A Nondeterministic View on Nonclassical Negations

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Abstract

We investigate two large families of logics, differing from each other by the treatment of negation. The logics in one of them are obtained from the positive fragment of classical logic (with or without a propositional constant **ff** for "the false") by adding various standard Gentzen-type rules for negation. The logics in the other family are similarly obtained from LJ^+, the positive fragment of intuitionistic logic (again, with or without **ff**). For all the systems, we provide simple semantics which is based on non-deterministic four-valued or three-valued structures, and prove soundness and completeness for all of them. We show that the role of each rule is to reduce the degree of nondeterminism in the corresponding systems. We also show that all the systems considered are decidable, and our semantics can be used for the corresponding decision procedures. Most of the extensions of LJ^+ (with or without **ff**) are shown to be conservative over the underlying logic, and it is determined which of them are not.

1 Introduction

From both classical and constructive points of view, the question whether we accept or reject a given sentence φ depends on the data (or information) we have concerning it. The data might be positive (in which case we might say that φ is *supported*), or negative (in which case we might say that φ is *questioned*). This intuition may be formally reflected by the use of *four* truth values: t, $\top$, f and $\bot$, where we expect a valuation v in $\{t, f, \top. \bot\}$ to satisfy:

- $v(\varphi) = t$ if φ is supported but not questioned.
- $v(\varphi) = f$ if φ is questioned but not supported.
- $v(\varphi) = \top$ if φ is both supported and questioned.
- $v(\varphi) =\bot$ if φ is neither supported nor questioned.

Driven by considerations of this sort, and following previous works by Dunn, Belnap suggested (in [Bel77b, Bel77a]) the use for computers of logics based on these four truth-values. He went on to propose a specific four-valued *matrix* for this task (see 2.2 below). By this Belnap (implicitly) accepted the classical extensionality principle, according to which the truth-value of a compound formula is completely determined by the truth-values of its immediate subformulas. In particular: negation has a fully deterministic interpretation in Dunn-Belnap's matrix. But is this interpretation the only plausible one? The answer depends, of course, on the intuitive meaning of $\neg$ in the context of the four values. The most natural such interpretation is perhaps that $\neg$ represents, within the language, the idea of negative data: $\neg\varphi$ should mean: "φ is questioned". This implies that supporting $\neg\varphi$ and questioning φ should amount to the same thing. Hence we get:

- $v(\varphi) = t$ if φ is supported and $\neg\varphi$ is not.
- $v(\varphi) = f$ if $\neg\varphi$ is supported and φ is not.
- $v(\varphi) = \top$ if both φ and $\neg\varphi$ are supported.
- $v(\varphi) =\perp$ if neither φ nor $\neg\varphi$ are supported.

Given the truth value of φ, what do these principles tell us about the truth-value of its negation? Well, it is easy to see that they dictate the following derived principles (and nothing stronger, as long as we do not introduce additional assumptions concerning supporting or questioning):

- If $v(\varphi) = t$ then $v(\neg\varphi) \in \{f, \perp\}$.
- If $v(\varphi) = f$ then $v(\neg\varphi) \in \{t, \top\}$.
- If $v(\varphi) = \top$ then $v(\neg\varphi) \in \{t, \top\}$.
- If $v(\varphi) =\perp$ then $v(\neg\varphi) \in \{f, \perp\}$.

It follows that the truth-value of φ does not fully determine the truth-value of $\neg\varphi$. Hence *nondeterministic* semantics seems to be appropriate here. A similar conclusion may be obtained if we examine the expected behavior of disjunction and conjunction. Thus in Dunn-Belnap's matrix we have $\top\vee\perp= t$, which might seems strange (and perhaps unintuitive). Again, such peculiarities can be overcome if one uses nondeterministic semantics.

In this paper we explore the application of these ideas for large families of logics. We concentrate on logics which are easily and naturally defined by using Gentzen-type systems with various standard, very common, rules for negation. The differences between the different logics we investigate concern:

The underlying logic : We consider two main possibilities: positive classical logic, and positive intuitionistic logic (also called minimal logic). In both cases we consider a pure subcase, in which the falsehood constant **ff** is not included, and subcases in which it is added (with appropriate rules) to the language (it turns out that such an addition has practically no effects on our results).

The rules for negation : In all the logics we consider, these are taken from a list of rules, given below, which includes the two standard classical rules for negation, as well as the most common standard rules for combining negation with other connectives.

Below we provide simple non-deterministic semantics for all the $24 \cdot 2^{10}$ different nonclassical systems we consider, and prove their soundness and completeness with respect to these semantics. The main insight we get is that the role of each rule is to reduce the degree of nondeterminism of some connective by restricting the allowed outputs of its application in some cases. We also show that all the systems we consider are decidable, and that our semantics can be used for the corresponding decision procedures. In the case of the extensions of LJ^+ (with or without **ff**) we show that most of them are conservative over the underlying logic, and determine which of them are not.

2 Preliminaries

From now on (unless otherwise stated), all formulas are assumed to be (depending on the context) either in the propositional language based in $\{\vee, \wedge, \supset, \neg\}$, or on that based on $\{\vee, \wedge, \supset, \neg, \mathbf{ff}\}$. We use p, q, r to denote atomic formulas, $A, B, C, \psi, \varphi, \phi$ to denote arbitrary formulas, and Γ, Δ to denote finite sets of formulas. A sequent has the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite sets of formulas. Following tradition, we write Γ, φ and Γ, Δ for $\Gamma \cup \{\varphi\}$ and $\Gamma \cup \Delta$, respectively. By a (propositional) *logic* we shall mean a pair $\langle \mathcal{L}, \vdash \rangle$, in which $\mathcal{L}$ is a propositional language, and $\vdash$ is a consequence relation on the set of formulas of $\mathcal{L}$.

2.1 The Logics and the Associated Proof Systems

2.1.1 The Standard Positive Logics

We start by presenting Gentzen-type systems for the four logics which we use as bases. To see what is the essence of the differences between those logics, we use (cut-free) multiple-conclusion versions for *all* of them, including the constructive ones.

THE SYSTEM LK^+

Axioms: $A \Rightarrow A$

Structural Rules: Cut, Weakening

Logical Rules:

$$(\supset\Rightarrow)\quad \frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \supset B, \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \supset B, \Delta}\quad (\Rightarrow\supset)$$

$$(\wedge \Rightarrow)\quad \frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}\quad (\Rightarrow \wedge)$$

$$(\vee \Rightarrow)\quad \frac{\Gamma, A \Rightarrow \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B}\quad (\Rightarrow \vee)$$

THE SYSTEM LK: This is the system obtained from LK^+ by adding the following axiom:

$$\mathbf{ff} \Rightarrow$$

THE SYSTEMS LJ^+ and LJ: These are the systems obtained from LK^+ and LK (respectively) by weakening their $(\Rightarrow\supset)$ rule to:

$$(\Rightarrow\supset)\quad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \supset B}$$

Notes:

1. LK is a standard Gentzen-type calculus for the classical propositional logic (taken in the language of $\{\vee, \wedge, \supset, \mathbf{ff}\}$), while LK^+ is its purely positive fragment. The system LJ is a sequent calculus for the propositional intuitionistic logic, while LJ^+ is its purely positive fragment. The four systems are sound and complete for the corresponding logics, and admit cut-elimination (see [Tak75]).

2. In both LK and LJ it is possible to define the usual negation connective of the corresponding logics by letting $\sim\varphi =_{Df} \varphi \supset \mathbf{ff}$ (for intuitionistic logic, this is in fact the common procedure).

We shall nevertheless take all four systems as "positive" logics, since our principal goal is to investigate the systems obtained from them by adding an independent negation connective $\neg$ to their languages.

2.1.2 Standard Rules for Negation and Corresponding Systems

The two standard Gentzen-type rules for classical negation are:

$$(\neg\Rightarrow)\quad \frac{\Gamma\Rightarrow\Delta,A}{\neg A,\Gamma\Rightarrow\Delta} \qquad\qquad \frac{A,\Gamma\Rightarrow\Delta}{\Gamma\Rightarrow\Delta,\neg A}\quad(\Rightarrow\neg)$$

Instead of these rules, many systems for classical or nonclassical logics employ rules for introducing combinations of negation with other connectives. The most common rules used for this task are the following:

$$(\neg\neg\Rightarrow)\quad \frac{A,\Gamma\Rightarrow\Delta}{\neg\neg A,\Gamma\Rightarrow\Delta} \qquad\qquad \frac{\Gamma\Rightarrow\Delta,A}{\Gamma\Rightarrow\Delta,\neg\neg A}\quad(\Rightarrow\neg\neg)$$

$$(\neg\supset\Rightarrow)\quad \frac{A,\neg B,\Gamma\Rightarrow\Delta}{\neg(A\supset B),\Gamma\Rightarrow\Delta} \qquad\qquad \frac{\Gamma\Rightarrow\Delta,A \qquad \Gamma\Rightarrow\Delta,\neg B}{\Gamma\Rightarrow\Delta,\neg(A\supset B)}\quad(\Rightarrow\neg\supset)$$

$$(\neg\vee\Rightarrow)\quad \frac{\Gamma,\neg A,\neg B\Rightarrow\Delta}{\Gamma,\neg(A\vee B)\Rightarrow\Delta} \qquad\qquad \frac{\Gamma\Rightarrow\Delta,\neg A \qquad \Gamma\Rightarrow\Delta,\neg B}{\Gamma\Rightarrow\Delta,\neg(A\vee B)}\quad(\Rightarrow\neg\vee)$$

$$(\neg\wedge\Rightarrow)\quad \frac{\Gamma,\neg A\Rightarrow\Delta \qquad \Gamma,\neg B\Rightarrow\Delta}{\Gamma,\neg(A\wedge B)\Rightarrow\Delta} \qquad\qquad \frac{\Gamma\Rightarrow\Delta,\neg A,\neg B}{\Gamma\Rightarrow\Delta,\neg(A\wedge B)}\quad(\Rightarrow\neg\wedge)$$

Now, in Gentzen's original formulation ([Gen69]) the rules $(\wedge\Rightarrow)$ and $(\Rightarrow\vee)$ were split into two rules, each with one side formula only. To make our investigation finer, we do the same here for $(\neg\vee\Rightarrow)$, $(\Rightarrow\neg\wedge)$ and $(\neg\supset\Rightarrow)$. Thus instead of these three rules, we consider the following six:

$$(\neg\supset\Rightarrow)_1\quad \frac{A,\Gamma\Rightarrow\Delta}{\neg(A\supset B),\Gamma\Rightarrow\Delta} \qquad\qquad \frac{\neg B,\Gamma\Rightarrow\Delta}{\neg(A\supset B),\Gamma\Rightarrow\Delta}\quad(\neg\supset\Rightarrow)_2$$

$$(\neg\vee\Rightarrow)_1\quad \frac{\Gamma,\neg A\Rightarrow\Delta}{\Gamma,\neg(A\vee B)\Rightarrow\Delta} \qquad\qquad \frac{\Gamma,\neg B\Rightarrow\Delta}{\Gamma,\neg(A\vee B)\Rightarrow\Delta}\quad(\neg\vee\Rightarrow)_2$$

$$(\Rightarrow\neg\wedge)_1\quad \frac{\Gamma\Rightarrow\Delta,\neg A}{\Gamma\Rightarrow\Delta,\neg(A\wedge B)} \qquad\qquad \frac{\Gamma\Rightarrow\Delta,\neg B}{\Gamma\Rightarrow\Delta,\neg(A\wedge B)}\quad(\Rightarrow\neg\wedge)_2$$

Definition 2.1 1. The set NR is the union of the following sets of rules:

$$NR_1=\{(\neg\Rightarrow),(\Rightarrow\neg)\}$$

$$NR_2 = \{(\neg\neg \Rightarrow), (\Rightarrow \neg\neg), (\Rightarrow \neg \supset), (\Rightarrow \neg\vee), (\neg\wedge \Rightarrow)\}$$

$$NR_3 = \{(\neg \supset\Rightarrow)_1, (\neg \supset\Rightarrow)_2, (\neg\vee \Rightarrow)_1, (\neg\vee \Rightarrow)_2, (\Rightarrow \neg\wedge)_1, (\Rightarrow \neg\wedge)_2\}$$

2. For $\mathbf{L} \in \{LK, LK^+, LJ, LJ^+\}$ and $S \subseteq NR$, we denote by $\mathbf{L}(S)$ the system obtained from $\mathbf{L}$ by adding the rules in S.

