Two Types of Multiple-Conclusion Systems

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Abstract

Hypersequents are finite sets of ordinary sequents. We show that multiple-conclusion sequents and single-conclusion hypersequents represent two different natural methods of switching from a single-conclusion calculus to a multiple-conclusion one. The use of multiple-conclusion sequents corresponds to using a multiplicative disjunction, while the use of single-conclusion hypersequents corresponds to using an additive one. Moreover: each of the two methods is usually based on a different natural semantic idea and accordingly leads to a different class of algebraic structures. In the cases we consider here the use of multiple-conclusion sequents corresponds to focusing the attention on structures in which there is a full symmetry between the sets of designated and antidesignated elements. The use of single-conclusion hypersequents, on the other hand, corresponds to the use of structures in which all elements except one are designated. Not surprisingy, the use of multiple-conclusion hypersequents corresponds to the use of structures which are both symmetrical and with a single nondesignated element.

Keywords: Hypersequents, Consequence relations, Substructural logics, Relevance logics

1 Introduction

Hypersequents ([7]) are essentially finite sets of ordinary sequents. As such they are usually taken as a generalization of Gentzen's sequents, and calculi which manipulate hypersequents are usually classified as a generalization of Gentzen-type calculi (this indeed is how they were presented in [7] and previous papers). This picture is not absolutely correct, though. Gentzen's multiple-conclusion sequents as introduced in [14] were themselves a generalization. They generalize his much more natural *single-conclusion* sequents. Now hypersequents in which all the components are single-conclusion sequents form in fact another, *different* generalization of this basic data structure, orthogonal in a way to Gentzen's multiple-conclusion sequents. Employing hypersequents in which the components are multiple-conclusion sequents provides, of course, the most natural generalization of both methods.

Our main thesis in this research is that multiple-conclusion sequents and single-conclusion hypersequents represent two different methods of switching from a single-conclusion calculus to a multiple-conclusion one, and that both are *natural*. In fact, from the proof-theoretical point of view each of these methods corresponds to the use of a different type of disjunction. That of multiple-conclusion sequents corresponds (in the terminology of [15]) to the use of the multiplicative disjunction, while that of single-conclusion hypersequents corresponds to the use of the additive disjunction. Moreover: each of the two methods is usually based on a different natural semantic idea and accordingly leads to a different class of algebraic structures. Now in classical logic there is no difference between the multiplicative disjunction and the additive one. Hence one might expect the differences between the two types of generalizations of basic sequents to be reflected only relative to proper substructural logics. This indeed is the case. While in the switch from intuitionistic logic to classical logic the two methods give equivalent results, this is not so with respect to weaker substructural logics.

The main part of this paper will be devoted to showing the different effects of applying the two methods of obtaining classical logic from intuitionistic logic to substructural logics which lack the full power of weakening. We show that the use of multiple-conclusion sequents corresponds to focusing the attention on structures in which each element is either designated or antidesignated ([16]), and there is a full *symmetry* between the sets of designated and antidesignated elements. The use of single-conclusion hypersequents, on the other hand, corresponds to the use of structures in which all elements except one are designated (It is interesting to note, that in the class of Heyting Algebras, the algebraic structures which correspond to intuitionistic logic, the two-valued classical algebra is the only one which falls under either category). Finally (and not surprisingly) the common extension to a calculus of multiple-conclusion hypersequents corresponds to the use of structures which are both symmetrical and with a single nondesignated element[1].

To make the presentation shorter and simpler, we choose to work with one particular system. Most of the proof-theoretical and the abstract semantic results can however easily be generalized to other substructural systems of this sort. The system we choose is the one which is obtained from the multiplicative-additive fragment of Intuitionistic Linear Logic by adding to it the contraction rule as well as its converse. There are two reasons for this choice. First, the availability of these two rules means that in this logic we are really dealing with *sets* of premises rather than with multisets (as in Linear Logic and in R). This is much closer in spirit to intuitionistic and classical logic, the connections between which we are trying to generalize. Second, in this case the multiple-conclusion generalizations lead to algebraic structures which are simple, useful and illuminating (this does not seem to be the case with close relatives, like the multiplicative fragment of R, for which only abstract algebraic semantics is known).

Two other interesting phenomena that are revealed in our case study and deserve mentioning are the following. First, the use of multiple-conclusion hypersequents allows us to get *strong* completeness theorems in cases in which the use of ordinary (multiple-conclusion) sequents provides only weak completeness. Second: there are in general two main methods of defining a consequence relation, given some Gentzen-type calculus. In calculi of sequents these methods usually lead to different consequence relations in case additive connectives are included (and this is indeed what happens here). In contrast, for the hypersequential systems we study here the two consequence relations are identical even in the presence of the additive connectives!

2 The General Proof-Theoretical Framework

In this section we describe the four types of Gentzen-type calculi we are about to employ in this paper. We start with some very general definitions.

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[1] Note that being an antidesignated element and being a nondesignated one is not the same thing. In our main structures each nondesignated element will be antidesignated, but not vice versa.

In what follows L is a fixed language.

Definition 2.1 *A (Tarskian)* finitary Consequence Relation *(CR) is a relation* $\vdash$ *between theories (i.e., sets of formulas) in* L *and formulas in* L *such that:*

(i) $\{A\} \vdash A$

(ii) If $T \vdash A$ *and* $T \subseteq T^*$ *then* $T^* \vdash A$

(iii) If $T \vdash A$ *and* $T \cup \{A\} \vdash B$ *then* $T \vdash B$.

(iv) $T \vdash A$ *only if* $\Gamma \vdash A$ *for some finite subset* Γ *of* T.

Definition 2.2 *1. A single-conclusion sequent (*s-sequent*, in short) is a syntactic structure of the form* $\Gamma \Rightarrow A$*, where* Γ *and* A *are, respectively, a multiset of formulas and a formula of* L.

2. An s-sequential calculus *is an axiomatic system* G *for s-sequents which satisfies the following conditions:*
Reflexivity: $\vdash_G A \Rightarrow A$ *for all* A.
Cut: $\{(\Gamma_1, A, \Gamma_2 \Rightarrow B),\ (\Delta \Rightarrow A)\} \vdash_G \Gamma_1, \Delta, \Gamma_2 \Rightarrow B$.

NOTE

It is possible to use lists instead of multisets in the last definition, without affecting much the next definition and the proposition that follows it. The Cut condition was formulated, in fact, so that this may be done easily. Still, it is much easier to generalize it to the multiple-conclusion case in a natural way if we use multisets rather than lists. Since we assume the permutation rule in all the systems considered in this paper, the present definition suffices for our needs.

Definition 2.3 *Let* G *be an s-sequential calculus. The external and the internal CRs which are induced by* G *are defined as follows:*

The external CR: $T \vdash^e_G B$ *if* $(\Rightarrow A_1), \ldots, (\Rightarrow A_n) \vdash_G \Rightarrow B$ *for some* $A_1, \ldots, A_n$ *in* T.

The internal CR: $T \vdash^i_G B$ *if there exists a multiset* Γ *of elements of* T *so that* $\vdash_G \Gamma \Rightarrow B$.

Proposition 2.4 *1.* $\vdash^e_G$ *and* $\vdash^i_G$ *are CRs.*

2. If $T \vdash^i_G B$ *then* $T \vdash^e_G B$.

3. If all the rules of G *are pure (or "multiplicative") then* $T \vdash^i_G B$ *whenever* $T \vdash^e_G B$ *(a rule of an s-sequential calculus is pure if whenever* $\Gamma \Rightarrow A$ *can be inferred by it from* $\Gamma_i \Rightarrow B_i$ $(i = 1, \ldots, n)$ *then* $\Gamma, \Gamma'_1, \ldots, \Gamma'_n \Rightarrow A$ *can also be inferred by it from* $\Gamma_i, \Gamma'_i \Rightarrow B_i$ $(i = 1, \ldots, n)$*).*

PROOF. It is obvious that $\vdash^e_G$ is a CR. That so is also $\vdash^i_G$ follows from the two conditions in the definition of an s-sequential calculus. The second part of the proposition follows by cuts, while the third is a consequence of the reflexivity condition and the purity of the rules. Details are left for the reader. ∎

Definition 2.5 *1. A multiple-conclusion sequent (*m-sequent*, in short) is a structure of the form* $\Gamma \Rightarrow \Delta$*, where* Γ *and* Δ *are multisets of formulas of* L.

