A Relativistic Zeno Effect

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Abstract
A Zenonian supertask involving an infinite number of identical colliding balls is generalized to include balls with different masses. Under the restriction that the total mass of all the balls is finite, classical mechanics leads to velocities that have no upper limit. Relativistic mechanics results in velocities bounded by that of light, but energy and momentum are not conserved, implying indeterminism. By entertaining the possibility that the missing energy and momentum are carried away by a photon, however, one can restore the conservation laws and determinism.

1 Classical Zeno process

It is a commonplace that, in the mechanics of a system composed of massive, elastically colliding particles, the conservation laws of momentum and energy may be applied, either individually to each pair of particles as they collide, or to the whole system. In the former case, momentum and energy are conserved at each collision, and in the latter the total momentum (the vector sum of all the individual momenta) and the total energy (the scalar sum of all the energies) are constant in time.

However, consider the infinite set of points on a straight line, $x_n = 2^{-n}$, where $n = 0, 1, 2, ...$, which we shall call 'Zeno points'. Suppose that there are equal point masses, 'Zeno balls', at each of these Zeno points. If all the balls are at rest except the zeroth one, which has a negative velocity, it will collide after a finite time with the first ball, transferring all its momentum to that ball (the collision is presumed to be perfectly elastic). This first ball then collides with its neighbour to the left, which will subsequently collide with *its* left neighbour, and so on *ad infinitum* until, after a finite time has elapsed, all the balls have been briefly in motion, but all have been brought to rest. Since there is no longer any motion, the momentum and kinetic energy of the original ball have been lost. Momentum and energy conservation have been violated!

Such is the scenario described by Pérez Laraudogoitia (1995); and, invoking the time-reversal invariance of Newtonian mechanics, this author proclaimed the demise of classical determinism. At any time after the balls have been brought to rest, the time reversal transformation does not change the configuration, and Newton's laws of motion yield as a viable evolution the simple possibility that all the balls remain at rest for all time. However, the time inverse of Laraudogoitia's scenario is also consistent with Newtonian mechanics, and so is a non-denumerable infinity of other evolutions.

Do these results endanger our classical view of the world, or is the problem merely academic? One might ask if there are ingredients in Laraudogoitia's model that are irreductibly 'unphysical', warranting its relegation to the decent obscurity of a mathematical curiosity. Is the requirement that the balls are point masses, without extension, such an unphysical requirement? As Laraudogoitia points out, the condition of zero extension can removed by supposing the nth Zeno ball to be a sphere of radius 2^{-n-2}. These non-punctual balls will not touch their neighbours, either in the initial, or in the final configurations, and the conclusions remain unaffected. The size of the balls has no positive lower bound, but there is no finite n such that the nth ball has zero radius. It is true that one can find an n so large that this nth ball is smaller than a hydrogen atom, or for that matter smaller than an electron; but outlawing the configuration on this ground would come at an unacceptable cost, for it would be tantamount to claiming that the discrete or atomic nature of matter could be deduced from the law of conservation of momentum.

The total mass of all the balls is infinite. Is this perhaps the key unphysical feature that leads to the loss of momentum? It is known that momentum is not conserved in collisions involving an infinitely massive body. This motivates the consideration of a new model in which the mass of each Zeno ball is only one half that of its neighbour to the right. The total mass of all the balls is now finite, for if $m_n = 2^{-n} m_0$ is the mass of the nth ball, the total mass is

$$\sum_{n=0}^{\infty} m_n = \sum_{n=0}^{\infty} 2^{-n} m_0 = 2m_0\,.$$

Initially, all the balls are at rest except the zeroth one, which has a negative velocity, u_0, so it collides after a finite time with the ball at x_1, after which we suppose its velocity to be v_0.

For $n = 1, 2, 3, \ldots$, let u_n be the velocity of the nth Zeno ball, after it has been struck by the $(n-1)$st Zeno ball from its right, and let v_n be its velocity after it has struck the $(n+1)$st Zeno ball to its left. At a time after the first, but before the second collision of the nth ball, the balls labelled $p = 0, 1, 2, \ldots, n-1$ have velocity v_p, the nth ball has velocity u_n, and all the others are still at rest. The total momentum of all the balls at such a time is therefore

$$\sum_{p=0}^{n-1} m_p v_p + m_n u_n\,,$$

and, by conservation of momentum, this must be equal to $m_0 u_0$, the initial momentum. As time goes on, more and more balls partake in the motion, and, after a finite time, they have all collided, and then

$$m_0 u_0 = \sum_{p=0}^{\infty} m_p v_p + \lim_{n\to\infty} m_n u_n \,. \qquad (1)$$

It is important to include the limit term on the right, but it is crucial to the conservation of momentum that it vanish. Only then is the sum of the final momenta of all the balls equal to the original momentum of the first one. One can check, from the conservation of energy and momentum at each collision, that $u_n = \left(\frac{4}{3}\right)^n u_0$, and hence

$$\lim_{n\to\infty} m_n u_n = \lim_{n\to\infty} \left(\tfrac{1}{2}\right)^n \left(\tfrac{4}{3}\right)^n m_0 u_0 = 0\,,$$

i.e. momentum is indeed conserved. Similarly, one can show that energy is conserved in this model.

