \; h_j(\kappa) < h_i(\kappa).$$

Since A is stationary in ω_1, fix a stationary set $A_1 \subseteq A$ and $n_1 < \omega$ such that:

$$\forall i \in A_1 \; \forall j < i \; \forall \kappa \in X_{n_1} \; h_j(\kappa) < h_i(\kappa).$$

Then in particular:

Statement 3.4. *For all $j < i$ in A_1 and for all κ in X_{n_1}, $h_j(\kappa) < h_i(\kappa)$.*

Since $\{a_i : i \in A_1\}$ is a subset of $\{b_j : j < \beta\}$, we can define $g : A_1 \to \beta$ by letting $g(i)$ be an ordinal less than β such that $a_i = b_{g(i)}$.

Claim 3.5. *There is a club set $C \subseteq \omega_1$ such that for all $i < j$ in C, if i is in A_1 then $\beta_i \leq g(i) < \beta_j$.*

Proof. Let C_1 be the set of α in ω_1 such that for all i in $\alpha \cap A_1$, $g(i) < \beta_\alpha$. It follows easily from the fact the map $\alpha \mapsto \beta_\alpha$ is increasing that C_1 is a club set.

Assume for a contradiction that there does not exist a club $C \subseteq C_1$ consisting of i such that, if i is in A_1, then $\beta_i \leq g(i)$. Then there is a stationary set $A' \subseteq A_1$ such that $g(i) < \beta_i$ for all i in A'. If ν is any limit ordinal in A', then $g(\nu) < \beta_\nu = \sup_{i<\nu} \beta_i$, so there is $i_\nu < \nu$ such that $g(\nu) < \beta_{i_\nu}$. By Fodor's Lemma, there is a stationary set $A'' \subseteq A'$ and $\gamma < \omega_1$ such that for all ν in A'', $g(\nu) < \beta_\gamma$. But $a_\nu = b_{g(\nu)}$. So the set $\{a_\nu : \nu \in A''\}$ is a stationary subset of $P_{\omega_1}(N)$ contained in $\{b_i : i < \beta_\gamma\}$, contradicting the minimality of β. □

Fix a club C as in Claim 3.5. We now thin out $A_1 \cap C$ to A_2. For each i in $A_1 \cap C$, let i^* be the least ordinal in $A_1 \cap C$ above i. Given i in $A_1 \cap C$, we know that $\beta_i \leq g(i) < \beta_{i^*}$. Therefore $g(i) + 1 \leq \beta_{i^*}$. So by property (II) in the construction of T, we have:

$$\chi_{a_i} = \chi_{b_{g(i)}} <_D f_{G(g(i)+1)} \leq_D f_{G(\beta_{i^*})} = h_{i^*},$$

and therefore

$$\chi_{a_i} <_D h_{i^*}.$$

Choose $m(i) < \omega$ so that for all κ in $X_{m(i)}$, $\chi_{a_i}(\kappa) < h_{i^*}(\kappa)$. Fix a stationary set $A_2 \subseteq A_1 \cap C$ and $n_2 < \omega$ such that for all i in A_2, $m(i) = n_2$. Then we have:

Statement 3.6. *For all i in A_2 and κ in X_{n_2}, $\chi_{a_i}(\kappa) < h_{i^*}(\kappa)$.*

Let $X = X_{n_1} \cap X_{n_2}$, which is in D. Since A_2 is a stationary subset of ω_1, let x be a closed set of ordinals in A_2 with order type $\omega + 1$, and let $\nu = \max(x)$. Since the sequence $\langle a_i : i < \omega_1 \rangle$ is increasing and continuous, $a_\nu = \bigcup \{a_i : i \in x \cap \nu\}$. Therefore for all κ in X,

$$\sup(a_\nu \cap \kappa) = \sup\{\sup(a_i \cap \kappa) : i \in x \cap \nu\}.$$

In other words:

Statement 3.7. *For all κ in X, $\chi_{a_\nu}(\kappa) = \sup\{\chi_{a_i}(\kappa) : i \in x \cap \nu\}$.*

Consider i in $x \cap \nu$. Then for all κ in X,

$$\chi_{a_i}(\kappa) < h_{i^*}(\kappa) < h_\nu(\kappa).$$

The first inequality is by Statement 3.6, and the second inequality is by Statement 3.4, noting that $i^* < \nu$. So for all i in $x \cap \nu$ and κ in X,

$$\chi_{a_i}(\kappa) < h_\nu(\kappa).$$

Since this inequality holds for all i in $x \cap \nu$, by Statement 3.7 we get that for all κ in X,

$$\chi_{a_\nu}(\kappa) \leq h_\nu(\kappa).$$

Now $a_\nu = b_{g(\nu)}$, so for all κ in X,

$$\chi_{b_{g(\nu)}}(\kappa) \leq h_\nu(\kappa).$$

By the property of C, $\beta_\nu \leq g(\nu)$, and by definition, $h_\nu = f_{G(\beta_\nu)}$. Hence $h_\nu \leq_D f_{G(g(\nu))}$. Therefore

$$\chi_{b_{g(\nu)}} \leq_D f_{G(g(\nu))}.$$

On the other hand, by property (II) in the construction of T,

$$f_{G(g(\nu))} <_D \chi_{b_{g(\nu)}}.$$

These last two inequalities give a contradiction. $\square$

In the rest of the section, we consider situations in which the hypotheses of Theorem 3.2 hold.

Proposition 3.8. *Let λ be a singular cardinal and μ a regular cardinal larger than λ. Let $\mathfrak{a}$ be a set of regular cardinals cofinal in λ with $|\mathfrak{a}| < \min(\mathfrak{a})$. Suppose there exists a filter D on $\mathfrak{a}$ and a scale of length μ in $\prod \mathfrak{a}/D$. Assume there exist sequences*

$$\langle c_\alpha : \alpha \in \mu \cap \mathrm{cof}(\omega_1) \rangle, \quad \langle \mathcal{C}_\beta : \beta \in \mu \cap \mathrm{cof}(\omega) \rangle$$

such that:

1. *for all $\alpha \in \mu \cap \mathrm{cof}(\omega_1)$, c_α is a club subset of α with order type ω_1,*
2. *for all $\beta \in \mu \cap \mathrm{cof}(\omega)$, $\mathcal{C}_\beta$ is a family of less than μ many closed countable subsets of β,*
3. *for all $\alpha \in \mu \cap \mathrm{cof}(\omega_1)$, if β is a limit point of c_α, then $c_\alpha \cap \beta$ is in $\mathcal{C}_\beta$.*

Then there is an ω_1-better scale in $\prod \mathfrak{a}/D$ of length μ.

A similar statement was shown to be true in the proof of Theorem 4.1 of [6], using weak square $\square^*_\lambda$ in the particular case when $\mu = \lambda^+$. Our argument is based on their proof.