2. An m-sequential calculus *is an axiomatic system G for m-sequents which satisfies the Relexivity condition from the definition of an s-sequential calculus as well as the following version of the Cut condition:*

$$\{(\Gamma_1, \Rightarrow \Delta_1, A),\ (A, \Gamma_2 \Rightarrow \Delta_2)\} \vdash_G \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2.$$

3. The external and the internal CRs are defined for m-sequential calculi exactly as in the s-sequential case.

Proposition 2.6 *Proposition 2.4 is true also for m-sequential calculi (where the definition of a pure rule is changed in the obvious way).*

PROOF. Left for the reader. ∎

Definition 2.7 *1. A single-conclusion hypersequent (*s-hyperequent*, in short) is a syntactic structure of the form:*

$$\Gamma_1 \Rightarrow A_1 \mid \Gamma_2 \Rightarrow A_2 \mid \cdots \mid \Gamma_n \Rightarrow A_n$$

(where $\Gamma_i \Rightarrow A_i$ is an ordinary s-sequent).

*2. A multiple-conclusion hypersequent (*m-hypersequent*, in short) is a syntactic structure of the form $\Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$ (where $\Gamma_i \Rightarrow \Delta_i$ is an ordinary m-sequent).*

3. An s-hypersequential calculus *is an axiomatic system G for deriving s-hypersequents which satisfies the following conditions:*
Reflexivity: $\vdash_G A \Rightarrow A$ *for all A.*
Cut: $S_1 \mid T_1 \mid \Gamma_1, \Gamma_2 \Rightarrow B \mid S_2 \mid T_2$ *follows in G from the set* $\{(S_1 \mid \Gamma_1 \Rightarrow A \mid S_2),\ (T_1 \mid A, \Gamma_2 \Rightarrow B \mid T_2)\}$ *(where S_1, S_2, T_1, T_2 are (possibly empty) s-hypersequents).*
External Contraction: $S \mid T \mid T \mid U \vdash_G S \mid T \mid U$.
External Permutation: $S \mid T \mid W \mid U \vdash_G S \mid W \mid T \mid U$.

4. An m-hypersequential calculus *is an axiomatic system G for m-hypersequents which satisfies the above Reflexivity, External Contraction, and External Permutation conditions, and in which $S_1 \mid T_1 \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2 \mid S_2 \mid T_2$ follows from $\{(S_1 \mid \Gamma_1 \Rightarrow \Delta_1, A \mid S_2),\ (T_1 \mid A, \Gamma_2 \Rightarrow \Delta_2 \mid T_2)\}$ (where S_1, S_2, T_1, T_2 are (possibly empty) m-hypersequents).*

5. Let G be an (s- or m-) hypersequential calculus. The external CR, $\vdash^e_G$, is defined for G exactly as in the sequential case, while the internal CR, $\vdash^i_G$, is defined for G as follows: $T \vdash^i_G B$ iff there exist multisets $\Gamma_1, \ldots, \Gamma_n$ of elements of T such that $\vdash_G \Gamma_1 \Rightarrow B \mid \cdots \mid \Gamma_n \Rightarrow B$.

NOTE
The names "External Contraction" and "External Permutation" are also used for the obvious rules which correspond to the above conditions, while "External Weakening" is the rule which allows to add arbitrary components to a hypersequent. The three rules are usually called the *standard external structural rules* (see [7]).

Proposition 2.8 *Proposition 2.4 is true also for hypersequential calculi (where again the definition of a pure rule is changed so that side sequents are allowed).*

PROOF. Most of the parts are straightforward. The only problematic one is showing the transitivity conditions for $\vdash^i_G$. For this we have to show that if there are submultisets $\Gamma_1, \ldots \Gamma_n, \Delta_1, \ldots, \Delta_m, \Delta'_1, \ldots, \Delta'_p$ of T such that $\vdash_G \Gamma_1 \Rightarrow A \mid \cdots \mid \Gamma_n \Rightarrow A$ and $\vdash_G \Delta_1, A \Rightarrow B \mid \cdots \mid \Delta_m, A \Rightarrow B \mid \Delta'_1 \Rightarrow B \mid \cdots \mid \Delta'_p \Rightarrow B$ then $T \vdash^i_G B$. For this we prove by induction on n that if there are submultisets $\Gamma_1, \ldots \Gamma_n, \Gamma'_1, \ldots, \Gamma'_k, \Delta_1, \ldots, \Delta_m, \Delta'_1, \ldots, \Delta'_p$ of T such that $\vdash_G \Gamma_1 \Rightarrow A \mid \cdots \mid \Gamma_n \Rightarrow A \mid \Gamma'_1 \Rightarrow B \mid \cdots \mid \Gamma'_k \Rightarrow B$ and $\vdash_G \Delta_1, A \Rightarrow B \mid \cdots \mid \Delta_m, A \Rightarrow B \mid \Delta'_1 \Rightarrow B \mid \cdots \mid \Delta'_p \Rightarrow B$ then $T \vdash^i_G B$. We leave the details to the reader. ∎

3 Two Ways from Intuitionistic to Classical Logic

Let LJ be the Gentzen-type formulation of Propositional Intuitionistic Logic, in which the negation of a formula A is defined as $A \to \bot$ and the axioms are taken to be $A \Rightarrow A$ and $\bot \Rightarrow A$ (for all A). LJ is a paradigmatic example of an s-sequential calculus. The usual way of obtaining Classical Logic from it is to turn this s-sequential calculus into an m-sequential one LK. This is done by keeping in LK exactly the same axioms and logical rules as in LJ (but allowing in applications of the logical rules also extra side formulas on the r.h.s of the $\Rightarrow$), and by adding to the set of structural rules of LJ their r.h.s counterparts. The usual interpretation of an m-sequent $A_1, \ldots, A_n \Rightarrow B_1, \ldots, B_k$ of LK is the sentence $A_1 \& A_2 \& \cdots \& A_n \to B_1 \vee B_2 \vee \cdots \vee B_k$.

NOTE

There is a slight difference between this formulation of LK and the usual one, in that in the above version the r.h.s. of a sequent cannot be empty. We take, however, a sequent of the form $\Gamma \Rightarrow$ to be the same as $\Gamma \Rightarrow \bot$. It is straightforward to see that with this translation the above version and the little bit more standard one are equivalent.

We present now another method of obtaining classical logic from LJ, which uses an s-hypersequential calculus rather than an m-sequential one. Our starting point is that in all the hypersequential calculi that have been used in the past new components are added to a given hypersequent by some form of the splitting rule:

$$\frac{G|\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2|H}{G|\Gamma_1 \Rightarrow \Delta_1|\Gamma_2 \Rightarrow \Delta_2|H}$$

By trying to directly apply this scheme to s-hypersequents we get:

$$\frac{G|\Gamma_1, \Gamma_2 \Rightarrow A|H}{G|\Gamma_1 \Rightarrow A|\Gamma_2 \Rightarrow |H}$$

This, however, is not an s-hypersequent. Following, however, our understanding above that an empty succedent means $\bot$, we get the following rule:

$$\frac{G|\Gamma_1, \Gamma_2 \Rightarrow A|H}{G|\Gamma_1 \Rightarrow A|\Gamma_2 \Rightarrow \bot|H}$$

A cut of this with the axiom $\bot \Rightarrow B$ leads to the following rule:

$$\frac{G|\Gamma_1, \Gamma_2 \Rightarrow A|H}{G|\Gamma_1 \Rightarrow A|\Gamma_2 \Rightarrow B|H}$$

In order to have cut elimination, we adopt the last rule as the official rule of our s-hypersequential version of classical logic.[2]

Definition 3.1 *The s-hypersequential system LJ^h is defined as follows:*

Axioms: *As in LJ.*

Logical Rules: *The s-hypersequential versions of the logical rules of LJ.*

Structural Rules:

1. *The standard external structural rules.*
2. *The s-hypersequential versions of the structural rules of LJ.*
3. *Classical splitting (CS):*

$$\frac{G|\Gamma_1, \Gamma_2 \Rightarrow A|H}{G|\Gamma_1 \Rightarrow A|\Gamma_2 \Rightarrow B|H}$$

The proof of the following theorem is now straightforward:

Theorem 3.2 *1. LJ^h admits cut-elimination.*

2. LJ^h and LK are equivalent: the m-sequent $\Gamma_1, \dots, \Gamma_n \Rightarrow A_1, \dots, A_n$ is derivable in LK iff the s-hypersequent $\Gamma_1 \Rightarrow A_1 \mid \cdots \mid \Gamma_n \Rightarrow A_n$ is derivable in LJ^h.

Like the m-sequents of LK, the s-hypersequents of LJ^h also have a standard interpretation. The interpretation of the s-hypersequent $\Gamma_1 \Rightarrow A_1 \mid \cdots \mid \Gamma_n \Rightarrow A_n$ is $(\&\Gamma_1 \to A_1) \vee \cdots \vee (\&\Gamma_n \to A_n)$ (where $\&\Gamma$ is the conjunction of the sentences in Γ). We see, therefore, that multiplicity of conclusions in an m-sequent corresponds to an *internal disjunction* of formulas within a sequent, while the multiplicity of conclusions in an s-hypersequent corresponds to an *external disjunction* of the translations of the individual components. In classical logic these two disjunctions are identical, since $A \to B \vee C$ is equivalent in it to $(A \to B) \vee (A \to C)$. This phenomenon is due to the fact that all the internal standard structural rules (on the l.h.s. of a sequent) are available in intuitionistic logic — the single-conclusion logic from which classical logic is derived. One might expect therefore that things would be different if we start with a substructural ([10]) single-conclusion logic. In the next section we will investigate an interesting representative case of this sort.

