

Probability without certainty

Foundationalism and the Lewis-Reichenbach debate

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Abstract

Like many discussions on the pros and cons of epistemic foundationalism, the debate between C.I. Lewis and H. Reichenbach dealt with three concerns: the existence of basic beliefs, their nature, and the way in which beliefs are related. In this paper we concentrate on the third matter, especially on Lewis's assertion that a probability relation must depend on something that is certain, and Reichenbach's claim that certainty is never needed. We note that Lewis's assertion is *prima facie* ambiguous, but argue that this ambiguity is only apparent if probability theory is viewed within a modal logic. Although there are empirical situations where Reichenbach is right, and others where Lewis's reasoning seems to be more appropriate, it will become clear that Reichenbach's stance is the generic one. This follows simply from the fact that, if $P(E|G) > 0$ and $P(E|\neg G) > 0$, then $P(E) > 0$. We conclude that this constitutes a threat to epistemic foundationalism.

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1. Historical introduction

Epistemic foundationalism is making a comeback. It has been close to death's door ever since Wilfred Sellars' assault in 1956, but now it is steadily reviving, gaining support even from its erstwhile enemies. Naturally, its face has changed. It is no longer that of the Cartesian rationalist or the Schlick-minded positivist, but it nevertheless sings the song of the given with conviction, and moreover in a variety of ways (see Pryor, 2001, Triplett, 1990, for overviews.)

This development within epistemology makes it the more interesting to re-examine a discussion that took place fifty years ago between Clarence Irving Lewis and Hans Reichenbach. The debate was carried out publicly, in journals and at conferences, as well as privately, in letters and in conversations. It reached its climax at the forty-eighth meeting of the Eastern Division of the American Philosophical Association at Bryn Mawr College, December 29, 1951. There Reichenbach and Lewis, joined by Nelson Goodman, read papers on 'The Experiential Element in Knowledge'. These papers were subsequently published in the *The Philosophical Review* of April 1952. The debate itself came to a premature end with the sudden death of Reichenbach one year later, in April 1953. Neither of the two had succeeded in convincing the other and the matter has remained unsettled to the present day.

Like so many debates on the pros and cons of foundationalism, the debate between Lewis and Reichenbach deals with three questions:

1. 'Do basic beliefs exist?'

2. 'If yes, what is their nature?'
3. 'How do non-basic beliefs depend on basic beliefs, or more generally: What are the mutual relations between beliefs?'

Contemporary discussions of foundationalism often focus on the first two questions (see for example Bonjour, 2004). To these questions Reichenbach answers that basic beliefs do not exist, whereas Lewis, being a representative of so-called 'strong foundationalism', does think there are basic beliefs, namely in the form of reports about what one sees, hears, smells, tastes or feels. In the present paper we are however exclusively interested in the answers to the third question.

Both Lewis and Reichenbach claim that the significant relations between beliefs are probability relations, but they disagree as to the consequences and the meaning of this claim. How important this disagreement must have been to both appears from the twenty-five letters that they exchanged between August 1930 and December 1951. Of the seventeen letters that were written between 1930 and 1939, eight are about the meaning of probability relations, and seven deal with attempts to get Reichenbach's German book on probability theory translated into English. The eight letters that Lewis and Reichenbach moreover exchanged in 1951 are all about probability and the imminent meeting at Bryn Mawr.¹

Already in his *Mind and the World Order* of 1929, Lewis had claimed that statements of the form 'x is probable' only make sense if one assumes there to be a y that is certain (where x and y may be events, statements or beliefs). He writes for example:

the immediate premises are, very likely, themselves only probable, and perhaps in turn based upon premises only probable. Unless this backward-leading chain comes to rest finally in certainty, no probability-judgment can be valid at all. (Lewis, 1929, pp. 328-329)

On the basis of what Lewis writes in a letter of August 26, 1930, we can infer that Reichenbach had questioned this claim one month earlier (see Footnote 1). Obviously, Lewis was not convinced by Reichenbach's remarks, for in his *An Analysis of Knowledge and Valuation* of 1946 he stresses the point again:

If anything is to be probable, then something must be certain. The data which themselves support a genuine probability, must themselves be certainties. (Lewis, 1946, p.186)

At the meeting in Bryn Mawr, Lewis is still of the same opinion:

The supposition that the probability of anything whatever always depends on something else which is only probable itself, is flatly incompatible with the assignment of any probability at all. (Lewis, 1952, p.173)

¹ There have certainly been more letters, now apparently lost, for on August 26, 1930, Lewis replies to a letter that Reichenbach sent him on July 29th of that year. The University of Pittsburgh, which keeps the Lewis-Reichenbach correspondence, has however only twenty-five letters, and we are most grateful to Mr. L. Lugar and Ms. B. Arden for sending us copies.