Proof. Let $\vec{f} = \langle f_i : i < \mu \rangle$ be a scale in $\prod \mathfrak{a}/D$. We define by induction a cofinal subsequence $\vec{g} = \langle g_i : i < \mu \rangle$ of $\vec{f}$. Suppose $\langle g_i : i < \beta \rangle$ is defined for some $\beta < \mu$. If β is not a limit ordinal of cofinality ω, then choose g_β from the set $\{f_i : \beta \leq i < \mu\}$ so that $g_i \leq_D g_\beta$ for all $i < \beta$.

Suppose β is a limit ordinal of cofinality ω. For each c in $\mathcal{C}_\beta$, define g_c in $\prod \mathfrak{a}$ by letting, for κ in $\mathfrak{a}$, $g_c(\kappa) = \sup\{g_i(\kappa) : i \in c\}$. Note that $g_c(\kappa) \in \kappa$, since κ is regular and uncountable and c is countable. Since $|\mathcal{C}_\beta| < \mu$, choose g_β in the set $\{f_i : \beta \leq i < \kappa^+\}$ so that $g_c \leq_D g_\beta$ for all c in $\mathcal{C}_\beta$, and $g_i \leq_D g_\beta$ for all $i < \beta$. This completes the construction of $\vec{g}$.

To show $\vec{g}$ is an ω_1-better scale in $\prod \mathfrak{a}/D$, consider α in $\mu \cap \mathrm{cof}(\omega_1)$. Let d be the club set of limit points of c_α. Consider β in d. Since β is a limit point of c_α,

$c_\alpha \cap \beta$ is in $\mathcal{C}_\beta$. So by construction, $g_{(c_\alpha \cap \beta)} \leq_D g_\beta$. Fix a set A in D such that for all κ in A, $g_{(c_\alpha \cap \beta)}(\kappa) \leq g_\beta(\kappa)$. Then for all γ in $d \cap \beta$ and κ in A, γ is in $c_\alpha \cap \beta$, therefore $g_\gamma(\kappa) \leq g_{(c_\alpha \cap \beta)}(\kappa) \leq g_\beta(\kappa)$. ◻

When $\mu = \kappa^+$ is a successor cardinal, then the existence of the sequences described in the last proposition follows from weak square $\square^*_\kappa$.

Definition 3.9. *Let κ be an uncountable cardinal and $\nu \leq \kappa^+$. A sequence $\langle \mathcal{C}_\alpha : \alpha < \kappa^+ \rangle$ is a $\square_{\kappa,\nu}$-sequence if for all limit ordinals $\alpha < \kappa^+$:*

1. *$1 \leq |\mathcal{C}_\alpha| \leq \nu$,*
2. *for all c in $\mathcal{C}_\alpha$, c is a club subset of α, and if $\mathrm{cf}(\alpha) < \kappa$ then $\mathrm{o.t.}(c) < \kappa$,*
3. *for all c in $\mathcal{C}_\alpha$, if β is a limit point of c, then $c \cap \beta$ is in $\mathcal{C}_\beta$.*

We say that $\square_{\kappa,\nu}$ holds if there exists a $\square_{\kappa,\nu}$-sequence.

We refer to a $\square_{\kappa,\kappa}$-sequence as a weak square sequence, or a $\square^*_\kappa$-sequence. We say that $\square^*_\kappa$ holds if there exists a $\square^*_\kappa$-sequence. If $\square^*_\kappa$ holds, then there is a $\square^*_\kappa$-sequence $\langle \mathcal{C}_\alpha : \alpha < \kappa^+ \rangle$ such that for each limit ordinal $\alpha < \kappa^+$, there is a club c in $\mathcal{C}_\alpha$ with order type equal to $\mathrm{cf}(\alpha)$; for a proof of this fact, see page 176 of [7]. Both $\square_\kappa$ (that is, $\square_{\kappa,1}$) and $\square^*_\kappa$ are due to Jensen [10]. The definition of $\square_{\kappa,\nu}$, where $1 < \nu < \kappa$, is due to Schimmerling [13].

Lemma 3.10. *Let κ be an uncountable cardinal and suppose $\square^*_\kappa$ holds. Then there exist sequences*

$$\langle c_\alpha : \alpha \in \kappa^+ \cap \mathrm{cof}(\omega_1) \rangle, \quad \langle \mathcal{C}_\beta : \beta \in \kappa^+ \cap \mathrm{cof}(\omega) \rangle$$

such that:

1. *for all $\alpha \in \kappa^+ \cap \mathrm{cof}(\omega_1)$, c_α is a club subset of α with order type ω_1,*
2. *for all $\beta \in \kappa^+ \cap \mathrm{cof}(\omega)$, $\mathcal{C}_\beta$ is a family of less than κ^+ many closed countable subsets of β,*
3. *for all $\alpha \in \kappa^+ \cap \mathrm{cof}(\omega_1)$, if β is a limit point of c_α, then $c_\alpha \cap \beta$ is in $\mathcal{C}_\beta$.*

Proof. Fix a $\square^*_\kappa$-sequence $\langle \mathcal{D}_\alpha : \alpha < \kappa^+ \rangle$ with the property that for every limit ordinal $\alpha < \kappa^+$, there is a club in $\mathcal{D}_\alpha$ with order type $\mathrm{cf}(\alpha)$. Define $\langle c_\alpha : \alpha \in \kappa^+ \cap \mathrm{cof}(\omega_1) \rangle$ by choosing for each α in $\kappa^+ \cap \mathrm{cof}(\omega_1)$ a club set c_α in $\mathcal{D}_\alpha$ with order type ω_1. For each β in $\kappa^+ \cap \mathrm{cof}(\omega)$, define $\mathcal{C}_\beta$ as the collection of countable sets in $\mathcal{D}_\beta$. Then properties (1) and (2) are immediate, and (3) follows from Definition 3.9(3). ◻

Corollary 3.11. *Let λ be a singular cardinal with cofinality ω. Assume $2^\lambda = \lambda^+$ and $\square^*_\lambda$ holds. Then every stationary subset of $P_{\omega_1}(\lambda)$ contains a stationary subset which does not reflect to any set of size $\aleph_1$.*

Proof. It suffices to verify the hypotheses of Theorem 3.2 in the case $\mu = \lambda^+ = 2^\lambda$. We know that the club filter on $P_{\omega_1}(\lambda)$ is $(2^\lambda)^+$-generated. By Theorem 3.1, fix a set $\mathfrak{a}$ of regular uncountable cardinals cofinal in λ with order type ω and a scale $\vec{f} = \langle f_i : i < \lambda^+ \rangle$ in $\prod \mathfrak{a}/D$, where D is the cobounded filter on $\mathfrak{a}$. Since the order type of $\mathfrak{a}$ is ω, the cobounded filter on $\mathfrak{a}$ is countably generated. By $\square^*_\lambda$, Lemma 3.10, and Proposition 3.8, there is an ω_1-better scale of length λ^+ in $\prod \mathfrak{a}/D$. So all the hypotheses of Theorem 3.2 are true. ◻

We remark that it was pointed out by Toshimichi Usuba that assuming the weaker statement ADS_λ in place of $\square^*_\lambda$ in Corollary 3.11, one can prove that every stationary subset of $P_{\omega_1}(\lambda)$ contains a stationary subset which does not reflect to any set of size $\aleph_1$ which contains $\aleph_1$ ([16]). (See Section 4 of [6] for the definition of ADS_λ.)