4 $RM0_{im}$ and Its Extensions

In this section we apply the two processes that lead from intuitionistic logic to classical logic to another substructural logic which is single-conclusion in an essential way: $RM0_{im}$. This logic is based on $RM0_{\to}$ (see [1]) which is the minimal implicational logic for which the following deduction theorem obtains: there is a proof of $A \to B$ from the *set* Γ which *uses all* the formulas in Γ iff there is a proof of B from $\Gamma \cup \{A\}$ which uses all formulas in $\Gamma \cup \{A\}$.

In order to get $RM0_{im}$ we enrich the language of $RM0_{\to}$ by adding to it versions of the two other connectives which are characteristic for logics which are essentially

[2] Another reasonable possibility of splitting in the s-hypersequential case is to split only the antecedent. This leads to the rule: $\frac{G|\Gamma_1,\Gamma_2 \Rightarrow A|H}{G|\Gamma_1 \Rightarrow A|\Gamma_2 \Rightarrow A|H}$. In [5] it is shown that the use of this rule leads from intuitionistic logic to Godel-Dummet *intermediate* logic LC ([11]). This is a demonstration of the fact, that the use of s-hypersequents rather than m-sequents allows more possibilities and better insights.

single-conclusion: conjunction and absurdity ($\bot$).[3] In order to deal first with logics which are purely multiplicative, we use the multiplicative $\otimes$ for our conjunction. We shall denote by $\mathcal{L}_{im}$ (intuitionistic multiplicative) the language $\{\rightarrow, \otimes, \bot\}$. Later we shall investigate the effects of adding to $\mathcal{L}_{im}$ the multiplicative constant 1 of [15] (also denoted t in the relevance literature), as well as the additive connectives.

4.1 The Basic Logics and Their Proof Systems

Since we are interested in this paper in Gentzen-type systems, we present the various logics by using such systems (it is possible to present corresponding Hilbert-type systems as well, but this is not important for our present purposes). We remind the reader that the permutation rule is tacitly assumed in all the systems below.

(1) The *s*-sequential system $RM0_{im}$.

Axioms:

$$A \Rightarrow A \qquad\qquad \Gamma, \bot \Rightarrow A$$

Logical Rules:

$$(\rightarrow\Rightarrow)\ \frac{\Gamma \Rightarrow A \qquad B, \Delta \Rightarrow C}{A \rightarrow B, \Gamma, \Delta \Rightarrow C} \qquad\qquad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B}\ (\Rightarrow\rightarrow)$$

$$(\otimes \rightarrow)\ \frac{\Gamma, A, B, \Delta \Rightarrow C}{\Gamma, A \otimes B, \Delta \Rightarrow C} \qquad\qquad \frac{\Gamma_1 \Rightarrow A \qquad \Gamma_2 \Rightarrow B}{\Gamma_1, \Gamma_2 \Rightarrow A \otimes B}\ (\Rightarrow \otimes)$$

Structural Rules:

$$(C)\ \frac{\Gamma, A, A \Rightarrow B}{\Gamma, A \Rightarrow B} \qquad\qquad \frac{\Gamma \Rightarrow A \qquad \Delta \Rightarrow A}{\Gamma, \Delta \Rightarrow A}\ (M)$$

$$\frac{\Gamma_1 \Rightarrow A \qquad A, \Gamma_2 \Rightarrow B}{\Gamma_1, \Gamma_2 \Rightarrow B}\ (cut)$$

NOTE

The proof-theoretical properties of the implicational fragment of $RM0_{im}$ has first been investigated in [18]. For reasons that will be clarified later, we have followed that paper in choosing the mingle rule (M) as primitive. Another possibility is to choose instead Expansion (E): $\frac{\Gamma, A, \Delta \Rightarrow B}{\Gamma, A, A, \Delta \Rightarrow B}$. The two formulations are equivalent, and both admit cut-elimination.

The next system is the natural m-sequential extension of $RM0_{im}$.

[3] We could have easily included $\top$ as well, but this is definable as $\bot \rightarrow \bot$. In linear logic $\bot$ and $\top$ (or "0" and "$\top$" as in [15], but unlike [19]) are considered to be "additives". In [8] we explain why it is safe (and even preferable) to take them as a part of the "multiplicative" fragment.

(2) The m-sequential system RMI_{im}.

Axioms: $A \Rightarrow A$ $\quad\quad$ $\Gamma, \perp \Rightarrow \Delta \quad (\Delta \neq \emptyset)$

Logical Rules: The m-sequential versions of the logical rules of $RM0_{im}$.

Structural Rules: The m-sequential versions of the structural rules of $RM0_{im}$ as well as their duals:

$$\frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}$$

$$\frac{A, \Gamma_1 \Rightarrow \Delta_1 \qquad A, \Gamma_2 \Rightarrow \Delta_2}{A, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2} \qquad \frac{\Gamma_1 \Rightarrow \Delta_1, A \qquad \Gamma_2 \Rightarrow \Delta_2, A}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2, A}$$

$$\frac{\Gamma_1 \Rightarrow \Delta_1, A \qquad A, \Gamma_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

NOTES

1. Again, instead of the two (relevant) mingle rules we could have chosen to use the two m-sequential Expansion rules.
2. A natural step is to combine the two "mingle" rules of RMI into one by deleting the "A" which is common to the two premises. We get what is known as the "mix" rule of [15]. This leads to the following *stronger* system:

(3) The m-sequential system RM_{im}. This is the system which is obtained from RMI_{im} by replacing the two mingle rules by:

$$\frac{\Gamma_1 \Rightarrow \Delta_1 \qquad \Gamma_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2} \quad (\text{mix}) \ .$$

NOTE

Both RMI_{im} and RM_{im} as presented here are really the $\mathcal{L}_{im}$-fragments of the full systems RMI and RM (see [1], [13] for RM, [4] for RMI).

We turn now to the classical-like hypersequential extensions of these three logics.

(4) The s-hypersequential system $RM0^h_{im}$.

Axioms: As in $RM0_{im}$.

Logical Rules: The hypersequential versions of the logical rules of $RM0_{im}$.

Structural Rules:

(i) The standard external structural rules.

(ii) The hypersequential versions of the structural rules of $RM0_{im}$.

(iii) Classical Splitting (CS):

$$\frac{G|\Gamma_1, \Gamma_2 \Rightarrow A|H}{G|\Gamma_1 \Rightarrow A|\Gamma_2, \Delta \Rightarrow B|H}$$

NOTE
This version of Classical Splitting can be derived from the original one using cuts. If $\Delta = A_1, \ldots, A_k$ then by using the classical splitting rule of the last section, one can derive from $G|\Gamma_1, \Gamma_2 \Rightarrow A|H$ the hypersequent $G|\Gamma_1 \Rightarrow A|\Gamma_2 \Rightarrow A_1 \otimes A_2 \otimes \cdots \otimes A_k \to B|H$. Since $\vdash_{RM0_{im}} \Delta, A_1 \otimes A_2 \otimes \cdots \otimes A_k \to B \Rightarrow B$, the hypersequent $G|\Gamma_2 \Rightarrow A|\Delta, \Gamma_2 \Rightarrow B|H$ follows by a cut. We have chosen the present version in order to secure cut-elimination.

(5) The m-hypersequential RMI^h_{im}. This is the m-hypersequential extension of $RM0^h_{im}$. Thus, for example, classical splitting takes here the form:

$$\frac{G|\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2|H}{G|\Gamma_1 \Rightarrow \Delta_1|\Gamma, \Gamma_2 \Rightarrow \Delta_2, \Delta|H}$$

provided $\Delta_1 \neq \emptyset$ and $\Delta_2 \cup \Delta \neq \emptyset$.

NOTE
RMI^h_{im} is also (as its name suggests) the natural classical hypersequential extension of RMI_{im}.

(6) The m-hypersequential RM^h_{im}. This is the classical hypersequential extension of RM_{im}. Alternatively, it is the system which is obtained from RMI^h_{im} by replacing the two (hypersequential) mingle rules by the (hypersequential) mix rule.

NOTE
As noted above, $RM0_{im}$, RMI_{im}, and RM_{im} are either natural conservative extensions or else fragments of well known systems. RMI^h_{im} and RM^h_{im} are in turn the $\mathcal{L}_{im}$-fragments of the systems $SRMI^\perp$ and $SRM^\perp$ (respectively) of [8] (this follows either by cut-elimination or can be shown by semantical methods). The system $RM0^h_{im}$, on the other hand, is introduced and investigated here for the first time.

