First-Order Logical Duality

Steve Awodey   Henrik Forssell*

Department of Philosophy   Department of Philosophy

Carnegie Mellon University   Carnegie Mellon University

Pittsburgh, PA 15217   Pittsburgh, PA 15217

USA   USA

awodey@cmu.edu   henrikforssell@gmail.com

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Abstract

From a logical point of view, Stone duality for Boolean algebras relates theories in classical propositional logic and their collections of models. The theories can be seen as presentations of Boolean algebras, and the collections of models can be topologized in such a way that the theory can be recovered from its space of models. The situation can be cast as a formal duality relating two categories of syntax and semantics, mediated by homming into a common dualizing object, in this case 2.

In the present work, we generalize the entire arrangement from propositional to first-order logic. Boolean algebras are replaced by Boolean categories presented by theories in first-order logic, and spaces of models are replaced by topological groupoids of models and their isomorphisms. A duality between the resulting categories of syntax and semantics, expressed first in the form of a contravariant adjunction, is established by homming into a common dualizing object, now **Sets**, regarded once as a boolean category, and once as a groupoid equipped with an intrinsic topology.

*Corresponding author. Present address: Dept. of Informatics, University of Oslo, PO Box 1080 Blindern, N-0316 Oslo, Norway. Phone: (+47) 22 85 27 81. Fax: (+47) 22 85 24 01

The overall framework of our investigation is provided by topos theory. Direct proofs of the main results are given, but the specialist will recognize toposophical ideas in the background. Indeed, the duality between syntax and semantics is really a manifestation of that between algebra and geometry in the two directions of the geometric morphisms that lurk behind our formal theory. Along the way, we construct the classifying topos of a decidable coherent theory out of its groupoid of models via a simplified covering theorem for coherent toposes.

Keywords: First-order logic, duality; categorical logic; topos theory; topological semantics.

AMS classification codes: 03G30; 18B25; 18C10; 18C50; 18F20.

Contents

Introduction

We present an extension of Stone duality for Boolean algebras from classical propositional logic to classical first-order logic. In broad strokes, the leading idea is to take the traditional logical distinction between syntax and

semantics and analyze it in terms of the classical mathematical distinction between algebra and geometry, with syntax corresponding to algebra and semantics to geometry. Insights from category theory allow us to recognize a certain duality between the notions of algebra and geometry. We see a first glimpse of this in Stone's duality theorem for Boolean algebras, the categorical formulation of which states that a category of 'algebraic' objects (Boolean algebras) is the categorical dual of a category of 'geometrical' objects (Stone spaces). "Categorically dual" means that the one category is opposite to the other, in that it can be obtained (up to equivalence) from the other by formally reversing the morphisms. In a more far reaching manner, this form of algebra-geometry duality is exhibited in modern algebraic geometry as reformulated in the language of schemes in the Grothendieck school, e.g. in the duality between the categories of commutative rings and the category of affine schemes.

On the other hand, we are informed by the category theoretic analysis of logic that it is closely connected with algebra, in the sense that logical theories can be regarded as categories and suitable categories can be presented as logical theories. For instance, Boolean algebras can be seen as classical propositional theories, categories with finite products can be seen as equational theories, Boolean coherent categories as theories in classical first-order logic, and elementary toposes – e.g. the topos of sheaves on a space – as theories in higher-order intuitionistic logic. Thus the study of these algebraic objects has a logical interpretation and, vice versa, reasoning in or about logical theories has application in their corresponding algebraic objects. With the connection between algebra and logic in hand, instances of the algebra-geometry duality can be seen to manifest a syntax-semantics duality between an algebra of syntax and a geometry of semantics. This notion of syntax as 'dual to semantics' is, expectedly, one which ignores presentation and other features which, so to speak, models can not distinguish. In the propositional case, one passes from a propositional theory to a Boolean algebra by constructing the Lindenbaum-Tarski algebra of the theory, a construction which identifies provably equivalent formulas (and orders them by provable implication). Thus any two complete theories, for instance, are 'algebraically equivalent' in the sense of having isomorphic Lindenbaum-Tarski algebras. The situation is precisely analogous to a presentation of an algebra by generators and relations: a logical theory corresponds to such a presentation, and two theories are equivalent if they present 'the same' – i.e. isomorphic – algebras. A similar construction is used to obtain, for a classical first-order

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theory, its 'corresponding' Boolean coherent category, resulting in a similar notion of algebraic or categorical equivalence.

Given this connection between formal theories and categories, Stone duality manifests a syntax-semantics duality for propositional logic as follows. While a Boolean algebra can be regarded as a propositional theory modulo 'algebraic' equivalence, on the other hand a Stone space can be seen as a space of corresponding two-valued models of such a theory. A model of a propositional theory is of course just a valuation of the propositional letters, or equivalently, a Boolean homomorphic valuation of all formulas. Thus we obtain the set of models of the theory corresponding to a Boolean algebra by taking morphisms in the category of Boolean algebras from the given algebra into the two-element Boolean algebra, 2,

$$\mathrm{Mod}_{\mathcal{B}} \cong \mathrm{Hom}_{\mathbf{BA}}(\mathcal{B}, 2).$$ (1)

And with a suitable topology in place—given in terms of the elements of the Boolean algebra $\mathcal{B}$—we can retrieve $\mathcal{B}$ from the space of models $\mathrm{Mod}_{\mathcal{B}}$ by taking morphisms in the category of Stone spaces from it into the two-element Stone space, 2,

$$\mathcal{B} \cong \mathrm{Hom}_{\mathbf{Stone}}(\mathrm{Mod}_{\mathcal{B}}, 2)$$

Here, the two-element set, 2, is in a sense living a 'dual' life, and 'homming into 2' forms a contravariant adjunction between the 'syntactical' category of Boolean algebras and the category of topological spaces, which, moreover, becomes an equivalence once we restrict to the 'semantical' subcategory of Stone spaces.

$$\mathbf{BA} \underset{\mathrm{Hom}_{\mathbf{BA}}(-,2)}{\overset{\mathrm{Hom}_{\mathbf{Stone}}(-,2)}{\rightleftarrows}} \simeq \mathbf{Stone}$$

Our construction for first-order logic generalizes this set-up by, on the 'syntax' side, representing first-order theories by Boolean coherent categories. On the semantical side we have, for each theory, a space of models, augmented with a space consisting of the isomorphisms between those models, such that these spaces form a *topological groupoid*, that is to say, such that the composition, domain and codomain, inverse arrow and identity arrow maps are all continuous. Our 'semantic' side is, accordingly, a category consisting of topological groupoids and continuous homomorphisms between them. Where

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in Stone Duality one considers the lattice of open sets of a space in order to recover a Boolean algebra, we consider the topos (or 'generalized space') of so-called *equivariant sheaves* on a topological groupoid in order to recover a Boolean coherent category. In particular, we show that the topos of equivariant sheaves on the topological groupoid of models and isomorphisms of a theory is the so-called classifying topos of (the Morleyization of) the theory, from which it is known that the theory can be recovered up to a notion of equivalence. (Here we build upon earlier results in [1] to the effect that any such topos can be represented by a topological groupoid constructed from its points. Our construction differs from the one given there in choosing a simpler cover which is better suited for our purpose).

Our semantic representation of this topos can also be understood from the perspective of definable sets. Suppose we have a theory, $\mathbb{T}$, in first order logic or some fragment of it, and that $\phi(\vec{x})$ is some formula in the language of the theory. Then $\phi(\vec{x})$ induces a definable set functor,

$$[\![\phi(\vec{x})]\!] : \mathrm{Mod}_{\mathbb{T}} \longrightarrow \mathbf{Sets}$$

from the groupoid of $\mathbb{T}$-models to the category of sets, which sends a model $\mathbf{M}$ to the extension, $[\![\phi(\vec{x})]\!]^{\mathbf{M}}$, of $\phi(\vec{x})$ in $\mathbf{M}$. The question is, then, whether these definable set functors can somehow be characterized among all functors of the form $\mathrm{Mod}_{\mathbb{T}} \to \mathbf{Sets}$, so that the theory can be recovered from its models in terms of them. Notice, incidentally, that in case of a positive answer, the category of sets takes on the role of a dualizing object, in analogy with $\mathbf{2}$ for Stone duality. For the models of a theory can be seen as suitable functors from the algebraic representation of the theory, $\mathcal{C}_{\mathbb{T}}$, into $\mathbf{Sets}$, so that both obtaining the models from the theory and recovering the theory from the models is done by 'homming' into $\mathbf{Sets}$,

$$\mathrm{Mod}_{\mathbb{T}} \simeq \mathrm{Hom}(\mathcal{C}_{\mathbb{T}}, \mathbf{Sets})$$
$$\mathcal{C}_{\mathbb{T}} \simeq \mathrm{Hom}(\mathrm{Mod}_{\mathbb{T}}, \mathbf{Sets})$$

Here the hom-sets must be suitably restricted from all functors to just those preserving the relevant structure, the determination of which is part of the task at hand.

Now, positive, and elegant, answers to the question of the characterization of definable set functors exist, to begin with, for certain fragments of first-order logic. For algebraic theories—axiomatized only by equations in

languages with only function symbols (and equality)—the categories of models (algebras) have all limits and colimits, and *Lawvere duality* tells us that an algebraic theory $\mathbb{T}$ can be recovered (up to splitting of idempotents) from its category of models in the form of those functors $\mathrm{Mod}_{\mathbb{T}} \longrightarrow \mathbf{Sets}$ which preserve limits, filtered colimits, and regular epimorphisms (see [2],[3]). Expanding from the algebraic case, recall, e.g. from [4, D1.1.], that the *Horn* formulas over a first-order signature are those formulas which are constructed using only connectives $\top$ and $\wedge$. Allowing also existential quantification brings us to *regular* formulas. A Horn (regular) theory is one which can be axiomatized using sequents involving only Horn (regular) formulas. In between, a Cartesian theory is a regular theory which can be axiomatized using only formulas that are Cartesian relative to the theory, in the sense, briefly, that existential quantification does not occur except under a certain condition. Now, the category $\mathrm{Mod}_{\mathbb{T}}$ of models and homomorphisms of a Cartesian theory $\mathbb{T}$ has limits and filtered colimits (but not, in general, regular epis), and *Gabriel-Ulmer duality* (see e.g. [5]) informs us, among other things, that the definable set functors for Cartesian formulas (relative to $\mathbb{T}$) can be characterized as the limit and filtered colimit preserving functors $\mathrm{Mod}_{\mathbb{T}} \to \mathbf{Sets}$ (and that the theory can be recovered in terms of them). If we allow for unrestricted existential quantification and pass to regular logic, then categories of models need no longer have arbitrary limits. But they still have products and filtered colimits, and, as shown by M. Makkai [6], the definable set functors for regular formulas can now be characterized as those functors $\mathrm{Mod}_{\mathbb{T}} \to \mathbf{Sets}$ that preserve precisely that.

Adding the connectives $\bot$ and $\vee$ to regular logic gives us the fragment known as *coherent logic* (see [4, D1.1.]), in which a far greater range of theories can be formulated. The theory of fields, for instance, cannot be expressed as a regular theory (since the category of fields does not have arbitrary products), but it can be expressed as a coherent theory (see [4, D1.1.7.(h)]). (In fact, it is a *decidable* coherent theory, where "decidable" means, here, that there is an *inequality predicate*, in the sense of a coherent formula which is provably the complement of equality.) Moreover, any classical first-order theory can be *Morleyized* to yield a coherent theory with the same category of models, see [4, D1.5.13] (we take the morphisms between models of a classical first-order theory to be the elementary embeddings). Thus the categories of models of coherent theories can not, in general, be expected to have more structure than those for classical first-order theories. What they do have are ultra-products. Although ultra-products are not an intrin-

sic feature of categories of models (for coherent theories), in the sense that they are not a categorical invariant, Makkai [7] shows that model categories and the category of sets can be equipped with a notion of ultra-product structure—turning them into so-called *ultra-categories*—which allows for the characterization of definable set functors as those functors that preserve this additional structure. Moreover, this approach can be modified in the case of classical first-order theories so that only the ultra-*groupoids* of models and isomorphisms, equipped with ultra-product structure, need be considered, see [8].