We can now prove the consistency of PFA with dense non-reflection in $P_{\omega_1}(\lambda)$, where λ is a singular cardinal with cofinality ω. This will follow immediately from Corollary 3.11 and a theorem of Magidor [12].

Theorem 3.12 (Magidor)**.** *Suppose PFA is consistent. Then PFA is consistent with the statement that $2^\alpha = \alpha^+$ for all $\alpha \geq \omega_1$ and $\square_{\kappa,\aleph_2}$ holds for all $\kappa \geq \omega_2$.*

Corollary 3.13. *Suppose PFA is consistent. Then PFA is consistent with the statement that for every singular cardinal λ with cofinality ω, every stationary subset of $P_{\omega_1}(\lambda)$ contains a stationary subset which does not reflect to any set of size $\aleph_1$.*

We end this section by describing circumstances in which Theorem 3.2 applies, without assuming $2^\lambda = \lambda^+$. First we need to review some facts from pcf theory without proof.

Let $\mathfrak{a}$ be a set of regular cardinals with $|\mathfrak{a}|^+ < \min(\mathfrak{a})$. For any cardinal κ, $J_{<\kappa}(\mathfrak{a})$ denotes the ideal consisting of sets $b \subseteq \mathfrak{a}$ such that for any ultrafilter D on $\mathfrak{a}$ with $b \in D$, the cofinality of $(\prod \mathfrak{a}, \leq_D)$ is less than κ. Clearly $J_{<\kappa_1}(\mathfrak{a}) \subseteq J_{<\kappa_2}(\mathfrak{a})$ for cardinals $\kappa_1 < \kappa_2$. If κ is a limit cardinal, then $J_{<\kappa}(\mathfrak{a})$ is the union of the ideals $J_{<\mu}(\mathfrak{a})$, where $\mu < \kappa$ is a cardinal; see page 211 of [5].

Recall that $\mathrm{pcf}(\mathfrak{a})$ denotes the set of regular cardinals κ for which there exists a filter D on $\mathfrak{a}$ and a scale of length κ in $\prod \mathfrak{a}/D$. The set $\mathrm{pcf}(\mathfrak{a})$ always has a maximum element; see 1.8 of [5].

Here are the facts from pcf theory we will use:

Fact 3.14. *Let κ be a regular cardinal.*

(1) κ is in $\mathrm{pcf}(\mathfrak{a})$ iff $J_{<\kappa^+}(\mathfrak{a}) \setminus J_{<\kappa}(\mathfrak{a})$ is non-empty.

(2) If b is in $J_{<\kappa^+}(\mathfrak{a}) \setminus J_{<\kappa}(\mathfrak{a})$, then there is a scale of length κ in $\prod \mathfrak{a}/D$, where D is the filter on $\mathfrak{a}$ generated by the set b together with the dual filter of $J_{<\kappa}(\mathfrak{a})$.

Fact 3.15. *If $\mathfrak{a}$ is an interval of regular cardinals, then so is $\mathrm{pcf}(\mathfrak{a})$.*

Fact 3.16. *There exists a sequence $\langle b_\kappa : \kappa \in \mathrm{pcf}(\mathfrak{a})\rangle$ of subsets of $\mathfrak{a}$ such that for each κ in $\mathrm{pcf}(\mathfrak{a})$, $J_{<\kappa^+}(\mathfrak{a})$ is the ideal on $\mathfrak{a}$ generated by $J_{<\kappa}(\mathfrak{a}) \cup \{b_\kappa\}$.*

Fact 3.17. *Let λ be a singular cardinal with cofinality ω. Assume there is $\omega_1 \leq \delta < \lambda$ such that there are only countably many regular cardinals between δ and λ. Let $\mathfrak{a} = (\delta, \lambda) \cap \mathrm{Reg}$. Then $\max(\mathrm{pcf}(\mathfrak{a}))$ is equal to the cofinality of the partial ordering $([\lambda]^{\aleph_0}, \subseteq)$.*

Fact 3.14 is proven as Lemma 1.4 and Corollary 4.4 of [5]. Facts 3.15 and 3.16 are proven as 2.2 and 7.9 of [5] respectively. Fact 3.17 is proven as Theorem 5.11 of [1].

The *pcf conjecture* is the statement that for any set of regular cardinals $\mathfrak{a}$ with $|\mathfrak{a}| < \min(\mathfrak{a})$, $\mathrm{pcf}(\mathfrak{a})$ has size $|\mathfrak{a}|$. It is not known whether the pcf conjecture is a theorem of ZFC.

Lemma 3.18. *Let $\mathfrak{a}$ be a countable set of regular cardinals with $\omega_1 < \min(\mathfrak{a})$ such that $\mathrm{pcf}(\mathfrak{a})$ is countable. Then for every κ in $\mathrm{pcf}(\mathfrak{a})$, there is a countably generated filter D on $\mathfrak{a}$ and a scale of length κ in $\prod \mathfrak{a}/D$.*

Proof. Note that for all κ in $\mathrm{pcf}(\mathfrak{a})$, $J_{<\kappa^+}(\mathfrak{a})$ is countably generated. This can be proven by an easy induction using the fact that $\mathrm{pcf}(\mathfrak{a})$ is countable, together with Fact 3.14(1) and Fact 3.16. Let κ be in $\mathrm{pcf}(\mathfrak{a})$. Let D_κ be the dual filter of $J_{<\kappa^+}(\mathfrak{a})$. Clearly D_κ is countably generated. By Fact 3.14(1), fix a set b in $J_{<\kappa^+}(\mathfrak{a}) \setminus J_{<\kappa}(\mathfrak{a})$. Then by Fact 3.14(2), there is a scale of length κ in $\prod \mathfrak{a}/D$, where D is the countably generated filter on $\mathfrak{a}$ generated by the set b together with D_κ. $\square$

Corollary 3.19. *Let λ be a singular strong limit cardinal of cofinality ω. Suppose there is $\omega_1 \leq \delta < \lambda$ such that there are only countably many regular cardinals between δ and λ. Let $\mathfrak{a} = (\delta, \lambda) \cap \mathrm{Reg}$. Assume $\mathrm{pcf}(\mathfrak{a})$ is countable. Then $2^\lambda = \kappa^+$ for some cardinal $\kappa \geq \lambda$. Assume that $\square^*_\kappa$ holds. Then every stationary subset of $P_{\omega_1}(\lambda)$ has a stationary subset which does not reflect to any set of size $\aleph_1$.*