Since all the six systems we have just introduced are purely multiplicative, there is no difference between the internal and the external consequence relations which are induced by them. Accordingly, we shall denote both by $\vdash_{RM0_{im}}, \vdash_{RM^h_{im}}$, etc. Figure 1 displays the obvious relations between these six systems in the form of a lattice (ordered by inclusion). We shall see below that it reflects the *exact* relationships which exist between these systems.

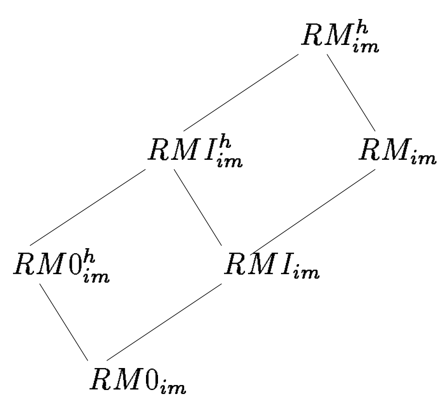

FIGURE 1

Proposition 4.1 *All the six systems above are different from each other. Moreover: no inclusion relations, other than those shown in Figure 1 (or which follow from it by transitivity), hold among them.*

PROOF. Obviously it is enough to show the following:

(i) $RM0^h_{im} \not\subseteq RM_{im}$
(ii) $RM_{im} \not\subseteq RMI^h_{im}$
(iii) $RMI_{im} \not\subseteq RM0^h_{im}$

Now (i) follows from the fact that $A \otimes B \vdash_{RM0^h_{im}} A$ (since $\vdash_{RM0^h_{im}} A \otimes B \Rightarrow A | \Rightarrow A$, using CS on $A \Rightarrow A$ and the $(\otimes \Rightarrow)$ rule), but $A \otimes B \not\vdash_{RM_{im}} A$ (this can easily be shown by using the cut-elimination theorem or by using Sugihara matrix, which is characteristic for RM [1]). For (ii) we note that $((A \to (B \to B)) \to A) \to A$ can easily be proved in RM_{im} but it is not provable in RMI^h_{im} (this is shown, using a semantic method, after Theorem 5.19 below). Similarly, for (iii) we have that $((B \to B) \to (A \to A)) \to (((A \to B) \to A) \to A)$ is a theorem of RMI_{im} but not of $RM0^h_{im}$ (this again, is shown below by semantic method. See the example after Theorem 5.18). ∎

Theorem 4.2 *The cut-elimination theorem is valid for all the systems above.*

The proof of this theorem uses the "history" technique from [3] and is quite long. We omit the details.

NOTES

1. In the last proof we gave explicit examples of *theorems* of RM_{im} and RMI_{im} which are not theorems of RMI^h_{im} and $RM0^h_{im}$ (respectively). We did *not* provide such an example for $RM0^h_{im}$ and RM_{im}. The reason is that it does not exist! At the next section it is shown that RM_{im} and RM^h_{im} have the same set of theorems, and similarly for RMI_{im} and RMI^h_{im}. The differences in these cases are due *only to the consequence relations.* On the other hand, the set of theorems of $RM0^h_{im}$ strictly contains that of $RM0_{im}$: In [8], p. 203 there is a proof of $((A \to B) \to A) \to A \otimes ((A \to B) \to (A \to B))$ which is really in $RM0^h_{im}$. It is easy, however, to see that this is not a theorem of $RM0_{im}$.
2. The fact that $A \otimes B \vdash_{RM0^h_{im}} A$ (and $A \otimes B \vdash_{RM0^h_{im}} B$) means that $\otimes$ behaves as an "extensional" conjunction in the three hypersequential systems we consider here, and so it has been justified to select it as the counterpart of classical and intuitionistic conjunction.[4]
3. The cut-elimination theorem can be used to show many important proof-theoretical properties of the above systems. It would be easier, however, to use for this the semantics of these systems. This will be developed in the next section.

4.2 Disjunction Connectives and Translations

It is straightforward to translate single-conclusion sequents into formulas:

[4] This idea has already been explored, but only in the presence of an involutive negation, in [8] and [9].

Definition 4.3 *Let $\Gamma = A_1, \ldots, A_n$. The translation of the sequent $\Gamma \Rightarrow B$ is the formula $A_1 \otimes A_2 \otimes \cdots \otimes A_n \rightarrow B$ (and just B if $n = 0$). We denote this translation by $\phi_{\Gamma \Rightarrow B}$.*

It is very easy to show that if s is a single-conclusion sequent then s and $\Rightarrow \phi_s$ follow from each other in $RM0_{im}$ (or any extension). In contrast, no translation of $\Gamma \Rightarrow \Delta$ seems to be available in $\mathcal{L}_{im}$ (unless Δ is a singleton). In order to be able to translate multiple-conclusion sequents, we should enrich the language with a multiplicative disjunction $+$. The corresponding Gentzen-type rules are:

$$(+\Rightarrow)\ \frac{A, \Gamma_1 \Rightarrow \Delta_1 \qquad B, \Gamma_2 \Rightarrow \Delta_2}{A + B, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2} \qquad\qquad \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A + B}\ (\Rightarrow +)$$

It is not difficult to see that the cut elimination theorem still obtains when we extend the various systems above with $+$. Hence all these extensions are conservative. Now in the presence of $+$ we have the following obvious translation for multiple-conclusion sequents:

Definition 4.4 *Let $\Gamma = A_1, \ldots, A_n$ and $\Delta = B_1, \ldots, B_k$ $(k > 0)$. $\phi_{\Gamma \Rightarrow \Delta}$, the translation of the sequent $\Gamma \Rightarrow \Delta$, is the formula:*

$$A_1 \otimes A_2 \otimes \cdots \otimes A_n \rightarrow B_1 + B_2 + \cdots + B_k$$

.

Proposition 4.5 *Let s be a multiple-conclusion sequent. Then s and $\Rightarrow \phi_s$ follow from each other in $RM0_{im}$, enriched with the two rules for $+$ (or any extension of this system).*

NOTE

It is possible to conservatively add to all the m-sequential and m-hypersequential systems we consider here also multiplicative negation without losing cut-elimination. Most of the results are preserved. Those that are not are noted below.

It follows that the multiplicity of formulas in the succedent of an m-sequent corresponds to the multiplicative disjunction of these formulas. We now show that the multiplicity of sequents in an hypersequent corresponds to the *additive* disjunction of its components. For this we enrich the language with the connective $\vee$, together with the appropriate version (in each case) of the following s-sequential rules:

$$\frac{A, \Gamma \Rightarrow C \qquad B, \Gamma \Rightarrow C}{A \vee B, \Gamma \Rightarrow C} \qquad\qquad \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B} \qquad\qquad \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \vee B}$$

By using the techniques of [3] and [6] it is possible to show that cut-elimination is preserved by this addition in all the cases we consider. It follows that the systems we get are all conservative extensions of the corresponding multiplicative systems (this fact can also be demonstrated using the semantical results of the next section).

With $\vee$ in our disposal we can now translate hypersequents into formulas:

Definition 4.6 *Let $H = \Gamma_1 \Rightarrow \Delta_1 | \cdots | \Gamma_n \Rightarrow \Delta_n$ be a hypersequent. ϕ_H, its translation, is $\phi_{\Gamma_1 \Rightarrow \Delta_1} \vee \cdots \vee \phi_{\Gamma_n \Rightarrow \Delta_n}$, where $\phi_{\Gamma_i \Rightarrow \Delta_i}$ is the translation of $\Gamma_i \Rightarrow \Delta_i$ as defined above (note that for the m-hypersequential system we need to use also $+$, the multiplicative disjunction, for translating each component).*

Proposition 4.7 *Let G be any of the 3 hypersequential systems above. Then H and $\Rightarrow \phi_H$ follow from each other in G (where H is an arbitrary hypersequent).*

PROOF. Since $\Gamma \Rightarrow \Delta$ and $\Rightarrow \phi_{\Gamma\Rightarrow\Delta}$ follow from each other already in the s-sequential system on which G is based, all we really need to show is that $\Rightarrow A \vee B$ and $\Rightarrow A \mid \Rightarrow B$ follow from each other in the basic hypersequential system with $\vee$. Well, the fact that $\Rightarrow A \vee B$ follows there from $\Rightarrow A \mid \Rightarrow B$ is trivial (using the rules for $\vee$ and external contraction), while the converse follows by a cut from the fact that $A \vee B \Rightarrow A \mid \Rightarrow B$ is provable (note the crucial role of the CS rule in deriving this sequent!). ∎

The upshot is that the two methods of extending a single-conclusion system into a multiple-conclusion one correspond to the two natural disjunctions that we have in substructural logics.