3. For $\mathbf{L} \in \{LK, LJ\}$ and $S \subseteq NR$, we denote by $\mathbf{L}^f(S)$ the system obtained from $\mathbf{L}(S)$ by adding the axiom:

$$\Rightarrow \neg\mathbf{ff}$$

Historical Notes: Some of the logics introduced in Definition 2.1 have already been studied in the literature. Thus $LJ^+(NR_1 \cup NR_2)$ and $LJ^+(NR - \{(\Rightarrow \neg)\})$ are respectively identical with the logics $\mathbf{N}^-$ and $\mathbf{N}$ of Nelson ([AN84]) and Kutschera ([vK69]) — see [Wan93] for further details and references. $LK^+(NR - \{(\Rightarrow \neg)\})$ is equivalent to the logic LPF of the VDM project ([Jon86]). $LK^+(NR_1 \cup NR_2)$ is the logic of the bilattice $\mathcal{FOUR}$ (see subsection 2.2). The logics $LK(\{(\Rightarrow \neg)\})$ and $LK(NR - \{(\neg \Rightarrow\})$ were introduced in [Bat80], where they were called PI and PI^s, respectively. Later Batens changed their names to $CLuN$ and $CLuNs$, respectively (see e.g. [Bat00]). $LK^+(NR - \{(\neg \Rightarrow\})$ was independently introduced (together with the 3-valued deterministic semantics described in subsection 2.2) in [Avr86, Avr91, Roz89]. In [Avr91] it was called PAC (this name was adopted in [CM02]). $LK(NR - \{(\neg \Rightarrow\})$ was originally introduced in [Sch60]. Later it was reintroduced (together with its 3-valued deterministic semantics) in [DdC70, D'O85], where it was called J_3 (see also [Eps90]), while in [CM02] it was called $LFI1$. The system $LK^+(\{(\Rightarrow \neg), (\neg\neg \Rightarrow)\})$ is the logic C_{min} studied in [CM99]. $LK^+(\{(\Rightarrow \neg), (\Rightarrow \neg\neg), (\Rightarrow \neg\vee)\})$ was again introduced in [Bat80], under the name PI^*. $LJ^+(\{(\Rightarrow \neg), (\neg\neg \Rightarrow)\})$ is Raggio's formulation (in [Rag68]) of da Costa's famous logic C_ω (see [dC74]).

2.1.3 Corresponding Hilbert-type Systems

Some of the logics mentioned above have been originally introduced using Hilbert-type systems. Such systems can be easily given for every system $\mathbf{L}(S)$ or $\mathbf{L}^f(S)$. We start with some standard Hilbert-type system $H\mathbf{L}$ for $\mathbf{L}$ (having MP as the only rule of inference), and add to it the axioms from the list below, which correspond to the negation rules in S. In the case of $\mathbf{L}^f(S)$ we add also the axiom $A(\Rightarrow \neg\mathbf{ff})$. Here is the list of axioms that correspond to our 13 rules and to $\Rightarrow \neg\mathbf{ff}$:

$A(\neg \Rightarrow)$**:** $\quad \neg\varphi \supset (\varphi \supset \psi)$

$A(\Rightarrow \neg)$: $\varphi \vee \neg\varphi$

$A(\neg\neg \Rightarrow)$: $\neg\neg\varphi \supset \varphi$

$A(\Rightarrow \neg\neg)$: $\varphi \supset \neg\neg\varphi$

$A(\neg \supset\Rightarrow)_1$: $\neg(\varphi \supset \psi) \supset \varphi$

$A(\neg \supset\Rightarrow)_2$: $\neg(\varphi \supset \psi) \supset \neg\psi$

$A(\Rightarrow \neg \supset)$: $(\varphi \wedge \neg\psi) \supset \neg(\varphi \supset \psi)$

$A(\neg\vee \Rightarrow)_1$: $\neg(\varphi \vee \psi) \supset \neg\varphi$

$A(\neg\vee \Rightarrow)_2$: $\neg(\varphi \vee \psi) \supset \neg\psi$

$A(\Rightarrow \neg\vee)$: $(\neg\varphi \wedge \neg\psi) \supset \neg(\varphi \vee \psi)$

$A(\neg\wedge \Rightarrow)$: $\neg(\varphi \wedge \psi) \supset (\neg\varphi \vee \neg\psi)$

$A(\Rightarrow \neg\wedge)_1$: $\neg\varphi \supset \neg(\varphi \wedge \psi)$

$A(\Rightarrow \neg\wedge)_2$: $\neg\psi \supset \neg(\varphi \wedge \psi)$

$A(\Rightarrow \neg\mathbf{ff})$: $\neg\mathbf{ff}$

Definition 2.2 For $S \subseteq NR$, let $H\mathbf{L}(S)$ be the system obtained from $H\mathbf{L}$ (where $\mathbf{L}$ is in $\{LK, LK^+, LJ, LJ^+\}$) by adding to it the axioms which correspond to the rules in S, and let $H\mathbf{L}^f(S)$ be $H\mathbf{L}(S)+A(\Rightarrow \neg\mathbf{ff})$.

Theorem 2.3 *$H\mathbf{L}(S)$ and $\mathbf{L}(S)$ are strongly equivalent for every $\mathbf{L} \in \{LK, LK^+, LJ, LJ^+\}$ and $S \subseteq NR$: If T is a set of sentences, Δ a finite set of sentences, and ψ_Δ is a disjunction of all the sentences in Δ, then $T \vdash_{H\mathbf{L}(S)} \psi_\Delta$ iff there is a finite subset Γ of T such that $\vdash_{\mathbf{L}(S)} \Gamma \Rightarrow \Delta$. Similar relations hold between $H\mathbf{L}^f(S)$ and $\mathbf{L}^f(S)$.*

Proof: Standard. ■

2.2 The Bilattice $\mathcal{FOUR}$

The Logic $LK^+(NR_2 \cup NR_3)$ has a well-known characteristic matrix, based on the four values $t, f, \bot$, and $\top$. In its best known presentation, this matrix was described and motivated by Belnap in [Bel77b, Bel77a], following works and ideas of Dunn (see e.g. [Dun76]). To motivate the design of this structure, Dunn and Belnap employ two natural orderings of the truth values: the "truth" partial order $\leq_t$, and the "knowledge" partial order $\leq_k$. According to $\leq_t$, f is the minimal element, t is the maximal one, and $\bot, \top$ are two intermediate values, which are incomparable. According to $\leq_k$ (originally due to Scott), $\bot$ is the minimal element, $\top$ – the maximal one, and t, f are the intermediate values (see Figure 1).

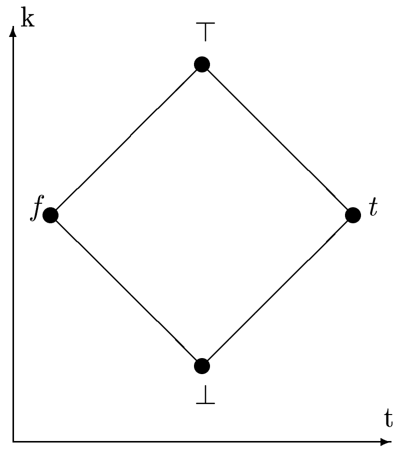

Figure 1: *FOUR*

Both $(\{t, f, \top, \bot\}, \leq_t)$ and $(\{t, f, \top, \bot\}, \leq_k)$ are lattices, and the lattice operations of the first are used to provide the semantics of $\vee$ and $\wedge$. In addition, there is a negation operation which is an involution w.r.t. $\leq_t$ and is monotone w.r.t. $\leq_k$ (there is exactly one such operation: its details are given below).[1] Dunn-Belnap's structure is nowadays known also as the basic (distributive) *bilattice*, and its logic — as the basic logic of (distributive) bilattices (see [Gin87, Gin88, Fit90b, Fit90a, Fit91, Fit94, AA96, AA98]). In [AA96] Belnap's matrix was extended with an appropriate implication connective. The resulting structure is described in the next definition.

Definition 2.4 The matrix $\mathcal{FOUR} = \langle \mathcal{T}, \mathcal{D}, \mathcal{O} \rangle$ is given by:

- $\mathcal{T} = \{t, \top, \bot, f\}$
- $\mathcal{D} = \{t, \top\}$
- The operations in $\mathcal{O}$ are defined by:

   1. $\neg t = f, \;\; \neg f = t, \;\; \neg\top = \top, \;\; \neg\bot = \bot$
   2. $a \vee b = sup_{\leq_t}(a, b), \;\; a \wedge b = inf_{\leq_t}(a, b)$

[1]Belnap allowed only the use of operations which are monotone with respect to $\leq_k$. The implication $\supset$ we use below does not have this property, and so $\leq_k$ has little role in the semantics of our extensions of LK^+. Surprisingly, it has great importance for the semantics of our extensions of LJ^+ (see section 4).

3. $a \supset b \;= \begin{cases} b & \text{if } a \in \mathcal{D} \\ t & \text{if } a \notin \mathcal{D} \end{cases}$

Theorem 2.5

1. ([AA96]) $LK^+(NR_2 \cup NR_3)$ *is sound and complete w.r.t to* $\mathcal{FOUR}$.

2. ([Avr91])[2] $LK^+(NR_2 \cup NR_3 \cup \{(\neg \Rightarrow)\})$ *is sound and complete w.r.t to the three-valued* $\{t, f, \bot\}$*-submatrix of* $\mathcal{FOUR}$*, while* $LK^+(NR_2 \cup NR_3 \cup \{(\Rightarrow \neg)\})$ *is sound and complete w.r.t to its* $\{t, f, \top\}$*-submatrix.*

3 Semantics in the Classical Case

3.1 Nondeterministic Matrices

Our main semantic tool in what follows will be the following generalization of the concept of a matrix from [AL04, AL01]:

Definition 3.1

1. A *non-deterministic matrix* (*Nmatrix* for short) for a propositional language $\mathcal{L}$ is a tuple $\mathcal{M} = \langle \mathcal{T}, \mathcal{D}, \mathcal{O} \rangle$, where:

   (a) $\mathcal{T}$ is a non-empty set of *truth values.*

   (b) $\mathcal{D}$ is a non-empty proper subset of $\mathcal{T}$.

   (c) For every n-ary connective $\diamond$ of $\mathcal{L}$, $\mathcal{O}$ includes a corresponding n-ary function $\widetilde{\diamond}$ from $\mathcal{T}^n$ to $2^{\mathcal{T}} - \{\emptyset\}$.

   We say that $\mathcal{M}$ is *(in)finite* if so is $\mathcal{T}$.

2. Let $\mathcal{F}$ be the set of formulas of $\mathcal{L}$. A *(legal) valuation* in an Nmatrix $\mathcal{M}$ is a function $v : \mathcal{F} \to \mathcal{T}$ that satisfies the following condition for every n-ary connective $\diamond$ of $\mathcal{L}$ and $\psi_1, \ldots, \psi_n \in \mathcal{F}$:

$$v(\diamond(\psi_1, \ldots, \psi_n)) \in \widetilde{\diamond}(v(\psi_1), \ldots, v(\psi_n))$$

3. A valuation v in an Nmatrix $\mathcal{M}$ is a *model* of (or *satisfies*) a formula ψ in $\mathcal{M}$ (notation: $v \models^{\mathcal{M}} \psi$) if $v(\psi) \in \mathcal{D}$. v is a *model* of a set Γ of formulas in $\mathcal{M}$ (notation: $v \models^{\mathcal{M}} \Gamma$) if it satisfies every formula in Γ.

[2] The two parts of this item are proved together in [Avr91]. However, each of them alone has been discovered and proved in many other papers. See the historical notes at the end of subsection 2.1.2 for relevant references.

4. $\vdash_{\mathcal{M}}$, the consequence relation induced by the Nmatrix $\mathcal{M}$, is defined as follows:

   $\Gamma \vdash_{\mathcal{M}} \Delta$ if for every v such that $v \models^{\mathcal{M}} \Gamma$ there exists $\varphi \in \Delta$ such that $v \models^{\mathcal{M}} \varphi$

5. A logic $\mathbf{L} = \langle \mathcal{L}, \vdash \rangle$ is *sound* for an Nmatrix $\mathcal{M}$ (where $\mathcal{L}$ is the language of $\mathcal{M}$) if $\vdash \subseteq \vdash_{\mathcal{M}}$. $\mathbf{L}$ is *complete* for $\mathcal{M}$ if $\vdash \supseteq \vdash_{\mathcal{M}}$. $\mathcal{M}$ is *characteristic* for $\mathbf{L}$ if $\mathbf{L}$ is both sound and complete for it (i.e.: if $\vdash = \vdash_{\mathcal{M}}$).

Note: We shall identify an ordinary (deterministic) matrix with an Nmatrix $\mathcal{M} = \langle \mathcal{T}, \mathcal{D}, \mathcal{O} \rangle$ such that the functions in $\mathcal{O}$ always return singletons.

The following Definition is a refinement of the notion of "refinement" used in [Avr03]:

Definition 3.2 Let $\mathcal{M}_1 = \langle \mathcal{T}_1, \mathcal{D}_1, \mathcal{O}_1 \rangle$ and $\mathcal{M}_2 = \langle \mathcal{T}_2, \mathcal{D}_2, \mathcal{O}_2 \rangle$ be Nmatrices for a language $\mathcal{L}$. $\mathcal{M}_2$ is called a *refinement* of $\mathcal{M}_1$ if $\mathcal{T}_2 \subseteq \mathcal{T}_1$, $\mathcal{D}_2 = \mathcal{D}_1 \cap \mathcal{T}_2$, and $\widetilde{\diamond}_{\mathcal{M}_2}(\vec{x}) \subseteq \widetilde{\diamond}_{\mathcal{M}_1}(\vec{x})$ for every n-ary connective $\diamond$ of $\mathcal{L}$ and every $\vec{x} \in \mathcal{T}_2^n$.