Reichenbach denies this view – in letters, in Chapter 8 of his *Wahrscheinlichkeitslehre*, and finally at the Bryn Mawr conference. For him, Lewis’s claim that probabilities presuppose certainties (cases where the probability value is one) is “just one of those fallacies in which probability theory is so rich” (Reichenbach, 1952, p.152). In an attempt to understand the root of the fallacy he writes:

We argue: if events are merely probable, the statement about their probability must be certain, because ... Because of what? I think there is tacitly a conception involved according to which knowledge is to be identified with certainty, and probable knowledge appears tolerable only if it is embedded in a framework of certainty. This is a remnant of rationalism. (Reichenbach, 1952, p.152)

Lewis, in turn, rejects the accusation of being an old fashioned rationalist and replies that, on the contrary, his position is an attempt to save empiricism from “a modernized coherence theory” like that of his opponent (Lewis, 1952, p.171, p.173).

The interesting thing about this disagreement is that both protagonists have intuitions that are sound. As we will show, there are cases that support Reichenbach and there are also cases where Lewis’s position seems to be appropriate. This fact would surely have surprised both, for Lewis held that Reichenbach’s position was inconsistent, while Reichenbach believed that only trivial examples could be instances of Lewis’s claim.

In the next section, we will indicate that Lewis’s position obscures an ambiguity that we will remove by using results of Halpern and of Meyer and van den Hoek. In Sect. 3 we explain how Bertrand Russell supported Lewis, against Reichenbach, and how the latter exposed an error in Russell’s use of the probability calculus. In Sect. 4 we apply the corrected analysis to two empirical situations, relegating the general treatment to the Appendix. By proving that the unconditional probability $P(E)$ is positive if two conditional probabilities, $P(E|G)$ and $P(E|\neg G)$, are themselves positive, we demonstrate that Reichenbach’s position is the generic one, that of Lewis being a special case.

2. Ambiguity in certainty

Nowhere in the correspondence between Lewis and Reichenbach is there any discussion on how probability should be interpreted. This is strange, since Reichenbach was a founder of the frequency interpretation², and Lewis, being a co-founder of modal logic, must have had subjectivistic proclivities. Even so, their major point of dissension is not at all tied to a specific interpretation: Lewis’s assertion and Reichenbach’s denial that probabilities require certainties can be equally expressed in an objective or a subjective language. For the purpose of this article, however, we shall generally express ourselves in subjectivistic terms.

Lewis’s claim that probability judgements only make sense if they are finally rooted in certainties might look ambiguous at first sight. It could mean, for example, ‘The probability of x given y is 0.3, and moreover y is certain’. But it might also be construed

² In 1915, long before Von Mises’ works appeared, Reichenbach introduced his own version of a frequency interpretation—cf. Reichenbach and Cohen 1978, Vol. II, p. 410.

as meaning that the sentence ‘ $p(x) = 0.3$ ’ is certain and hence that $p(p(x) = 0.3) = 1$. Lewis phrases his claim mostly in terms of the second, but sometimes in terms of the first meaning. This might indicate that he suspected the two meanings to be equivalent, and thus the ambiguity to be only apparent. Lewis never proves the equivalence between the two meanings, but it is not difficult to construct a proof on his behalf. In the first place, if y is certain, then ‘ $p(x) = p(x|y) = 0.3$ ’ is true, and thus $p(p(x) = 0.3) = 1$. In the other direction, the second-order probability assignment $p(p(x) = 0.3) = 1$ is equivalent to $p(x) = 0.3$, and hence to $p(x|y) = 0.3$ for *any* y such that $p(y)=1$.