Our approach, similarly, relies on equipping the models of a theory with external structure, but in our case the structure is topological. We, too, restrict consideration to groupoids of models and isomorphisms, instead of categories of models and homomorphisms or elementary embeddings. We carry our construction out for decidable coherent theories, corresponding to (small) decidable coherent categories ("decidable" meaning, in the categorical setting, that diagonals are complemented). As we remarked, the theory of fields is a notable example of such a theory, and the decidable coherent theories do include all classical first-order theories in the sense that the Morleyization of a classical theory is decidable coherent. Accordingly, our construction restricts to the classical first-order case, corresponding to Boolean coherent theories.

The first part of the construction (Section 1) concerns the characterization of definable set functors for a theory and the recovery of the theory from its groupoid of models in terms of them. The idea is that definable sets can be characterized as being, in a sense, *compact*; not by regarding each individual set as compact, but by regarding the definable set functor as being a compact object in a suitable category. Pretend, for a moment, that the models of a theory $\mathbb{T}$ form a set and not a proper class, and suppose, for simplicity, that the models are all disjoint. A definable set functor from the groupoid of $\mathbb{T}$-models and isomorphisms,

$$[\![\phi(\vec{x})]\!]^{(-)} : \mathrm{Mod}_{\mathbb{T}} \longrightarrow \mathbf{Sets}$$

can, equivalently, be considered as a set (indexed) over the set $(\mathrm{Mod}_{\mathbb{T}})_0$ of models,

$$\coprod_{\mathbf{M}\vDash\mathbb{T}} [\![\phi(\vec{x})]\!]^{\mathbf{M}} \xrightarrow{\ p\ } (\mathrm{Mod}_{\mathbb{T}})_0 \tag{2}$$

with $p^{-1}(\mathbf{M}) = [\![\phi(\vec{x})]\!]^{\mathbf{M}}$, together with an action on this set by the set

$(\mathrm{Mod}_{\mathbb{T}})_1$ of isomorphisms,

$$(\mathrm{Mod}_{\mathbb{T}})_1 \times_{(\mathrm{Mod}_{\mathbb{T}})_0} \coprod_{\mathbf{M} \vDash \mathbb{T}} [\![\phi(\vec{x})]\!]^{\mathbf{M}} \xrightarrow{\ \alpha\ } \coprod_{\mathbf{M} \vDash \mathbb{T}} [\![\phi(\vec{x})]\!]^{\mathbf{M}} \tag{3}$$

such that for any $\mathbb{T}$-model isomorphism, $\mathbf{f} : \mathbf{M} \to \mathbf{N}$, and element, $\vec{m} \in [\![\phi(\vec{x})]\!]^{\mathbf{M}}$, we have $\alpha(\mathbf{f}, \vec{m}) = \mathbf{f}(\vec{m}) \in [\![\phi(\vec{x})]\!]^{\mathbf{N}}$. Now, if the set of $\mathbb{T}$-models and the set of isomorphisms are topological spaces forming a topological groupoid, then we can ask for the collection

$$\coprod_{\mathbf{M} \vDash \mathbb{T}} [\![\phi(\vec{x})]\!]^{\mathbf{M}}$$

of elements of the various definable sets to be a space, in such a way that the projection function p in (2) is a local homeomorphism, and such that the action α in (3) is continuous. This makes definable set functors into equivariant sheaves on the groupoid, and we show that in the topos of all such sheaves they can be characterized as the compact decidable objects (up to a suitable notion of equivalence).

The second part (Section 2) concerns the construction, based on the representation result of the first part, of a duality between the category of decidable coherent categories (representing theories in first-order logic) and the category of topological groupoids of models. Specifically, we construct an adjunction between the category of decidable coherent categories and a category of 'coherent' topological groupoids, such that the counit component of the adjunction is an equivalence, up to pretopos completion. As a technical convenience, we introduce a size restriction both on theories and their models (corresponding to the pretence, above, that the collection of models of a theory forms a set). The restriction, given a theory, to a set of models large enough for our purposes can be thought of as akin to the fixing of a 'monster' model for a complete theory, although in our case a much weaker saturation property is asked for, and a modest cardinal bound on the size of the models is sufficient.

In summary, we present a 'syntax-semantics' duality which shows how to recover a coherent decidable or a classical first-order theory from its models. Compared with the duality theory of Makkai [7, 8], we give an alternative notion of external structure with which to equip the models, which in our case is topological instead of based on ultra-products. This permits the use of topos theory in establishing the main results, and in particular results in

a semantic construction of the classifying topos of the theory. Finally, our construction restricts to classical Stone duality in the propositional case.

Many more details of the results contained herein can be found in the second author's doctoral dissertation [9].

1 The Representation Theorem

1.1 Theories and Models

We show how to recover a classical, first-order theory from its groupoid of models and model-isomorphisms, bounded in size and equipped with topological structure. We present this from a logical perspective, that is, from the perspective of the syntax and model theory of first-order theories. One can, of course, go back and forth between this perspective and the categorical perspective of decidable or Boolean coherent categories and set-valued coherent functors. Section 2 briefly outlines the translation between the two, and presents a duality between the 'syntactical' category of theories and a 'semantical' category of model-groupoids. In categorical terms, the purpose of the current section is to show that the topos of coherent sheaves on a decidable coherent category can be represented as the topos of equivariant sheaves on a topological groupoid of 'points', or set-valued coherent functors, and invertible natural transformations. This builds upon earlier results in [1] and [10] to the effect that a coherent topos can be represented by a topological groupoid constructed from its points (our construction differs from the one given in *loc.cit.* in choosing a simpler cover which is better suited for our purpose).

Let Σ be a (first-order, possibly many-sorted) signature. Recall that a formula over Σ is *coherent* if it is constructed using only the connectives $\top$, $\wedge$, $\exists$, $\bot$, and $\vee$. We consider formulas in suitable contexts, $[\vec{x} \mid \phi]$, where the context $\vec{x}$ is a list of distinct variables containing (at least) the free variables of ϕ. A sequent, $\phi \vdash_{\vec{x}} \psi$—where $\vec{x}$ is a suitable context for both ϕ and ψ—is coherent if both ϕ and ψ are coherent. Henceforth we shall not be concerned with axiomatizations, and so we consider a (coherent) *theory* to be a deductively closed set of (coherent) sequents.

Let $\mathbb{T}$ be a coherent (alternatively first-order) theory over a signature, Σ. Recall that the *syntactic category*, $\mathcal{C}_{\mathbb{T}}$, of $\mathbb{T}$ has as objects equivalence classes of coherent (alt. first-order) formulas in context, e.g. $[\vec{x} \mid \phi]$, which is

equivalent to a formula in context, $[\vec{y} \mid \psi]$, if the contexts are α-equivalent and $\mathbb{T}$ proves the formulas equivalent[1], i.e. $\mathbb{T}$ proves the following sequents.

$$\phi \vdash_{\vec{x}} \psi[\vec{x}/\vec{y}]$$
$$\psi[\vec{x}/\vec{y}] \vdash_{\vec{x}} \phi$$

An arrow between two objects, say $[\vec{x} \mid \phi]$ and $[\vec{y} \mid \psi]$ (where we may assume that $\vec{x}$ and $\vec{y}$ are distinct), consists of a class of $\mathbb{T}$-provably equivalent formulas in context, say $[\vec{x}, \vec{y} \mid \sigma]$, such that $\mathbb{T}$ proves that σ is a functional relation between ϕ and ψ:

$$\sigma \vdash_{\vec{x},\vec{y}} \phi \wedge \psi$$
$$\phi \vdash_{\vec{x}} \exists \vec{y}. \, \sigma$$
$$\sigma \wedge \sigma(\vec{z}/\vec{y}) \vdash_{\vec{x},\vec{y},\vec{z}} \vec{y} = \vec{z}$$

If $\mathbb{T}$ is a coherent theory, then $\mathcal{C}_{\mathbb{T}}$ is a coherent category. If $\mathbb{T}$, in addition, has an inequality predicate (for each sort), that is, a formula with two free variables (of that sort), $x \neq y$, such that $\mathbb{T}$ proves

$$x \neq y \wedge x = y \vdash_{x,y} \bot$$
$$\top \vdash_{x,y} x \neq y \vee x = y$$

then $\mathcal{C}_{\mathbb{T}}$ is *decidable*, in the sense that for each object, A, the diagonal, $\Delta : A \rightarrowtail A \times A$, is complemented as a subobject. We call a coherent theory which has an inequality predicate (for each sort) a *decidable* coherent theory for that reason (and with apologies for overloading the term). Finally, if $\mathbb{T}$ is a first-order theory, then $\mathcal{C}_{\mathbb{T}}$ is a Boolean coherent category, i.e. a coherent category such that every subobject is complemented.

Conversely, given a coherent category, $\mathcal{C}$, one can construct the coherent *theory, $\mathbb{T}_{\mathcal{C}}$, of* $\mathcal{C}$ by having a sort for each object and a function symbol for each arrow, and taking as axioms all sequents which are true under the canonical interpretation of this language in $\mathcal{C}$ (again, see [4] for details). A coherent decidable category allows for the construction of a coherent decidable theory (including an inequality predicate for each sort), and Boolean coherent $\mathcal{C}$ allows for the construction of a first-order $\mathbb{T}_{\mathcal{C}}$. Thus we can turn theories

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[1]See [4, D1] for further details. Note that we, unlike [4], choose to identify $\mathbb{T}$-provably equivalent formulas. The reason is that they define exactly the same sets, i.e. the same definable set functors.

into categories and categories back into theories. It is in this sense that we say that (decidable) coherent categories represent (decidable) coherent theories, and Boolean coherent categories represent first-order theories. (Since Boolean coherent categories are, of course, coherent, building the Boolean coherent syntactical category of a classical first-order theory and then taking its coherent internal theory will produce a decidable coherent theory with the same models as the original classical one; thus yielding an alternative, but less economical, way of *Morleyizing* a classical theory than the one presented in [4, D1.5.13].) We show how to recover a theory from its models in the sense that we recover its syntactic category, up to pretopos completion. Roughly, the pretopos completion of a theory is the theory equipped with disjoint sums and quotients of equivalence relations, see e.g. [8]. A theory and its pretopos completion have the same models in (the pretopos) **Sets**.

The category of models and homomorphisms of a coherent theory $\mathbb{T}$ is equivalent to the category of coherent functors from $\mathcal{C}_{\mathbb{T}}$ into the category **Sets** of sets and functions and natural transformations between them,

$$\mathrm{Mod}_{\mathbb{T}} \simeq \mathrm{Hom}(\mathcal{C}_{\mathbb{T}}, \mathbf{Sets})$$

and the same holds for models in an arbitrary coherent category, $\mathcal{E}$,

$$\mathrm{Mod}_{\mathbb{T}}(\mathcal{E}) \simeq \mathrm{Hom}(\mathcal{C}_{\mathbb{T}}, \mathcal{E})$$

Indeed, this is the universal property that characterizes $\mathcal{C}_{\mathbb{T}}$. The same is true for classical first-order theories if "homomorphism" is replaced by "elementary embedding" (Note that the elementary embeddings between models of a classical first-order theory coincide with the homomorphisms between models of its Morleyization.) We pass freely between considering models traditionally as structures and algebraically as functors. In passing, we note that decidability for coherent theories can be characterized semantically:

Lemma 1.1.1 *Let $\mathbb{T}$ be a coherent theory over a signature Σ, and $\mathrm{Mod}_{\mathbb{T}}$ the category of $\mathbb{T}$-models and homomorphisms. Then $\mathbb{T}$ is decidable (i.e. has an inequality predicate for each sort) if and only if for every $\mathbb{T}$-model homomorphism, $\mathbf{f} : \mathbf{M} \to \mathbf{N}$ and every sort A of Σ, the component function $f_A : [\![A]\!]^{\mathbf{M}} \to [\![A]\!]^{\mathbf{N}}$ is injective.*

PROOF This follows from a slight rewriting of the proof of [4, D3.5.1].   ⊣

Given a coherent theory $\mathbb{T}$ taking sheaves on $\mathcal{C}_\mathbb{T}$ equipped with the coherent coverage (finite epimorphic covering families) results in a topos $\mathrm{Sh}(\mathcal{C}_\mathbb{T})$ with the universal property that the category of $\mathbb{T}$-models in any topos $\mathcal{E}$ is equivalent to the category of geometric morphisms from $\mathcal{E}$ to $\mathrm{Sh}(\mathcal{C}_\mathbb{T})$ and geometric transformations between them,

$$\mathrm{Mod}_\mathbb{T}(\mathcal{E}) \simeq \mathrm{Hom}(\mathcal{E}, \mathrm{Sh}(\mathcal{C}_\mathbb{T}))$$

The topos $\mathrm{Sh}(\mathcal{C}_\mathbb{T})$ is known as the *classifying topos* of $\mathbb{T}$ (see [4, D3]).