In Laraudogoitia's model of equal masses, all the velocities v_n vanish, and all the velocities u_n are equal to u_0. In this case the series in Eq.(1) is zero and the limit term is $m_0 u_0$. Being non-zero, this term implies momentum nonconservation.

At first sight, it looks as though we may have located the source of the paradox in Laraudogoitia's model, namely the infinite total mass. However the matter is not so simple, for the new model has its own unphysical feature. For in it $v_n = \frac{1}{3}\left(\frac{4}{3}\right)^n u_0$, so the final velocities of the Zeno balls, after their collisions, have no upper bound. As the motion progresses from right to left, the balls get lighter and lighter, but they move faster and faster. Newton's mechanics may have been provisionally reprieved[1]; but should one not be using relativistic mechanics long before considering balls moving with speeds approaching that of light?

2 Relativistic Zeno process

The relativistic equations for the conservation of momentum and energy just before and just after the nth collision lead to the recurrence relation

$$\varepsilon(u_{n+1}) = \frac{1 + 2\,\varepsilon(u_n)}{2 + \varepsilon(u_n)}\,\varepsilon(u_n)\,, \qquad (2)$$

[1] It can be proved that, whenever the total mass is finite, momentum is conserved in classical mechanics. However, a similar statement for energy conservation is not true.

with the notation

$$\varepsilon(u) = \sqrt{\frac{1-u}{1+u}}\,,$$

where the speed of light has been set equal to unity. This nonlinear recurrence relation cannot be resolved in closed form, but, given that $u_0 \neq 0$, one can show that the sequence $\varepsilon(u_n)$, $n = 0, 1, 2, ...$, is monotonically decreasing to zero, corresponding to light velocity.

In terms of the Lorentz factor,

$$\gamma(u) = (1-u^2)^{-\frac{1}{2}} = \tfrac{1}{2}\left[\varepsilon(u) + \varepsilon^{-1}(u)\right], \tag{3}$$

the total energy of the balls, after the first, but before the second collision of the nth ball, is

$$m_0[\gamma(u_0) + 1] = \sum_{p=0}^{n-1} m_p\gamma(v_p) + m_n\gamma(u_n) + \sum_{p=n+1}^{\infty} m_p\,.$$

In the limit $n \to \infty$, this becomes

$$m_0[\gamma(u_0) + 1] = \sum_{p=0}^{\infty} m_p\gamma(v_p) + \lim_{n\to\infty} m_n\gamma(u_n)\,,$$

which is the relativistic analogue of Eq.(1). Conservation of energy would be guaranteed if the limit term on the right were to vanish. However,

$$m_n\gamma(u_n) = \tfrac{1}{2}m_n\left[\varepsilon(u_n) + \varepsilon^{-1}(u_n)\right],$$

and for N sufficiently large, with $n \geq N$, Eq.(2) reduces to the linear form $\varepsilon(u_{n+1}) \approx \frac{1}{2}\varepsilon(u_n)$. By iteration down to N one finds $\varepsilon(u_n) \approx 2^{N-n}\varepsilon(u_N)$, and so the energy that escapes to infinity is

$$\begin{aligned} E_\infty &= \lim_{N\to\infty}\lim_{n\to\infty} 2^{-n-1}m_0\left[2^{N-n}\varepsilon(u_N) + 2^{-N+n}\varepsilon^{-1}(u_N)\right] \\ &= \lim_{N\to\infty} 2^{-N-1}\varepsilon^{-1}(u_N)\,, \end{aligned} \tag{4}$$

and this is not zero. A similar analysis shows that momentum is also not conserved.

The energy loss, E_∞, can be computed numerically for different values of the initial velocity from Eqs.(2) and (4). As a percentage of the energy at the beginning, $m_0\gamma(u_0)$, we find that only 0.14% of the energy is lost if the zeroth ball has initially one tenth of the velocity of light. For smaller, nonrelativistic velocities the percentage loss, while not strictly zero, is very small. If the zeroth ball moves initially with one half of the velocity of light, the energy loss is still only 7.5%.

One has to go to ultra-relativistic velocities before the loss becomes large. For example, if $u_0 = 0.99$, then the energy that is carried off to infinity is 80% of the initial energy.