Proposition 3.3 *If $\mathcal{M}_2$ is a refinement of $\mathcal{M}_1$ then $\vdash_{\mathcal{M}_1} \subseteq \vdash_{\mathcal{M}_2}$. Hence if $\mathbf{L}$ is sound for $\mathcal{M}_1$ then $\mathbf{L}$ is also sound for $\mathcal{M}_2$.*

Proof: Suppose $\Gamma \vdash_{\mathcal{M}_1} \Delta$. We show that $\Gamma \vdash_{\mathcal{M}_2} \Delta$. So assume that v is a model of Γ in $\mathcal{M}_2$. Since $\mathcal{T}_2 \subseteq \mathcal{T}_1$, and $\widetilde{\diamond}_{\mathcal{M}_2}(\vec{x}) \subseteq \widetilde{\diamond}_{\mathcal{M}_1}(\vec{x})$ for every n-ary connective $\diamond$ of $\mathcal{L}$ and every $\vec{x} \in \mathcal{T}_2^n$, v is a legal valuation in $\mathcal{M}_1$. Since $\mathcal{D}_2 \subseteq \mathcal{D}_1$, v is actually a model of Γ in $\mathcal{M}_1$. This and the fact that $\Gamma \vdash_{\mathcal{M}_1} \Delta$ imply that $v(\varphi) \in \mathcal{D}_1$ for some $\varphi \in \Delta$. But $v(\varphi)$ is also in $\mathcal{T}_2$, and so $v(\varphi) \in \mathcal{D}_1 \cap \mathcal{T}_2 = \mathcal{D}_2$. Hence v is a model in $\mathcal{M}_2$ of some element of Δ. ■

3.2 Nondeterministic Four-Valued Semantics

Classical Logic has, of course, the semantics of the usual two-valued deterministic matrix. This semantics can, however, be easily generalized as follows.

Definition 3.4 1. Let $\mathcal{M} = \langle \mathcal{T}, \mathcal{D}, \mathcal{O} \rangle$ be an Nmatrix for a language which includes that of LK^+. We say that $\mathcal{M}$ is *suitable* for LK^+ if the following conditions are satisfied:

- If $a \in \mathcal{D}$ and $b \in \mathcal{D}$ then $a \widetilde{\wedge} b \subseteq \mathcal{D}$
- If $a \notin \mathcal{D}$ then $a \widetilde{\wedge} b \subseteq \mathcal{T} - \mathcal{D}$
- If $b \notin \mathcal{D}$ then $a \widetilde{\wedge} b \subseteq \mathcal{T} - \mathcal{D}$

- If $a \in \mathcal{D}$ then $a \widetilde{\vee} b \subseteq \mathcal{D}$
- If $b \in \mathcal{D}$ then $a \widetilde{\vee} b \subseteq \mathcal{D}$
- If $a \notin \mathcal{D}$ and $b \notin \mathcal{D}$ then $a \widetilde{\vee} b \subseteq \mathcal{T} - \mathcal{D}$

- If $a \notin \mathcal{D}$ then $a \widetilde{\supset} b \subseteq \mathcal{D}$
- If $b \in \mathcal{D}$ then $a \widetilde{\supset} b \subseteq \mathcal{D}$
- If $a \in \mathcal{D}$ and $b \notin \mathcal{D}$ then $a \widetilde{\supset} b \subseteq \mathcal{T} - \mathcal{D}$

2. Let $\mathcal{M} = \langle \mathcal{T}, \mathcal{D}, \mathcal{O} \rangle$ be an Nmatrix for a language which includes the language of LK. We say that $\mathcal{M}$ is *suitable* for LK if it is suitable for LK^+, and the following condition is satisfied:

   - $\widetilde{\mathbf{ff}} \subseteq \mathcal{T} - \mathcal{D}$

Theorem 3.5 *LK (LK^+) is sound for any Nmatrix $\mathcal{M}$ which is suitable for it. Moreover: it is complete for the relevant fragment of $\mathcal{M}$.*

Proof: We leave the easy proof for the reader. ■

Convention: For convenience, we henceforth usually employ the same symbol for a connective and for the corresponding nondeterministic operation in a given Nmatrix. We also denote by the same symbol (usually $\mathcal{O}$) the set of connectives of a language $\mathcal{L}$ and the corresponding set of operations of an Nmatrix for $\mathcal{L}$.

We turn now to Nmatrices for logics with negation which are based on the basic four truth values described in the introduction.

Definition 3.6 Let $\mathcal{M}_P$ ($\mathcal{M}_P^{\mathbf{ff}}$) be the following Nmatrix $\langle \mathcal{T}, \mathcal{D}, \mathcal{O} \rangle$:

- $\mathcal{T} = \{t, \top, f, \bot\}$
- $\mathcal{D} = \{t, \top\}$
- $a \supset b = \begin{cases} \mathcal{D} & b \in \mathcal{D} \text{ or } a \in \mathcal{T} - \mathcal{D} \\ \mathcal{T} - \mathcal{D} & \text{otherwise} \end{cases}$

  $a \vee b = \begin{cases} \mathcal{D} & a \in \mathcal{D} \text{ or } b \in \mathcal{D} \\ \mathcal{T} - \mathcal{D} & \text{otherwise} \end{cases}$

$$a \wedge b = \begin{cases} \mathcal{D} & a \in \mathcal{D}, b \in \mathcal{D} \\ \mathcal{T} - \mathcal{D} & \text{otherwise} \end{cases}$$

$$\neg t = \neg \perp = \mathcal{T} - \mathcal{D} \qquad \neg f = \neg \top = \mathcal{D}$$

$$(\mathbf{ff} = \mathcal{T} - \mathcal{D})$$

Proposition 3.7 *LK^+ (LK) is sound for any refinement of $\mathcal{M}_P$ ($\mathcal{M}_P^{\mathbf{ff}}$).*

Proof: This follows from Theorem 3.5. ∎

Note: Since the ordinary two-valued matrix is a refinement of $\mathcal{M}_P$ ($\mathcal{M}_P^{\mathbf{ff}}$) and is complete for LK^+ and LK, so are $\mathcal{M}_P$ and $\mathcal{M}_P^{\mathbf{ff}}$ (and this is obviously true for every refinement of them).

3.3 Effects of the Negation Rules

We turn now to the effects of the various negation rules. We shall show that to each of them corresponds a condition which leads to a certain refinement of $\mathcal{M}_P$ (or $\mathcal{M}_P^{\mathbf{ff}}$). These conditions are independent of each other, but never contradict each other. To see how these conditions are obtained, take $(\neg \supset \Rightarrow)_1$ as an example. This rule is equivalent to the validity of $\neg(\varphi \supset \psi) \vdash \varphi$. It means therefore that $v(\neg(\varphi \supset \psi)) \notin \mathcal{D}$ if $v(\varphi) \notin \mathcal{D}$. Since $v(\varphi \supset \psi)$ itself should be in $\mathcal{D}$ if $v(\varphi) \notin \mathcal{D}$, it follows by $\mathcal{M}_P$'s truth tables for negation that $v(\varphi \supset \psi)$ should be t if $v(\varphi) \notin \mathcal{D}$. This is therefore the condition that corresponds to this rule, and it turns 8 (out of the many more) possible nondeterministic choices in $\mathcal{M}_P$ (or $\mathcal{M}_P^{\mathbf{ff}}$) to deterministic ones. Similar analysis can be done for the other rules. The resulting list of conditions is listed in the next Definition.

Definition 3.8 1. The refining conditions induced by the negation rules are:

- **$C(\neg \Rightarrow)$:** Use only t, f and $\perp$
- **$C(\Rightarrow \neg)$:** Use only t, f and $\top$
- **$C(\neg\neg \Rightarrow)$:** $\neg f = \{t\}$, $\neg \perp = \{\perp\}$
- **$C(\Rightarrow \neg\neg)$:** $\neg t = \{f\}$, $\neg \top = \{\top\}$
- **$C(\neg\vee \Rightarrow)_1$:** $x \vee y = \{sup_t(x, y)\}$ if $x \in \{t, \perp\}$
- **$C(\neg\vee \Rightarrow)_2$:** $x \vee y = \{sup_t(x, y)\}$ if $y \in \{t, \perp\}$
- **$C(\Rightarrow \neg\vee)$:** $x \vee y = \{sup_t(x, y)\}$ if $x \in \{f, \top\}$, $y \in \{f, \top\}$
- **$C(\neg\wedge \Rightarrow)$:** $x \wedge y = \{inf_t(x, y)\}$ if $x \in \{t, \perp\}$, $y \in \{t, \perp\}$

$C(\Rightarrow \neg\wedge)_1$: $x \wedge y = \{inf_t(x,y)\}$ if $x \in \{f, \top\}$

$C(\Rightarrow \neg\wedge)_2$: $x \wedge y = \{inf_t(x,y)\}$ if $y \in \{f, \top\}$

$C(\neg \supset\Rightarrow)_1$: $x \supset y = \{t\}$ if $x \notin \mathcal{D}$

$C(\neg \supset\Rightarrow)_2$: $x \supset y = \{t\}$ if $y = t$ or $x \notin \mathcal{D}$ and $y = \bot$

$x \supset y = \{y\}$ if $y = t$ or $x \in \mathcal{D}$ and $y = \bot$

$C(\Rightarrow \neg \supset)$: $x \supset y = \{y\}$ if $x \in \mathcal{D}$ and $y \in \{f, \top\}$

2. For $S \subseteq NR$, let $C(S) = \{Cr \mid r \in S\}$

From now until the end of this subsection, we shall concentrate on the language without **ff**.

Definition 3.9 For $S \subseteq NR$, let $\mathcal{M}_P[S]$ be the weakest refinement of $\mathcal{M}_P$ in which the conditions in $C(S)$ are all satisfied.[3]

Proposition 3.10 *If $S \subseteq NR$ then $LK^+(S)$ is sound for $\mathcal{M}_P[S]$.*

Proof: We show, by way of example, that $(\neg\vee \Rightarrow)_1$ is valid in any refinement $\mathcal{M}$ of $\mathcal{M}_P$ in which the condition $C(\neg\vee \Rightarrow)_1$ is satisfied. So assume that v is a valuation in $\mathcal{M}$ such that $v(\neg A) \notin \mathcal{D}$. Then $v(A) \in \{t, \bot\}$. Condition $C(\neg\vee \Rightarrow)_1$ entails that in this case also $v(A \vee B) \in \{t, \bot\}$, and so $v(\neg(A \vee B)) \notin \mathcal{D}$. Hence $\neg(A \vee B) \vdash_{\mathcal{M}} \neg A$. ∎

Theorem 3.11 *If $S \subseteq NR$ then $LK^+(S)$ is strongly complete for $\mathcal{M}_P[S]$.*

Proof: Using Theorem 2.3, it suffices to show that if $\mathbf{T}$ is a theory and φ_0 is a sentence such that $\mathbf{T} \nvdash_{HLK^+(S)} \varphi_0$, then there exists a model of $\mathbf{T}$ in $\mathcal{M}_P[S]$ which is not a model of φ_0. For this extend $\mathbf{T}$ to a maximal theory $\mathbf{T}^*$ such that $\mathbf{T}^* \nvdash_{HLK^+(S)} \varphi_0$. Since $HLK^+(S)$ is an extension of HLK^+ having only MP as a rule of inference, $\mathbf{T}^*$ has the following properties:

1. $\psi \notin \mathbf{T}^*$ iff $\psi \supset \varphi_0 \in \mathbf{T}^*$.

2. If $\psi \notin \mathbf{T}^*$ then $\psi \supset \varphi \in \mathbf{T}^*$ for every sentence φ.

3. $\varphi \vee \psi \in \mathbf{T}^*$ iff either $\varphi \in \mathbf{T}^*$ or $\psi \in \mathbf{T}^*$.

4. $\varphi \wedge \psi \in \mathbf{T}^*$ iff both $\varphi \in \mathbf{T}^*$ and $\psi \in \mathbf{T}^*$.

5. $\varphi \supset \psi \in \mathbf{T}^*$ iff either $\varphi \notin \mathbf{T}^*$ or $\psi \in \mathbf{T}^*$.

[3] It is easy to see that the conditions in NR cannot cause any conflict, so $\mathcal{M}_P[S]$ is well-defined.

(The proofs of 1–5 are all standard: Property 1 follows from the maximality property of $\mathbf{T}^*$ and the deduction theorem. Property 2 is proved first for $\psi = \varphi_0$ using Property 1 and the tautology $((\varphi_0 \supset \varphi) \supset \varphi_0) \supset \varphi_0$, and then deduced for all $\psi \notin \mathbf{T}^*$ using Property 1. Properties 3–5 are easy corollaries of Properties 1, 2, and the fact that $\mathbf{T}^* \not\vdash_{HLK^+(S)} \varphi_0$).

Define a valuation v in $\mathcal{M}_P[S]$ as follows:

$$v(\psi) = \begin{cases} \bot & \psi \notin \mathbf{T}^*, \neg\psi \notin \mathbf{T}^* \\ f & \psi \notin \mathbf{T}^*, \neg\psi \in \mathbf{T}^* \\ t & \psi \in \mathbf{T}^*, \neg\psi \notin \mathbf{T}^* \\ \top & \psi \in \mathbf{T}^*, \neg\psi \in \mathbf{T}^* \end{cases}$$

We shall now show that v is a legal valuation, i.e. it respects the interpretations of the connectives in $\mathcal{M}_P[S]$. Properties 3–5 of $\mathbf{T}^*$ easily imply that v respects the basic constraints concerning the positive connectives. As for the basic constraints concerning negation, we have:

- Assume $v(\psi) \in \{t, \bot\}$. By definition, this implies $\neg\psi \notin \mathbf{T}^*$, and so $v(\neg\psi) \in \{f, \bot\}$.
- Assume $v(\psi) \in \{f, \top\}$. By definition, this implies $\neg\psi \in \mathbf{T}^*$, and so $v(\neg\psi) \in \{t, \top\}$.