1.2 Stone Representation

Let $\mathbb{T}$ be a classical first-order theory or a decidable coherent theory. We cut down to a *set* of $\mathbb{T}$-models by choosing an regular cardinal, κ, such that $\mathbb{T}$ (as a deductively closed set of sequents) is of cardinality $< \kappa$. Denote by $\mathbf{Sets}_\kappa$ the category of sets of size (hereditarily) less than κ – or, as we shall say briefly, κ-*small* sets – and by $X_\mathbb{T}$ the set of $\mathbb{T}$-models in $\mathbf{Sets}_\kappa$. This set of models is large enough for our purposes in that, using Deligne's Theorem (and thus the Axiom of Choice), the coherent functors from the coherent category $\mathcal{C}_\mathbb{T}$ to $\mathbf{Sets}_\kappa$ *jointly reflect covers* with respect to the coherent coverage on $\mathcal{C}_\mathbb{T}$ and the canonical coverage on $\mathbf{Sets}_\kappa$. Precisely:

Lemma 1.2.1 *For any family $\{f_i : C_i \to C \mid i \in I\}$ in $\mathcal{C}_\mathbb{T}$, if for all coherent functors $F : \mathcal{C} \longrightarrow \mathbf{Sets}_\kappa$, we have that the $F(f_i)$ are jointly surjective, then there exists $i_1, \ldots, i_n$ such that $\{f_{i_1}, \ldots, f_{i_n}\}$ cover C in $\mathcal{C}_\mathbb{T}$.*

For a first-order theory, this comes to saying that for any $\mathbb{T}$-type p, there exists a model $\mathbf{M}$ in $X_\mathbb{T}$ such that $\mathbf{M}$ realizes p. We say that $X_\mathbb{T}$ is a *saturated set of models* for $\mathbb{T}$.

Next, for $[\vec{x} \mid \phi] \in \mathcal{C}_\mathbb{T}$, the definable set functor given by ϕ restricts to a functor

$$[\![\vec{x} \mid \phi]\!]^{(-)} : X_\mathbb{T} \longrightarrow \mathbf{Sets}$$
$$\mathbf{M} \longmapsto [\![\vec{x} \mid \phi]\!]^{\mathbf{M}}$$

which, following the equivalence $\mathbf{Sets}^{X_\mathbb{T}} \simeq \mathbf{Sets}/X_\mathbb{T}$, corresponds to the set over $X_\mathbb{T}$:

$$[\![\vec{x} \mid \phi]\!]_{X_\mathbb{T}} := \left\{ \langle \mathbf{M}, \vec{b} \rangle \;\middle|\; \mathbf{M} \in X_\mathbb{T}, \vec{b} \in [\![\vec{x} \mid \phi]\!]^{\mathbf{M}} \right\} \xrightarrow{\;\pi_1\;} X_\mathbb{T}$$

Where π_1 projects out the model $\mathbf{M}$. Note the notation "$[\![\vec{x} \mid \phi]\!]_{X_\mathbb{T}}$" for the set on the left, which we shall make extensive use of below. The mapping $[\vec{x} \mid \phi] \mapsto (\pi_1 : [\![\vec{x} \mid \phi]\!]_{X_\mathbb{T}} \to X_\mathbb{T})$ gives us the object part of a functor,

$$\mathcal{M}_d : \mathcal{C}_\mathbb{T} \longrightarrow \mathbf{Sets}/X_\mathbb{T}$$

(which sends an arrow of $\mathcal{C}_\mathbb{T}$ to the obvious function over $X_\mathbb{T}$).

Proposition 1.2.2 (Stone representation for coherent categories) *The functor*
$$\mathcal{M}_d : \mathcal{C}_\mathbb{T} \longrightarrow \mathbf{Sets}/X_\mathbb{T}$$

is coherent and reflects covers with respect to the coherent coverage on $\mathcal{C}_\mathbb{T}$ and the canonical coverage on $\mathbf{Sets}/X_\mathbb{T}$. As a consequence, $\mathcal{M}_d$ is conservative, that is, $\mathcal{M}_d$ is faithful and reflects isomorphisms.

PROOF Considering each $\mathbb{T}$-model $\mathbf{M}$ as a coherent functor from $\mathcal{C}_\mathbb{T}$ to $\mathbf{Sets}$, we have a commuting triangle:

Then $\mathcal{M}_d$ is coherent since all $\mathbf{M} \in X_\mathbb{T}$ are coherent, and $\mathcal{M}_d$ reflects covers since the $\mathbf{M} \in X_\mathbb{T}$ jointly reflect covers. $\dashv$

Let $G_\mathbb{T}$ be the set of isomorphisms between models in $X_\mathbb{T}$, giving us a groupoid,

where c is composition of arrows; i sends an arrow to its inverse; s sends an arrow to its source/domain and t to its target/codomain; and Id sends an object to its identity arrow. By equipping $X_\mathbb{T}$ with the logical topology defined below, and then introducing continuous $G_\mathbb{T}$-actions, we will make the objects in the image of $\mathcal{M}_d$—that is, the definable set functors—compact and generating, and the embedding full. That is, we factor $\mathcal{M}_d$, first, through the category of sheaves on $X_\mathbb{T}$ (equipped with the logical topology) and, second,

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through the category of equivariant sheaves, or sheaves with a continuous $G_{\mathbb{T}}$-action (u^* and v^* are forgetful functors):

$$
\begin{array}{ccc}
& \mathbf{Sets}/X_{\mathbb{T}} & \mathbf{Sets}/X_{\mathbb{T}} \\
\mathcal{M}_d \nearrow & \uparrow u^* & u \downarrow \quad \searrow m_d \\
\mathcal{C}_{\mathbb{T}} \xrightarrow{\mathcal{M}} \mathrm{Sh}(X_{\mathbb{T}}) & & \mathrm{Sh}(X_{\mathbb{T}}) \xrightarrow{m} \mathrm{Sh}(\mathcal{C}_{\mathbb{T}}) \\
\mathcal{M}^\dagger \searrow & \uparrow v^* & v \downarrow \quad \nearrow m^\dagger \\
& \mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}}) & \mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}})
\end{array}
$$

The diagram on the right then shows the induced geometric morphisms. Our main result of Section 1 (Theorem 1.6.11) is that $\mathcal{M}^\dagger$ is full, faithful, and cover reflecting, and that $\mathcal{C}_{\mathbb{T}}$ generates $\mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}})$ (as a full subcategory), whence $m^\dagger$ is an equivalence:

$$
\mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}}) \simeq \mathrm{Sh}(\mathcal{C}_{\mathbb{T}})
$$

1.3 Definable Sets are Sheaves on a Space of Models

We introduce the following 'logical' topology on the set $X_{\mathbb{T}}$ of $\mathbb{T}$-models.

Definition 1.3.1 The *logical topology* on $X_{\mathbb{T}}$ is defined by taking as basic open sets those of the form

$$
\langle\!\langle [\vec{x} \mid \phi], \vec{b} \rangle\!\rangle := \left\{ \mathbf{M} \in X_{\mathbb{T}} \;\middle|\; \vec{b} \in [\![\vec{x} \mid \phi]\!]^{\mathbf{M}} \right\} \subseteq X_{\mathbb{T}}
$$

for $[\vec{x} \mid \phi] \in \mathcal{C}_{\mathbb{T}}$ and $b \in \mathbf{Sets}_\kappa$, with $\vec{b}$ the same length as $\vec{x}$.

In Section 2.1 we will give a more intrinsic specification, in terms of the objects and morphisms of a decidable coherent category, rather than in terms of the formulas of a decidable coherent theory.

Next, we factor $\mathcal{M}_d : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathbf{Sets}/X_{\mathbb{T}}$ through $\mathrm{Sh}(X_{\mathbb{T}})$ by making each $[\![\vec{x} \mid \phi]\!]_{X_{\mathbb{T}}}$ into a sheaf on $X_{\mathbb{T}}$ with respect to the following topological structure. We shall use $*$ to denote concatenation of tuples,

$$
\langle a_1, \ldots, a_n \rangle * \langle b_1, \ldots, b_m \rangle = \langle a_1, \ldots, a_n, b_1, \ldots, b_m \rangle.
$$

Definition 1.3.2 For an object $[\vec{x} \mid \phi]$ of $\mathcal{C}_{\mathbb{T}}$, the *logical topology* on the set

$$
[\![\vec{x} \mid \phi]\!]_{X_{\mathbb{T}}} = \left\{ \langle \mathbf{M}, \vec{a} \rangle \;\middle|\; \mathbf{M} \in X_{\mathbb{T}}, \vec{a} \in [\![\vec{x} \mid \phi]\!]^{\mathbf{M}} \right\}
$$

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is given by basic opens of the form

$$\langle [\vec{x}, \vec{y} \mid \psi], \vec{b} \rangle := \left\{ \langle \mathbf{M}, \vec{a} \rangle \;\middle|\; \vec{a} * \vec{b} \in [\![\vec{x}, \vec{y} \mid \phi \wedge \psi]\!]^{\mathbf{M}} \right\}$$

(where $\vec{b}$ is of the same length as $\vec{y}$)

For any object $[\vec{x} \mid \phi]$ in $\mathcal{C}_{\mathbb{T}}$, we now have the following:

Lemma 1.3.3 *The projection* $\pi_1 : [\![\vec{x} \mid \phi]\!]_{X_{\mathbb{T}}} \to X_{\mathbb{T}}$ *is a local homeomorphism.*

PROOF First, the projection is continuous. For let a basic open $\langle [\vec{y} \mid \psi], \vec{b} \rangle \subseteq X_{\mathbb{T}}$ be given. Then

$$\pi_1^{-1} \left(\langle [\vec{y} \mid \psi], \vec{b} \rangle \right) = \langle [\vec{x}, \vec{y} \mid \psi], \vec{b} \rangle \subseteq [\![\vec{x} \mid \phi]\!]_{X_{\mathbb{T}}}$$

Next, the projection is open. For given a basic open $\langle [\vec{x}, \vec{y} \mid \psi], \vec{b} \rangle \subseteq [\![\vec{x} \mid \phi]\!]_{X_{\mathbb{T}}}$ we have

$$\pi_1 \left(\langle [\vec{x}, \vec{y} \mid \psi], \vec{b} \rangle \right) = \langle [\vec{y} \mid \exists \vec{x}. \phi \wedge \psi], \vec{b} \rangle \subseteq X_{\mathbb{T}}$$

which is open. Finally, let $\langle \mathbf{M}, \vec{a} \rangle \in [\![\vec{x} \mid \phi]\!]_{X_{\mathbb{T}}}$ be given. Then

$$\langle \mathbf{M}, \vec{a} \rangle \in V := \langle [\vec{x}, \vec{y} \mid \vec{x} = \vec{y}], \vec{a} \rangle \subseteq [\![\vec{x} \mid \phi]\!]_{X_{\mathbb{T}}}$$

and $\langle \mathbf{N}, \vec{a'} \rangle \in V$ if and only if $\vec{a} = \vec{a'}$. Thus $\pi_1 \restriction_V$ is injective. We now have that $\pi_1 \restriction_V \colon V \to \pi_1(V)$ is continuous, open, and bijective, and therefore a homeomorphism.   ⊣