A more elaborate version of this proof can be obtained within the framework of modal logic, in particular of epistemic dynamic or KD logic.³ Joseph Y. Halpern has explained how modal logic can be used in relation to belief, in the language of probability theory (Halpern, 1991, theorem 4.1). Indeed, probability theory corresponds to the system KD45, which according to Halpern has been identified as “perhaps the most appropriate [logic] for belief, [providing] a complete axiomatization for reasoning about certainty” (Halpern, 1991, p. 1). As Meyer and van der Hoek have shown, in KD45 every formula is equivalent to a formula without nestings of modal operators (Meyer and van der Hoek, 1995, Remark 1.7.6.4.2). The following two implications, generalizations of axioms 4 and 5 from KD45, are valid for all events x and all probabilities α :

$$\begin{aligned} p(x) = \alpha &\rightarrow p(p(x) = \alpha) = 1 \\ p(x) \neq \alpha &\rightarrow p(p(x) \neq \alpha) = 1 \end{aligned}$$

Using these results, we can show that $p(p(x) = 0.3) = 1$ is equivalent to $p(x) = 0.3$. From right to left this is simply an instantiation of the first implication. From left to right the detailed argument runs as follows: by using Kolmogorov’s axioms, we see easily that $p(p(x) = 0.3) = 1 \rightarrow p(p(x) \neq 0.3) \neq 1$. Moreover, the instantiation of the second implication, $p(x) \neq 0.3 \rightarrow p(p(x) \neq 0.3) = 1$, implies $p(p(x) \neq 0.3) \neq 1 \rightarrow p(x) = 0.3$ by contraposition. Thus $p(p(x) = 0.3) = 1 \rightarrow p(x) = 0.3$.⁴

3. Lewis, Reichenbach and Russell

As Reichenbach remarks, and as Lewis acknowledges, a more detailed version of Lewis’s argument was spelled out by Bertrand Russell in his *Human Knowledge* of 1948. According to Russell, any merely probabilistic statement (let us say a statement with probability $p < 1$) must have a ground, i.e. it must be conditioned by a further statement. This further statement may be certain or it may itself be probabilistic. In the first case, we have arrived at a firm basis; in the latter case (if this further statement has probability $q_1 < 1$), it must in turn be conditioned by a further statement, and if this too is probabilistic (let us say with probability $q_2 < 1$), then it also must be conditioned, etc. Can this process go on for ever, never reaching firm ground? Russell’s answer is that it cannot. For the

³ We thank Barteld Kooi for pointing this out and for helping us to construct the extended proof.

⁴ It may seem awkward to assign probabilities to assignments of probabilities, but from a logical point of view there is nothing untoward.

probability of the original statement in this case is given by the product $pq_1q_2 \dots$ *ad infinitum*. And since all the factors in the product are less than one, the outcome “may be expected to be zero” (Russell, 1948, p.434). In other words, if an uncertain statement is supported by an infinite sequence of further uncertain statements, the unconditional probability of the original statement has ineluctably dwindled away to nothing.

Clearly, if Russell’s argument were valid, it would form a solid strut for Lewis’s position. But is it valid? The argument of Russell hinges on two assumptions. First, if a statement with probability p is conditioned by a statement with probability q_1 , then the probability of the original statement is given by the product pq_1 . Second, if the product consists of an infinite number of factors, all less than unity, then it converges to zero. Reichenbach, in his rebuttal of Russell’s argument, only discusses the first assumption. Before proceeding with Reichenbach’s reaction to this first assumption, let us say a few words about the second one.

Contrary to what Russell intimates, it is not true that, if all factors are less than one, an infinite product would necessarily converge to zero; in symbols: $\forall n: q_n < 1$ does not imply $\prod_n q_n = 0$. Such a product can be non-zero. A necessary – although by no means sufficient – condition for this to happen is that the factors q_n tend to 1 as n tends to infinity. Of course, a mathematically erudite philosopher like Russell must have known that an infinite product of factors, all smaller than one, can converge to a number greater than zero.⁵ This may well have been his reason for writing that the outcome “may be expected to be zero”, rather than that it “*must* be zero”, which would have been incorrect. Why then does he deliberately give the mistaken impression that the product would be zero? We can only guess at the answer. Perhaps he thought these technical subtleties would be misplaced in such a book as *Human Knowledge*, aimed as it was at a non-specialist market. Or perhaps he judged them to be simply irrelevant to the main point.