Proof. Since λ is a strong limit cardinal of cofinality ω, $2^\lambda = \lambda^\omega$ and $\lambda^\omega = \mathrm{cf}([\lambda]^{\aleph_0}) \cdot 2^\omega = \mathrm{cf}([\lambda]^{\aleph_0})$. Combining these facts with Fact 3.17, we get that $2^\lambda = \max(\mathrm{pcf}(\mathfrak{a}))$. Now $\mathfrak{a}$ is an interval of regular cardinals, so $\mathrm{pcf}(\mathfrak{a})$ is as well by Fact 3.15. As $\mathrm{pcf}(\mathfrak{a})$ is countable, $2^\lambda = \max(\mathrm{pcf}(\mathfrak{a}))$ is less than $\lambda^{+\omega_1}$. Since there are no regular limit cardinals between λ and $\lambda^{+\omega_1}$, $2^\lambda = \kappa^+$ for some $\kappa \geq \lambda$. By Lemma 3.18, there is a countably generated filter D on $\mathfrak{a}$ and a scale of length 2^λ in $\prod \mathfrak{a}/D$. So assuming $\square^*_\kappa$ holds, it follows from Lemma 3.10 and Proposition 3.8 that there is an ω_1-better scale of length 2^λ in $\prod \mathfrak{a}/D$. $\square$

4. Adding Non-Reflecting Sets for a Singular Cardinal

Let λ be a singular cardinal with cofinality ω. By Section 1, we can add a non-reflecting stationary subset of $P_{\omega_1}(\lambda)$, using for example conditions of size less than ω_2. But this poset will add many subsets of ω_2, and so λ will not be a strong limit in the extension. In this section we present a $(\lambda+1)$-strategically closed forcing poset for adding a non-reflecting stationary subset of $P_{\omega_1}(\lambda)$. Using a suitable product of this forcing, we obtain dense non-reflection in $P_{\omega_1}(\lambda)$ with a forcing poset which does not add subsets to λ.

Fix a set $\mathfrak{a}$ of regular cardinals larger than ω_1 with order type ω which is cofinal in λ. For functions f and g in $\prod \mathfrak{a}$, we write $g \leq^* f$ to indicate $g \leq_D f$, where D is the cobounded filter on $\mathfrak{a}$, and similarly for $g <^* f$. Recall that for a countable set $b \subseteq \lambda$, χ_b is the function in $\prod \mathfrak{a}$ defined by letting $\chi_b(\kappa) = \sup(b \cap \kappa)$ for all κ in $\mathfrak{a}$.

Define $\mathbb{P}$ as the set of pairs $\langle X, f\rangle$ satisfying:

- $X \subseteq P_{\omega_1}(\lambda)$,
- $|X| \leq \lambda$,
- f is in $\prod \mathfrak{a}$,
- for all a in X, $\chi_a <^* f$,
- for any set N in $[\lambda]^{\aleph_1}$, $P_{\omega_1}(N) \cap X$ is non-stationary in $P_{\omega_1}(N)$.

We let $\langle Y, g\rangle \leq \langle X, f\rangle$ if:

- $X \subseteq Y$,
- $f \leq^* g$,
- for all b in $Y \setminus X$, $f \leq^* \chi_b$.

Note that $\mathbb{P}$ has size 2^λ, and so is $(2^\lambda)^+$-c.c.

Proposition 4.1. *The forcing poset $\mathbb{P}$ is $(\lambda+1)$-strategically closed.*

Proof. Since λ is singular, by an easy argument it suffices to show that $\mathbb{P}$ is ν-strategically closed for all $\nu < \lambda$. So let $\nu < \lambda$ be given. We describe a strategy for Player II. At successor stages, Player II just repeats Player I's last play. Suppose $\delta < \nu$ is a limit ordinal and a sequence of plays $\langle \langle X_i, f_i \rangle : 0 < i < \delta \rangle$ has been determined. Define $X_\delta = \bigcup_{0<i<\delta} X_i$. Define f_δ by letting $f_\delta(\kappa) = 0$ if $\kappa \in \mathfrak{a} \cap \nu$, and $f_\delta(\kappa) = (\sup_{0<i<\delta} f_i(\kappa)) + 1$ if κ is in $\mathfrak{a} \setminus \nu$.

We claim that $\langle X_\delta, f_\delta \rangle$ is a condition in $\mathbb{P}$ which is below $\langle X_i, f_i \rangle$ for all $0 < i < \delta$. Thus the game continues through all stages less than ν. We will prove that for any N in $[\lambda]^{\aleph_1}$, $P_{\omega_1}(N) \cap X_\delta$ is non-stationary in $P_{\omega_1}(N)$. The remaining properties are easy to check.

Suppose for a contradiction N is in $[\lambda]^{\aleph_1}$ and $P_{\omega_1}(N) \cap X_\delta$ is stationary in $P_{\omega_1}(N)$. Fix an increasing and continuous sequence $\langle a_i : i < \omega_1 \rangle$ of countable sets with union equal to N. Then there is a stationary set $A \subseteq \omega_1$ such that $\{a_i : i \in A\} \subseteq X_\delta$.

Claim 4.2. *The cofinality of δ is ω_1.*

The proof is similar to the proof of Claim 3.3.

Fix an increasing and continuous sequence $\langle \beta_i : i < \omega_1 \rangle$ cofinal in δ. Define $g : A \to \omega_1$ by letting $g(i)$ be the least ordinal such that a_i is in $X_{\beta_{g(i)}}$. Note that $g(i)$ is always a successor ordinal, since the map $j \mapsto \beta_j$ is normal and $X_\xi = \bigcup_{j<\xi} X_j$ for every limit ordinal $\xi < \delta$.

Claim 4.3. *There is a club $C \subseteq \omega_1$ such that for all $i < j$ in C, if i is in A then $i < g(i) < j$.*

The proof is similar to the proof of Claim 3.5.

Fix a club C as in Claim 4.3. We thin out $A \cap C$ to A_1. Consider i in $A \cap C$. Let i^* be the least ordinal in $A \cap C$ above i. Then $\beta_i < \beta_{g(i)} < \beta_{i^*}$. Since a_i is in $X_{\beta_{g(i)}}$, by the definition of $\mathbb{P}$ we have that $\chi_{a_i} <^* f_{\beta_{g(i)}} \leq^* f_{\beta_{i^*}}$. So there is $\kappa(i)$ in $\mathfrak{a}$ such that that for all κ in $\mathfrak{a} \setminus \kappa(i)$, $\chi_{a_i}(\kappa) < f_{\beta_{i^*}}(\kappa)$. Since $\mathfrak{a}$ is countable and $A \cap C$ is stationary, we can fix a stationary set $A_1 \subseteq A \cap C$ and κ_1 in $\mathfrak{a}$ so that for all i in A_1, $\kappa(i) = \kappa_1$. Then for all i in A_1 and κ in $\mathfrak{a} \setminus \kappa_1$, $\chi_{a_i}(\kappa) < f_{\beta_{i^*}}(\kappa)$.