Lemma 1.3.4 *Given an arrow*

$$[\vec{x}, \vec{y} \mid \sigma] : [\vec{x} \mid \phi] \longrightarrow [\vec{y} \mid \psi]$$

in $\mathcal{C}_{\mathbb{T}}$, *the corresponding function* $f_\sigma : [\![\vec{x} \mid \phi]\!]_{X_{\mathbb{T}}} \to [\![\vec{y} \mid \psi]\!]_{X_{\mathbb{T}}}$ *is continuous.*

PROOF Given a basic open $\langle [\vec{y}, \vec{z} \mid \xi], \vec{c} \rangle \subseteq [\![\vec{y} \mid \psi]\!]_{X_{\mathbb{T}}}$, then

$$f_\sigma^{-1} \left(\langle [\vec{y}, \vec{z} \mid \xi], \vec{c} \rangle \right) = \langle [\vec{x}, \vec{z} \mid \exists \vec{y}. \sigma \wedge \xi], \vec{c} \rangle \qquad\qquad\qquad ⊣$$

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Proposition 1.3.5 *The functor $\mathcal{M}_d : \mathcal{C}_\mathbb{T} \longrightarrow \mathbf{Sets}/X_\mathbb{T}$ factors through the category $\mathrm{Sh}(X_\mathbb{T})$ of sheaves as*

$$
\begin{array}{ccc}
 & \mathbf{Sets}/X_\mathbb{T} & \\
 \mathcal{M}_d \nearrow & \Big\uparrow u^* & \\
 \mathcal{C}_\mathbb{T} \xrightarrow[\ \mathcal{M}\ ]{} & \mathrm{Sh}(X_\mathbb{T}) &
\end{array}
$$

where $u^* : \mathrm{Sh}(X_\mathbb{T}) \longrightarrow \mathbf{Sets}/X_\mathbb{T}$ is the forgetful (inverse image) functor. Moreover, $\mathcal{M}$ is coherent and reflects covers.

PROOF $\mathcal{M}$ is obtained by Lemma 1.3.3 and Lemma 1.3.4. Since $\mathcal{M}_d$ is coherent and the forgetful functor u^* reflects coherent structure, $\mathcal{M}$ is coherent. Since u^* preserves covers (being geometric) and $\mathcal{M}_d$ reflects them, $\mathcal{M}$ reflects covers. $\dashv$

1.4 $\mathbb{G}_\mathbb{T}$ is an Open Topological Groupoid

Consider now the set $G_\mathbb{T}$ of $\mathbb{T}$-model isomorphisms between the models in $X_\mathbb{T}$. Such an isomorphism, $\mathbf{f} : \mathbf{M} \to \mathbf{N}$, consists of a family of bijections, $f_A : [\![x : A \mid \top]\!]^\mathbf{M} \to [\![x : A \mid \top]\!]^\mathbf{N}$, indexed by the sorts of $\mathbb{T}$, subject to the usual conditions ensuring that $\mathbf{f}$ is an invertible homomorphism of $\mathbb{T}$-models. We equip $G_\mathbb{T}$ with a topology to make the groupoid,

$$
G_\mathbb{T} \times_{X_\mathbb{T}} G_\mathbb{T} \xrightarrow{\ c\ } G_\mathbb{T} \overset{i}{\underset{\substack{s \\ \text{—Id—} \\ t}}{\rightleftarrows}} X_\mathbb{T}
$$

of $\mathbb{T}$-models and isomorphisms a topological groupoid. (For shorter notation we write "$\mathbb{G}_\mathbb{T}$", or "$G_\mathbb{T} \rightrightarrows X_\mathbb{T}$" if we want to display the set of objects and the set of arrows of the groupoid.)

Definition 1.4.1 The *logical topology* on $G_\mathbb{T}$ is defined by taking as subbasic open sets those of the form

- $s^{-1}\left(\langle\!\langle [\vec{x} \mid \phi], \vec{a} \rangle\!\rangle\right) = \left\{ \mathbf{f} \in G_\mathbb{T} \ \big| \ \vec{a} \in [\![\vec{x} \mid \phi]\!]^{s(\mathbf{f})} \right\}$

- $\langle\!\langle B : b \mapsto c \rangle\!\rangle := \left\{ \mathbf{f} \in G_\mathbb{T} \ \big| \ b \in [\![x : B \mid \top]\!]^{s(\mathbf{f})} \wedge f_B(b) = c \right\}$, where B is a sort of $\mathbb{T}$.

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- $t^{-1}\left(\langle[\vec{x}\mid\psi],\vec{e}\rangle\right)=\left\{\mathbf{f}\in G_{\mathbb{T}}\mid \vec{e}\in[\![\vec{x}\mid\psi]\!]^{t(\mathbf{f})}\right\}$

The most readable form to present a basic open set U is as an array displaying the 'source condition', the 'preservation condition', and the 'target condition', e.g.:

$$U=\begin{pmatrix}[\vec{x}\mid\phi],\vec{a}\\ \vec{B}:\vec{b}\mapsto\vec{c}\\ [\vec{y}\mid\psi],\vec{d}\end{pmatrix}$$

$$=\left\{\mathbf{f}:\mathbf{M}\Rightarrow\mathbf{N}\mid \vec{a}\in[\![\vec{x}\mid\phi]\!]^{\mathbf{M}}\wedge\vec{b}\in[\![\vec{x}:\vec{B}\mid\top]\!]^{\mathbf{M}}\wedge f_{\vec{B}}(\vec{b})=\vec{c}\wedge\vec{d}\in[\![\vec{y}\mid\psi]\!]^{\mathbf{N}}\right\}$$

Lemma 1.4.2 *With respect to the logical topologies on $G_{\mathbb{T}}$ and $X_{\mathbb{T}}$, the groupoid*

is a topological groupoid (i.e. the source, target, identity, inverse, and composition maps are all continuous).

PROOF Straightforward verification. ⊣

It is clear that if we are presented with a basic open set

$$\langle[\vec{y}:\vec{B}\mid\phi],\vec{b}\rangle\subseteq X_{\mathbb{T}}\text{ or }\langle[\vec{x}:\vec{A},\vec{y}:\vec{B}\mid\psi],\vec{b}\rangle\subseteq[\![\vec{x}:\vec{A}\mid\phi]\!]_{X_{\mathbb{T}}}$$

we can assume without loss of generality that, for $i\neq j$, $B_i=B_j$ implies $b_i\neq b_j$. We say that $\langle[\vec{y}:\vec{B}\mid\phi],\vec{b}\rangle$ is presented in *reduced form* if this condition is satisfied. It is clear that, as long as we are careful, we can replace elements in a model by switching to an isomorphic model. We write this out as a technical lemma for reference.

Lemma 1.4.3 *Let a list of sorts $\vec{A}$ of $\mathbb{T}$ and two tuples $\vec{a}$ and $\vec{b}$ of $\mathbf{Sets}_{\kappa}$ be given, of the same length as $\vec{A}$, and satisfying the requirement that whenever $i\neq j$, $A_i=A_j$ implies $a_i\neq a_j$ and $b_i\neq b_j$. Then for any $\mathbf{M}\in X_{\mathbb{T}}$, if $\vec{a}\in[\![\vec{x}:\vec{A}\mid\top]\!]^{\mathbf{M}}$, there exists an $\mathbf{N}\in X_{\mathbb{T}}$ and an isomorphism $\mathbf{f}:\mathbf{M}\to\mathbf{N}$ in $G_{\mathbb{T}}$ such that $f_{\vec{A}}(\vec{a})=\vec{b}$.*

Proposition 1.4.4 *The groupoid $\mathbb{G}_{\mathbb{T}}$ is an open topological groupoid.*

PROOF It remains (by Lemma 1.4.2) to verify that the source map is open, from which it follows that the target map is open as well. Let a basic open subset

$$V = \begin{pmatrix} [\vec{x} : \vec{A} \mid \phi], \vec{a} \\ \vec{B} : \vec{b} \mapsto \vec{c} \\ [\vec{y} : \vec{D} \mid \psi], \vec{d} \end{pmatrix}$$

of $G_{\mathbb{T}}$ be given, and suppose $\mathbf{f} : \mathbf{M} \to \mathbf{N}$ is in V. We must find an open neighborhood around $\mathbf{M}$ which is contained in $s(V)$. We claim that

$$U = \langle [\vec{x} : \vec{A}, \vec{y} : \vec{D}, \vec{z} : \vec{B} \mid \phi \wedge \psi], \vec{a} * f_{\vec{D}}^{-1}(\vec{d}) * \vec{b} \rangle$$

does the trick. Clearly, $\mathbf{M} \in U$. Suppose $\mathbf{K} \in U$. Consider the tuples $f_{\vec{D}}^{-1}(\vec{d}) * \vec{b}$ and $\vec{d} * \vec{c}$ together with the list of sorts $\vec{D} * \vec{B}$. Since $f_{\vec{D}*\vec{B}}$ sends the first tuple to the second, we can assume that the conditions of Lemma 1.4.3 are satisfied (or a simple rewriting will see that they are), and so there exists a $\mathbb{T}$-model $\mathbf{L}$ and an isomorphism $\mathbf{g} : \mathbf{K} \to \mathbf{L}$ such that $g \in V$. So $U \subseteq s(V)$. ⊣

1.5 Definable Sets as Equivariant Sheaves

Recall that if $\mathbb{H}$ is an arbitrary topological groupoid, which we also write as $H_1 \rightrightarrows H_0$, the topos of *equivariant sheaves* (or *continuous actions*) on $\mathbb{H}$, written $\mathrm{Sh}(\mathbb{H})$ or $\mathrm{Sh}_{H_1}(H_0)$, consists of the following [4, B3.4.14(b)], [11], [12]. An object of $\mathrm{Sh}(\mathbb{H})$ is a pair $\langle a : A \to H_0, \alpha \rangle$, where a is a local homeomorphism (that is, an object of $\mathrm{Sh}(H_0)$) and $\alpha : H_1 \times_{H_0} A \to A$ is a continuous function from the pullback (in **Top**) of a along the source map $s : H_1 \to H_0$ to A such that

$$a(\alpha(f, x)) = t(f)$$

and satisfying the axioms for an action:

(i) $\alpha(1_h, x) = x$ for $h \in H_0$.

(ii) $\alpha(g, \alpha(f, x)) = \alpha(g \circ f, x)$.

For illustration, it follows that for $f \in H_1$, $\alpha(f, -)$ is a bijective function from the fiber over $s(f)$ to the fiber over $t(f)$. An arrow

$$h : \langle a : A \to H_0, \alpha \rangle \longrightarrow \langle b : B \to H_0, \beta \rangle$$

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is an arrow of $\mathrm{Sh}(H_0)$,

$$A \xrightarrow{\quad h \quad} B$$
$$a \searrow \qquad \swarrow b$$
$$H_0$$

which commutes with the actions:

$$
\begin{array}{ccc}
H_1 \times_{H_0} A & \xrightarrow{\alpha} & A \\
{}^{1_{H_1} \times_{H_0} h} \Big\downarrow & & \Big\downarrow {}^{h} \\
H_1 \times_{H_0} B & \xrightarrow{\beta} & B
\end{array}
$$

We now return to the definable set functors, $[\![\vec{x} \mid \phi]\!]^{(-)} : \mathrm{Mod}_{\mathbb{T}} \longrightarrow \mathbf{Sets}$. Ignoring the isomorphisms between the $\mathbb{T}$-models for the moment, we have described such a functor – restricted to κ-small models – first as a set and then (introducing topological structure) as a sheaf over $X_{\mathbb{T}}$. The action of the functor on the model isomorphisms can now be introduced as an action of the groupoid on the sheaf, as follows.