It remains to show that v respects the conditions induced by the rules in S:

$C(\Rightarrow \neg)$: Assume $(\Rightarrow \neg) \in S$. Then $\varphi \vee \neg\varphi \in \mathbf{T}^*$, and so Property 3 above entails that for every φ, either $\varphi \in \mathbf{T}^*$, or $\neg\varphi \in \mathbf{T}^*$. Hence there is no φ such that $v(\varphi) = \bot$.

$C(\neg \Rightarrow)$: Assume $(\neg \Rightarrow) \in S$. Then, by the corresponding axiom and the fact that $\varphi_0 \notin \mathbf{T}^*$, there is no φ such that both $\varphi \in \mathbf{T}^*$ and $\neg\varphi \in \mathbf{T}^*$. Hence there is no φ such that $v(\varphi) = \top$.

$C(\Rightarrow \neg\neg)$: Assume $(\Rightarrow \neg\neg) \in S$. Then $\neg\neg\varphi \in \mathbf{T}^*$ whenever $\varphi \in \mathbf{T}^*$. This easily implies (by the definition of v) that if $v(\varphi) = t$ then $v(\neg\varphi) = f$, while $v(\varphi) = \top$ then $v(\neg\varphi) = \top$.

$C(\neg\neg \Rightarrow)$: Assume $(\neg\neg \Rightarrow) \in S$. Then $\neg\neg\varphi \notin \mathbf{T}^*$ whenever $\varphi \notin \mathbf{T}^*$. This easily entails (by the definition of v) that $v(\varphi) = f$ implies $v(\neg\varphi) = t$, while $v(\varphi) = \bot$ implies $v(\neg\varphi) = \bot$.

$C(\neg\vee \Rightarrow)_1$: Assume $(\neg\vee \Rightarrow)_1 \in S$. Then $\neg(\varphi \vee \psi) \notin \mathbf{T}^*$ if $\neg\varphi \notin \mathbf{T}^*$. Hence if $v(\varphi) \in \{t, \bot\}$ then $v(\neg(\varphi \vee \psi)) \in \{t, \bot\}$. Now if $v(\varphi) = t$ then $\varphi \in \mathbf{T}^*$, and so $\varphi \vee \psi \in \mathbf{T}^*$. Hence in this case $v(\varphi \vee \psi) = t$. If $v(\varphi) = \bot$ then $\varphi \notin \mathbf{T}^*$, and so $\varphi \vee \psi \in \mathbf{T}^*$ iff $\psi \in \mathbf{T}^*$ (iff $v(\psi) \in \mathcal{D}$). It follows that in this case $v(\varphi \vee \psi) = t$ if $v(\psi) \in \mathcal{D}$, and $v(\varphi \vee \psi) = \bot$ otherwise. In all cases we find that $v(\varphi \vee \psi) = sup_t(v(\varphi), v(\psi))$ if $v(\varphi) \in \{t, \bot\}$.

$C(\neg\vee \Rightarrow)_2$: Similar.

$C(\Rightarrow \neg\vee)$**:** Assume $(\Rightarrow \neg\vee) \in S$. Then $\neg(\varphi\vee\psi) \in \mathbf{T}^*$ if $\neg\varphi \in \mathbf{T}^*$ and $\neg\psi \in \mathbf{T}^*$. Thus $\neg(\varphi\vee\psi) \in \mathbf{T}^*$ if $v(\varphi) \in \{f, \top\}$ and $v(\psi) \in \{f, \top\}$. Hence $v(\varphi \vee \psi)$ is either f or $\top$, depending whether $\varphi \vee \psi \in \mathbf{T}^*$ or not. Since $\varphi \vee \psi \in \mathbf{T}^*$ iff either $\varphi \in \mathbf{T}^*$ or $\psi \in \mathbf{T}^*$, we obviously have $v(\varphi \vee \psi) = sup_t(v(\varphi), v(\psi))$ in the case under discussion.

We leave the proofs of the conditions corresponding to the remaining rules to the reader.

Now $v(\psi) \in \mathcal{D}$ iff $\psi \in \mathbf{T}^*$. Hence $v(\psi) \in \mathcal{D}$ for every $\psi \in \mathbf{T}$, while $v(\varphi_0) \notin \mathcal{D}$. It follows that v is indeed a model of $\mathbf{T}$ which is not a model of $v(\varphi_0)$. ■

We next apply our soundness and completeness results to derive three important properties of the systems considered above.

Theorem 3.12 *$LK^+(S)$ admits cut-elimination for any $S \subseteq NR$.*

Proof: If $S \cap NR_1 = \emptyset$ then the proof is a straightforward adaption of Gentzen's original proof for LK ([Gen69, Tak75]) [4]. The case when $S \cap NR_1 = \{(\Rightarrow \neg)\}$ was proved (using our 3-valued non-deterministic semantics) in [Avr03]. A completely analogous semantic proof can be given in the dual case where $S \cap NR_1 = \{(\neg \Rightarrow)\}$. ■

Our main tool for the next two theorems is the following Definition (from [Avr03]) and the corresponding simple Lemma, the trivial proof of which we leave to the reader:

Definition 3.13 Let $\mathcal{L}$ be a propositional language, and let $\mathcal{M} = \langle \mathcal{T}, \mathcal{D}, \mathcal{O} \rangle$ be an Nmatrix for $\mathcal{L}$. A *semivaluation* in $\mathcal{M}$ is any function $v' : \mathcal{F}' \rightarrow \mathcal{T}$ such that $\mathcal{F}'$ is a set of formulas of $\mathcal{L}$ which is closed under subformulas, and v' respects $\mathcal{M}$ (in the sense that $\diamond(\psi_1, \ldots, \psi_n) \in \mathcal{F}'$ implies $v'(\diamond(\psi_1, \ldots, \psi_n)) \in \tilde{\diamond}(v(\psi_1), \ldots, v(\psi_n)))$.

Lemma 3.14 *Any semivaluation in $\mathcal{M}$ can be extended to a valuation v in $\mathcal{M}$.*

Theorem 3.15 *$LK^+(S)$ is decidable for every $S \subseteq NR$.*

Proof: Let $\Gamma \Rightarrow \Delta$ be a sequent of the language of $LK^+(S)$. Let $\mathcal{F}'$ be the set of all subformulas of formulas in $\Gamma \Rightarrow \Delta$. To decide whether $\Gamma \Rightarrow \Delta$ is provable in $LK^+(S)$, check whether for every $v' : \mathcal{F}' \rightarrow \{t, f, \top, \bot\}$ which is a semivaluation in $\mathcal{M}_P[S]$, either $v'(\varphi) \notin \mathcal{D}$ for some $\varphi \in \Gamma$, or $v'(\varphi) \in \mathcal{D}$ for some $\varphi \in \Delta$. By Lemma 3.14 (together with the soundness and completeness of

[4] Such an adaption is not so easy if $S \cap NR_1$ is a singleton since the case in which the cut formula is of the form $\neg\varphi$ causes then difficulties.

$LK^+(S)$ with respect to $\mathcal{M}_P[S]$), this is indeed sufficient. Since the number of such semivaluations is finite, this is a decision procedure. ■

We turn to the question: Which of the various logics we have considered are actually different from each other? Well, if $NR_1 \subseteq S$ then $LK^+(S)$ is just classical logic, and so all the other rules are derivable in it. It is also easy to see that $(\neg \supset\Rightarrow)_1$ is derivable from $(\neg \Rightarrow)$ in the context of LK^+. The next theorem shows that these are the only dependencies in NR (and so there are $5 \cdot 2^{10}$ different nonclassical logics of the form $LK^+(S)$, where $S \subseteq NR$).

Theorem 3.16 *Let $\mathcal{S}_0 = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$, where:*

$$\mathcal{S}_1 = \{S \mid S \subseteq NR, \quad S \cap NR_1 = \emptyset\}$$

$$\mathcal{S}_2 = \{S \mid S \subseteq NR, \quad S \cap NR_1 = \{(\Rightarrow \neg)\}\}$$

$$\mathcal{S}_3 = \{S \mid S \subseteq NR, \quad S \cap NR_1 = \{(\neg \Rightarrow)\}, \quad (\neg \supset\Rightarrow)_1 \notin S\}$$

Then if $S \in \mathcal{S}_0$ then $LK^+(S)$ is strictly weaker than classical logic. Moreover: if $S_1, S_2 \in \mathcal{S}_0$ and $S_1 \neq S_2$, then $LK^+(S_1) \neq LK^+(S_2)$

Proof: It is easy to see that $A(\neg \Rightarrow)$ is not valid in $\mathcal{M}_P[(NR - \{(\neg \Rightarrow)\})]$, and $A(\Rightarrow \neg)$ is not valid in $\mathcal{M}_P[(NR - \{(\Rightarrow \neg)\})]$. [5] Hence $LK^+(S)$ is strictly weaker than classical logic if $S \in \mathcal{S}_0$, and $A(\neg \Rightarrow)$ is not derivable in $LK^+(NR - \{(\neg \Rightarrow)\})$, while $A(\Rightarrow \neg)$ is not derivable in $LK^+(NR - \{(\Rightarrow \neg)\})$. To show that $LK^+(S_1) \neq LK^+(S_2)$ if $S_1, S_2 \in \mathcal{S}_0$ and $S_1 \neq S_2$, it suffices therefore to show the following: If $r \in NR_2 \cup NR_3$, and $A(r)$ is the corresponding Hilbert-type axiom, then $A(r)$ is not valid in $\mathcal{M}_P[S_r^1]$, where $S_r^1 = NR - \{r, (\neg \Rightarrow)\}$, and if $r \neq (\neg \supset\Rightarrow)_1$ then $A(r)$ is not valid in $\mathcal{M}_P[S_r^2]$ either, where $S_r^2 = NR - \{r, (\Rightarrow \neg)\}$. For this it suffices (by Lemma 3.14) to provide appropriate semivaluations v_1 and v_2 which refute $A(r)$ in S_r^1 and S_r^2, respectively. Here is a list of such refuting semivaluations in each case:

$r = (\neg\neg \Rightarrow)$:

$$\begin{array}{lll} v_1(p) = f & v_1(\neg p) = \top & v_1(\neg\neg p) = \top \\ v_2(p) = \bot & v_2(\neg p) = f & v_2(\neg\neg p) = t \end{array}$$

$r = (\Rightarrow \neg\neg)$:

$$\begin{array}{lll} v_1(p) = \top & v_1(\neg p) = t & v_1(\neg\neg p) = f \\ v_2(p) = t & v_2(\neg p) = \bot & v_2(\neg\neg p) = \bot \end{array}$$

[5] These two Nmatrices are actually famous ordinary matrices, and are respectively the $\{t, f, \top\}$-submatrix of $\mathcal{FOUR}$ and the $\{t, f, \bot\}$-submatrix of $\mathcal{FOUR}$ mentioned in Theorem 2.5.

$r = (\neg\vee \Rightarrow)_1$:

$$\begin{array}{lllll} v_1(p) = t & v_1(q) = \top & v_1(p \vee q) = \top & v_1(\neg(p \vee q)) = \top & v_1(\neg p) = f \\ v_2(p) = \bot & v_2(q) = f & v_2(p \vee q) = f & v_2(\neg(p \vee q)) = t & v_2(\neg p) = \bot \end{array}$$

$r = (\neg\vee \Rightarrow)_2$:

$$\begin{array}{lllll} v_1(p) = \top & v_1(q) = t & v_1(p \vee q) = \top & v_1(\neg(p \vee q)) = \top & v_1(\neg q) = f \\ v_2(p) = f & v_2(q) = \bot & v_2(p \vee q) = f & v_2(\neg(p \vee q)) = t & v_2(\neg q) = \bot \end{array}$$

$r = (\Rightarrow \neg\vee)$:

$$\begin{array}{lllll} v_1(p) = v_1(q) = \top & v_1(p \vee q) = t & v_1(\neg p) = v_1(\neg q) = \top & v_1(\neg p \wedge \neg q) = \top & v_1(\neg(p \vee q)) = f \\ v_2(p) = v_2(q) = f & v_2(p \vee q) = \bot & v_2(\neg p) = v_2(\neg q) = t & v_2(\neg p \wedge \neg q) = t & v_2(\neg(p \vee q)) = \bot \end{array}$$

$r = (\neg\wedge \Rightarrow)$:

$$\begin{array}{lllll} v_1(p) = v_1(q) = t & v_1(\neg p) = v_1(\neg q) = f & v_1(p \wedge q) = \top & v_1(\neg(p \wedge q)) = \top & v_1(\neg p \vee \neg q) = f \\ v_2(p) = v_2(q) = \bot & v_2(\neg p) = v_2(\neg q) = \bot & v_2(p \wedge q) = f & v_2(\neg(p \wedge q)) = t & v_2(\neg p \vee \neg q) = \bot \end{array}$$

$r = (\Rightarrow \neg\wedge)_1$:

$$\begin{array}{lllll} v_1(p) = \top & v_1(q) = t & v_1(p \wedge q) = t & v_1(\neg p) = \top & v_1(\neg(p \wedge q)) = f \\ v_2(p) = f & v_2(q) = \bot & v_2(p \wedge q) = \bot & v_2(\neg p) = t & v_2(\neg(p \wedge q)) = \bot \end{array}$$