The reason for the breakdown of the conservation laws is that, although the masses get smaller and smaller as n increases, their velocities approach more and more closely that of light, so that the Lorentz factor (3) increases without limit. The product of the decreasing masses and increasing Lorentz factor turns out to have a finite, nonzero limit, and this accounts for the fact that the sum of the final energies of all the balls is less than the original energy of the zeroth ball. Because of the nonvanishing limit term (4), energy and momentum are carried off to infinity. In the classical model of the previous section, we recall that the final velocities increased without bound, but not sufficiently quickly to offset the decrease in the masses of the balls. In relativistic mechanics, the matter is different, for there is a delicate balance between the rate of increase of the Lorentz factor and the rate of decrease of the masses, resulting in energy and momentum nonconservation.

3 Mechanical supertasks

In the relativistic model of Sect. 2, both energy and momentum disappear at infinity, and moreover to the same extent, for the missing energy, $\lim_{n\to\infty} m_n\gamma(u_n)$, and the missing momentum, $\lim_{n\to\infty} m_n\gamma(u_n)u_n$, are both equal to the nonzero $\lim_{N\to\infty} 2^{-N-1}\varepsilon^{-1}(u_N)$. Since a photon has precisely this property of equality between its energy and its momentum (in units in which the speed of light is unity), it follows that a possible way to rescue the conservation laws is to posit the creation of a photon[2] at the moment that the Zeno process is completed, with just the right frequency to take away the energy and momentum in question. Thanks to this mechanism, indeterminism has been removed as well. For at any time after the creation of the photon, the time-reversal operation involves not only the reversal of the momenta of all the balls, but also that of the photon. In the time-reversed scenario, the photon is absorbed, initiating the infinity of collisions which finally ensures the concentration of all the energy and momentum in the zeroth ball.

In the original supertask of Laraudogoitia, in which all the balls have the same mass, a similar rescue operation is blocked by the fact that the momentum and energy of a ball are not equal. The only way to carry off the missing energy and momentum (in his model, remember, this means all the energy and all the momentum) is to suppose that a new ball, equal in mass to all the others, is miraculously created, at just the right place and at just the right time, in order

[2]Or two photons, if one takes the model sufficiently seriously to insist on conservation of angular momentum too.

to carry off the energy and momentum. Far-fetched though it may seem, this is one of the scenarios envisaged in the recent literature on the subject (Pérez Laraudogoitia, Bridger and Alper, 2002).

The reasoning behind this *creatio ex nihilo* is another scenario, introduced by Bridger and Alper (2000). All the Zeno balls are at rest, but another ball, of the same mass as that of each Zeno ball, approaches from the left of them all. When the new ball reaches the accumulation point of mass at the origin, from the left, Bridger and Alper argue that it can neither continue moving to the right, nor come to rest. Indeed, it cannot move to any point to the right of the origin, for to do that it would have had to collide with an equally massive, stationary ball, which would have brought it to rest. But any such ball has an infinite number of twins to its left, which should have stopped the moving ball before it reached the ball in question. It is suggested that the new ball cannot stop at the origin either, because there is no Zeno ball at this location to stop it. Since the new ball cannot be at the origin, nor to its right, nor anywhere else, it follows that it is nowhere, i.e. it has ceased to exist!

By a combination of the time reversal of this process with Laraudogoitia's original supertask (called ST), a new scenario dubbed TRST′ is described:

"However, unlike ST, at the completion of TRST′, an unnamed particle appears at the origin at t = 1 , moving toward the left. ... In this scenario, energy and momentum are conserved. Unfortunately, at the present time we do not know how or even whether this process can be analyzed from the global standpoint." (Pérez Laraudogoitia, Bridger and Alper, 2002, p. 186.)

Concerning Alpert and Bridger's contention that the new ball cannot be brought to rest at the origin because there is no Zeno ball with which it could collide, that depends on how one understands the meaning of the word 'collision'. One says that two bodies collide at a given time if at least one point of one body, and one point of the other, have zero spatial separation at that time, rather than that the two points be coincident. This is motivated by the physical idea of interaction by a short-ranged force, in the limit that the range is taken to zero. When the rightmost point of the new ball reaches the origin, its separation from a Zeno ball (indeed an infinite number of them) is zero. The set of Zeno balls is an open collection of spheres, the origin being a limit point of that set. When the new ball encroaches upon that point, the measure of its separation from the set of Zeno balls is null. Moreover, the centre-of-mass of the set of Zeno balls *is* the origin, and since the mass is infinite, the new ball must simply be reflected with unchanged speed, as with any elastic collision against an infinite mass. Energy is conserved but momentum is not.