Definition 1.5.1 For each $[\vec{x} \mid \phi] \in \mathcal{C}_{\mathbb{T}}$ the function

$$\theta_{[\vec{x} \mid \phi]} : G_{\mathbb{T}} \times_{X_{\mathbb{T}}} [\![\vec{x} \mid \phi]\!]_{X_{\mathbb{T}}} \longrightarrow [\![\vec{x} \mid \phi]\!]_{X_{\mathbb{T}}} \tag{4}$$

is defined by $\langle \mathbf{f}, \langle s(\mathbf{f}), \vec{a} \rangle \rangle \mapsto \langle t(\mathbf{f}), \mathbf{f}(\vec{a}) \rangle$. (The subscript on θ will usually be left implicit.)

Lemma 1.5.2 *The pair $\langle \mathcal{M}([\vec{x} \mid \phi]), \theta \rangle$ is an object of $\mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}})$, i.e. the function*

$$\theta : G_{\mathbb{T}} \times_{X_{\mathbb{T}}} [\![\vec{x} \mid \phi]\!]_{X_{\mathbb{T}}} \longrightarrow [\![\vec{x} \mid \phi]\!]_{X_{\mathbb{T}}}$$

is a continuous action of $G_{\mathbb{T}}$ on $[\![\vec{x} \mid \phi]\!]_{X_{\mathbb{T}}}$.

PROOF We verify that θ is continuous. Let a basic open

$$U = \langle [\vec{x} : \vec{A}, \vec{y} : \vec{B} \mid \psi], \vec{b} \rangle \subseteq [\![\vec{x} : \vec{A} \mid \phi]\!]_{X_{\mathbb{T}}}$$

be given, and suppose $\theta(\mathbf{f}, \langle \mathbf{M}, \vec{a} \rangle) = \langle \mathbf{N}, f_{\vec{A}}(\vec{a}) \rangle \in U$ for $\mathbf{M}, \mathbf{N} \in X_{\mathbb{T}}$ and $f : \mathbf{M} \to \mathbf{N}$ in $G_{\mathbb{T}}$. Then we can specify an open neighborhood around $\langle \mathbf{f}, \langle \mathbf{M}, \vec{a} \rangle \rangle$ which θ maps into U as:

$$\langle \mathbf{f}, \langle \mathbf{M}, \vec{a} \rangle \rangle \in \left(\overline{\vec{B} : f_{\vec{B}}^{-1}(\vec{b}) \mapsto \vec{b}} \right) \times_{X_{\mathbb{T}}} \langle [\vec{x} : \vec{A}, \vec{y} : \vec{B} \mid \psi], f_{\vec{B}}^{-1}(\vec{b}) \rangle$$

$$\dashv$$

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1.6 Stable Subsets

For a subobject (represented by an inclusion) $[\vec{x} \mid \xi] \hookrightarrow [\vec{x} \mid \phi]$ in $\mathcal{C}_{\mathbb{T}}$, the open subset $[\![\vec{x} \mid \xi]\!]_{X_{\mathbb{T}}} \subseteq [\![\vec{x} \mid \phi]\!]_{X_{\mathbb{T}}}$ is closed under the action θ in the usual sense that $\theta(a) \in [\![\vec{x} \mid \xi]\!]_{X_{\mathbb{T}}}$ for any point $a \in [\![\vec{x} \mid \xi]\!]_{X_{\mathbb{T}}}$. For an object, $\langle A \to X_{\mathbb{T}}, \alpha \rangle$, of $\mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}})$, we call a subset, $S \subseteq A$, that is closed under the action of $G_{\mathbb{T}}$ *stable*, so as to reserve "closed" to mean topologically closed. We claim that the only stable opens of $[\![\vec{x} \mid \phi]\!]_{X_{\mathbb{T}}}$ come from subobjects of $[\vec{x} \mid \phi]$ as joins. Specifically:

Lemma 1.6.1 *Let* $[\vec{x} : \vec{A} \mid \phi]$ *in* $\mathcal{C}_{\mathbb{T}}$ *and* U *a basic open subset of* $[\![\vec{x} : \vec{A} \mid \phi]\!]_{X_{\mathbb{T}}}$ *of the form*

$$U = \langle [\vec{x} : \vec{A}, \vec{y} : \vec{B} \mid \psi], \vec{b} \rangle$$

be given. Then the stabilization (closure) of U *under the action* θ *of* $G_{\mathbb{T}}$ *on* $\langle [\vec{x} : \vec{A} \mid \phi] \rangle$ *is a subset of the form* $[\![\vec{x} : \vec{A} \mid \xi]\!]_{X_{\mathbb{T}}} \subseteq [\![\vec{x} : \vec{A} \mid \phi]\!]_{X_{\mathbb{T}}}$.

PROOF We can assume without loss that U is in reduced form. Let φ be the formula expressing the conjunction of inequalities $y_i \neq y_j$ for all pairs of indices $i \neq j$ such that $B_i = B_j$ in $\vec{B}$. We claim that the stabilization of U is $[\![\vec{x} : \vec{A} \mid \xi]\!]_{X_{\mathbb{T}}}$ where ξ is the formula $\exists \vec{y} : \vec{B}. \phi \wedge \psi \wedge \varphi$. First, $[\![\vec{x} : \vec{A} \mid \xi]\!]_{X_{\mathbb{T}}}$ is a stable set containing U. Next, suppose $\langle \mathbf{M}, \vec{a} \rangle \in [\![\vec{x} : \vec{A} \mid \xi]\!]_{X_{\mathbb{T}}}$. Then there exists $\vec{c}$ such that $\vec{a} * \vec{c} \in [\![\vec{x} : \vec{A}, \vec{y} : \vec{B} \mid \phi \wedge \psi \wedge \varphi]\!]^{\mathbf{M}}$. Then $\vec{b}$ and $\vec{c}$ (with respect to $\vec{B}$) satisfy the conditions of Lemma 1.4.3, so there exists a $\mathbb{T}$-model $\mathbf{N}$ with isomorphism $\mathbf{f} : \mathbf{M} \to \mathbf{N}$ such that $f_{\vec{B}}(\vec{c}) = \vec{b}$. Then $\theta(\mathbf{f}, \langle \mathbf{M}, \vec{a} \rangle) \in U$, and hence $\langle \mathbf{M}, \vec{a} \rangle$ is in the stabilization of U. $\dashv$

Definition 1.6.2 We call a subset of the form $[\![\vec{x} \mid \xi]\!]_{X_{\mathbb{T}}} \subseteq [\![\vec{x} \mid \phi]\!]_{X_{\mathbb{T}}}$, for a subobject

$$[\vec{x} \mid \xi] \hookrightarrow [\vec{x} \mid \phi]$$

in $\mathcal{C}_{\mathbb{T}}$, a *definable* subset of $[\![\vec{x} \mid \phi]\!]_{X_{\mathbb{T}}}$.

Corollary 1.6.3 *Any open stable subset of* $[\![\vec{x} : \vec{A} \mid \phi]\!]_{X_{\mathbb{T}}}$ *is a union of definable subsets.*

We also note the following:

Lemma 1.6.4 *Let* $\langle [\vec{x} : \vec{A} \mid \phi], \vec{a} \rangle$ *be a basic open of* $X_{\mathbb{T}}$ *in reduced form. Then there exists a sheaf* $\mathcal{M}([\vec{x} : \vec{A} \mid \xi])$ *and a (continuous) section*

$$s : \langle [\vec{x} : \vec{A} \mid \phi], \vec{a} \rangle \longrightarrow [\![\vec{x} : \vec{A} \mid \xi]\!]_{X_{\mathbb{T}}}$$

such that $[\![\vec{x} : \vec{A} \mid \xi]\!]_{X_{\mathbb{T}}}$ is the stabilization of the open set $s(\langle [\vec{x} : \vec{A} \mid \phi] \rangle) \subseteq [\![\vec{x} : \vec{A} \mid \xi]\!]_{X_{\mathbb{T}}}$.

PROOF Let φ be the formula expressing the inequalities $x_i \neq x_j$ for all pairs of indices $i \neq j$ such that $A_i = A_j$ in $\vec{A}$. Let $\xi := \phi \wedge \varphi$ and consider the function $s : \langle [\vec{x} : \vec{A} \mid \phi], \vec{a} \rangle \longrightarrow [\![\vec{x} : \vec{A}|\xi]\!]_{X_{\mathbb{T}}}$ defined by $\mathbf{M} \mapsto \langle \mathbf{M}, \vec{a} \rangle$. The image of s is the open set $\langle [\vec{x} : \vec{A}, \vec{y} : \vec{A} \mid \vec{x} = \vec{y}], \vec{a} \rangle$, so s is a (continuous) section. And by the proof of Lemma 1.6.1, the stabilization of $\langle [\vec{x} : \vec{A}, \vec{y} : \vec{A} \mid \vec{x} = \vec{y}], \vec{a} \rangle$ is exactly $[\![\vec{x} : \vec{A} \mid \xi]\!]_{X_{\mathbb{T}}}$. $\dashv$

Consider now the topos of equivariant sheaves $\mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}})$. For an arrow, $f : C \to D$, of $\mathcal{C}_{\mathbb{T}}$, clearly the function $\mathcal{M}(f) : \mathcal{M}(C) \to \mathcal{M}(D)$ commutes with the actions θ_C and θ_D, so that, by Lemma 1.5.2, we have a functor $\mathcal{M}^{\dagger} : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}})$ which factors $\mathcal{M} : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}})$ through $\mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}})$:

$$
\begin{array}{ccc}
 & \mathrm{Sh}(X_{\mathbb{T}}) & \\
\overset{\mathcal{M}}{\nearrow} & \uparrow v^{*} & \\
\mathcal{C}_{\mathbb{T}} \underset{\mathcal{M}^{\dagger}}{\longrightarrow} & \mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}}) &
\end{array} \tag{5}
$$

where v^{*} is the forgetful functor. We call the image of $\mathcal{M}^{\dagger}$ the *definable* objects and arrows of $\mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}})$. Since $\mathcal{M}$ is coherent and the forgetful functor v^{*} reflects coherent structure, $\mathcal{M}^{\dagger}$ is coherent. Therefore, (5) induces a commuting diagram of geometric morphisms:

$$
\begin{array}{ccc}
 & \mathrm{Sh}(X_{\mathbb{T}}) & \\
\overset{m}{\swarrow} & \downarrow v & \\
\mathrm{Sh}(\mathcal{C}_{\mathbb{T}}) \underset{m^{\dagger}}{\longleftarrow} & \mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}}) &
\end{array}
$$

where $m^{\dagger}$ is a surjection because m is. We state these facts for reference:

Lemma 1.6.5 $\mathcal{M}^{\dagger} : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}})$ *is coherent, conservative (i.e. faithful and reflects isomorphisms), and reflects covers.*

The remainder of this section is devoted to establishing that the geometric morphism

$$
m^{\dagger} : \mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}}) \longrightarrow \mathrm{Sh}(\mathcal{C}_{\mathbb{T}})
$$

is an equivalence. The main remaining step is to establish that the definable objects generate $\mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}})$ (Corollary 1.6.10). First, it is a known fact that any equivariant sheaf on an open topological groupoid has an open action (see e.g. [11]):

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Lemma 1.6.6 *For any object in* $\mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}})$,

$$\left\langle\, R \xrightarrow{\ r\ } X_{\mathbb{T}} \,,\rho \,\right\rangle$$

the projection $\pi_2 : G_{\mathbb{T}} \times_{X_{\mathbb{T}}} R \longrightarrow R$ *is open.*

PROOF By Proposition 1.4.4, since pullback preserves open maps of spaces.⊣

Corollary 1.6.7 *For any object* $\langle r : R \to X_{\mathbb{T}}, \rho \rangle$ *in* $\mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}})$, *the action*

$$\rho : G_{\mathbb{T}} \times_{X_{\mathbb{T}}} R \longrightarrow R$$

is open. Consequently, the stabilization of any open subset of R is again open.