However this may be, Russell’s first assumption is even more puzzling. As Reichenbach is quick to point out, it is simply not true that the probability in question is given by pq_1 . If a statement has probability p on condition that a second statement is true, but probability p' when that second statement is false, then the probability of the first statement is given by $pq_1 + p'(1 - q_1)$, where q_1 is the probability that the second statement is true. In modern notation, the unconditional probability that E occurs is

$$P(E) = P(E|G_1) P(G_1) + P(E|\neg G_1) P(\neg G_1), \quad (1)$$

where the unconditional probability that the ground, G_1 , occurs is $q_1 = P(G_1)$. The conditional probabilities that E occurs when G_1 respectively occurs and does not occur are $p = P(E|G_1)$ and $p' = P(E|\neg G_1)$ ⁶. If G_1 itself is conditioned by the ground G_2 , and so

⁵ An example of such convergence is $q_n = 2^{-w}$, where $w = 2^n$, for which $\prod_n q_n = 1/2$. In general, given that $0 < q_n < 1$, we may write $\prod_n q_n = \exp [- \sum_n | \log q_n |]$, the product and sum being from $n = 1$ to $n = \infty$, so the convergence of the sum is the necessary and sufficient condition that the infinite product of the q_n be non-zero.

⁶ Jeffrey conditionalization (Jeffrey, 1965, Chap. 11) amounts to the use of Eq. (1), which is itself a consequence of the probability calculus, under changing subjective estimates for $P(G_1)$, in which the conditional probabilities $P(E|G_1)$ and $P(E|\neg G_1)$ do not change. Jeffrey was primarily interested in the effect of changing subjective estimates for $P(G_1)$ on the basis of new information that replaces old information: we discuss Jeffrey’s position more fully in our “Probability all the way up (or no probability at all)”, forthcoming in *Synthese*.

on, the formula must be iterated, and this produces a much more complicated regression than the simple product that Russell had envisaged. There are correspondingly more elaborate ways of avoiding the conclusion that the unconditional probability of E must vanish if the chain of conditional probabilities is unending.⁷

Lewis appears however not to be impressed by Reichenbach's amendment to Russell's simple product of probabilities. Apparently failing to see the relevance of the second term in Eq.(1), Lewis states flatly:

I disbelieve that this will save his point. For that, I think he must prove that, where any regress of probability-values is involved, the progressively qualified fraction measuring the probability of the quaesitum will converge to some determinable value other than zero; and I question whether such a proof can be given (Lewis, 1952, p.172).

In fact it is not difficult to meet Lewis's challenge and prove indeed that a "progressively qualified fraction measuring the probability of the quaesitum" can converge to some value other than zero. In the next section we give a simple example of a "regress of probability-values" yielding a number that is not zero. The regression takes the form of a sum consisting in an infinite number of terms. In addition, we also formulate another example, one that is more in line with Lewis's intuitions. Both examples are couched in the language of statistics. Readers associating this with an objective interpretation of probability can appeal to Miller's Principle (Halpern, 1991, p. 14), in the form usually called the Principal Principle, in order to make the connection between objective chance and the subjective interpretation to which we adhere in this paper.

4. Two medical examples

Let $P(E)$ in Eq.(1) stand for the probability that a man will suffer from prostatic cancer, and $\alpha = P(E|G_1)$ for the conditional probability that he will have the complaint, given that his father did so. Since not all prostatic cancer patients have fathers with a similar affliction, $\beta = P(E|\neg G_1)$ is not zero. On the other hand, a man is more likely to contract prostatic cancer if his father had it than if he did not. Thus $\alpha > \beta > 0$, and empirical values of α and β have been estimated from the study of large populations. We may rewrite Eq.(1) as

$$P(E) = \alpha P(G_1) + \beta [1 - P(G_1)] = \beta + [\alpha - \beta] P(G_1) .$$

$P(G_1)$ is the probability that the father had prostatic cancer, and of course this probability can be in turn conditioned by the fact that *his* father did, or did not similarly suffer. Thus

⁷ Reichenbach 1952. Reichenbach stresses the same point in his letter to Russell, written in 1949 and published in Reichenbach and Cohen 1978, Volume II, pp. 405-411. Wesley Salmon, who was Reichenbach's student at the time, remembers having been taken through Russell's argument in Reichenbach's class (Salmon, 1978, p. 73).