$r = (\Rightarrow \neg\wedge)_2$:

$$\begin{array}{lllll} v_1(p) = t & v_1(q) = \top & v_1(p \wedge q) = t & v_1(\neg q) = \top & v_1(\neg(p \wedge q)) = f \\ v_2(p) = \bot & v_2(q) = f & v_2(p \wedge q) = \bot & v_2(\neg q) = t & v_2(\neg(p \wedge q)) = \bot \end{array}$$

$r = (\Rightarrow \neg \supset)_1$:

$$v_1(p) = f \quad v_1(q) = \top \quad v_1(p \supset q) = \top \quad v_1(\neg(p \supset q)) = \top$$

$r = (\Rightarrow \neg \supset)_2$:

$$\begin{array}{lllll} v_1(p) = t & v_1(q) = t & v_1(p \supset q) = \top & v_1(\neg(p \supset q)) = \top & v_1(\neg q) = f \\ v_2(p) = t & v_2(q) = \bot & v_2(p \supset q) = f & v_2(\neg(p \supset q)) = t & v_2(\neg q) = \bot \end{array}$$

$r = (\Rightarrow \neg \supset)$:

$$\begin{array}{llllll} v_1(p) = \top & v_1(q) = \top & v_1(\neg q) = \top & v_1(p \wedge \neg q) = \top & v_1(p \supset q) = t & v_1(\neg(p \supset q)) = f \\ v_2(p) = t & v_2(q) = f & v_2(\neg q) = t & v_2(p \wedge \neg q) = t & v_2(p \supset q) = \bot & v_2(\neg(p \supset q)) = \bot \end{array}$$

It is easy to check that each of these semivaluations indeed satisfies all the constraints which correspond to the rules in $NR_2 \cup NR_3$ except for the relevant one, as well as $C(\Rightarrow \neg)$ (in the case of v_1), or $C(\neg \Rightarrow)$ (in the case of v_2) ■

3.4 Adding The Propositional Constant ff

For $S \subseteq NR$, let $\mathcal{M}_P^{\mathbf{ff}}[S]$ be the weakest refinement of $\mathcal{M}_P^{\mathbf{ff}}$ in which the conditions in $C(S)$ are all satisfied. All the theorems of the previous subsection concerning the systems $LK^+(S)$ and their relations with the Nmatrices $\mathcal{M}_P[S]$ are true (with practically the same proofs) for the systems $LK(S)$ and for their relations with the Nmatrices $\mathcal{M}_P^{\mathbf{ff}}[S]$. Note also that if $(\neg \Rightarrow) \notin S$ then the valuation that assigns $\top$ to every sentence is legal in $\mathcal{M}_P[S]$, but not in $\mathcal{M}_P^{\mathbf{ff}}[S]$. Hence in this case no counterpart of **ff** is definable in $LK^+(S)$, and $LK(S)$ is a proper extension of $LK^+(S)$. In contrast, if $(\neg \Rightarrow) \in S$ then **ff** can be interpreted as $\varphi \wedge \neg\varphi$ (for some φ), and so there is no real difference between $LK^+(S)$ and $LK(S)$ in this case.

Turning our attentions to the systems $LK^f(S)$ $(S \subseteq NR)$, let $\mathcal{M}_P^F[S]$ be the weakest refinement of $\mathcal{M}_P^{\mathbf{ff}}$ which satisfies the conditions in $C(S)$ together with the condition:

- $\widetilde{\mathbf{ff}} = \{f\}$

Again, all the theorems of the previous subsection concerning the systems $LK^+(S)$ and their relations with the Nmatrices $\mathcal{M}_P[S]$ are true for the systems $LK^f(S)$ and for their relations with the Nmatrices $\mathcal{M}_P^F[S]$. However, this time $\Rightarrow \neg\mathbf{ff}$ is derivable in $LK(S)$ if $(\Rightarrow \neg) \in S$, but not if $(\Rightarrow \neg) \notin S$ (take $v(\mathbf{ff}) = v(\neg\mathbf{ff}) = \perp$). Hence if $(\Rightarrow \neg) \notin S$ then $LK^f(S)$ is a new logic (and all the logics of the form $LK^f(S)$, where $(\Rightarrow \neg) \notin S$, are different from each other and from all the logics we have considered above).

4 Semantics in the Intuitionistic Case

4.1 General Semantics

The previous section was devoted to extensions of positive classical logics. However, LJ^+ might be a better starting point for investigating negations (and it is certainly the natural basis for investigating *constructive* negations). One reason is that the valid sentences of LJ^+ are all intuitively correct. LK^+, in contrast, includes counterintuitive tautologies like $(A \wedge B \supset C) \supset (A \supset C) \vee (B \supset C)$ or $A \vee (A \supset B)$. Moreover: the classical natural deduction rules for the positive connectives ($\wedge, \vee$ and $\supset$) define the intuitionistic positive logic LJ^+, not the classical one. It is only with the aid of the classical rules for (the classical) negation that one can prove the counterintuitive positive tautologies mentioned above.

Now, it is well known that it is impossible to conservatively add to the intuitionistic positive logic a negation which is both explosive (i.e.: $\neg A, A \vdash B$ for all A, B) and satisfies the law of excluded middle LEM. With such an addition we get classical logic. The intuitionists indeed reject LEM, retaining the explosive nature of negation (which is usually defined using the constant **ff** and implication). In this section we shall see that this is not the only possible choice. The main problem we shall solve in it is: Which of the logics $LJ^+(S)$ ($S \subseteq NR$) is conservative over LJ^+? (and similarly for LJ). We believe that each such logic is entitled to be called "a (constructive) logic with a constructive negation".

As in the case of LK^+ (or LK) , we start with generalizing the standard, two-valued semantics of LJ^+ (or LJ). Recall that this semantics is usually provided by the class of all Kripke frames of the form $\mathcal{W} = \langle W, \leq, v \rangle$ [6], where $\langle W, \leq \rangle$ is a nonempty partially ordered set (of "worlds"), and $v : W \times \mathcal{F} \to \mathcal{T}$ (where $\mathcal{F}$ is the set of formulas of the language) satisfies the following conditions:

1. If $y \geq x$ and $v(x, \varphi) = t$ then $v(y, \varphi) = t$.[7]
2.
   - $v(x, \varphi \wedge \psi) = t$ iff $v(x, \varphi) = t$ and $v(x, \psi) = t$
   - $v(x, \varphi \vee \psi) = t$ iff $v(x, \varphi) = t$ or $v(x, \psi) = t$
   - $v(x, \mathbf{ff}) = f$ (if **ff** is in the language).
3. $v(x, \varphi \supset \psi) = t$ iff $v(y, \psi) = t$ for every $y \geq x$ such that $v(y, \varphi) = t$

Obviously, if $\mathcal{W} = \langle W, \leq, v \rangle$ is a frame, then for every $x \in W$ the function $\lambda\varphi.v(x, \varphi)$ behaves like an ordinary classical valuation with respect to all the connectives except $\supset$. The treatment of $\supset$ is indeed what distinguishes between classical logic and intuitionistic logic. This observation leads to the following nondeterministic generalization of Kripke frames for intuitionistic logic:

Definition 4.1 Let $\supset$ be one of the connectives of a propositional language $\mathcal{L}$, and let $\mathcal{M} = \langle \mathcal{T}, \mathcal{D}, \mathcal{O} \rangle$ be an Nmatrix for $\mathcal{L} - \{\supset\}$. Denote by $\mathcal{F}$ be the set of formulas of $\mathcal{L}$. An $\mathcal{M}$-*frame* for $\mathcal{L}$ is a triple $\mathcal{W} = \langle W, \leq, v \rangle$ such that:

1. $\langle W, \leq \rangle$ is a nonempty partially ordered set
2. $v : W \times \mathcal{F} \to \mathcal{T}$ satisfies the following conditions:

[6] In the literature by a "frame" one usually means just the pair $\langle W, \leq \rangle$. Here we have found it convenient to use this technical term differently, so that the valuation v is an integral part of it.

[7] For the language of LJ it suffices to demand this condition for atomic formulas only; then one can prove that every formula has this property. This is not the case for the nondeterministic generalizations we present below.

- The persistence condition: if $y \geq x$ and $v(x, \varphi) \in \mathcal{D}$ then $v(y, \varphi) \in \mathcal{D}$
- For every $x \in W$, $\lambda\varphi.v(x, \varphi)$ is a legal $\mathcal{M}$-valuation.
- $v(x, \varphi \supset \psi) \in \mathcal{D}$ iff $v(y, \psi) \in \mathcal{D}$ for every $y \geq x$ such that $v(y, \varphi) \in \mathcal{D}$

We say that a formula φ is *true* in a world $x \in W$ of a frame $\mathcal{W}$ if $v(x, \varphi) \in \mathcal{D}$. A sequent $\Gamma \Rightarrow \Delta$ is *valid* in $\mathcal{W}$ if for every $x \in W$ there is either $\varphi \in \Gamma$ such that φ is not true in x, or $\psi \in \Delta$ such that ψ is true in x.

Note: Obviously, if $\mathcal{M}_1$ is a refinement of $\mathcal{M}_2$, then any $\mathcal{M}_1$-frame is also an $\mathcal{M}_2$-frame, and every sequent valid in $\mathcal{M}_2$ is also valid in $\mathcal{M}_1$.

Definition 4.2 1. Let $\mathcal{M} = \langle \mathcal{T}, \mathcal{D}, \mathcal{O} \rangle$ be an Nmatrix for a language which includes the language of LJ^+. We say that $\mathcal{M}$ is *suitable* for LJ^+ if the following conditions are satisfied:

- If $a \in \mathcal{D}$ and $b \in \mathcal{D}$ then $a \wedge b \subseteq \mathcal{D}$
- If $a \notin \mathcal{D}$ then $a \wedge b \subseteq \mathcal{T} - \mathcal{D}$
- If $b \notin \mathcal{D}$ then $a \wedge b \subseteq \mathcal{T} - \mathcal{D}$

- If $a \in \mathcal{D}$ then $a \vee b \subseteq \mathcal{D}$
- If $b \in \mathcal{D}$ then $a \vee b \subseteq \mathcal{D}$
- If $a \notin \mathcal{D}$ and $b \notin \mathcal{D}$ then $a \vee b \subseteq \mathcal{T} - \mathcal{D}$

- If $b \in \mathcal{D}$ then $a \supset b \subseteq \mathcal{D}$
- If $a \in \mathcal{D}$ and $b \notin \mathcal{D}$ then $a \supset b \subseteq \mathcal{T} - \mathcal{D}$

2. Let $\mathcal{M} = \langle \mathcal{T}, \mathcal{D}, \mathcal{O} \rangle$ be an Nmatrix for a language which includes the language of LJ. We say that $\mathcal{M}$ is *suitable* for LJ if it is suitable for LJ^+, and the following condition is satisfied:

- $\mathbf{ff} \subseteq \mathcal{T} - \mathcal{D}$

Note: An Nmatrix which is suitable for LJ^+ (LJ) is also suitable for LK^+ (LK) iff it satisfies just one more condition: If $a \notin \mathcal{D}$ then $a \supset b \subseteq \mathcal{D}$.

Theorem 4.3 *Assume $\mathcal{W}$ is an $\mathcal{M}$-frame, where $\mathcal{M}$ is suitable for LJ^+ (LJ). Then any sequent provable in LJ^+ (LJ) is valid in $\mathcal{W}$.*

Proof: Again, we leave the easy proof to the reader. ■

From now on we shall concentrate on the systems $LJ^+(S)$ ($S \subseteq NR$). Like in the classical case (see Subsection 3.4), obtaining similar results for $LJ(S)$ and $LJ^f(S)$ causes no further difficulties.

Definition 4.4 Let $\mathcal{M}_{IP}$ be the following Nmatrix $\langle \mathcal{T}, \mathcal{D}, \mathcal{O} \rangle$ for the language $\{\neg, \wedge, \vee, \supset\}$:

- $\mathcal{T} = \{t, \top, f, \bot\}$
- $\mathcal{D} = \{t, \top\}$
- $a \supset b = \begin{cases} \mathcal{D} & b \in \mathcal{D} \\ \mathcal{T} - \mathcal{D} & b \notin \mathcal{D}, a \in \mathcal{D} \\ \mathcal{T} & a, b \in \mathcal{T} - \mathcal{D} \end{cases}$

  $a \vee b = \begin{cases} \mathcal{D} & a \in \mathcal{D} \text{ or } b \in \mathcal{D} \\ \mathcal{T} - \mathcal{D} & \text{otherwise} \end{cases}$

  $a \wedge b = \begin{cases} \mathcal{D} & a, b \in \mathcal{D} \\ \mathcal{T} - \mathcal{D} & \text{otherwise} \end{cases}$

  $\neg t = \neg \bot = \mathcal{T} - \mathcal{D} \quad \neg f = \neg \top = \mathcal{D}$

Note: The only difference between $\mathcal{M}_{IP}$ and $\mathcal{M}_P$ is that in $\mathcal{M}_{IP}$ we have $a \supset b = \mathcal{T}$ in case $a, b \in \mathcal{T} - \mathcal{D}$, while in $\mathcal{M}_P$ $a \supset b = \mathcal{D}$ in this case.