PROOF Let a basic open $V \times_{X_{\mathbb{T}}} U \subseteq G_{\mathbb{T}} \times_{X_{\mathbb{T}}} R$ be given (so that $U \subseteq R$ and $V \subseteq G_{\mathbb{T}}$ are open). Observe that, since the inverse map $i : G_{\mathbb{T}} \longrightarrow G_{\mathbb{T}}$ is a homeomorphism, $i(V)$ is open, and

$$
\begin{aligned}
\rho(V \times_{X_{\mathbb{T}}} U) \ &= \{ y \in R \mid \exists \langle f, x \rangle \in V \times_{X_{\mathbb{T}}} U.\, \rho(f,x) = y \} \\
&= \{ y \in R \mid \exists f^{-1} \in i(V).\, s(f^{-1}) = r(y) \wedge \rho(f^{-1}, y) \in U \} \\
&= \pi_2(\rho^{-1}(U) \cap (i(V) \times_{X_{\mathbb{T}}} R))
\end{aligned}
$$

is open by Lemma 1.6.6. Finally, for any open $U \subseteq R$, the stabilization of U is $\rho(G_{\mathbb{T}} \times_{X_{\mathbb{T}}} U)$. ⊣

Lemma 1.6.8 *For any object* $\langle\, R \xrightarrow{\ r\ } X_{\mathbb{T}} \,,\rho \,\rangle$ *in* $\mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}})$, *and any element $x \in R$, there exists a basic open* $\langle [\vec{x} : \vec{A} \mid \phi], \vec{a} \rangle \subseteq X_{\mathbb{T}}$ *and a section $v :$* $\langle [\vec{x} : \vec{A} \mid \phi], \vec{a} \rangle \to R$ *containing x such that for any* $\mathbf{f} : \mathbf{M} \to \mathbf{N}$ *in $G_{\mathbb{T}}$ such that* $\mathbf{M} \in \langle [\vec{x} : \vec{A} \mid \phi], \vec{a} \rangle$ *and $f_{\vec{A}}(\vec{a}) = \vec{a}$ (thus $\mathbf{N}$ is also in $\langle [\vec{x} : \vec{A} \mid \phi], \vec{a} \rangle$), we have $\rho(\mathbf{f}, v(\mathbf{M})) = v(\mathbf{N})$.*

PROOF Given $x \in R$, choose a section $s : \langle [\vec{y} : \vec{B} \mid \psi], \vec{b} \rangle \to R$ such that $x \in s(\langle [\vec{y} : \vec{B} \mid \psi], \vec{b} \rangle)$. Pull the open set $s(\langle [\vec{y} : \vec{B} \mid \psi], \vec{b} \rangle)$ back along the continuous action ρ,

$$
\begin{array}{ccc}
V & \longrightarrow & s(\langle [\vec{y} : \vec{B} \mid \psi], \vec{b} \rangle) \\
{\scriptstyle \subseteq}\big\downarrow & & {\scriptstyle \subseteq}\big\downarrow \\
G_{\mathbb{T}} \times_{X_{\mathbb{T}}} R & \xrightarrow{\ \ \rho\ \ } & R
\end{array}
$$

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to obtain an open set V containing $\langle 1_{r(x)}, x \rangle$. Since V is open, we can find a box of basic opens around $\langle 1_{r(x)}, x \rangle$ contained in V:

$$\langle 1_{r(x)}, x \rangle \in W := \begin{pmatrix} [\vec{z} : \vec{C} \mid \xi], \vec{c} \\ \vec{K} : \vec{k} \mapsto \vec{k} \\ [\vec{z'} : \vec{C'} \mid \eta], \vec{c'} \end{pmatrix} \times_{X_{\mathbb{T}}} v'(U_{[\vec{y'} : \vec{D} \mid \theta], \vec{d}}) \subseteq V$$

where v' is a section $v' : \langle [\vec{y'} : \vec{D} \mid \theta], \vec{d} \rangle \to R$ with x in its image. Notice that the preservation condition of W (i.e. $\vec{K} : \vec{k} \mapsto \vec{k}$) must have the same sets on both the source and the target side, since it is satisfied by $1_{r(x)}$. Now, restrict v' to the subset

$$U := \langle [\vec{z} : \vec{C}, \vec{z''} : \vec{K}, \vec{z'} : \vec{C'}, \vec{y'} : \vec{D} \mid \xi \wedge \eta \wedge \theta], \vec{c} * \vec{k} * \vec{c'} * \vec{d} \rangle$$

to obtain a section $v = v' \restriction_U : U \to R$. Notice that $x \in v(U)$. Furthermore, $v(U) \subseteq s(\langle [\vec{y} : \vec{B} \mid \psi], \vec{b} \rangle)$, for if $v(\mathbf{M}) \in v(U)$, then $\langle 1_{\mathbf{M}}, v(\mathbf{M}) \rangle \in W$, and so $\rho(\langle 1_{\mathbf{M}}, v(\mathbf{M}) \rangle) = v(\mathbf{M}) \in s(\langle [\vec{y} : \vec{B} \mid \psi], \vec{b} \rangle)$. Finally, if $\mathbf{M} \in U$ and $\mathbf{f} : \mathbf{M} \to \mathbf{N}$ is an isomorphism in $G_{\mathbb{T}}$ such that

$$f_{\vec{C} * \vec{K} * \vec{C'} * \vec{D}}(\vec{c} * \vec{k} * \vec{c'} * \vec{d}) = \vec{c} * \vec{k} * \vec{c'} * \vec{d}$$

then $\langle \mathbf{f}, v(\mathbf{M}) \rangle \in W$, and so $\rho(\mathbf{f}, v(\mathbf{M})) \in s(\langle [\vec{y} : \vec{B} \mid \psi], \vec{b} \rangle)$. But we also have $v(\mathbf{N}) \in v(U) \subseteq s(\langle [\vec{y} : \vec{B} \mid \psi], \vec{b} \rangle)$, and $r(\rho(\mathbf{f}, v(\mathbf{M}))) = r(v(\mathbf{N}))$, so $\rho(\mathbf{f}, v(\mathbf{M})) = v(\mathbf{N})$. $\dashv$

Lemma 1.6.9 *For any object in* $\mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}})$,

$$\langle\, R \xrightarrow{\; r \;} X_{\mathbb{T}} \,, \rho \rangle$$

and any element $x \in R$, there exists a morphisms of $\mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}})$ *with definable domain and with x in its image.*

PROOF First, we construct a function over $X_{\mathbb{T}}$ with definable domain and with x in its image. Choose a section $v : \langle [\vec{x} : \vec{A} \mid \phi], \vec{a} \rangle \to R$ with the property described in Lemma 1.6.8 such that $x \in v(\langle [\vec{x} : \vec{A} \mid \phi], \vec{a} \rangle)$. We can assume that $\langle [\vec{x} : \vec{A} \mid \phi], \vec{a} \rangle$ is on reduced form. Then, by Lemma 1.6.4 there exists an object $[\vec{x} : \vec{A} \mid \xi]$ in $\mathcal{C}_{\mathbb{T}}$ and a section $s : \langle [\vec{x} : \vec{A} \mid \phi], \vec{a} \rangle \to [\![\vec{x} : \vec{A} \mid \xi]\!]_{X_{\mathbb{T}}}$ such that $[\![\vec{x} : \vec{A} \mid \xi]\!]_{X_{\mathbb{T}}}$ is the stabilization of $s(\langle [\vec{x} : \vec{A} \mid \phi], \vec{a} \rangle)$. Define a mapping $\hat{v} : [\![\vec{x} : \vec{A} \mid \xi]\!]_{X_{\mathbb{T}}} \to R$ as follows: for an element $\langle \mathbf{N}, \vec{c} \rangle \in [\![\vec{x} : \vec{A} \mid \xi]\!]_{X_{\mathbb{T}}}$,

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there exists $\langle \mathbf{M}, \vec{a} \rangle \in s(\langle [\vec{x} : \vec{A} \mid \phi], \vec{a} \rangle) \subseteq [\![\vec{x} : \vec{A} \mid \xi]\!]_{X_{\mathbb{T}}}$ and $\mathbf{f} : \mathbf{M} \to \mathbf{N}$ in $G_{\mathbb{T}}$ such that $f_{\vec{A}}(\vec{a}) = \vec{c}$. Set $\hat{v}(\langle \mathbf{N}, \vec{c} \rangle) = \rho(\mathbf{f}, v(\mathbf{M}))$. We verify that $\hat{v}$ is well defined: suppose $\langle \mathbf{M}', \vec{a} \rangle \in s(\langle [\vec{x} : \vec{A} \mid \phi], \vec{a} \rangle) \subseteq [\![\vec{x} : \vec{A} \mid \xi]\!]_{X_{\mathbb{T}}}$ and $\mathbf{g} : \mathbf{M}' \to \mathbf{N}$ in $G_{\mathbb{T}}$ is such that $g_{\vec{A}}(\vec{a}) = \vec{c}$. Then $\mathbf{g}^{-1} \circ \mathbf{f} : \mathbf{M} \to \mathbf{M}'$ sends $\vec{a} \in [\![\vec{x} : \vec{A} \mid \phi]\!]^{\mathbf{M}}$ to $\vec{a} \in [\![\vec{x} : \vec{A} \mid \phi]\!]^{\mathbf{M}'}$, and so by the choice of section $v : \langle [\vec{x} : \vec{A} \mid \phi], \vec{a} \rangle \to R$, we have that $\rho(\mathbf{g}^{-1} \circ \mathbf{f}, v(\mathbf{M})) = v(\mathbf{M}')$. But then

$$\rho(\mathbf{g}, v(\mathbf{M}')) = \rho(\mathbf{g}, \rho(\mathbf{g}^{-1} \circ \mathbf{f}, v(\mathbf{M}))) = \rho(\mathbf{f}, v(\mathbf{M}))$$

so the value of $\hat{v}$ at $\langle \mathbf{N}, \vec{c} \rangle$ is indeed independent of the choice of $\langle \mathbf{M}, \vec{a} \rangle$ and $\mathbf{f}$. Finally, the following triangle commutes,

$$
\begin{array}{ccc}
[\![\vec{x} : \vec{A} \mid \xi]\!]_{X_{\mathbb{T}}} & \xrightarrow{\hat{v}} & R \\
& \searrow^{s} \quad \nearrow_{v} & \\
& \langle [\vec{x} : \vec{A} \mid \phi], \vec{a} \rangle &
\end{array}
\tag{6}
$$

and so x is in the image of $\hat{v}$.

Second, we verify that the function $\hat{v} : [\![\vec{x} : \vec{A} \mid \xi]\!]_{X_{\mathbb{T}}} \to R$ is the underlying function of a morphism,

$$
\begin{array}{ccc}
[\![\vec{x} : \vec{A} \mid \xi]\!]_{X_{\mathbb{T}}} & \xrightarrow{\hat{v}} & R \\
& \searrow^{p} \quad \swarrow_{r} & \\
& X_{\mathbb{T}} &
\end{array}
$$

of $\mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}})$, where the action on $[\![\vec{x} : \vec{A} \mid \xi]\!]_{X_{\mathbb{T}}}$ is denoted θ (recall 4 on page 19). The definition of $\hat{v}$ makes it straightforward to see that $\hat{v}$ commutes with the actions θ and ρ of $[\![\vec{x} : \vec{A} \mid \xi]\!]_{X_{\mathbb{T}}}$ and R, respectively. Remains to show that $\hat{v}$ is continuous. Recall the triangle (6). Let $y \in \hat{v}([\![\vec{x} : \vec{A} \mid \xi]\!]_{X_{\mathbb{T}}})$ be given, and suppose U is a open neighborhood of y. By Corollary 1.6.7, we can assume that $U \subseteq \hat{v}([\![\vec{x} : \vec{A} \mid \xi]\!]_{X_{\mathbb{T}}})$. Suppose $y = \hat{v}(\langle \mathbf{N}, \vec{c} \rangle) = \rho(\mathbf{f}, v(\mathbf{M}))$ for a $\mathbf{f} : \mathbf{M} \to \mathbf{N}$ such that $\theta(\mathbf{f}, s(\mathbf{M})) = \langle \mathbf{N}, \vec{c} \rangle$. We must find an open neighborhood W around $\langle \mathbf{N}, \vec{c} \rangle$ such that $\hat{v}(W) \subseteq U$. First, define the open neighborhood $T \subseteq G_{\mathbb{T}} \times_{X_{\mathbb{T}}} R$ around $\langle \mathbf{f}, v(\mathbf{M}) \rangle$ by

$$T := \rho^{-1}(U) \cap \left(G_{\mathbb{T}} \times_{X_{\mathbb{T}}} v(\langle [\vec{x} : \vec{A} \mid \phi], \vec{a} \rangle) \right)$$