$$P(G_1) = \beta + [\alpha - \beta] P(G_2),$$

where $P(G_2)$ is the probability that the man's paternal grandfather contracted prostatic cancer. Evidently this "regress of probability-values", to use Lewis's words, can be continued *ad infinitum*, yielding

$$\begin{aligned} P(E) &= \beta + [\alpha - \beta] [\beta + [\alpha - \beta][\beta + \dots]] \\ &= \beta + \beta[\alpha - \beta] + \beta[\alpha - \beta]^2 + \dots \\ &= \beta \sum_n [\alpha - \beta]^n, \end{aligned}$$

the sum being from $n = 0$ to $n = \infty$. Since $0 < \alpha - \beta < 1$, the geometric series is convergent, and its sum yields

$$P(E) = \beta / [1 - \alpha + \beta].$$

This is certainly non-zero, for example if $\alpha = 3/4$ and $\beta = 3/8$, we compute $P(E) = 3/5$. We conclude that Lewis's 'disbelief' is rather shallow.

To summarize, the question 'Can a probability statement make sense without presupposing a certainty?' is answered by Lewis, with support from Russell, in the following way: 'No, it cannot. For if it could, this would necessarily lead to an infinite product, and the probability would be zero.' We have shown that Lewis's answer is incorrect by giving an example in which the iteration produces an infinite sum of products that yields a probability different from zero.

Nevertheless, there is a core of truth in Lewis's intuition. For there is a scenario to which Lewis's reasoning applies. Rather than the example given above, in which $P(E)$ is equal to the *sum* of products of probabilities, consider cases in which $P(E)$ is a *single* product of factors, each less than one. This scenario is perhaps somewhat exceptional, but it does show that an infinite iteration can lead to literally nothing, and hence must come to a stop in order to make sense. In such a case, one could reasonably claim that Lewis's intuition is sound.

An example in this class is furnished by a genetic condition that can only be inherited from a mother who carries the gene in question. It is believed that some mitochondrial disorders are of this kind. Suppose the conditional probability that a girl has the gene, given that her mother carries it, is $p = P(E|G_1)$. Then the unconditional probability that the child has the gene is $P(E) = pP(G_1)$, where $P(G_1)$ is the probability that the mother carries the gene. This follows from the fact that $P(\neg G_1) = 0$, i.e. it is impossible for the girl to have the gene if her mother does not. Also $P(G_1) = pP(G_2)$, where $P(G_2)$ is the probability that the maternal grandmother carried the gene. Iterating this procedure, we find

$$P(E) = p^n P(G_n),$$

where $P(G_n)$ is the probability that the great-great-grandmother in the n th generation on the mother's side carried the gene. Since p^n goes to zero as n tends to infinity, we have here indeed a case of the sort for which Russell's argument, and Lewis's intuition, are relevant. The only way to prevent $P(E) = 0$ is to suppose that there exists an n such that the n th great-great-grandmother had the gene, not on the basis of a probability argument,

but for some reason implying the breakdown, in that generation, of the rule that the gene can only be inherited. Such a reason might be, for example, the occurrence of a mutation. Since $P(E)$ is known to be non-zero, it must be true that the girl had a female ancestor who acquired the gene otherwise than through inheritance. This certainty of non-inheritance in some generation may be taken to be an example of what Lewis had in mind when he claimed that a chain of probabilities must be supported by a certain ground. On the other hand, since there must have been a generation in which inheritance of the gene did not take place, presumably the probability of a woman's having the gene in *any* generation, when her mother lacked it, must after all be nonzero. In this way, Reichenbach could expropriate this example too as grist to his mill.

The situation in which $P(E)$ is forced to be zero, unless there is a certain ground, *can* take place, but only if two very special requirements are fulfilled. First, for every n , the conditional probability that the ground, G_n , occurs if *its* ground, G_{n+1} , does *not* occur, must be zero. Second, the special cases given by $\forall n: q_n < 1$ but $\prod_n q_n \neq 0$ (see our discussion of Russell's second assumption in the previous section) must be excluded. In all other cases, the contention of Reichenbach that no certain ground is needed to allow $P(E)$ to be non-zero is correct.