There has been some disagreement about whether Laraudogoitia's original supertask should be counted as being truly 'Newtonian' or not (Alper and Bridger, 1998; Earman and Norton, 1998; Alper, Bridger, Earman, and Norton, 2000; Pérez Laraudogoitia, Bridger and Alper, 2002). However, everyone agrees that the model is unphysical. Suppose that the theoretician conveniently turns gravitation off, so there is not an infinitely massive, infinitely extensive black hole centred at the origin. Even if she agrees to ignore the atomic nature of matter, and, more awkwardly, the quantum nature of energy and momentum, there is still the question whether she should allow impulsive forces that have no upper bound. Suppose she does! In such a world, Laraudogoitia is quite right to point out that energy and momentum would not be conserved.

As has been shown in Sect. 2, violation of energy and momentum conservation is not limited to the case that the total mass is infinite. Indeed, when the total mass of the Zeno balls is finite, energy and momentum can still escape to infinity. In the model of this paper, it is not really necessary to turn off gravity, for the mass enclosed in a sphere of radius 2^{-n}, centred on the origin, is only $2^{-n+1}m_0$, and since the Schwarzschild radius of this mass is proportional to the mass itself, with Newton's gravitational constant as the very small coefficient of proportionality, it is clear that the present model would only be slightly perturbed by the presence of gravity. To extremely good approximation, general relativistic considerations may be neglected. They do not affect the finding that momentum and energy are lost at infinity.

Special relativistic corrections are small if the initial velocity, u_0, is very much less than the velocity of light. In that case energy and momentum are conserved to a high degree of approximation. However, when u_0 is comparable to light velocity the violation of the conservation laws is appreciable. In this sense the present model is a more convincing case for the breakdown in the conservation laws than was Laraudogoitia's model, given that the latter could not tolerate the reintroduction of gravity, even of the Newtonian kind, let alone that of Einstein. For in that model the infinite mass centred at the origin would lead to infinite gravitational forces that would completely destroy the validity of the mechanical model as presented.

In this paper, special relativity has been taken into account, but it was deemed permissible to neglect gravitational effects. It was suggested that the missing energy and momentum could be carried off by a photon (or photons), a move that also could remove the indeterminacy. However, in the attempt to improve the credentials of the model, one might try to specify a mechanism whereby the photons are created. Relativistic quantum field theory provides such a mechanism, but at a price that calls the whole venture into question. For relativistic quantum field theory is a marriage of special relativity and quantum mechanics, and the latter theory does not tolerate an infinite set of moving balls, of smaller and smaller spatial dimension. Indeed, a rough estimate of where the model must break down is given by equating the de Broglie wavelength of a moving Zeno ball with its distance to the next ball. As $n \to \infty$, the momentum of the nth Zeno ball tends to a constant, which depends only on the initial velocity, u_0, so the de Broglie wavelength has a limiting value too. Intervals between balls which are smaller than this make no sense, so the infinite sequence of Zeno balls also makes no sense, according to quantum mechanics. Thus, even in principle, the infinity of Zeno balls cannot be physically implemented, and in that sense the breakdown of the conservation laws is no direct threat to our physical view of the world. Nevertheless, the question of the internal consistency of mechanics with determinism and with the conservation laws of energy and momentum is of great theoretical interest. We must conclude that there is no such consistency, and that the limitation to finite total mass is insufficient to restore it.

To answer the question: 'What is the fundamental reason for the violation of the conservation laws in the models of the Zeno balls?', one can scarcely do better than return to the original Zeno paradox in the variant called 'Achilles and the Tortoise'. The source of the problem is the infinite number of subintervals, and in particular the fact that there is no last interval. To be assured that Achilles will draw abreast of the tortoise, one needs to postulate that the hero's position is a continuous function of the time. To be quite sure that he pulls ahead of the beast, one needs to add the postulate that his speed is a continuous function of the time (assuredly, these postulates become deductions if one is told that the fleet-footed Greek runs at a constant speed for all relevant times).

The Zeno balls resemble the variant of the conundrum called the 'staccato run', in which Achilles stops for a short time at each Zeno point before running on. An even better analogy would be a relay race, in which a clone of Achilles is stationed at each Zeno point. The original Achilles runs from the zeroth Zeno point and passes the baton to his clone at the first Zeno point, who then runs to the next clone, passing on the baton, and so forth. It is now more awkward to claim that the position of the baton is a continuous function of the time, and that its speed is continuous, for there is no carrier at the limit point. Read now 'Zeno ball' for 'Achilles clone', and 'momentum' for 'baton', and the problem is not so much that momentum has disappeared in a puff of metaphysical smoke, as that there is no ball at the limit point to carry it away. If one waves a wand and creates a ball at the right time and the right place, or, with less strain on the credulity, a quantum of light (in the case of finite total mass), then there is a carrier to ensure the safe passage of the momentum through the limit point.

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