From the homeomorphism $v(\langle [\vec{x} : \vec{A} \mid \phi], \vec{a} \rangle) \cong s(\langle [\vec{x} : \vec{A} \mid \phi], \vec{a} \rangle)$ we obtain a homeomorphism $G_{\mathbb{T}} \times_{X_{\mathbb{T}}} v(\langle [\vec{x} : \vec{A} \mid \phi], \vec{a} \rangle) \cong G_{\mathbb{T}} \times_{X_{\mathbb{T}}} s(\langle [\vec{x} : \vec{A} \mid \phi], \vec{a} \rangle)$. Set

$T' \subseteq G_{\mathbb{T}} \times_{X_{\mathbb{T}}} s(\langle\!\langle[\vec{x}:\vec{A} \mid \phi], \vec{a}\rangle\!\rangle)$ to be the open subset corresponding to T under this homeomorphism,

$$\langle \mathbf{f}, v(\mathbf{M})\rangle \in T \subseteq G_{\mathbb{T}} \times_{X_{\mathbb{T}}} v(\langle\!\langle[\vec{x}:\vec{A} \mid \phi], \vec{a}\rangle\!\rangle)$$
$$\cong$$
$$\langle \mathbf{f}, s(\mathbf{M})\rangle \in T' \subseteq G_{\mathbb{T}} \times_{X_{\mathbb{T}}} s(\langle\!\langle[\vec{x}:\vec{A} \mid \phi], \vec{a}\rangle\!\rangle)$$

Then $\langle \mathbf{N}, \vec{c}\rangle = \theta(\mathbf{f}, s(\mathbf{M})) \in \theta(T')$, and by Corollary 1.6.7, $\theta(T')$ is open. We claim that $\hat{v}(\theta(T')) \subseteq U$: for suppose $\langle \mathbf{g}, s(\mathbf{P})\rangle \in T'$. Then $\langle \mathbf{g}, v(\mathbf{P})\rangle \in T \subseteq \rho^{-1}(U)$, and so $\hat{v}(\theta(\mathbf{g}, s(\mathbf{P}))) = \rho(\langle \mathbf{g}, v(\mathbf{P})\rangle) \in U$. Thus $\theta(T')$ is the required W. $\dashv$

Corollary 1.6.10 *The definable objects generate the topos* $\mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}})$.

We are thus in a position to conclude:

Theorem 1.6.11 *For a decidable coherent theory* $\mathbb{T}$ *with a saturated set of* κ*-small models* $X_{\mathbb{T}}$, *we have an equivalence of toposes,*

$$\mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}}) \simeq \mathrm{Sh}(\mathcal{C}_{\mathbb{T}}).$$

PROOF Since, by Corollary 1.6.10, the definable objects form a generating set, the full subcategory of definable objects is a site for $\mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}})$ when equipped with the canonical coverage inherited from $\mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}})$ (see e.g. [4, C2.2.16]). We argue first that $\mathcal{M}^{\dagger} : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}})$ is full: because $\mathcal{M}^{\dagger}$ is coherent (Lemma 1.6.5), definable objects are decidable. Therefore, any graph of a morphism between definable objects is complemented. Because $\mathcal{M}^{\dagger}$ reflects covers and any subobject of a definable object is a join of definable subobjects (Lemma 1.6.3), definable objects are compact in $\mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}})$ (in the sense that any covering family of subobjects has a finite covering subfamily). But then every complemented subobject of a definable object is a finite join of definable subobjects, and therefore definable. Hence $\mathcal{M}^{\dagger}$ is full. By Lemma 1.6.5, $\mathcal{M}^{\dagger}$ is also faithful. Finally, the canonical coverage inherited from $\mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}})$ coincides with the coherent coverage since $\mathcal{M}^{\dagger}$ reflects covers precisely with respect to the canonical coverage on $\mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}})$ and the coherent coverage on $\mathcal{C}_{\mathbb{T}}$. Therefore, $\mathcal{C}_{\mathbb{T}}$ equipped with the coherent coverage is a site for $\mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}})$, so $\mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}}) \simeq \mathrm{Sh}(\mathcal{C}_{\mathbb{T}})$. $\dashv$

Remark 1.6.12 An alternate proof of Theorem 1.6.11, following the lines of [10], is given in [9, Chapter 3]. It proceeds by showing that the spatial covering

$$m : \mathrm{Sh}(X_{\mathbb{T}}) \longrightarrow \mathrm{Sh}(\mathcal{C}_{\mathbb{T}})$$

of Section 1.2 is an open surjection and thus, by results of [13], an effective descent morphism. The groupoid representation $\mathrm{Sh}_{G_{\mathbb{T}}}(X_{\mathbb{T}}) \simeq \mathrm{Sh}(\mathcal{C}_{\mathbb{T}})$ then follows from descent theory.

2 Duality

2.1 Representation Theorem for Decidable Coherent Categories

Since one can pass back and forth between coherent theories and categories by taking the theories of categories and the syntactic categories of theories, Proposition 1.2.2 translates to a representation result for decidable coherent categories, in terms of groupoids of $\mathbf{Sets}_{\kappa}$-valued coherent functors and invertible natural transformations between them. We spell this representation out, including a more direct characterization of the topology on the set of $\mathbf{Sets}_{\kappa}$-valued coherent functors (Definition 2.1.2).

Let $\mathcal{D}$ be a (small) decidable coherent category, that is, a category with finite limits, images, stable covers, finite unions of subobjects, and complemented diagonals ([4, A1.4]). For a (regular) cardinal κ, we say that $\mathcal{D}$ *has a saturated set of κ-small models* if the coherent functors from $\mathcal{D}$ to the category of (hereditarily) κ-small sets,

$$\mathcal{D} \longrightarrow \mathbf{Sets}_{\kappa}$$

jointly reflect covers, in the sense, again, that for any family of arrows $f_i : C_i \to C$ in $\mathcal{D}$, if for all $M : \mathcal{D} \longrightarrow \mathbf{Sets}_{\kappa}$ in $X_{\mathcal{D}}$

$$\bigcup_{i \in I} \mathrm{Im}\,(M(f_i)) = M(C)$$

then there exists $f_{i_1}, \ldots, f_{i_n}$ such that $\mathrm{Im}\,(f_{i_1}) \vee \ldots \vee \mathrm{Im}\,(f_{i_n}) = C$.

Definition 2.1.1 Let **dCoh** be the category of small decidable coherent categories with coherent functors between them. For κ a (regular) cardinal,

let $\mathbf{dCoh}_\kappa$ be the full subcategory of those categories which have a saturated set of κ-small models, i.e. such that the coherent functors to $\mathbf{Sets}_\kappa$ reflect covers.

Note that any coherent category which is of cardinality $< \kappa$ is in $\mathbf{dCoh}_\kappa$, as are all distributive lattices.

Definition 2.1.2 For $\mathcal{D}$ in $\mathbf{dCoh}_\kappa$:

1. Let $X_\mathcal{D}$ be the set of coherent functors from $\mathcal{D}$ to $\mathbf{Sets}_\kappa$,

$$X_\mathcal{D} = \mathrm{Hom}_{\mathbf{dCoh}}(\mathcal{D}, \mathbf{Sets}_\kappa).$$

2. Let $G_\mathcal{D}$ be the set of invertible natural transformations between functors in $X_\mathcal{D}$, with s and t the source and target, or domain and codomain, maps,

$$s, t : G_\mathcal{D} \rightrightarrows X_\mathcal{D}$$

Denote the resulting groupoid by $\mathbb{G}_\mathcal{D}$.

3. The *coherent topology* on $X_\mathcal{D}$ is given by taking as a subbasis the collection of sets of the form,

$$\langle \vec{f}, \vec{a} \rangle = \langle \langle f_1 : A \to B_1, \ldots, f_n : A \to B_n \rangle, \langle a_1, \ldots, a_n \rangle \rangle$$
$$= \{ M \in X_\mathcal{D} \mid \exists x \in M(A). \, M(f_1)(x) = a_1 \wedge \ldots \wedge M(f_n)(x) = a_n \}$$

for a finite span of arrows

in $\mathcal{D}$ and $a_1, \ldots, a_n \in \mathbf{Sets}_\kappa$. Let the *coherent topology* on $G_\mathcal{D}$ be the coarsest topology such that $s, t : G_\mathcal{D} \rightrightarrows X_\mathcal{D}$ are both continuous and all sets of the form

$$\langle A, a \mapsto b \rangle = \{ \mathbf{f} : M \to N \mid a \in M(A) \wedge f_A(a) = b \}$$

are open, for A an object of $\mathcal{D}$ and $a, b \in \mathbf{Sets}_\kappa$.

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Remark 2.1.3 Note that if $\mathcal{D}$ is a Boolean algebra and we require coherent functors into **Sets** to send the terminal object to the distinguished terminal object $\{\star\}$ in **Sets**, then $X_{\mathcal{D}}$ is the Stone space of $\mathcal{D}$.

For $\mathcal{D}$ in $\mathbf{dCoh}_\kappa$, we have the decidable coherent theory $\mathbb{T}_{\mathcal{D}}$ of $\mathcal{D}$, and its syntactic category, $\mathcal{C}_{\mathbb{T}_{\mathcal{D}}}$ (as described in Section 1.1). Sending an object, D, in $\mathcal{D}$ to the object $[x : D \mid \top]$ in $\mathcal{C}_{\mathbb{T}_{\mathcal{D}}}$, and an arrow $f : C \to D$ to $[x : C, y : D \mid f(x) = y]$, defines a functor

$$\eta_{\mathcal{D}} : \mathcal{D} \longrightarrow \mathcal{C}_{\mathbb{T}_{\mathcal{D}}}$$

which is one half of an equivalence, the other half being the (or a choice of) canonical $\mathbb{T}_{\mathcal{D}}$-model in $\mathcal{D}$.

Now, any $\mathbb{T}_{\mathcal{D}}$-model, $\mathbf{M}$, in $\mathbf{Sets}_\kappa$ can be seen as a coherent functor, $\mathbf{M} : \mathcal{C}_{\mathbb{T}_{\mathcal{D}}} \longrightarrow \mathbf{Sets}_\kappa$. Composition with $\eta_{\mathcal{D}}$

induces restriction functions

commuting with source and target (as well as composition and insertion of identities) maps.