The two examples that we have described above really do occur in the empirical world. That makes them more appealing than the examples in much of Reichenbach's writing, for these are seldom more than purely mathematical possibilities. More importantly, the example of the men suffering from prostatic cancer helps us to appreciate what the essential point in Reichenbach's rectification of Russell's error is (although it is not clear whether Reichenbach himself thought of the matter in this way). For if neither $P(E|G_1)$ nor $P(E|\neg G_1)$ is zero, then $P(E)$ cannot be zero either, whatever the value of $P(G_1)$ is. This follows directly from Eq.(1), for either $P(G_1)$ vanishes, or it does not. If $P(G_1) = 0$, then $P(\neg G_1) = 1$, and therefore $P(E) = P(E|\neg G_1) > 0$. If $P(G_1) \neq 0$, then $P(E) \geq P(E|G_1) P(G_1) > 0$. Hence $P(E) \neq 0$ is a necessary consequence of the non-vanishing of $P(E|G_1)$ and $P(E|\neg G_1)$, and no regress of probabilities is needed to reach this conclusion. More generally, if $P(E|\neg G_1)$ is zero, a regress must be started, but it will stop as soon as a further ground is found for which the probabilities conditioned by both the occurrence and the absence of its ground are non-zero. Details are given in the Appendix.

It is enough to know that two conditional probabilities are non-zero, in order to be able to conclude that the corresponding unconditional probability is non-zero too. Hence the point is not so much that Reichenbach's infinite iteration converges to a non-zero number, whereas Russell's usually converges to zero, although both statements are true. The real hub around which the argument rotates is the fact that something can be said about an unconditional probability (in this case that it is non-zero) on the basis of conditional probabilities alone.

5. Conclusions

In his major epistemological work, written during his stay in Istanbul, Reichenbach found an apt metaphor to summarize his probabilistic world view:

All we have is an elastic net of probability relations, floating in open space.
(Reichenbach, 1938, p.192)

Lewis, using a somewhat different image, comments upon Reichenbach's view with evident relish:

...the probabilistic conception strikes me as supposing that if enough probabilities can be got to lean against one another they can all be made to stand up. I suggest that, on the contrary, unless some of them can stand up alone, they will all fall flat. (Lewis, 1952, p.173)

We have shown that the controversy is more readily bridgeable than these quotations might seem to suggest. For whether or not a sequence of probability judgements about probability judgements tends to zero depends on the case. Sometimes an infinite number of (conditional) probabilities would not lead to disaster; but sometimes the structure can only be prevented from falling flat if one probability can stand up alone. However, we have also shown that, if $P(E|G_1) > 0$ and $P(E|\neg G_1) > 0$ then $P(E) > 0$, and this makes Reichenbach's position the more generic one.

We believe this finding to be a source of difficulty for those who have recently sought to reanimate epistemic foundationalism. As Richard Fumerton notes in his entry on foundationalism in the Stanford Encyclopedia for Philosophy, echoing ideas of J.S. Mill, there is a real sense in which one does not significantly extend one's knowledge by working towards conclusions that are implicitly contained in the premises. "To advance beyond foundations", Fumerton remarks, "we will inevitably need to employ non-deductive reasoning and ... that will ultimately require us to have noninferential (direct) knowledge of propositions describing probability connections between evidence and conclusions." Fumerton even goes so far as to write that one might have a priori knowledge of such probability connections, although he admits, tongue in cheek, that the view is not overly popular (Fumerton, 2005).

It is of course true that knowledge typically grows by employing non-deductive forms of reasoning. However this does not mean that ultimately we must have noninferential knowledge of probability connections. The view that probabilistic reasoning requires direct knowledge of probability statements reflects precisely Lewis's standpoint; and intuitively attractive though it may be, it leads to serious difficulties, as we have seen. These difficulties need not drive everyone immediately into Hans Reichenbach's stable, but they do seem to threaten some of epistemic foundationalism's new breeds.