Proposition 4.5 *Let $\mathcal{M}$ be a refinement of $\mathcal{M}_{IP}$. Then LJ^+ is sound for every $\mathcal{M}$-frame*

Proof: This follows from Theorem 4.3. ■

Proposition 4.6 *Let $\mathcal{M}$ be a refinement of $\mathcal{M}_{IP}$. Then the persistence condition in the definition of an $\mathcal{M}$-frame (see Definition 4.1) can be replaced by the following* monotonicity condition*:*

- If $x \leq y$ then $v(x, \varphi) \leq_k v(y, \varphi)$

Proof: Assume the persistence condition, and let $x \leq y$. We show that $v(x, \varphi) \leq_k v(y, \varphi)$. There are 4 cases to consider:

$v(x, \varphi) = \bot$**:** This case is trivial.

$v(x, \varphi) = t$**:** In this case $v(y, \varphi) \in \{t, \top\}$ by persistence, whence $v(x, \varphi) \leq_k v(y, \varphi)$.

$v(x, \varphi) = f$**:** In this case $v(x, \neg\varphi) \in \mathcal{D}$, whence $v(y, \neg\varphi) \in \mathcal{D}$ by persistence. This is possible only if $v(y, \varphi) \in \{f, \top\}$, and so again $v(x, \varphi) \leq_k v(y, \varphi)$.

$v(x, \varphi) = \top$**:** In this case both $v(x, \varphi)$ and $v(x, \neg\varphi)$ are in $\mathcal{D}$. Hence both $v(y, \varphi)$ and $v(y, \neg\varphi)$ are in $\mathcal{D}$ by persistence. This is possible only if $v(y, \varphi) = \top$ as well.

The converse — that the monotonicity condition implies the persistence condition — is trivial. ■

Proposition 4.6 implies that if $\mathcal{M}$ is a refinement of $\mathcal{M}_{IP}$ then an $\mathcal{M}$-frame can be defined as a triple $\mathcal{W} = \langle W, \leq, v\rangle$ such that:

1. $\langle W, \leq\rangle$ is a nonempty partially ordered set.
2. $v : W \times \mathcal{F} \to \mathcal{T}$ satisfies:
   - For every φ the function $\lambda x.v(x, \varphi)$ is $\leq_k$- monotonic.
   - For every $x \in W$, $\lambda\varphi.v(x, \varphi)$ is a legal $\mathcal{M}$-valuation.
   - $v(x, \varphi \supset \psi) \in \mathcal{D}$ iff $v(y, \psi) \in \mathcal{D}$ for every $y \geq x$ such that $v(y, \varphi) \in \mathcal{D}$.

4.2 Effects of the Negation Rules

We now turn to the effects of the various negation rules in the context of our semantics for LJ^+ and its extensions. We shall see that again each of them has a corresponding a condition leading to a certain refinement of $\mathcal{M}_{IP}$ on which the corresponding frames are based. With only two exceptions, the conditions are identical to those we have in the classical case (and are again independent of each other, and never contradict each other).

Definition 4.7 For $r \in NR$, define $C^I r$ as follows:

- $C^I(\neg \supset\Rightarrow)_1$: If $x \notin \mathcal{D}$ then $x \supset y \subseteq \{t, \bot\}$
- $C^I(\neg \supset\Rightarrow)_2$: If $y \in \{t, \bot\}$ then $x \supset y \subseteq \{t, \bot\}$
- $C^I r = Cr$ otherwise.

Definition 4.8

1. For $S \subseteq NR$, let $C^I(S) = \{C^I r \mid r \in S\}$.
2. For $S \subseteq NR$, let $\mathcal{M}_{IP}[S]$ be the weakest refinement of $\mathcal{M}_{IP}$ in which the conditions in $C^I(S)$ are satisfied.

Theorem 4.9 *If $S \subseteq NR$ then $LJ^+(S)$ is sound and strongly complete for $\mathcal{M}_{PI}[S]$-frames: $\mathbf{T} \vdash_{LJ^+(S)} \psi$ iff for every $\mathcal{M}_{PI}[S]$-frame $\mathcal{W} = \langle W, \leq, v\rangle$, and every $x \in W$, if $v(x,\varphi) \in \mathcal{D}$ for every $\varphi \in \mathbf{T}$ then also $v(x,\psi) \in \mathcal{D}$.*

Proof: The easy proof of soundness is left to the reader.

To prove completeness of $LJ^+(S)$, define (as usual) a *prime* theory of $LJ^+(S)$ to be a set of sentences $\mathbf{T}$ closed under $\vdash_{HLJ^+(S)}$ and such that if $\varphi \vee \psi \in \mathbf{T}$ then either $\varphi \in \mathbf{T}$ or $\psi \in \mathbf{T}$. Since $HLJ^+(S)$ is an extension by axioms of $LJ^+(S)$, it has the property that if $\mathbf{T} \not\vdash_{HLJ^+(S)} \psi$ then there is a prime extension $\mathbf{T}^*$ of $\mathbf{T}$ such that $\psi \notin \mathbf{T}^*$. Define a canonical frame $\mathcal{W} = \langle W, \leq, v\rangle$ as follows:

- W is the set of prime theories of $LJ^+(S)$.
- $\leq = \subseteq$
- $v(\mathbf{T},\psi) = \begin{cases} \bot & \psi \notin \mathbf{T}, \neg\psi \notin \mathbf{T} \\ f & \psi \notin \mathbf{T}, \neg\psi \in \mathbf{T} \\ t & \psi \in \mathbf{T}, \neg\psi \notin \mathbf{T} \\ \top & \psi \in \mathbf{T}, \neg\psi \in \mathbf{T} \end{cases}$

Obviously, $v(\mathbf{T},\psi) \in \mathcal{D}$ iff $\psi \in \mathbf{T}$. With this fact, the proof that v satisfies the persistence condition, as well as the basic conditions concerning the positive connectives, is like in the standard proofs of the completeness of HLJ^+ (using its canonical model). The definition of v immediately implies that v satisfies the basic conditions concerning $\neg$. Finally, the proof that for every $\mathbf{T} \in W$, $\lambda\varphi.v(\mathbf{T},\varphi)$ respects the constraints imposed by the conditions in $C^I(S)$ is like in the proof of the completeness of $HLK^+(S)$ (Theorem 3.11). Hence $\mathcal{W}$ is an $\mathcal{M}_{PI}[S]$-frame.

Assume now that $\mathbf{T} \not\vdash_{HLJ^+(S)} \psi_0$. Then there exists $\mathbf{T}^* \in W$ such that $\mathbf{T} \subseteq \mathbf{T}^*$ and $\psi_0 \notin \mathbf{T}^*$. Hence $v(\mathbf{T}^*,\varphi) \in \mathcal{D}$ for every $\varphi \in \mathbf{T}$, while $v(\mathbf{T}^*,\psi_0) \notin \mathcal{D}$. ■

4.3 What Combinations of Rules are Admissible?

Proposition 4.9 does not have much value in itself. Indeed, it does not guarantee that $LJ^+(S)$ is conservative over LJ^+, and neither does it provide a decision procedure for $LJ^+(S)$. The reason is that a valuation is an infinite object. Now, to provide a countermodel v for a formula ψ, all one needs to do in the case of valuations in finite matrices or Nmatrices is to give the truth-values that v assigns to subformulas of ψ. However, here it is not clear that such a partial description would suffice. Indeed, in the proof of the next theorem we give an example in which this is *not* the case.

Semantics based on the idea of valuations might be called *effective* if such a phenomenon does not occur. Below we define this intuitive idea in exact terms:

Definition 4.10 Let $\mathcal{M} = \mathcal{M}_{IP}[S]$ for some $S \subseteq NR$. An $\mathcal{M}$-*semiframe* is a triple $\mathcal{W} = \langle W, \leq, v' \rangle$ such that:

1. $\langle W, \leq \rangle$ is a nonempty partially ordered set.

2. $v' : W \times \mathcal{F}' \to \mathcal{T}$ is a partial valuation such that:

   - $\mathcal{F}'$ is a subset of $\mathcal{F}$ which is closed under subformulas.
   - v' satisfies the monotonicity condition: if $y \geq x$ and $\varphi \in \mathcal{F}'$, then $v'(x, \varphi) \leq_k v'(y, \varphi)$.
   - v' respects $\mathcal{M}$: If $\diamond(\psi_1, \ldots, \psi_n) \in \mathcal{F}'$, then $v'(x, \diamond(\psi_1, \ldots, \psi_n)) \in \tilde{\diamond}(v'(x, \psi_1), \ldots, v'(x, \psi_n))$.
   - If $\varphi \supset \psi \in \mathcal{F}'$ then $v'(x, \varphi \supset \psi) \in \mathcal{D}$ iff $v'(y, \psi) \in \mathcal{D}$ for every $y \geq x$ such that $v'(y, \varphi) \in \mathcal{D}$.

Definition 4.11 $\mathcal{M}_{IP}[S]$ is called effective if for any $\mathcal{M}_{IP}[S]$- semiframe $\langle W, \leq, v' \rangle$ there exists an $\mathcal{M}_{IP}[S]$-frame $\langle W, \leq, v \rangle$ such that v extends v'.

The two crucial problems we are going to solve now are:

1. For which S is $LJ^+(S)$ conservative over LJ^+?

2. For which S is $\mathcal{M}_{IP}[S]$ effective?

We start with the second problem.

Theorem 4.12 *Let $S \subseteq NR$. Then $\mathcal{M}_{IP}[S]$ is effective iff either $\{(\Rightarrow \neg), (\neg \supset \Rightarrow)_1\} \not\subseteq S$ or $\{(\Rightarrow \neg), (\neg \Rightarrow)\} \subseteq S$.*

Proof: If $\{(\Rightarrow \neg), (\neg \Rightarrow)\} \subseteq S$ then neither $\bot$ nor $\top$ are available, and so the monotonicity condition for v means that $v(x, \varphi) = v(y, \varphi)$ if $x \leq y$, and that for every x, $\lambda\varphi.v(x, \varphi)$ is a classical valuation (in practical terms, this means that W can be taken to be a singleton, and the semantics reduces to the classical one). Hence the theorem is trivial in this case (and follows from lemma 3.14).

To show that if $\{(\Rightarrow \neg), (\neg \Rightarrow)\} \not\subseteq S$ then the condition that $\{(\Rightarrow \neg), (\neg \supset \Rightarrow)_1\} \not\subseteq S$ is necessary for effectiveness, take for example $W = \{a, b\}$ with $a < b$, and define $v'(a, p) = v'(a, q) = v'(b, q) = f$, $v'(b, p) = \top$. Then v' respects the monotonicity condition, but if $\{(\Rightarrow \neg), (\neg \supset \Rightarrow)_1\} \subseteq S$ then there is no extension v of v' such that $\langle W, \leq v\rangle$ is an $\mathcal{M}_{IP}[S]$-frame: $v(a, p \supset q)$ should on the one hand be f according to the definition of an $\mathcal{M}_{IP}$-frame (because the presence of $(\Rightarrow \neg)$ implies that $\perp$ is not available), while according to $C^I(\neg \supset\Rightarrow)_1$ it should be t (again, because $\perp$ is not available).

Assume next that neither $\{(\Rightarrow \neg), (\neg \Rightarrow)\} \subseteq S$ nor $\{(\Rightarrow \neg), (\neg \supset\Rightarrow)_1\} \subseteq S$. We show that $\mathcal{M}_{IP}[S]$ is effective. So let $\langle W, \leq, v'\rangle$ (where $v' : W \times \mathcal{F}' \to \mathcal{T}$) be a semiframe. We extend it to a frame $\langle W, \leq, v\rangle$ by defining v inductively as follows:

- $v(x, \psi) = v'(x, \psi)$ if $\psi \in \mathcal{F}'$
- $v(x, p) = t$ if p is atomic, $p \notin \mathcal{F}'$
- $v(x, \neg\psi) = \neg_{\mathcal{FOUR}} v(x, \psi)$ if $\psi \notin \mathcal{F}'$, where $\neg_{\mathcal{FOUR}}$ is $\mathcal{FOUR}$'s negation (see Definition 2.4)
- $v(x, \psi_1 \vee \psi_2) = sup_t(v(x, \psi_1), v(x, \psi_2))$ if $\psi_1 \vee \psi_2 \notin \mathcal{F}'$
- $v(x, \psi_1 \wedge \psi_2) = inf_t(v(x, \psi_1), v(x, \psi_2))$ if $\psi_1 \wedge \psi_2 \notin \mathcal{F}'$
- If $\psi_1 \supset \psi_2 \notin \mathcal{F}'$ then there are two cases:
  - If $(\Rightarrow \neg) \notin S$ then

$$v(x, \psi_1 \supset \psi_2) = \begin{cases} v(x, \psi_2) & \text{if } v(x, \psi_1) \in \mathcal{D} \\ \perp & \text{if } v(x, \psi_1) \notin \mathcal{D},\ \exists y \geq x. v(y, \psi_1) \in \mathcal{D} \wedge v(y, \psi_2) \notin \mathcal{D} \\ t & \text{otherwise} \end{cases}$$