By the stationarity of A_1, let x be a closed subset of A_1 with order type $\omega+1$, and let $\nu = \max(x)$. As $\langle a_i : i < \omega_1 \rangle$ is increasing and continuous, $a_\nu = \bigcup \{a_i : i \in x \cap \nu\}$. It follows that for all κ in $\mathfrak{a}$, $\chi_{a_\nu}(\kappa) = \sup\{\chi_{a_i}(\kappa) : i \in x \cap \nu\}$.

Consider κ in $\mathfrak{a}$ which is larger than κ_1 and ν. Then for all i in $x \cap \nu$, $i^* < \nu$, so $\chi_{a_i}(\kappa) < f_{\beta_{i^*}}(\kappa) \leq \sup_{0<j<\beta_\nu} f_j(\kappa)$. Since this is true for all i in $x \cap \nu$, $\chi_{a_\nu}(\kappa) \leq \sup_{0<j<\beta_\nu} f_j(\kappa)$. By the definition of f_{β_ν}, $\sup_{0<j<\beta_\nu} f_j(\kappa) < f_{\beta_\nu}(\kappa)$. So $\chi_{a_\nu} <^* f_{\beta_\nu}$. On the other hand, since $g(\nu) > \nu$, a_ν is in $X_{\beta_{g(\nu)}} \setminus X_{\beta_\nu}$. By the definition of the ordering on $\mathbb{P}$ we have $f_{\beta_\nu} \leq^* \chi_{a_\nu}$, which is a contradiction. □

Let $\dot{T}$ be a $\mathbb{P}$-name for the set $\bigcup \{X : \exists f \ \langle X, f \rangle \in \dot{G}\}$.

We claim that for any stationary set $S \subseteq P_{\omega_1}(\lambda)$, $\mathbb{P}$ forces $\dot{T} \cap \check{S}$ is stationary. For suppose p is in $\mathbb{P}$ and p forces $\dot{F} : [\lambda]^{<\omega} \to \lambda$ is a function. Since $\dot{F}$ is a name for a subset of the ground model of size λ and $\mathbb{P}$ is $(\lambda+1)$-strategically closed, there is $q \leq p$ and $F : [\lambda]^{<\omega} \to \lambda$ such that q forces $\dot{F} = \check{F}$. Let $q = \langle X, f \rangle$. Since S is stationary, we can choose a in S such that a is closed under F and $f \leq^* \chi_a$. Fix g in $\prod \mathfrak{a}$ such that $f <^* g$ and $\chi_a <^* g$. Then $\langle X \cup \{a\}, g \rangle$ is a condition in $\mathbb{P}$ which is below q and forces a is in $\dot{T} \cap \check{S}$ and is closed under $\dot{F}$.

Finally, we show $\mathbb{P}$ forces $\dot{T}$ does not reflect to any set of size $\aleph_1$. Suppose for a contradiction p is in $\mathbb{P}$ and $\dot{N}$ is a $\mathbb{P}$-name such that p forces $\dot{N}$ is in $[\lambda]^{\aleph_1}$ and

$P_{\omega_1}(\dot{N}) \cap \dot{T}$ is stationary. Since $\mathbb{P}$ is $(\lambda+1)$-strategically closed, there is $q \leq p$ and N in $[\lambda]^{\aleph_1}$ such that q forces $\dot{N} = \check{N}$. Let $q = \langle X, f \rangle$. Define g in $\prod \mathfrak{a}$ by letting $g(\kappa) = (\max\{f(\kappa), \sup(N \cap \kappa)\}) + 1$ for κ in $\mathfrak{a}$. Let $r = \langle X, g \rangle$. Clearly r is a condition and $r \leq q$. We claim that r forces $P_{\omega_1}(N) \cap \dot{T} = P_{\omega_1}(N) \cap X$, which is a contradiction since $P_{\omega_1}(N) \cap X$ is non-stationary. If not, then there is $\langle Y, h \rangle \leq \langle X, g \rangle$ and y in $Y \setminus X$ which is in $P_{\omega_1}(N)$. By the definition of the ordering on $\mathbb{P}$, $g \leq^* \chi_y$. But for all κ in $\mathfrak{a}$, since $y \subseteq N$ we have $\chi_y(\kappa) = \sup(y \cap \kappa) \leq \sup(N \cap \kappa) < g(\kappa)$. So $\chi_y <^* g$, which is a contradiction.

As in Section 1, we can obtain a model satisfying dense non-reflection in $P_{\omega_1}(\lambda)$ using a product forcing of the above poset. Assume $2^\lambda = \lambda^+$. Define $\mathbb{Q}$ as the product forcing consisting of partial functions $p : \lambda^{++} \to \mathbb{P}$ with domain of size less than λ^+, ordered in the usual way. Since $\mathbb{P}$ is $(\lambda+1)$-strategically closed, it is easy to show that $\mathbb{Q}$ is $(\lambda+1)$-strategically closed by using Player II's strategy separately on each coordinate. Since $2^\lambda = \lambda^+$, $\mathbb{P}$ has size λ^+, and a straightforward Δ-system argument shows that $\mathbb{Q}$ is λ^{++}-c.c. For all $\alpha < \lambda^{++}$, $\mathbb{Q}$ can be factored as $\mathbb{Q}_\alpha \times \mathbb{Q}^\alpha$ as in Section 1. The fact that $\mathbb{Q}$ is λ^{++}-c.c. implies that any stationary subset of $P_{\omega_1}(\lambda)$ in $V^{\mathbb{Q}}$ appears in $V^{\mathbb{Q}_\alpha}$ for some $\alpha < \lambda^{++}$. Thus the same argument as given in Section 1 shows that $\mathbb{Q}$ forces dense non-reflection in $P_{\omega_1}(\lambda)$.

5. Global Dense Non-Reflection

In this final section we prove the following theorem.

Theorem 5.1. *Suppose PFA holds. Then there is a class-sized ω_2–strategically closed partial order $\mathcal{P}$ preserving ZFC + PFA and forcing GCH above ω and dense non-reflection in $P_{\omega_1}(\lambda)$ for every cardinal $\lambda \geq \omega_2$ such that λ is either regular or of countable cofinality. Furthermore, $\mathcal{P}$ preserves cofinalities if GCH above ω holds in the ground model.*

We will sketch two proofs of Theorem 5.1. Before starting it will be convenient to fix some pieces of notation. Suppose $\mathcal{P}$ is a partial order and $\dot{X}$ is a $\mathcal{P}$–name for a subset of some ordinal α. We will say that $\dot{X}$ is a nice $\mathcal{P}$–name for a subset of α in case it consists of pairs of the form $\langle p, \check{\xi} \rangle$, with $p \in \mathcal{P}$ and $\check{\xi}$ the canonical name for an ordinal $\xi \in \alpha$.