4.3 Adding Additive Conjunction and Identity

In this subsection we briefly discuss what happens if we add to the language and our systems the other standard connectives of substructural logics which fit a single-conclusion framework: the additive (or extensional) conjunction ($\wedge$) and the propositional constant 1 (or t, as in the literature of relevance logic). Their basic s-sequential rules are:

$$\frac{A, \Gamma \Rightarrow C}{A \wedge B, \Gamma \Rightarrow C} \qquad \frac{B, \Gamma \Rightarrow C}{A \wedge B, \Gamma \Rightarrow C} \qquad \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}$$

$$\Rightarrow 1 \qquad \frac{\Gamma \Rightarrow C}{1, \Gamma \Rightarrow C}$$

we will denote the systems which result from the six systems of Figure 1 by the addition of the appropriate versions of these rules and the rules for $\vee$ by $RM0_i$, $RM0_i^h$ etc.

4.3.1 The Consequence Relations

With respect to the consequence relations there is a significant difference between the sequential calculi and the hypersequential ones. Relative to the sequential calculi $\vdash_G^e$ and $\vdash_G^i$ are not identical anymore. Thus $A, B \vdash_G^e A \wedge B$ in all systems, but it is easy to see that $A, B \not\vdash_G^i A \wedge B$ when G is one of the sequential systems. Relative to the hypersequential systems the two consequence relations are still the same, despite the impurity of the additive rules. For example, the fact that $A, B \vdash^i_{RM0_i^h} A \wedge B$ can be seen as follows: By CS we can infer from $A \Rightarrow A$ $A, B \Rightarrow A \wedge B \mid \Rightarrow A$. The hypersequent $A, B \Rightarrow A \wedge B \mid \Rightarrow B$ can similarly be proved from $B \Rightarrow B$. Finally, from the two last hypersequents it is possible to infer $A, B \Rightarrow A \wedge B \mid \Rightarrow A \wedge B$.

In the general case one should prove the following:

Theorem 4.8 *Let $\Delta = A_1, \ldots, A_n$ and suppose that*

$$\{(\Rightarrow A_1), \ldots, (\Rightarrow A_n)\} \vdash_{RM0_i^h} \Gamma_1 \Rightarrow B_1 \mid \cdots \mid \Gamma_k \Rightarrow B_k.$$

Then

$$\vdash_{RM0_i^h} \Delta, \Gamma_1 \Rightarrow B_1 \mid \Gamma_1 \Rightarrow B_1 \mid \cdots \mid \Delta, \Gamma_k \Rightarrow B_k \mid \Gamma_k \Rightarrow B_k.$$

Similar theorems obtain for RMI_i^h and RM_i^h.

PROOF. By induction on the proof of $\Gamma_1 \Rightarrow B_1|\cdots|\Gamma_k \Rightarrow B_k$ from the given assumptions. We do one case as an example. Suppose the last step in the given proof is inferring $A \vee B, \Gamma \Rightarrow C$ from $A, \Gamma \Rightarrow C$ and $B, \Gamma \Rightarrow C$ (for simplicity, we omit irrelevant components of the hypersequents). By induction hypothesis $A, \Gamma \Rightarrow C|\Delta, A, \Gamma \Rightarrow C$ is provable. From this we can infer by CS $A, \Gamma \Rightarrow C|A, \Gamma \Rightarrow C|\Delta, A \vee B, \Gamma \Rightarrow C$ and then $A, \Gamma \Rightarrow C|\Delta, A \vee B, \Gamma \Rightarrow C$ follows by an external contraction (we assume here that Δ is nonempty, since the theorem is trivial otherwise). $B, \Gamma \Rightarrow C|\Delta, A \vee B, \Gamma \Rightarrow C$ can be proved similarly, and from these two sequents $A \vee B, \Gamma \Rightarrow C|\Delta, A \vee B, \Gamma \Rightarrow C$ immediately follows. ∎

NOTE
The use of CS is essential in this proof, and the theorem is not always valid if weaker versions of splitting are used.

4.3.2 Cut-elimination and conservation results

Using the techniques of [3] and [6] it is possible to show that the cut elimination theorem obtain for $RM0_i$, $RM0_i^h$, RM_i and RM_i^h. It follows that they all are conservative extensions of the corresponding multiplicative systems.

With respect to the remaining two systems (RMI_i and RMI_i^h) things are more complicated. We have already noted that if we add to RMI_{im} and RMI_{im}^h only the additive disjunction $\vee$ then we still have cut-elimination (and the resulting systems are conservative over their multiplicative fragments). If, however, we add either $\wedge$ or 1 then this is not the case anymore. For example: two cuts of $1, p \Rightarrow p$ and $1, q \Rightarrow q$ with $\Rightarrow 1, 1$ yield $p, q \Rightarrow p, q$. This sequent obviously does not have a cut free proof even in RMI_i^h, and it is not provable in either RMI_{im} or in RMI_{im}^h. It *is* provable, however, in RM_m or RM_i^h. It is not difficult, in fact, to see that RMI_i and RM_i are equivalent, and so are RMI_i^h and RM_i^h.

Without entering into details, we note that the reason for the difference here between $\vee$ and $\wedge$ is that applications of the impure $(\Rightarrow \wedge)$ rule can be done without any side formulas, while this is not possible in our systems in applications of the dual $(\vee \Rightarrow)$ (since the succedents are never empty). When an internal negation is added this difference disappears. On the other hand it is proved in [6] that if we limit $(\Rightarrow \wedge)$ so that the presence of at least one side formula is required (i.e., it is not allowed to infer $\Rightarrow A \wedge B$ from $\Rightarrow A$ and $\Rightarrow B$) then we do have cut elimination (even in the presence of an internal negation). It is here where the use of the Mingle rules rather than Expansion becomes crucial.[5]

5 Corresponding Algebraic Structures

In this section we present the algebraic semantics of the systems above. We start with $RM0_{im}$. Its algebraic semantics is very abstract, and so not too useful. Still, it serves as the common basis for the much more illuminating semantics that we construct for its extensions later.

[5] It is interesting to note also that conjunction turns out to be problematic in a similar way also in certain important intermediate logics (See [5]).

Definition 5.1 *An $RM0_{im}$-structure is a tuple $\overline{S} = \langle S, \leq, \perp, \top, \otimes, \to, D\rangle$ such that:*

1. *$\langle S, \leq, \perp, \top\rangle$ is a nontrivial bounded poset.*
2. *$\otimes$ is an associative, commutative and idempotent operation on S.*
3. *$\to$ is residuation operation w.r.t. $\otimes$:*

$$a \otimes b \leq c \Leftrightarrow a \leq b \to c$$

4. *D is a cone in S: $D \subseteq S$, and*

$$a \in D,\ a \leq b \Rightarrow b \in D$$

5. *$a \leq b \Leftrightarrow a \to b \in D$*

Lemma 5.2 *In every $RM0_{im}$-structure:*

1. *$\top \in D,\ \perp \notin D$*
2. *$a \leq b \Rightarrow a \otimes c \leq b \otimes c$ (i.e. $\otimes$ is order preserving)*
3. *$a \in D,\ b \in D \Rightarrow a \otimes b \in D$ and $a \otimes b = \sup(a, b)$*
4. *D is an upper semilattice with a top element ($\top$)*
5. *$a \in D,\ a \leq b \Rightarrow a \to b = b$*
6. *$a \otimes \perp = \perp \otimes a = \perp$*
7. *$\perp \to a = \top$*
8. *$a \in D \Rightarrow a \to \perp = \perp$*

Definition 5.3 *An $RM0_{im}$-model of a formula φ in $\mathcal{L}_{im}$ is a pair $\langle \overline{S}, v\rangle$ where $\overline{S}$ is an $RM0_{im}$-structure and v is an operations-respecting valuation in S such that $v(\varphi) \in D$. An $RM0_{im}$-model of a theory $\mathcal{T}$ is an $RM0_{im}$-model of each element of $\mathcal{T}$.*

The following theorem can easily be proved with the help of the previous lemma and the use of Lindenbaum Algebras:

Theorem 5.4 Soundness and Strong Completeness: *$\mathcal{T} \vdash_{RM0_{im}} \varphi$ iff every $RM0_{im}$-model of $\mathcal{T}$ is an $RM0_{im}$-model of φ.*

NOTES

1. It can be shown that every $RM0_{im}$-structure can be embedded in a lattice which is also an $RM0_{im}$-structure. Hence it is possible to make the last completeness theorem w.r.t. the narrower class of what might be called $RM0_{im}$-lattices (i.e. $RM0_{im}$-structures in which $\langle S, \leq\rangle$ is a lattice).
2. If we assume that $D = \{\top\}$ we get a sound and strongly complete semantics of intuitionistic logic (in the language $\mathcal{L}_{im}$). It is easy to see that in this case $a \otimes b$ is the meet of a and b, and so $\langle S, \leq\rangle$ is a lower semilattice. If we demand it also to actually be a *lattice* we get another characterization of what is known as Heyting Algebras (it is easy to see that the various conditions would indeed force this lattice to be distributive).