  - If $(\Rightarrow \neg) \in S$ but $(\neg \Rightarrow) \notin S$ and $(\neg \supset\Rightarrow)_1 \notin S$ then

$$v(x, \psi_1 \supset \psi_2) = \begin{cases} t & \text{if } v(x, \psi_2) = t \\ f & \text{if } \exists y \geq x. v(y, \psi_1) \in \mathcal{D} \wedge v(y, \psi_2) \notin \mathcal{D} \\ \top & \text{otherwise} \end{cases}$$

  (note that if $(\Rightarrow \neg) \notin S$ then $\perp$ is available, while if $(\Rightarrow \neg) \in S$ then $(\neg \Rightarrow) \notin S$, and so $\top$ is available. These facts justify their use in the definition of $v(x, \psi_1 \supset \psi_2)$)

We prove now by induction on the complexity of ψ that $v(x, \psi)$ is well-defined for every $x \in W$, and that $\lambda x.v(x, \psi)$ is monotonic. This follows from our assumption on v' if $\psi \in \mathcal{F}'$, and is trivial if ψ is atomic. The cases where ψ is of one of the forms $\neg\psi_1$, $\psi_1 \vee \psi_2$, or $\psi_1 \wedge \psi_2$ follow easily from the induction hypothesis concerning ψ_1, ψ_2, and the monotonicity of the operations $\neg, \vee$ and $\wedge$ in $\mathcal{FOUR}$. It remains to prove the case where $\psi = \psi_1 \supset \psi_2$, and $\psi \notin \mathcal{F}'$. Now, a problem with the coherence of the definition of $v(x,\psi)$ may occur in this case only if $(\Rightarrow\neg)\in S$, and $v(x,\psi_2)=t$. However, by induction hypothesis for ψ_2, if $v(x,\psi_2)=t$ then $v(z,\psi_2)\in\mathcal{D}$ for all $z\geq x$, and so only the first clause in the definition of $v(x,\psi_1\supset\psi_2)$ is applicable, implying that $v(x,\psi)$ is well-defined in this case too. We show now that under the same assumptions concerning ψ, $v(x,\psi)\leq_k v(y,\psi)$ if $y\geq x$. There are two cases to consider:

- Assume that $(\Rightarrow\neg)\notin S$. If $v(x,\psi_1)\in\mathcal{D}$ then by the induction hypothesis also $v(y,\psi_1)\in\mathcal{D}$ and $v(x,\psi_2)\leq_k v(y,\psi_2)$. Hence in this case $v(x,\psi)=v(x,\psi_2)\leq_k v(y,\psi_2)=v(y,\psi)$. If $v(x,\psi_1)\notin\mathcal{D}$ and $v(x,\psi)=\perp$ then trivially $v(x,\psi)\leq_k v(y,\psi)$. Finally, if $v(x,\psi_1)\notin\mathcal{D}$ and $v(x,\psi)=t$ then the definition of v and the fact that $y\geq x$ imply that $v(y,\psi)\in\mathcal{D}$, and so again $v(x,\psi)\leq_k v(y,\psi)$.

- Assume that $(\Rightarrow\neg)\in S$. Then $v'(z,\varphi)\neq\perp$ for every $z\in W$ and every $\varphi\in\mathcal{F}'$, and so from the definition of v it follows that $v(z,\varphi)\neq\perp$ for every $z\in W$ and *every* φ. Hence $v(z,\varphi)\notin\mathcal{D}$ iff $v(z,\varphi)=f$, and $v(z,\varphi)\neq t$ iff $v(z,\varphi)\in\{f,\top\}$. Therefore, the induction hypothesis implies that if $v(z,\psi_2)\neq t$ for some z then $v(w,\psi_2)\neq t$ for every $w\geq z$. Hence if $v(x,\psi)=f$ then $v(y,\psi)\neq t$, and so $v(x,\psi)\leq_k v(y,\psi)$. If $v(x,\psi)=t$ then $v(x,\psi_2)=t$, and so by the induction hypothesis $v(y,\psi_2)\in\mathcal{D}$, and also $v(z,\psi_2)\in\mathcal{D}$ for every $z\geq y$. Thus by the definition of v $v(y,\psi)\in\mathcal{D}$ in this case, and so $v(x,\psi)=t\leq_k v(y,\psi)$. Finally, assume that $v(x,\psi)=\top$. Therefore $v(x,\psi_2)\neq t$ and $\forall z\geq x.v(z,\psi_1)=f\vee v(z,\psi_2)\in\mathcal{D}$. Since $y\geq x$, the first fact implies that also $v(y,\psi_2)\neq t$, whence $v(y,\psi)\neq t$. The second fact implies that also $\forall z\geq y.v(z,\psi_1)=f\vee v(z,\psi_2)\in\mathcal{D}$, whence $v(y,\psi)\neq f$. In consequence $v(y,\psi)=\top$, and so $v(x,\psi)\leq_k v(y,\psi)$.

The monotonicity of v, together with its definition and our assumption about v', easily imply that $v(x,\psi_1\supset\psi_2)\notin\mathcal{D}$ iff $\exists y\geq x.v(y,\psi_1)\in\mathcal{D}\wedge v(y,\psi_2)\notin\mathcal{D}$, and that $\mathcal{W}=\langle W,\leq,v\rangle$ is an $\mathcal{M}_{IP}$-frame. The assumption about v' and the definition of v also easily imply that every condition in $C^I(S)$ concerning the operations $\neg,\vee,\wedge$ is satisfied in $\mathcal{W}$. We show that this is the case also with respect to the other conditions:

- Assume that $(\Rightarrow\neg)\in S$. Then $v'(z,\varphi)\neq\perp$ for every $z\in W$ and every $\varphi\in\mathcal{F}'$. Hence by induction on the structure of φ we have $v(z,\varphi)\neq\perp$ for every $z\in W$ and every φ.

- Assume that $(\neg\Rightarrow)\in S$. Then $v'(z,\varphi)\neq\top$ for every $z\in W$ and every $\varphi\in\mathcal{F}'$. Hence by induction on the structure of φ we have $v(z,\varphi)\neq\top$ for every $z\in W$ and every φ.

- Assume that $(\Rightarrow \neg \supset)_1 \in S$. Then $(\Rightarrow \neg) \notin S$. Hence our assumption concerning v', and the definition of v trivially implies in this case that if $v(\psi_1) \notin \mathcal{D}$ then $v(x, \psi_1 \supset \psi_2) \in \{t, \perp\}$.

- Assume that $(\Rightarrow \neg \supset)_2 \in S$. We show that if $v(\psi_2) \in \{t, \perp\}$ then $v(x, \psi_1 \supset \psi_2) \in \{t, \perp\}$. This is trivial from our assumption concerning v' if $\psi_1 \supset \psi_2 \in \mathcal{F}'$, and from the definition of $v(x, \psi_1 \supset \psi_2)$ if $\psi_1 \supset \psi_2 \notin \mathcal{F}'$ (note that if $(\Rightarrow \neg) \in S$ then the condition just means that $v(x, \psi_1 \supset \psi_2) = t$ if $v(\psi_2) = t$).

- Assume that $(\neg \supset \Rightarrow) \in S$. We show that if $v(x, \psi_1) \in \mathcal{D}$ and $v(x, \psi_2) \in \{f, \top\}$ then $v(x, \psi_1 \supset \psi_2) = v(x, \psi_2)$. This is obvious if $\psi_1 \supset \psi_2 \in \mathcal{F}'$, or if $(\Rightarrow \neg) \notin S$. If $(\Rightarrow \neg) \in S$ and $\psi_1 \supset \psi_2 \notin \mathcal{F}'$ then the definition of v implies first of all that $v(x, \psi_1 \supset \psi_2) \in \{f, \top\}$ if $v(x, \psi_2) \in \{f, \top\}$. If in addition $v(x, \psi_1) \in \mathcal{D}$ then this fact, together with the definition of v and its monotonicity, implies that indeed $v(x, \psi_1 \supset \psi_2) = v(x, \psi_2)$.

It follows that $\mathcal{W}$ is an $\mathcal{M}_{IP}[S]$-frame as required. ■

Notes:

1. The definition of a semiframe includes the monotonicity condition rather than the persistence condition. This is important, since Theorem 4.12 is not true if the monotonicity condition is replaced by the persistence condition. Thus if we take $W = \{a, b\}$ with $a < b$, and define $v'(a, p) = f, v'(b, p) = t$ then the persistence condition is met, but there is no way to extend v' to an appropriate v, because we should have $v(a, \neg p) \in \mathcal{D}$ while $v(b, \neg p) \notin \mathcal{D}$, contradicting the persistence condition. The two conditions are equivalent for *full* valuations (by Proposition 4.6), but not for partial ones!

2. The problem with the combination $\{(\Rightarrow \neg), (\neg \supset \Rightarrow)_1\}$ is that the condition imposed by $(\neg \supset \Rightarrow)_1$ is not consistent with the condition of k-monotonicity in case $\perp$ is not available.

We can at last turn to the problem: for which S is $LJ^+(S)$ conservative over LJ^+?

Definition 4.13 $SN = NR - \{(\Rightarrow \neg)\}$ $\quad SP = NR - \{(\neg \Rightarrow), (\neg \supset \Rightarrow)_1\}$.

Theorem 4.14 *Assume that $S \subseteq NR$. Then $LJ^+(S)$ is a conservative extension of LJ^+ iff neither $\{(\Rightarrow \neg), (\neg \Rightarrow)\} \subseteq S$ nor $\{(\Rightarrow \neg), (\neg \supset \Rightarrow)_1\} \subseteq S$ (i.e., iff either $S \subseteq SN$ or $S \subseteq SP$).*

Proof: The two conditions are necessary, since $\not\vdash_{LJ^+} p \vee (p \supset q)$, but for every φ, ψ:

$$\varphi \vee \neg\varphi,\ \neg\varphi \supset (\varphi \supset \psi) \vdash_{LJ^+} \varphi \vee (\varphi \supset \psi)$$

$$(\varphi \supset \psi) \vee \neg(\varphi \supset \psi),\ \neg(\varphi \supset \psi) \supset \varphi \vdash_{LJ^+} \varphi \vee (\varphi \supset \psi)$$

To show that the two conditions together are also sufficient, it suffices to show that both $LJ^+(SN)$ and $LJ^+(SP)$ are conservative over LJ^+. So let ψ be a sentence in the language of LJ^+ which is not provable in LJ^+. We show that ψ is provable in neither $LJ^+(SN)$ nor $LJ^+(SP)$. Since $\not\vdash_{LJ^+} \psi$, there is an *ordinary* two-valued Kripke frame $\langle W, \leq, u\rangle$ (where $u : W \times \mathcal{F} \to \{t, f\}$) in which ψ is not valid (i.e. $u(x_0, \psi) = f$ for some $x_0 \in W$). Now we define the corresponding semiframes for $LJ^+(SN)$ and $LJ^+(SP)$. Let $\mathcal{F}'$ be the set of formulas in the language of LJ^+.

$LJ^+(SN)$: Define v'_N on $W \times \mathcal{F}'$ by:

$$v'_N(x, \varphi) = \begin{cases} t & \text{if } u(x, \varphi) = t \\ \bot & \text{if } u(x, \varphi) = f \end{cases}$$

It is straightforward to check that $\langle W, \leq, v'_N\rangle$ is an $\mathcal{M}_{IP}[SN]$-semiframe (note that any condition concerning $\neg$ is vacuously satisfied, since there is no sentence of the form $\neg\varphi$ in $\mathcal{F}'$).

$LJ^+(SP)$: Define v'_P on $W \times \mathcal{F}'$ by:

$$v'_P(x, \varphi) = \begin{cases} \top & \text{if } u(x, \varphi) = t \\ f & \text{if } u(x, \varphi) = f \end{cases}$$

Again, it is straightforward to check that $\langle W, \leq, v'_P\rangle$ is an $\mathcal{M}_{IP}[SP]$-semiframe.

By Theorem 4.12, $\langle W, \leq, v'_N\rangle$ and $\langle W, \leq, v'_P\rangle$ can respectively be extended to an $\mathcal{M}_{IP}[SN]$-frame $\langle W, \leq, v_N\rangle$ and an $\mathcal{M}_{IP}[SP]$-frame $\langle W, \leq, v_P\rangle$. Since $v_N(x_0, \psi) = v'_N(x_0, \psi) = \bot$, ψ is not valid in $\langle W, \leq, v'_N\rangle$, and so it is not provable in $LJ^+(SN)$. Similarly, $v_P(x_0, \psi) = v'_P(x_0, \psi) = f$. Hence ψ is not valid in $\langle W, \leq, v'_P\rangle$, and so is not provable in $LJ^+(SP)$. ■

Corollary 4.15 Suppose $S_1 \subseteq NR$, and $\{(\Rightarrow \neg), (\neg \Rightarrow)\} \not\subseteq S_1$. Then:

1. If also $\{(\Rightarrow \neg), (\neg \supset\Rightarrow)_1\} \not\subseteq S_1$ then $LJ^+(S_1) \neq LK^+(S_1)$.
2. If $\{(\Rightarrow \neg), (\neg \supset\Rightarrow)_1\} \subseteq S_1$ (or $\{(\Rightarrow \neg), (\neg \Rightarrow)\} \subseteq S_1$) then $LJ^+(S_1) = LK^+(S_1)$.
3. If $S_2 \subseteq NR$, $S_1 \neq S_2$ then $LJ^+(S_1) \neq LK^+(S_2)$.
4. If $S_2 \subseteq NR$, $S_1 \neq S_2$ then $LJ^+(S_1) \neq LJ^+(S_2)$.