When dealing with set–forcing, the following slightly nonstandard notion of two–step iteration will simplify the parts of the proof of Theorem 5.1 in which we need to compute cardinalities of partial orders: Suppose $\mathcal{P}$ is a poset. If $\dot{\mathcal{Q}}_0$ is a $\mathcal{P}$–name for a poset, then it is clear that, for some ordinal α, the two–step iteration $\mathcal{P} * \dot{\mathcal{Q}}_0$ (in the standard sense) is isomorphic to one of the form $\mathcal{P} * \dot{\mathcal{Q}}$ in which $\dot{\mathcal{Q}}$ is forced to consist of subsets of α. And furthermore, it is clear that this second iteration has a dense suborder consisting of pairs $\langle p, \dot{q} \rangle$ such that $\dot{q}$ is a nice $\mathcal{P}$–name for a subset of α. When $\dot{\mathcal{Q}}$ is a $\mathcal{P}$–name for a collection of subsets of a minimal ordinal α, we will define the two–step iteration $\mathcal{P} * \dot{\mathcal{Q}}$ as the suborder of the corresponding two–step iteration $\mathcal{I}$, taken in the standard sense, consisting precisely of the pairs $\langle p, \dot{q} \rangle \in \mathcal{I}$ such that $\dot{q}$ is a nice $\mathcal{P}$–name for a subset of α. The above remark shows that we do not lose any generality by doing so. Hence, we may and will identify every two–step iteration in the standard sense with an iteration (in the new sense) of the form $\mathcal{P} * \dot{\mathcal{Q}}$ for which there is a minimal α such that $\dot{\mathcal{Q}}$ is forced to consist of subsets of α.

The proof of Theorem 5.1 will involve reverse Easton iterations. By such an iteration we mean any forcing iteration in which direct limits are taken at all regular cardinals and inverse limits are taken everywhere else. In other words, a reverse Easton iteration is a forcing iteration of the form $\langle \mathcal{P}_\xi : \xi \in \Omega \rangle$, allowing Ω to be either an ordinal or ORD, based on a sequence $\langle \dot{\mathcal{Q}}_\xi : \xi \in \Omega \rangle$ of names such that each $\dot{\mathcal{Q}}_\xi$ is a $\mathcal{P}_\xi$–name for a poset, and such that $sup(supp(p) \cap \overline{\xi}) < \overline{\xi}$ whenever $\overline{\xi} \in \Omega$ is a regular cardinal, $\xi_0 \in \Omega$, and p is a condition in $\mathcal{P}_{\xi_0}$.[4] Every class forcing $\mathcal{P}$ preserves all of the ZFC axioms except possibly for the Power Set Axiom and the Axiom scheme of Replacement. In the case that $\mathcal{P}$ is the direct limit of a reverse Easton iteration as above with the additional property that for every λ there is some ξ such that the tail forcing $\mathcal{P}/G$ is forced to be λ–distributive in $V[G]$ for each $\mathcal{P}_\xi$–generic filter G, then $\mathcal{P}$ preserves these remaining axioms as well (see Section 2.2 of [9]).

The following easy general fact will be useful.

Lemma 5.2. *Let α be a regular cardinal, let Ω be* ORD *or a member of it, let $\xi_0 \in \Omega$, and let $\langle \mathcal{P}_\xi : \xi \leq \Omega \rangle$ be a forcing iteration, based on a sequence $\langle \dot{\mathcal{Q}}_\xi : \xi \in \Omega \rangle$ of names such that, for every $\xi \in [\xi_0, \Omega)$, $\dot{\mathcal{Q}}_\xi$ is $< \alpha$–strategically closed in $V^{\mathcal{P}_\xi}$. Suppose that $\mathcal{P}_{\xi_0}$ has the α–chain condition. Suppose in addition that $\{supp(q) \backslash \xi_0 : q \in \mathcal{P}_\Omega\}$ is closed under unions of $\subseteq$–increasing sequences of length less than α.*

Then, $\mathcal{P}_\Omega / G$ is $< \alpha$–strategically closed in $V[G]$ for every $\mathcal{P}_{\xi_0}$–generic filter G over V.

Let us assume PFA. By first forcing with an ω_2–directed closed forcing $\mathcal{P}^0$ if necessary we may assume that GCH holds above ω. $\mathcal{P}^0$ is the direct limit of a reverse Easton iteration as above in which, at each step ξ, we force with trivial forcing (that is, with $\{\emptyset\}$) unless ξ is an infinite V–cardinal above ω_1, in which case $\dot{\mathcal{Q}}_\xi$ is, in $V^{\mathcal{P}_\xi}$, the forcing for adding a subset of ξ by initial segments.

If $\xi \geq \omega_2$ is a V–cardinal and G is $\mathcal{P}_\xi$–generic over V, then $\mathcal{P}_{\xi+1}/G$ forces $|P(\gamma)^{V[G]}| \leq \xi$ for every $\gamma < \xi$. And, if in addition ξ is such that $|\mathcal{P}_{\xi^+}| \leq \xi$ (which will happen whenever ξ is a strong limit), then $\mathcal{P}^0/G'$ is easily seen to be ξ^+–directed closed in $V[G']$ for every $\mathcal{P}_{\xi^+}$–generic G'. It follows that $\mathcal{P}^0$ preserves ZFC and forces GCH above ω.

By a theorem from [11], any ω_2-directed closed forcing poset preserves PFA. Clearly $\mathcal{P}^0$ is ω_2–directed closed. If $\dot{\mathcal{R}}$ is a $\mathcal{P}^0$–name for a proper poset and $\dot{D}_i$ (for $i < \omega_1$) are names for dense subsets of $\dot{\mathcal{R}}$, then there is some ξ large enough so that all of these names are in fact $\mathcal{P}_\xi$–names, such that each $\dot{D}_i$ is a $\mathcal{P}_\xi$–name for a dense subset of $\dot{\mathcal{R}}$ and such that, in addition, $\dot{\mathcal{R}}$ is a $\mathcal{P}_\xi$–name for a proper poset. This is true because properness is a local condition (in other words, a poset $\mathbb{P}$ is proper if and only if $\mathbb{P}$ is proper in $H(\theta)$ for any large enough θ with $\mathbb{P} \in H(\theta)$), and for every θ there is a ξ_0 such that forcing with $\mathcal{P}^0/\dot{G}_{\xi_0}$ over $V^{\mathcal{P}_{\xi_0}}$ leaves $H(\theta)$ unchanged. But $\mathcal{P}_\xi$ is an ω_2–directed closed poset and PFA holds in V, which implies that in $V^{\mathcal{P}_\xi}$, and therefore in $V^{\mathcal{P}^0}$, there is a filter of $\dot{\mathcal{R}}$ intersecting each $\dot{D}_i$.