As was noted above, the value of this semantics for $RM0_{im}$-structures is rather limited (although the fact mentioned in note 1 can, e.g., be used to show a proof-theoretical result: that the addition of the additive disjunction with its rules to $RM0_{im}$ is a conservative extension). As promised, we turn now to show that its two "classical", multiple-conclusion extensions do have *concrete* semantics, with a "surprising" value. The structures which are involved are based on D (the set of designated values) in an essential way. We start with RMI_{im}. The structures which correspond to this system have already been introduced and thoroughly investigated in [4]. Here we only give a description (which is not identical to that in [4] but is obviously equivalent to it) and the relevant results.

Definition 5.5 *A proper symmetrical lattice (a proper S-lattice, in short) is a tuple* $\overline{S} = \langle S, \leq, \perp, \top, D\rangle$ *such that*

1. $\langle S, \leq, \perp, \top\rangle$ *is a nontrivial bounded poset.*
2. D *is a subset of* S *such that:*
   (a) D *is an upper semilattice w.r.t.* $\leq$.
   (b) $a \in D,\ a \leq b \Rightarrow b \in D$ *(in particular,* $\top \in D$*).*
   (c) $a \in D,\ a \leq b,\ a \leq c \Rightarrow b \leq c$ *or* $c \leq b$.[6]
3. *There is a function* $\lambda a.a^+$ *from* $\overline{D}$ *(the complement of* D *w.r.t.* S*) onto* D *such that*
   (a) *If* $a, b \in \overline{D}$ *then* $a \leq b$ *iff* $b^+ \leq a^+$ *(in particular,* $\lambda a.a^+$ *is 1-1).*
   (b) *If* $a \in \overline{D}$ *and* $b \in D$ *then* $a \leq b$ *iff either* $a^+ \leq b$ *or* $b \leq a^+$ *(in particular,* $a < a^+$ *for all* $a \in \overline{D}$*).*

NOTE

It is not difficult to show that proper S-lattices are structures which are constructed as follows: Given an upper semilattice $\langle D, \leq_D\rangle$ which has a top element and which satisfies condition (2)(iii) from the last definition, we make a mirror-image $\langle \overline{D}, \leq_{\overline{D}}\rangle$ of it and then "glue" $\overline{D}$ and D together so that $a < a^+$ for each $a \in \overline{D}$ (where a^+ is the element from D of which a is the mirror image). For example, if D is the inverse tree of Figure 2, then $\overline{D}$ is the tree of Figure 3, and by "gluing" them we get the proper S-lattice of Figure 4:

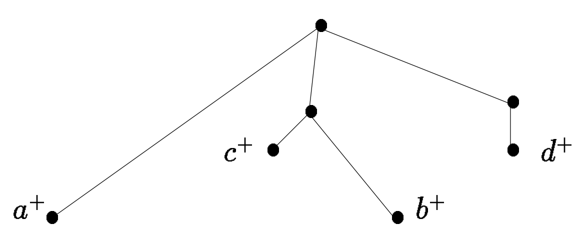

FIGURE 2

[6] In case D is finite, condition (iii) together with the fact that $\top \in D$ implies condition (i), and it means that $\langle D, \geq\rangle$ is a tree. In the general case conditions (i)-(iii) can be taken as a generalization of the notion of a tree which is slightly weaker than the usual one.

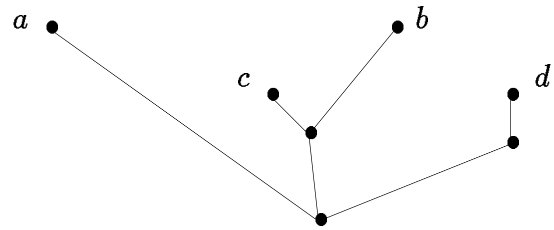

FIGURE 3

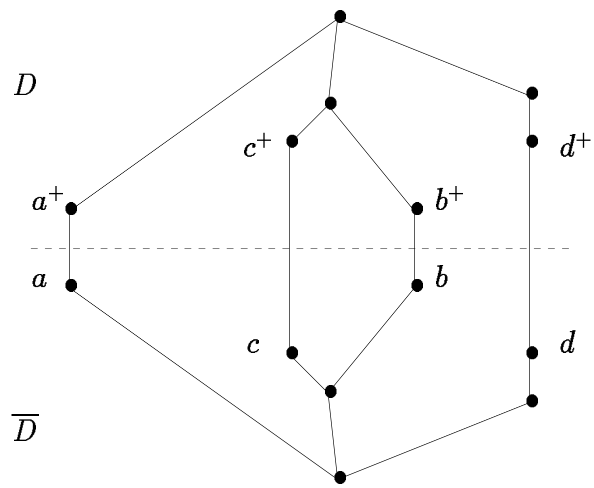

FIGURE 4

It is also not difficult to see that by using (3)(ii) (from Definition 5.5) as a definition we get the minimal poset which extends both $\langle D, \leq\rangle$, $\langle \overline{D} \leq_{\overline{D}}\rangle$ and such that $a < a^+$ for all $a \in \overline{D}$.

Definition 5.6 *Let $\overline{S}$ be a proper S-lattice. Define*

$$|a| = \begin{cases} a & a \in D \\ a^+ & a \in \overline{D} \end{cases}$$

Theorem 5.7 *A proper S-lattice is indeed a lattice.*

PROOF. It is straightforward to see that for all $a, b \in S$

$$a \vee b = \begin{cases} b & a \leq b \\ a & b \leq a \\ \sup(|a|, |b|) & \text{otherwise} \end{cases} \qquad a \wedge b = \begin{cases} a & a \leq b \\ b & b \leq a \\ \sup(|a|, |b|)^{\perp} & \text{otherwise} \end{cases}$$

where if $a \in D$ then $a^{\perp}$ denotes the unique $x \in \overline{D}$ s.t. $a = x^+$ (the assumption that $\langle D, \leq\rangle$ is tree-like is crucial here!). ∎

Definition 5.8 *Let $\overline{S}$ be a proper S-lattice, $a, b \in S$. define:*

$$a \to b = \begin{cases} \sup(|a|, |b|) & a \leq b \\ \\ \sup(|a|, |b|)^{\perp} & a \not\leq b \end{cases}$$

$$a \otimes b = \begin{cases} \min(a, b) & |a| = |b| \\ a & |b| < |a| \\ b & |a| < |b| \\ \sup(|a|, |b|) & \text{otherwise} . \end{cases}$$

Theorem 5.9 *If $\overline{S}$ is a proper S-lattice then $\overline{S}$ together with $\to$ and $\otimes$ is an $RM0_{im}$-structure.*

The proof of this theorem is implicit in [4].

NOTE

As can be expected in structures which correspond to a multiple-conclusion logic, in proper S-lattices there is a very natural way to define a De-Morgan negation and an internal (multiplicative) disjunction:

$$\sim a = \begin{cases} a^{+} & a \in \overline{D} \\ \\ a^{\perp} & a \in D \end{cases}$$

$$a + b = \sim (\sim a \otimes \sim b).$$

Before showing a soundness and completeness theorem for RMI_{im} relative to proper S-lattices, let us generalize these structures a bit:

Definition 5.10 *A symmetrical lattice (S-lattice, in short) is defined exactly as a proper S-lattice, except that the demand that $\lambda a \in \overline{D}.a^{+}$ is surjective is relaxed to:*
($*$) If $a \in D$ is not in the image (under $\lambda x.x^{+}$) of any $x \in \overline{D}$, then a is a minimal element of D.

¿From an intuitive point of view, the definition of an S-lattice means that when we "glue" together D and its mirror image we may identify minimal elements of D with the corresponding elements of its image. If we return, e.g., to Figures 2 and 3, then by identifying a with a^{+}, b with b^{+} we get:

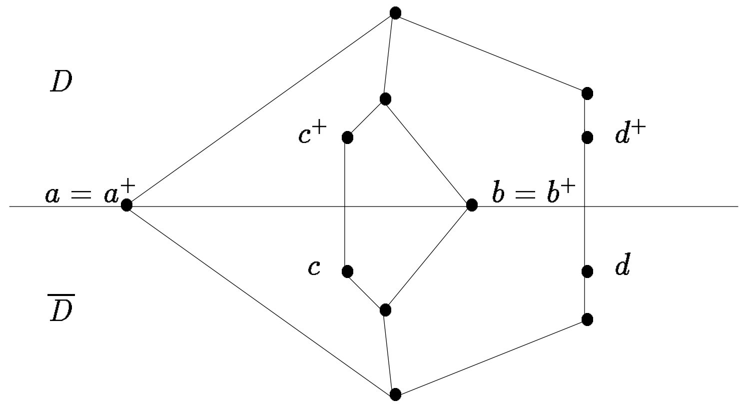

FIGURE 5

Note that it is not necessary to identity all minimal elements of D with the corresponding elements of $\overline{D}$. Thus in Figure 5 c^+ and d^+ are elements of D which are not identified with c and d (respectively).