Appendix

In this appendix, we treat the general case in which the probability of the occurrence of an event is conditioned by a ground, whose occurrence is in turn conditioned by a further ground, and so on *ad infinitum*. For notational convenience, we replace E in Eq.(1) by G_0 , and rewrite that equation in the form

$$\begin{aligned} P(G_0) &= P(G_0|G_1) P(G_1) + P(G_0|\neg G_1) [1 - P(G_1)] \\ &= P(G_0|\neg G_1) + P(G_0||G_1) P(G_1) , \end{aligned} \tag{2}$$

where we have introduced the notation

$$P(A||B) \equiv P(A|B) - P(A|\neg B)$$

a quantity that R.C. Jeffrey, following a suggestion of Carnap, has dubbed the ‘relevance’ of B to A (Jeffrey 1965, 170, 181). The idea is that G_1 is a ground for the occurrence of G_0 , so that normally $P(G_0|G_1) > P(G_0|\neg G_1)$; in words, G_0 is more likely to occur if G_1 has occurred than if it has not. Note that $P(G_0|G_1) = P(G_0|\neg G_1)$ is not strictly excluded, but this is the limit in which the probability for the occurrence of G_0 is not affected by the presence or absence of G_1 . In this case one would hardly call G_1 a ground for G_0 at all. For technical reasons we do not wish to exclude this limiting case; but we can exclude $P(G_0|G_1) < P(G_0|\neg G_1)$ on purely conventional grounds: if we thought that G_1 was a ground for G_0 , but found that $P(G_0|\neg G_1)$ is greater than $P(G_0|G_1)$, we were simply mistaken, for properly speaking $\neg G_1$ is a ground for G_0 , and not G_1 , so all we need to do is to rename $\neg G_1$ as G_1 , and G_1 as $\neg G_1$. Accordingly, we can always arrange that

$$P(G_0||G_1) \equiv P(G_0|G_1) - P(G_0|\neg G_1) \geq 0, \quad (3)$$

by suitable nomenclature of G_1 and $\neg G_1$.

Suppose now that G_2 is a ground for G_1 , so that

$$P(G_1) = P(G_1|\neg G_2) + P(G_1||G_2) P(G_2) \quad (4)$$

Inserting this into Eq.(2), we obtain the iterated form

$$P(G_0) = P(G_0|\neg G_1) + P(G_0||G_1) P(G_1|\neg G_2) + P(G_0||G_1) P(G_1||G_2) P(G_2),$$

and of course a formula similar to Eq.(4) can be used for $P(G_2)$, if G_3 is a ground for G_2 , and so on for any number of steps. After N iterations, the formula reads

$$P(G_0) = \sum_{n=0}^N Q_n P(G_n|\neg G_{n+1}) + Q_{N+1} P(G_{N+1}), \quad (5)$$

where $Q_0 = 1$ and

$$Q_n = \prod_{m=0}^{n-1} P(G_m||G_{m+1})$$

for $n \geq 1$, as can be readily shown by mathematical induction.

If $P(G_n|\neg G_{n+1}) = 0$ for $n = 0, 1, 2, \dots, N$, then $P(G_n||G_{n+1}) = P(G_n|G_{n+1})$ for the same values of n , and so Eq.(5) reduces to

$$P(G_0) = P(G_0|G_1) P(G_1|G_2) \dots P(G_N|G_{N+1}) P(G_{N+1}), \quad (6)$$

the extreme situation that we may charitably assume Russell (and perhaps also Lewis) to have had in mind. Here the probability of the occurrence of G_0 is simply the product of the conditional probabilities of successive grounds. This product can fail to be zero in one

of two ways: either the product can have a finite number of terms, the final probability, $P(G_{N+1})=1$, neither having nor needing to have a ground, or the product might have an infinite number of terms, but the infinite product might converge to a non-zero value, as in the example that we cited in Footnote 5 (although this escape route is perhaps rather academic).

However, the generic situation is that in which $P(G_n|\neg G_{n+1})$ is not zero for some n , and Reichenbach was quite right to insist that the straightforward product of conditional probabilities, Eq.(6), is usually incorrect. Suppose first of all that $P(G_0|\neg G_1) \neq 0$. Then from the second line of Eq.(2), and the constraint (3), we have $P(G_0) \geq P(G_0|\neg G_1) > 0$. In this case, $P(G_0)$ has been shown to be non-zero, without any iteration at all. More generally, if $P(G_n|\neg G_{n+1}) = 0$, for $n = 0, 1, 2, \dots, N - 1$, but $P(G_N|\neg G_{N+1}) > 0$, we see from Eq.(5) that

$$P(G_0) \geq Q_N P(G_N|\neg G_{N+1}) = P(G_0|G_1) P(G_1|G_2) \dots P(G_{N-1}|G_N) P(G_N|\neg G_{N+1}) .$$

This product of a finite number of positive terms is non-zero, and no infinite regress need be hazarded to come to this conclusion.

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