It may be worth mentioning that the use of reverse Easton supports is not relevant for the task of preserving ZFC and PFA. In fact, the direct limit $\overline{\mathcal{P}^0}$ of the

[4]Where $supp(p)$, the support of p, is the set of all $\xi < \xi_0$ such that $p \restriction \xi$ does not force (in $\mathcal{P}_\xi$) that $p(\xi)$ is the weakest condition in $\dot{\mathcal{Q}}_\xi$.

iteration obtained if we use full supports in the above definition of $\mathcal{P}^0$ would do as well.[5] The main use of reverse Easton supports is in ensuring, in a GCH–context, that the relevant posets have the relevant chain condition. This is exemplified in the constructions we are going to consider next.

We are left with the task of showing, under the assumption that GCH holds above ω and that PFA holds, that there is a class forcing $\mathcal{P}$ as in the conclusion of Theorem 5.1 which moreover preserves cofinalities. We will give two constructions of such a $\mathcal{P}$. In what follows, for a regular cardinal ξ, we let $\mathbb{Q}(\xi)$ denote the product forcing from Section 1 for forcing dense non-reflection in $P_{\omega_1}(\xi)$ using conditions of size less than ξ.

First construction. Let $\mathcal{R}$ be the class of all regular cardinals $\xi \geq \omega_2$. For the first construction we build a reverse Easton iteration $\langle \mathcal{P}_\xi \,:\, \xi \in \mathrm{Ord}\rangle$, based on a sequence $\langle \dot{\mathcal{Q}}_\xi \,:\, \xi \in \mathrm{Ord}\rangle$ of names, such that each $\dot{\mathcal{Q}}_\xi$ is forced to be trivial forcing unless it is true in $V^{\mathcal{P}_\xi}$ that ξ is a regular cardinal above ω_1 and that $2^\zeta = \zeta^+$ for all ζ with $\omega_1 \leq \zeta \leq \xi$.

In that case, if it is not the case in $V^{\mathcal{P}_\xi}$ that ξ is the successor of a singular cardinal λ of countable cofinality, then $\dot{\mathcal{Q}}_\xi$ is forced to be $\mathbb{Q}(\xi)$.

In the other case, if it holds in $V^{\mathcal{P}_\xi}$ that $\xi = \lambda^+$ and λ is a singular cardinal of countable cofinality, then $\dot{\mathcal{Q}}_\xi$ is forced to be the two–step iteration $\mathfrak{a}(\lambda) * (\mathbb{Q}(\xi) \times \mathbb{Q})$, where $\mathfrak{a}(\lambda)$ is the atomic forcing that picks an ω–sequence $\mathfrak{a}$ cofinal in λ consisting of regular cardinals in $(\omega_1,\, \lambda)$,[6] and where $\mathbb{Q}$ is as in Section 4 for λ and $\mathfrak{a}$.[7]

Lemma 5.3. *For each regular cardinal $\xi \geq \omega_2$, $\mathcal{P}_\xi$ has size at most ξ.*

Proof. This can be proved by induction on ξ using GCH above ω. The result when ξ is inaccessible is straightforward as in that case $\mathcal{P}_\xi$ is the direct limit of $\langle \mathcal{P}_{\xi'} \,:\, \xi' < \xi\rangle$.

If $\xi = \xi_0^+$ for ξ_0 a singular cardinal and $ot(\mathcal{R} \cap \xi_0) = \bar{\xi} \leq \xi_0$, then $\mathcal{P}_\xi$ is (isomorphic to) $\mathcal{P}_{\xi_0}$, and $|\mathcal{P}_{\xi_0}|$ is then the cardinality of a collection of $2^{|\bar{\xi}|} = |\bar{\xi}|^+$–many sets of the form ${}^X(\bigcup_{\xi' < \xi_0} \mathcal{P}_{\xi'})$ for some $X \subseteq \bar{\xi}$. It follows that $|\mathcal{P}_\xi| = |\mathcal{P}_{\xi_0}| \leq \xi_0^+ = \xi$ since $\bigcup_{\xi' < \xi_0} \mathcal{P}_{\xi'}$ has size at most $sup(\mathcal{R} \cap \xi_0) = \xi_0$ by induction hypothesis.

Finally, if $\xi = \xi_0^+$ for ξ_0 regular, then $\mathcal{P}_\xi$ is isomorphic to $\mathcal{P}_{\xi_0} * \dot{\mathcal{Q}}_{\xi_0}$, with $|\mathcal{P}_{\xi_0}| \leq \xi_0$ by induction hypothesis and such that $\dot{\mathcal{Q}}_{\xi_0}$ is forced to be either the empty set or a subcollection of $\{\, {}^X(P_{\xi_0}(P_{\omega_1}(\xi_0))) \,:\, X \in ((\xi_0^{\aleph_0})^+)^{<\xi_0}\}$ or of the same cardinality as a subcollection of the product of $\{\, {}^X(P_{\xi_0}(P_{\omega_1}(\xi_0))) \,:\, X \in ((\xi_0^{\aleph_0})^+)^{<\xi_0}\}$ and $\{\, {}^X(P_{\lambda^+}(P_{\omega_1}(\lambda)) \times {}^\omega\lambda) \,:\, X \in (\lambda^{++})^\lambda\}$ for some $\lambda < \xi_0$. But in these two cases, $\xi_0^{<\xi_0} = \xi_0$ holds in $V^{\mathcal{P}_{\xi_0}}$. It follows, in either case, that there is a certain set $\mathcal{X}$ of size ξ_0^+ such that every nice $\mathcal{P}_{\xi_0}$–name for a condition in $\dot{\mathcal{Q}}_{\xi_0}$ can be coded as a function from $\mathcal{P}_{\xi_0}$ into $\mathcal{X}$. Hence, by $|\mathcal{P}_{\xi_0}| \leq \xi_0$ and $(\xi_0^+)^{\xi_0} = \xi_0^+$ we have that $|\mathcal{P}_\xi| = |\mathcal{P}_{\xi_0} * \dot{\mathcal{Q}}_{\xi_0}| = \xi_0^+ = \xi$ by our definition of two–step iteration. □

[5] But $\overline{\mathcal{P}^0}$ might collapse more cardinals than $\mathcal{P}^0$.

[6] That is, conditions in $\mathfrak{a}(\lambda)$ are sequences $\mathfrak{a}$ as above, and any two distinct conditions are incompatible.

[7] We will see that $\mathcal{P}$ preserves all regular cardinals as well as GCH above ω, but defining the iteration this way makes it clear at the present point that each $\dot{\mathcal{Q}}_\xi$ is in fact well–defined and makes it easy to find simple inductive proofs of the relevant facts about the iteration.

Note that, by Lemma 5.2, $\mathcal{P}$ is $<\omega_2$–strategically closed. In particular, it preserves ω_1 and ω_2. Also, since $\mathcal{P}_\xi$, for $\xi \in \mathcal{R}$, has size at most ξ and forces that $\dot{\mathcal{Q}}_\xi$ has the ξ^+-chain condition (by Propositions 1.11 and by the remarks at the end of Section 4) and each component $\dot{\mathcal{Q}}_\zeta$ on the tail is forced to be $<\zeta$–strategically closed (by Proposition 1.10 and again by the end of Section 4), we get the following result by Lemma 5.2.