The definitions of $|a|, a \vee b, a \wedge b, a \rightarrow b, a \otimes b$ and $a + b$ can now be extended to S-lattices without any change. The same applies to the definition of $\sim a$, provided we define $a^\perp$ to be a in case no $b \in \overline{D}$ exists s.t. $a = b^+$ (this can happen, recall, only if a is a minimal element of D). Theorems 5.7 and 5.9 remain then valid for S-lattices in general.

NOTE

It seems appropriate to call an element a such that $\sim a \in D$ *antidesignated.* In proper S-lattices an element is antidesignated iff it is not designated, but in general a minimal element of D can sometimes be antidesignated as well. Note that the symmetry in S-lattices is a symmetry between D and the set of antidesignated elements.

We now turn to the soundness and completeness results.

Definition 5.11 *A (proper) RMI_{im}-model of a sentence φ is a pair $\langle \overline{S}, v \rangle$ where $\overline{S}$ is a (proper) S-lattice and v is an operations-respecting valuation in it such that $v(\varphi) \in D$. A (proper) RMI_{im}-model of a theory $\mathcal{T}$ is an RMI_{im}-model of each element of $\mathcal{T}$.*

Theorem 5.12 Strong soundness and completeness of *RMI_{im}: $\mathcal{T} \vdash_{RMI_{im}} \varphi$ iff every RMI_{im}-model of $\mathcal{T}$ is an RMI_{im}-model of φ.*

PROOF. An analogous theorem was proved in [4] for RMI_m, the *full* multiplicative fragment of RMI (including negation and $+$).[7] The soundness part of the present theorem is an immediate corollary. By the cut-elimination theorem for RMI_m, so is also the completeness part, since it implies that if a sequent of the form $\Gamma \Rightarrow A$ is provable in RMI_m, and Γ, A are in $\mathcal{L}_{im}$ then $\Gamma \Rightarrow A$ is provable in RMI_{im}. ∎

[7] The propositional constant $\perp$ has not been considered there, but it is obvious that its addition causes no problem. Note also that RMI_m was called in [4] $RMI_{\stackrel{\sim}{\rightarrow}}$, following the names in [1].

NOTE
It seems difficult to prove the soundness part directly, since unless Δ is a singleton, no translation of $\Gamma \Rightarrow \Delta$ seems to be available in $\mathcal{L}_{im}$.

Theorem 5.13 Strong soundness and completeness of RMI_{im}, version II: $\mathcal{T} \vdash_{RMI_{im}} \varphi$ *iff every proper RMI_{im}-model of $\mathcal{T}$ is an RMI_{im}-model of φ.*

PROOF. Soundness follows immediately from the previous theorem. For completeness, assume $\mathcal{T} \not\vdash_{RMI_{im}} \varphi$. We construct a proper RMI_{im}-model of $\mathcal{T}$ which is not a model of φ. By the previous theorem there is an RMI_{im}-model $(\overline{S}, v)$ of $\mathcal{T}$ which is not a model of φ. Suppose $\overline{S} = \langle S, \leq, \perp, \top, D\rangle$. Let $\overline{S}^* = \langle S^*, \leq^*, \perp, \top, D\rangle$ be the unique proper S-lattice which is based on D. It is easy to see that the difference between S^* and S is that S^* is obtained from S by splitting each element a of D which is not in the image of $\lambda_{x.x^+}$ into two elements, a and a^+, so that a is now a new maximal element of $\overline{D}$, while a^+ replaces a in D (compare how the proper S-lattice of Figure 4 can be obtained from the improper S-lattice of Figure 5)[8]. Define now an assignment v^* on S^* so that $v^*(p) = v(p)$, unless $v(p)$ has been split, in which case $v^*(p) = (v(p))^+$. It is easy to see that for every A

$$v^*(A) = \begin{cases} (v(A))^+ & v(A) \text{ has been split} \\ v(A) & \text{otherwise} . \end{cases}$$

It follows that if $v(A)$ is not designated in $\overline{S}$ then $v^*(A)$ is not designated in $\overline{S^*}$ either (since only designated elements might have been split), while if $v(A)$ is designated in $\overline{S}$ then $v^*(A)$ is designated in $\overline{S^*}$ (since a^+ is designated). In particular $\langle \overline{S^*}, v^*\rangle$ is a proper model of $\mathcal{T}$ which is not a model of φ. ∎

NOTE
The last theorem is not valid when $\sim$ is present, since every proper model of $\{\sim A, A\}$ is trivially a proper model of B, but $\sim A, A \not\vdash_{RMI_{im}} B$. The reason is that while $a \to a = a \otimes a = a$ in case $a \in D$, this is never true for $\sim a$ in proper models (our theorem and proof remain valid, therefore, when we add & and $\vee$, since again $a \vee a = a\&a = a$ for all a. It remains valid, in fact, even if we add $+!$).

The above two theorems are *strong* completeness theorems for RMI_{im}. This means that they characterize the *consequence relation* which is associated with this logic. If we are interested only in *weak* completeness (i.e. characterizing the set of provable formulas) then just one, extremely simple S-lattice will do. This is the (improper!) denumerable S-lattice A_ω, in which $S = \{\top, \perp, I_1, I_2, I_3, \ldots\}$, $a \leq b$ iff either $a = \perp$ or $b = \top$ or $a = b$, $\overline{D} = \{\perp\}$, and $\perp^+ = \top$.

Theorem 5.14 Weak completeness of RMI_{im}: $\vdash_{RMI_{im}} \varphi$ *iff φ is valid in A_ω.*

PROOF. The corresponding result for the system RMI_m has been proved, using two different methods, in [2] and [4]. Its adaption to the *im*-language is done exactly as in the proof of 5.12. ∎

[8] Exact definitions of this "splitting" can be found in [4].

NOTE

Theorem 5.14 provides a decision procedure for RMI_{im} and a powerful tool for proving properties of it (for example the fact that the rule $\frac{A \otimes B}{A}$ is admissible in it is a trivial corollary of 5.14).

Before turning to $RM0^h_{im}$ a few words on RM_{im} are in order. It is not difficult to show that the corresponding semantics is that of linear S-lattices, i.e., S-lattices in which $\leq$ is a total order. Linear S-lattices are exactly what is usually called Sugihara Matrices ([1],[13]). Again we have the option to limit ourselves to proper linear S-lattices (which are Sugihara matrices without 0). For weak completeness just the 3-valued S-lattice suffices (This 3-valued structure is usually known as Sobociński 3-valued logic, and was first introduced in [17]. In [2] and [8] it is called A_1). As in the case of RMI_{im}, these results all follow from corresponding known results concerning RM_m of Sobociński ([17]), Meyer ([1]), Dunn ([12]), and the author of this paper.

We turn now to the semantics of $RM0^h_{im}$. Like RMI_{im}, the corresponding algebraic structures are based on D. The construction is, however, much simpler in this case.

Definition 5.15 *An im-F-structure (or just F-structure, in short[9]) is an $RM0_{im}$-structure $\overline{S}$ in which $S = D \cup \{\bot\}$.*

F-structures can rather easily be constructed and characterized. The following theorem means, in fact, that the complicated notion of an F-structure is equivalent to the rather simple one of a bounded upper semilattice.

Theorem 5.16 *In every F-structure $\overline{S}$ the carrier S is a bounded upper semilattice with respect to $\leq$. Conversely, let $\langle S, \leq, \bot, \top\rangle$ be a bounded upper semilattice s.t. $\bot \neq \top$. Then there is a unique way to turn it into an F-structure.*

PROOF. That in every F-structure $\overline{S}$, $\langle S, \leq, \bot, \top\rangle$ is a bounded upper semilattice follows from Lemma 5.2(4). For the converse, let $\overline{S} = \langle S, \leq, \bot, \top\rangle$ be a bounded upper semilattice such that $\bot \neq \top$. The definition of an F-structure and Lemma 5.2 together imply that the only possible way to turn it into an F-structure is to define:

$$D = S \perp \{\bot\}$$

$$a \otimes b = \begin{cases} \bot & a = \bot \vee b = \bot \\ \\ \sup(a,b) & \text{otherwise} \end{cases} \qquad a \to b = \begin{cases} \bot & a \not\leq b \\ \top & a = \bot \\ b & \text{otherwise} \end{cases}$$

It remains to show that with this definition $\langle S, \leq, \bot, \top, D, \otimes, \to\rangle$ is indeed an F-structure. This is straightforward. ∎

Definition 5.17 *An F-model of a sentence A is a pair $\langle \overline{S}, v\rangle$ where $\overline{S}$ is an F-structure and v is a valuation in S s.t. $v(A) \neq \bot$ (i.e., $v(A) \in D$). An F-model of a sequent s is an F-model of ϕ_s. An F-model of a hypersequent is an F-model of at least one of its components.*

Theorem 5.18 Strong soundness and completeness: *$\mathcal{T} \vdash_{RM0^h_{im}} \varphi$ iff every F-model of $\mathcal{T}$ is also an F-model of φ.*

[9] Note that the name "F-structures" has been used in [9] for a more general type of structures, in which $\otimes$ is not necessarily idempotent.