Proof:

1. Immediate from Theorem 4.14

2. This follows from the fact (shown in the proof of Theorem 4.14) that if $\{(\Rightarrow \neg), (\neg \supset\Rightarrow)_1\} \subseteq S$ then $\varphi \vee (\varphi \supset \psi)$ is provable in $LJ^+(S)$. Indeed, it is well-known that by adding $\varphi \vee (\varphi \supset \psi)$ to LJ^+ we get LK^+.

3. If $\{(\Rightarrow \neg), (\neg \Rightarrow)\} \subseteq S_2$ then $LJ^+(S_2) = LK^+(S_2) =$ classical logic, while $LK^+(S_1)$ (and so also $LJ^+(S_1)$) is strictly weaker than classical logic (by Theorem 3.16). Hence the claim is trivial in this case, and we may assume that also $\{(\Rightarrow \neg), (\neg \Rightarrow)\} \not\subseteq S_2$. Now if $\{(\Rightarrow \neg), (\neg \supset\Rightarrow)_1\} \not\subseteq S$ then the claim is again immediate from Theorem 4.14. Otherwise it follows from part 2 of this Theorem, and (the proof of) Theorem 3.16.

4. Again, we may assume that also $\{(\Rightarrow \neg), (\neg \Rightarrow)\} \not\subseteq S_2$. Hence it suffices to show that if $r \in NR$, $S \subseteq NR$, $r \notin S$, and $\{(\Rightarrow \neg), (\neg \Rightarrow)\} \not\subseteq S$, Then $A(r)$ is not provable in $LJ^+(S)$. In the proof of Theorem 3.16 this was shown even for $LK^+(S)$ in the case where $r \neq (\neg \supset\Rightarrow)_1$ or $(\neg \Rightarrow) \notin S$. For the remaining cases, it suffices to show that $\neg(p \supset q) \supset p$ is not a theorem of $NR - \{(\Rightarrow \neg), (\neg \supset\Rightarrow)_1\}$. For this, we construct a countermodel as follows. Let $\mathcal{F}' = \{p, q, p \supset q, \neg(p \supset q), \neg(p \supset q) \supset p\}$, $W = \{0, 1\}$, and let $\leq$ be the usual partial order on W. Define a semivaluation v' on $W \times \mathcal{F}'$ by:

   $$\begin{array}{lllll} v'(0,p) = \perp & v'(0,q) = f & v'(0,p \supset q) = f & v'(0,\neg(p \supset q)) = t & v'(0,\neg(p \supset q) \supset p) = \perp \\ v'(1,p) = t & v'(1,q) = f & v'(1,p \supset q) = f & v'(1,\neg(p \supset q)) = t & v'(1,\neg(p \supset q) \supset p) = t \end{array}$$

   It is straightforward to show that $\langle W, \leq, v' \rangle$ is an $\mathcal{M}_{IP}[NR - \{(\Rightarrow \neg), (\neg \supset\Rightarrow)_1\}]$-semiframe, and obviously $\neg(p \supset q) \supset p$ is not valid in this semiframe. Hence $\neg(p \supset q) \supset p$ is not a theorem of $NR - \{(\Rightarrow \neg), (\neg \supset\Rightarrow)_1\}$ by Theorems 4.9 and 4.12.

Consequently, there are $5 \cdot 2^{10}$ new different logics of the form $LJ^+(S)$, where $S \subseteq NR$, all of them conservative over LJ^+ (while the others belong to the family $\{LK^+(S) \mid S \subseteq NR\}$). ■

Discussion: It follows from Theorem 4.14 that $LJ^+(SN)$ and $LJ^+(SP)$ are the two maximal logics in the family $\{LJ^+(S) \mid S \subseteq NR\}$ which are conservative extensions of constructive positive logic. Now the first is the well-known system **N** of Nelson ([AN84]) and Kutschera ([vK69]). The system $LJ^+(SP)$, in contrast, has not been investigated before (to the best of our knowledge). However, it is a very attractive system for constructive negation. First: it is paraconsistent (i.e.: a single contradiction does not imply everything in it). Second: LEM is valid in it. In fact, $LJ^+(SP)$

is obtained from **N** by replacing two of its axioms by LEM. Now, while ELM is very intuitive, the two axioms it replaces are not. Indeed, one of them, $\neg\varphi \supset (\varphi \supset \psi)$, intuitively means that if φ is false then it implies everything. The second, $\neg(\varphi \supset \psi) \supset \varphi$, intuitively means that if there is something that φ does not imply, then φ should be true (i.e.: it cannot be false). Obviously, these two principles are very similar to each other (and are both counterintuitive). It is no wonder that from a constructive point of view, each of them is inconsistent with LEM, and is rejected in $LJ^+(SP)$. It is worth noting that in contrast, and despite the paraconsistent nature of $LJ^+(SP)$, the basic (and very intuitive) law of contradiction $\neg(\varphi \wedge \neg\varphi)$ *is* valid in it.

4.4 Decidability of the Systems

As noted above, Theorem 4.9 alone does not guarantee the decidability of the logics it deals with, since the countermodel it provides for unprovable sentences may be infinite (in fact, its proof constructs only an infinite countermodel). Still, it is possible to derive decidability results from it by using filtration techniques and Theorem 4.12. We prefer instead to present a more direct proof, which involves a weak form of the cut-elimination theorem for our logics.

Now it was shown in [Avr03] that in general the cut-elimination theorem does not hold for our Gentzen-type systems. Moreover: examples have been given there of a subset S of NR and a sequent which is provable in $LJ^+(S)$, but any proof of it there should contain a non-analytic cut (i.e. a cut in which the cut-formula is not a subformula of the sequent being proved). This is perhaps not surprising, since our logical rules themselves do not have the strict subformula property: some of them involve negations of subformulas of their conclusion which are not subformulas themselves. Therefore, it is reasonable to expect the same from cuts. This leads to the following theorem:

Theorem 4.16 *Assume that $S \subseteq NR$, and $\{(\Rightarrow \neg), (\neg \supset\Rightarrow)_1\} \not\subseteq S$. Then for every sequent s in the language of LJ^+ there is either a* finite *$\mathcal{M}_{PI}[S]$-frame in which s is not valid, or a proof in $LJ^+(S)$ in which every cut is either on a subformula of s or on a negation of such a subformula.*

Proof: Assume that s is a sequent which has no such proof in $LJ^+(S)$. By Theorem 4.12, it suffices to construct a finite $\mathcal{M}_{PI}[S]$-semiframe in which s is not valid. Let $\mathcal{F}'$ be the set of subformulas of s, and let $\mathcal{F}'' = \mathcal{F}' \cup \{\neg\psi \mid \psi \in \mathcal{F}'\}$. Call a proof in $LJ^+(S)$ an *s-proof* if every cut in it is on some formula in $\mathcal{F}''$. Let W be the set of all sequents which do not have s-proofs in $LJ^+(S)$, and the union of their two sides is $\mathcal{F}''$. W is of course finite. Obviously, if $\Gamma \Rightarrow \Delta$ does not have an s-proof in a $LJ^+(S)$, and $\psi \in \mathcal{F}''$, then either $\psi, \Gamma \Rightarrow \Delta$ or $\Gamma \Rightarrow \Delta, \psi$ does not have an s-proof in $LJ^+(S)$. It follows that any sequent which consists of elements of $\mathcal{F}''$ and has no s-proof in $LJ^+(S)$ can be extended to an element of W. In particular, s itself is a subsequent of some sequent $\Gamma_0 \Rightarrow \Delta_0 \in W$. Define now a partial order $\leq$ on W as follows: $\Gamma_1 \Rightarrow \Delta_1 \leq \Gamma_2 \Rightarrow \Delta_2$ if $\Gamma_1 \subseteq \Gamma_2$ (iff $\Delta_2 \subseteq \Delta_1$, since $\Gamma_1 \cup \Delta_1 = \Gamma_2 \cup \Delta_2 = \mathcal{F}''$). Finally, define $v' : W \times \mathcal{F}' \to \mathcal{T}$ by:

$$v(\Gamma \Rightarrow \Delta, \psi) = \begin{cases} \bot & \psi \in \Delta, \neg\psi \in \Delta \\ f & \psi \in \Delta, \neg\psi \in \Gamma \\ t & \psi \in \Gamma, \neg\psi \in \Delta \\ \top & \psi \in \Gamma, \neg\psi \in \Gamma \end{cases}$$

It is not difficult to see that v is well-defined (note that if $\Gamma \Rightarrow \Delta \in W$ and $\psi \in \mathcal{F}'$ then $\{\psi, \neg\psi\} \subseteq \Gamma \cup \Delta$), and that $\langle W, \leq, v\rangle$ is an $\mathcal{M}_{PI}[S]$-semiframe. We prove here the conditions concerning $\supset$:

- Suppose $\varphi \supset \psi \in \mathcal{F}'$, and $v(\Gamma \Rightarrow \Delta, \varphi \supset \psi) \in \mathcal{D}$. Then $\varphi \supset \psi \in \Gamma$. Let $\Gamma \Rightarrow \Delta \leq \Gamma' \Rightarrow \Delta'$, and suppose $v(\Gamma' \Rightarrow \Delta', \varphi) \in \mathcal{D}$. Then $\varphi \in \Gamma'$. Since also $\varphi \supset \psi \in \Gamma'$ (as $\Gamma \subseteq \Gamma'$), it is not possible that $\psi \in \Delta'$ (because otherwise $\Gamma' \Rightarrow \Delta'$ would have a cut-free proof, and so an s-proof). Hence $\psi \in \Gamma'$, and so $v(\Gamma' \Rightarrow \Delta', \psi) \in \mathcal{D}$ as well.

- Suppose $\varphi \supset \psi \in \mathcal{F}'$, and let $w = \Gamma \Rightarrow \Delta$. Assume that $v(w, \varphi \supset \psi) \notin \mathcal{D}$. Then $\varphi \supset \psi \in \Delta$. Thus $\Gamma \Rightarrow \varphi \supset \psi$ does not have an s-proof, whence $\Gamma, \varphi \Rightarrow \psi$ does not have an s-proof either. Since $\Gamma \cup \{\varphi, \psi\} \subseteq \mathcal{F}''$, by this virtue there is $w' = \Gamma' \Rightarrow \Delta' \in W$ such that $\Gamma \cup \{\varphi\} \subseteq \Gamma', \psi \in \Delta'$. Hence $w \leq w'$, and $v(w', \varphi) \in \mathcal{D}$, while $v(w', \psi) \notin \mathcal{D}$.

- Suppose $(\Rightarrow \neg \supset) \in S$. We prove that $C(\Rightarrow \neg \supset)$ is satisfied. Thus assume that $\varphi \supset \psi \in \mathcal{F}'$, and that $v(\Gamma \Rightarrow \Delta, \varphi) \in \mathcal{D}$, $v(\Gamma \Rightarrow \Delta, \psi) \in \{f, \top\}$. Then $\varphi \in \Gamma$ and $\neg\psi \in \Gamma$. Since $(\Rightarrow \neg \supset) \in S$, $\neg(\varphi \supset \psi)$ cannot be in Δ (otherwise $\Gamma \Rightarrow \Delta$ would have a cut-free proof). As $\neg(\varphi \supset \psi) \in \mathcal{F}''$, we get $\neg(\varphi \supset \psi) \in \Gamma$, and so $v(\Gamma \Rightarrow \Delta, \varphi \supset \psi) \in \{f, \top\}$. Given the global conditions concerning v and $\supset$ proved above, this implies $C(\Rightarrow \neg \supset)$.

- Suppose $(\neg \supset \Rightarrow)_1 \in S$. We prove that $C^I(\neg \supset \Rightarrow)_1$ is satisfied. Thus assume that $\varphi \supset \psi \in \mathcal{F}'$, and that $v(\Gamma \Rightarrow \Delta, \varphi) \notin \mathcal{D}$. Then $\varphi \in \Delta$ (by the definition of v). Since $(\neg \supset \Rightarrow)_1 \in S$, this yields $\neg(\varphi \supset \psi) \notin \Gamma$, whence $v(\Gamma \Rightarrow \Delta, \varphi \supset \psi) \in \{t, \bot\}$.

- We prove similarly that if $(\neg \supset \Rightarrow)_2 \in S$ then $C^I(\neg \supset \Rightarrow)_2$ is satisfied.

Now by definition of v and $\Gamma_0 \Rightarrow \Delta_0$, v refutes s in the world $\Gamma_0 \Rightarrow \Delta_0$ of W. Hence $\langle W, \leq, v\rangle$ is a finite countermodel of s as required. ■

Corollary 4.17 *$LJ^+(S)$ is decidable for every $S \subseteq NR$.*

Proof: If $\{(\Rightarrow \neg), (\neg \supset\Rightarrow)_1\} \subseteq S$ then $LJ^+(S) = LK^+(S)$ (Corollary 4.15), and so $LJ^+(S)$ is decidable in this case by Corollary 3.15. Otherwise Theorem 4.16 and its proof imply that in order to determine whether a given sequent s is provable in $LJ^+(S)$, one has to check at most 2^{2n} finite $\mathcal{M}_{IP}[S]$-semiframes, where n is the cardinality of $\mathcal{F}'$, the set of the subformulas of s, and the relevant semiframes are based on $\mathcal{F}'$. ■

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