Lemma 5.4. *For each infinite cardinal $\xi \geq \omega_1$, $\mathcal{P}_{\xi^+}$ has the ξ^+–chain condition and $\mathcal{P}$ factors as $\mathcal{P}_{\xi^+} * \dot{\mathcal{P}}^*$, with $\dot{\mathcal{P}}^*$ a $\mathcal{P}_{\xi^+}$–name for an $<\xi^+$–strategically closed forcing.*

By an inductive argument using Lemma 5.4 it follows that $\mathcal{P}$ preserves the regularity of all $\xi \in \mathcal{R}$.

As to the preservation of ZFC by $\mathcal{P}$, we only need to care about Replacement and the Power Set Axiom. But the preservation of these axioms follows also from Lemma 5.4 and the general facts in Section 2.2 of [9].

Using once more Lemmas 5.3 and 5.4 one can prove that $\mathcal{P}$ preserves GCH at uncountable regular cardinals. This implies that every singular cardinal λ remains strong limit after forcing with $\mathcal{P}$ and hence $(2^\lambda)^{V^\mathcal{P}} = (\lambda^{cf(\lambda)})^{V^\mathcal{P}} = (\lambda^{cf(\lambda)})^V = \lambda^+$ again by the relevant distributivity of the tail forcings. Hence, $\mathcal{P}$ preserves GCH above ω.

Also, from the results in Section 1 we know that dense non-reflection in $P_{\omega_1}(\lambda)$ holds in $V^{\mathcal{P}_{\lambda^+}}$ for every regular $\lambda \geq \omega_2$, which implies the same conclusion in $V^\mathcal{P}$ for every such λ by Lemma 5.4. And, for λ is a singular cardinal of countable cofinality, we know by Section 4 that dense non-reflection in $P_{\omega_1}(\lambda)$ holds in $V^{\mathcal{P}_{\lambda^+}}$. The reason why this is true is that the product $\dot{\mathcal{Q}}_\lambda$ is forcing–equivalent in $V^{\mathcal{P}_\lambda}$ to the two–step iteration $\mathbb{Q}(\lambda) * \tilde{\mathcal{Q}}_\lambda$, where $\tilde{\mathcal{Q}}_\lambda$ is the two–step iteration $\mathfrak{a}(\lambda) * \mathbb{Q}$, from the definition of the iteration $\langle \mathcal{P}_\xi \,:\, \xi \in \textsc{Ord}\rangle$, as defined in $(V^{\mathcal{P}_\xi})^{\mathbb{Q}(\lambda)}$.[8] Again by Lemma 5.4, we get then that dense non-reflection in $P_{\omega_1}(\lambda)$ holds in the final extension.

The preservation of PFA is similar to the proof that $\mathcal{P}^0$ preserves PFA, using also the proof of the results in Section 2. Given a proper poset $\dot{\mathbb{R}} \in V^\mathcal{P}$ and dense subsets $\dot{D}_i$ of $\dot{\mathbb{R}}$ for $i < \omega_1$, also in $V^\mathcal{P}$, we can find a high enough ξ such that the above names are in fact $\mathcal{P}_\xi$–names, such that all $\dot{D}_i$ are, in $V^{\mathcal{P}_\xi}$, dense subsets of $\dot{\mathbb{R}}$, and such that $\dot{\mathbb{R}}$ is, in $V^{\mathcal{P}_\xi}$, a proper poset. Now we can define a $\mathcal{P}_\xi * \dot{\mathbb{R}}$–name $\dot{\mathbb{S}}$ for a poset analogous to the name $\dot{\mathbb{S}}$ in the proof of Theorem 2.1. The corresponding version of Proposition 2.2 holds for this choice of $\mathcal{P}_\xi * \dot{\mathbb{R}} * \dot{\mathbb{S}}$ by essentially the same proof, using the fact that all $\dot{\mathcal{Q}}_\zeta$ are forced to be ω_1–closed (by Corollary 1.5 and by the proof of Proposition 4.1). Using this one can show, by essentially the same arguments as in the proof of Theorem 2.1, that in $V^{\mathcal{P}_\xi}$ there is a filter of $\dot{\mathbb{R}}$ intersecting all $\dot{D}_i$, but then of course the same is true in $V^\mathcal{P}$.

Second construction. For the second construction we start forcing, over our ground model V of PFA and GCH above ω, with Magidor's partial order – let us call it $\mathbb{M}$ – for getting $\square_{\kappa,\aleph_2}$ for all $\kappa \geq \omega_1$ while preserving PFA ([12]). $\mathbb{M}$ is the direct limit of a reverse Easton iteration $\langle \overline{\mathcal{P}}_\xi \,:\, \xi \in Ord\rangle$ on which nothing happens unless ξ is a successor cardinal κ^+ for $\kappa \geq \omega_1$. In that case we force at stage ξ of the

[8] This is true by the fact that $\mathbb{Q}(\lambda)$ is λ^+–distributive in $V^{\mathcal{P}_\lambda}$.

iteration with the natural forcing for adding a $\square_{\kappa,\aleph_2}$–sequence by initial segments. Arguments as in the first construction show that $\mathbb{M}$ preserves ZFC, GCH above ω and cofinalities, and that it forces $\square_{\kappa,\aleph_2}$ for all cardinals $\kappa \geq \omega_1$. Magidor [12] proved that $\mathbb{M}$ preserves PFA.

Let V_1 be the extension of V by $\mathbb{M}$. Now we build a reverse Easton iteration $\langle \tilde{\mathcal{P}}_\xi : \xi \in \textsc{Ord} \rangle$, based on a sequence $\langle \dot{\mathcal{Q}}_\xi : \xi \in \textsc{Ord} \rangle$ of names for posets. The definition of this sequence is like the definition of the sequence of names making up the iteration $\langle \mathcal{P}_\xi : \xi \in \textsc{Ord} \rangle$ in the first construction, except that now we only force with $\mathbb{Q}(\xi)$ when $\xi \geq \omega_2$ is a regular cardinal in $V^{\tilde{\mathcal{P}}_\xi}$. In other words, we do not take any steps now to force dense non-reflection in $P_{\omega_1}(\lambda)$ when λ is a singular cardinal of countable cofinality.

By the same arguments as in the first construction we can prove that the direct limit $\mathcal{P}$ of this iteration preserves ZFC, GCH above ω, cofinalities and PFA, and that it forces dense non-reflection in $P_{\omega_1}(\lambda)$ for all regular $\lambda \geq \omega_2$.

Finally, when λ is a singular cardinal of countable cofinality, dense non-reflection holds in $P_{\omega_1}(\lambda)$ in $V_1^{\mathcal{P}}$ by Corollary 3.11 because $\square_{\lambda,\aleph_2}$ obviously holds in this extension.

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