PROOF. For soundness it suffices to prove that if $\langle \overline{S}, v\rangle$ is an F-model of $\mathcal{T}$ and the hypersequent G follows in $RM0^h_{im}$ from $\{\Rightarrow A | A \in \mathcal{T}\}$ then $\langle \overline{S}, v\rangle$ is an F-model of G. This is obvious, except perhaps the validity of CS. So suppose $G|\Gamma_1, \Gamma_2 \Rightarrow A|H$ is true in $\langle \overline{S}, v\rangle$. If one of the components of G or of H is true in $\langle \overline{S}, v\rangle$ we are done. Otherwise $\Gamma_1, \Gamma_2 \Rightarrow A$ is true. If all the sentences in Γ_2 are true this entails that $\Gamma_1 \Rightarrow A$ is true. If not then $v(C) = \perp$ for some C in Γ_2, and so $v(\Gamma_2, \Delta \Rightarrow B) = \top$ for all Δ, B, by Lemma 5.2(7).

For the converse, suppose $\mathcal{T} \nvdash_{RM0^h_{im}} \varphi$. We construct a model $\langle \overline{S}, v\rangle$ of $\mathcal{T}$ in which φ is not true. For this extend $\mathcal{T}$ to a maximal theory $\mathcal{T}^*$ such that $\mathcal{T}^* \nvdash_{RM0^h_{im}} \varphi$. Obviously, $A \notin \mathcal{T}^*$ iff there exist $\Delta_1, \ldots, \Delta_k \subseteq \mathcal{T}^*$ such that

$$\vdash_{RM0^h_{im}} A, \Delta_1 \Rightarrow \varphi | \cdots | A, \Delta_j \Rightarrow \varphi | \Delta_{j+1} \Rightarrow \varphi | \cdots | \Delta_k \Rightarrow \varphi \ .$$

This easily entails, using cuts, that

(i) If $\mathcal{T}^* \vdash_{RM0^h_{im}} \Delta_1 \Rightarrow B_1 | \cdots | \Delta_k \Rightarrow B_k$ and $\Delta_1, \ldots, \Delta_k \subseteq \mathcal{T}^*$ then $B_i \in \mathcal{T}^*$ for some $1 \leq i \leq k$.

Since $RM0_{im}$ is contained in $RM0^h_{im}$, (i) entails that

(ii) If $\mathcal{T}^* \vdash_{RM0_{im}} C$ then $C \in \mathcal{T}^*$.

Since $\vdash_{RM0^h_{im}} \Rightarrow A | \Rightarrow A \to B$ (because from $A \Rightarrow A$ one can infer this hypersequent by CS and $(\Rightarrow\to)$), another corollary of (i) is:

(iii) For every A, B, either $A \in \mathcal{T}^*$ or $A \to B \in \mathcal{T}^*$.

Define now the Lindenbaum algebra $\overline{S}$ of $\mathcal{T}^*$ and the canonical valuation v in it in the usual way. Using (ii) it is easy to see that $\overline{S}$ is an $RM0_{im}$-structure and v is a model of $\mathcal{T}$ in it which is not a model of φ. It remains to show that $\overline{S}$ is actually an F-structure. This, however, easily follows from (iii). ∎

EXAMPLE

In the proof of 4.1 we used the fact that $((B \to B) \to (A \to A)) \to (((A \to B) \to A) \to A)$ is not provable in $RM0^h_{im}$. This can be shown now using the counter-model $\langle \overline{S}, v\rangle$, where $S = \{\perp, 1, 2, \top\}$, $\perp < 1 < 2 < \top$ and $v(A) = 2, v(B) = 1$.

We turn now to the semantics of the last two systems, RMI^h_{im} and RM^h_{im}. We start with RMI^h_{im}. Since it is an extension of both $RM0^h_{im}$ and RMI_{im}, it is sound w.r.t. the semantics of both. Now the simplest *infinite* structure which is both an S-lattice and an F-structure is A_ω (see theorem 5.14. In fact A_ω is the simplest infinite S-lattice as well as the simplest infinite F-structure!). It turns out that it is indeed a *strongly* characteristic matrix for RMI^h_{im}.

Theorem 5.19 Soundness and Strong Completeness for RMI^h_{im}: $\mathcal{T} \vdash_{RMI^h_{im}} \varphi$ *iff every model of $\mathcal{T}$ in A_ω is also a model of φ.*

PROOF. The corresponding result for the full multiplicative language was proved in theorem 8.10 of [8]. Again its adaption to the *im*-language is done exactly as in the proof of 5.12. (it is not too difficult also to prove it directly along the lines of the proof of 5.18). ∎

EXAMPLE

In the proof of 4.1 we used the fact that $((A \to (B \to B)) \to A) \to A$ is not provable in RMI^h_{im}. This is immediate from the last theorem, since by defining $v(A) = I_1$, $v(B) = I_2$ we get a counter-model of this formula in A_ω.

Theorems 5.19 should be compared with theorem 5.14. From the two theorems follow that RMI^h_{im} is a conservative extension of RMI_{im} (with respect to provability of sequents), and that the difference between the sequential calculus and the associated hypersequential one corresponds (in this case, at least) to the difference between strong completeness and week completeness.

Finally, RM^h_{im} is an extension of both $RM0^h_{im}$ and RM_{im}, and so it is sound w.r.t. the semantics of both. Now the *only* structures which are both linear S-lattices and also F-structures are the two-valued Boolean Algebra and A_1, the three-valued substructure of A_ω (see discussion after the proof of 5.14). This observation naturally leads to the following theorem, which again follows from a corresponding result in [8] for the full multiplicative language:

Theorem 5.20 Soundness and Strong Completeness for RM^h_{im}: $\mathcal{T} \vdash_{RM^h_{im}} \varphi$ *iff every model of $\mathcal{T}$ in A_1 is also a model of φ.*

Again the strong completeness of RM^h_{im} relative to A_1 should be compared with the weak completeness of RM_{im} relative to this matrix.

The algebraic semantics can help also to shed a new light on the difference that we have seen in the previous section between $\vee$ on the one hand and $\wedge$ and 1 on the other.

Let us start with $\vee$. From an algebraic point of view its rules clearly represent the operation of a join. To have an appropriate semantics for it we need therefore to use upper semilattices. But with the exception of $RM0_{im}$, the structures which correspond to our various systems *are* indeed upper semilattices. As for $RM0_{im}$, we noted already above that we could have demanded the corresponding structures to be lattices without losing completeness. Hence $\vee$ has an obvious interpretation in the structures we consider, relative to which its rules are sound. This fact alone immediately entails that its addition is conservative (w.r.t. all of our systems). It is not difficult to see also that the hypersequential systems with $\vee$ are in fact complete relative to this semantics (the fact that the two consequence relations are identical there is very important for the proof). As for the sequential ones— there is certainly no completeness if we use $\vdash^i$ (since $A \vee B \Rightarrow C$ does *not* follow according to it from $A \Rightarrow C$ and $B \Rightarrow C$), and I dont know whether the systems are complete with $\vdash^e$.

Things are more complicated if we add conjunction. Its rules partially correspond to the operation of a meet. Hence we should demand our structures to be lower semilattices in order to give it an appropriate interpretation. This is *not* enough, though. In order for instances of $(\Rightarrow \wedge)$ with no side formulas to be sound, the set D of designated elements should be closed under $\wedge$. The only type of S-lattices in which this is the case are the linear ones (i.e.: Sugihara matrices). This explains why RMI_{im} and RMI^h_{im} collapse to RM_{im} and RM^h_{im} (respectively) if we add $\wedge$ to them. The structures which corresponds to the other 4 systems either satisfy already the two demands or can be embedded in structures which do so. Hence the addition of $\wedge$ to them is conservative. As for completeness— it is not difficult to show that we have it in the hypersequential cases if we add $\wedge$ and its rules (or both $\wedge$ and $\vee$). On the other hand the distributive laws are valid in Sugihara matrices but cannot be proved in RM_i. Hence we dont even have weak completeness in the case of $RMI_i = RM_i$ (To get it we need another hypersequential system. See [3]). It is not clear whether we have weak completeness in the case of $RM0_i$.

Finally the rules for 1 mean that the set D should have a least element. Obviously, the situation w.r.t. this demand is similar to that concerning $\wedge$.

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Received March 10, 1998