Lemma 2.1.4 *The maps ϕ_0 and ϕ_1 are homeomorphisms of spaces.*

PROOF Any coherent functor $M : \mathcal{D} \longrightarrow \mathbf{Sets}_\kappa$ lifts to a unique $\mathbb{T}_{\mathcal{D}}$-model $\mathbf{M} : \mathcal{C}_{\mathbb{T}_{\mathcal{D}}} \longrightarrow \mathbf{Sets}_\kappa$, to yield an inverse $\psi_0 : X_{\mathcal{D}} \to X_{\mathbb{T}_{\mathcal{D}}}$ to ϕ_0. Similarly, an invertible natural transformation of functors $f : M \to N$ lifts to a unique

$\mathbb{T}_{\mathcal{D}}$-isomorphism $\mathbf{f} : \mathbf{M} \to \mathbf{N}$ to yield an inverse $\psi_1 : G_{\mathcal{D}} \to G_{\mathbb{T}_{\mathcal{D}}}$ to ϕ_1. We verify that these four maps are all continuous. For a subbasic open

$$U = \langle\!\langle \langle f_1 : A \to B_1, \ldots, f_n : A \to B_n \rangle, \langle a_1, \ldots, a_n \rangle \rangle\!\rangle \subseteq X_{\mathcal{D}}$$

we have

$$\phi_0^{-1}(U) = \langle\!\langle [y_1 : B_1, \ldots, y_n : B_n \mid \exists x : A. \bigwedge_{1 \le i \le n} f_i(x) = y_i], \vec{a} \rangle$$

so ϕ_0 is continuous. To verify that ψ_0 is continuous, there are two cases to consider, namely non-empty and empty context. For basic open

$$\langle\!\langle [x : A_1, \ldots, x_n : A_n \mid \phi], \langle a_1, \ldots, a_n \rangle \rangle\!\rangle \subseteq X_{\mathbb{T}_{\mathcal{D}}}$$

the canonical interpretation of $\mathbb{T}_{\mathcal{D}}$ in $\mathcal{D}$ yields a subobject of a product in $\mathcal{D}$,

$$[\![x : A_1, \ldots, x_n : A_n \mid \phi]\!] \rightarrowtail A_1 \times \ldots \times A_n \xrightarrow{\pi_i} A_i.$$

Choose a monomorphism $r : R \rightarrowtail A_1 \times \ldots \times A_n$ representing that subobject. Then

$$\psi_0^{-1}(\langle\!\langle [x : A_1, \ldots, x_n : A_n \mid \phi], \langle a_1, \ldots, a_n \rangle \rangle\!\rangle)$$
$$= \langle\!\langle \langle \pi_1 \circ r : R \to A_1, \ldots, \pi_n \circ r : R \to A_n \rangle, \langle a_1, \ldots, a_n \rangle \rangle\!\rangle$$

and it is clear that this is independent of the choice of product diagram and of representing monomorphism. For the empty context case, consider a basic open $U = \langle\!\langle [\mid \varphi], \star \rangle\!\rangle$, where φ is a sentence of $\mathbb{T}_{\mathcal{D}}$ and $\star$ is the element of the distinguished terminal object of **Sets** (traditionally $\star = \emptyset$, notice that any $\langle\!\langle [\mid \varphi], a \rangle\!\rangle$ with $a \ne \star$ is automatically empty). The canonical interpretation of φ in $\mathcal{D}$ yields a subobject of a terminal object, $[\![\varphi]\!] \rightarrowtail 1$. Choose a representative monomorphism $r : R \rightarrowtail 1$. Then, independently of the choices made,

$$\psi_0^{-1}(U) = \bigcup_{a \in \mathbf{Sets}_\kappa} \langle\!\langle r : R \to 1, a \rangle\!\rangle.$$

So ψ_0 is continuous. With ϕ_0 continuous, it is sufficient to check ϕ_1 on subbasic opens of the form $U = \langle\!\langle A, a \mapsto b \rangle\!\rangle \subseteq G_{\mathcal{D}}$. But

$$\phi_1^{-1}(U) = \left(\begin{array}{c} \overline{} \\ [x : A \mid \top] : a \mapsto b \\ \overline{} \end{array} \right)$$

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so ϕ_1 is continuous. Similarly, it is sufficient to check ψ_1 on subbasic opens of the form

$$U = \left(\ [x : A \mid \overline{\top}] : a \mapsto b \ \underline{} \right)$$

but $\psi_1^{-1}(U) = \langle A, a \mapsto b \rangle$, so ψ_1 is continuous. $\dashv$

Corollary 2.1.5 *Definition 2.1.2 yields, for a decidable coherent category $\mathcal{D}$, a topological groupoid $\mathbb{G}_{\mathcal{D}}$ such that*

$$\mathbb{G}_{\mathcal{D}} \cong \mathbb{G}_{\mathbb{T}_{\mathcal{D}}}$$

in the category **Gpd**.

We can now state the main representation result of this section.

Theorem 2.1.6 *For a decidable coherent category with a saturated set of κ-small models, the topos of coherent sheaves on $\mathcal{D}$ is equivalent to the topos of equivariant sheaves on the topological groupoid $\mathbb{G}_{\mathcal{D}}$ of models and isomorphisms equipped with the coherent topology,*

$$\mathrm{Sh}(\mathcal{D}) \simeq \mathrm{Sh}(\mathbb{G}_{\mathcal{D}}) \,.$$

PROOF The equivalence $\eta_{\mathcal{D}} : \mathcal{D} \longrightarrow \mathcal{C}_{\mathbb{T}_{\mathcal{D}}}$ yields an equivalence $\mathrm{Sh}(\mathcal{D}) \simeq \mathrm{Sh}(\mathcal{C}_{\mathbb{T}_{\mathcal{D}}})$, whence

$$\mathrm{Sh}(\mathcal{D}) \simeq \mathrm{Sh}(\mathcal{C}_{\mathbb{T}_{\mathcal{D}}}) \simeq \mathrm{Sh}(\mathbb{G}_{\mathbb{T}_{\mathcal{D}}}) \cong \mathrm{Sh}(\mathbb{G}_{\mathcal{D}})$$

by Theorem 1.6.11. $\dashv$

2.2 The Semantical Functor Mod

We proceed to construct a 'syntax-semantics' adjunction between the category $\mathbf{dCoh}_\kappa$ (syntax) and a subcategory of topological groupoids (semantics). Given a coherent functor

$$F : \mathcal{A} \longrightarrow \mathcal{D}$$

between two objects of $\mathbf{dCoh}_\kappa$, precomposition with F,

$$\mathcal{A} \xrightarrow{\ F\ } \mathcal{D} \overset{M}{\underset{N}{\underset{\Downarrow}{\rightrightarrows}}} \mathbf{Sets}_\kappa$$

yields a homomorphism of (discrete) groupoids

$$
\begin{array}{ccc}
G_{\mathcal{D}} & \xrightarrow{\;\;f_1\;\;} & G_{\mathcal{A}} \\
{\scriptstyle s}\big\downarrow\big\downarrow{\scriptstyle t} & & {\scriptstyle s}\big\downarrow\big\downarrow{\scriptstyle t} \\
X_{\mathcal{D}} & \xrightarrow[\;\;f_0\;\;]{} & X_{\mathcal{A}}
\end{array}
\tag{7}
$$

We verify that f_0 and f_1 are both continuous. For basic open

$$
U = \langle\!\langle \langle g_1 : A \to B_1, \ldots, g_n : A \to B_n \rangle, \langle a_1, \ldots, a_n \rangle \rangle\!\rangle \subseteq X_{\mathcal{A}},
$$

we see that

$$
f_0^{-1}(U) = \langle\!\langle \langle F(g_1) : FA \to FB_1, \ldots, F(g_n) : FA \to FB_n \rangle, \langle a_1, \ldots, a_n \rangle \rangle\!\rangle \subseteq X_{\mathcal{D}}.
$$

And for basic open $U = \langle\!\langle C, a \mapsto b \rangle\!\rangle \subseteq G_{\mathcal{A}}$, we see that

$$
f_1^{-1}(U) = \langle\!\langle F(C), a \mapsto b \rangle\!\rangle \subseteq G_{\mathcal{D}}
$$

Thus composition with F yields a morphism of topological groupoids,

$$
f : \mathbb{G}_{\mathcal{D}} \longrightarrow \mathbb{G}_{\mathcal{A}},
$$

and thereby we get a contravariant functor,

$$
\mathrm{Mod} : \mathbf{dCoh}_{\kappa}^{\mathrm{op}} \longrightarrow \mathbf{Gpd}.
$$

which we shall refer to as the *semantical* functor.

Summarizing, for any decidable coherent category $\mathcal{D}$, we take

$$
\mathrm{Mod}(\mathcal{D}) = \mathrm{Hom}_{\mathbf{dCoh}}(\mathcal{D}, \mathbf{Sets}_{\kappa}),
$$

regarded as a groupoid of natural isomorphisms and equipped with the coherent topology, as in Definition 2.1.1.

2.3 The Syntactical Functor Form

We construct an adjoint to the semantical functor Mod from a subcategory of **Gpd** containing the image of Mod. As in the propositional (distributive

lattices) case, there are various subcategories that will work for this; we choose one such that is convenient for the present purpose, namely those groupoids $\mathbb{G}$ which are *coherent*, in the sense that $\mathrm{Sh}(\mathbb{G})$ is a coherent topos, that is, has a coherent site of definition (see [4, D3.3]).

Recall that an object A in a topos is *compact* if every covering of it (in terms of morphisms or subobjects) has a finite subcovering ([4, D3.3.2]).

Definition 2.3.1 CohGpd is the subcategory of **Gpd** consisting of coherent groupoids and those morphisms $f : \mathbb{G} \longrightarrow \mathbb{H}$ which preserve compact objects, in the sense that the induced inverse image functor $f^* : \mathrm{Sh}(\mathbb{H}) \longrightarrow \mathrm{Sh}(\mathbb{G})$ sends compact objects to compact objects.

Remark 2.3.2 Recall (e.g. from [4]) that:

(i) An object C in a topos $\mathcal{E}$ is *coherent* if (1) it is compact; and (2) it is *stable*, in the sense that for any morphism $f : B \longrightarrow A$ with B compact, the domain K of the kernel pair of f,

$$K \underset{k_2}{\overset{k_1}{\rightrightarrows}} B \overset{f}{\longrightarrow} A$$

is again compact.

(ii) In a coherent topos, $\mathrm{Sh}(\mathcal{C})$ say, with $\mathcal{C}$ a small coherent category, the full subcategory, $\mathcal{D} \hookrightarrow \mathrm{Sh}(\mathcal{C})$, of coherent objects is a pretopos. $\mathcal{D}$ forms a coherent site for $\mathrm{Sh}(\mathcal{C})$; includes $\mathcal{C}$ (through the Yoneda embedding); and is a pretopos completion of $\mathcal{C}$. Thus one can recover $\mathcal{C}$ from $\mathrm{Sh}(\mathcal{C})$ up to pretopos completion as the coherent objects.

(iii) Any compact decidable object is coherent. The full subcategory of decidable objects in a coherent category is again a coherent category. Accordingly, the full subcategory of compact decidable objects in a coherent topos is a decidable coherent category.

By Theorem 2.1.6, $\mathrm{Mod}(\mathcal{D})$ is a coherent groupoid, for any $\mathcal{D}$ in $\mathbf{dCoh}_\kappa$, and we can recover $\mathcal{D}$ from $\mathrm{Mod}(\mathcal{D})$, up to pretopos completion, by taking the compact decidable objects in $\mathrm{Sh}(\mathrm{Mod}(\mathcal{D}))$. For arbitrary coherent groupoids, however, this procedure will yield an decidable coherent category, but not necessarily one in $\mathbf{dCoh}_\kappa$, i.e. not necessarily with a saturated set of smaller than κ models. However, one can use the groupoid, $\mathbf{Sets}^*_\kappa$ of smaller than κ sets and bijections to classify a suitable collection of objects, as we now proceed to describe.

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2.3.3   The Decidable Object Classifier

Definition 2.3.4 The topological groupoid $\mathbb{S}$ consists of (hereditarily) κ-small sets with bijections between them, equipped with topology as follows. The topology on the set of objects, S_0, is generated by the empty set and basic opens of the form

$$\langle a_1, \ldots, a_n \rangle := \{A \in \mathbf{Sets}_\kappa \mid a_1, \ldots, a_n \in A\}$$

while the topology on the set, S_1 of bijections between κ-small sets is the coarsest topology such that the source and target maps $s, t : S_1 \rightrightarrows S_0$ are both continuous, and containing all sets of the form

$$\langle a \mapsto b \rangle := \left\{ f : A \xrightarrow{\cong} B \text{ in } \mathbf{Sets}_\kappa \ \middle| \ a \in A \wedge f(a) = b \right\}$$

We recognize $\mathbb{S}$ as the groupoid of models and isomorphisms for the decidable coherent theory, $\mathbb{T}_{\neq}$, of equality and inequality (with the obvious signature and axioms).

Lemma 2.3.5 *There is an isomorphism $\mathbb{S} \cong \mathbb{G}_{\mathbb{T}_{\neq}}$ in $\mathbf{Gpd}$.*

PROOF Any set A in $\mathbf{Sets}_\kappa$ is the underlying set of a canonical $\mathbb{T}_{\neq}$-model, and any bijection $f : A \to B$ is the underlying function of a $\mathbb{T}_{\neq}$-model isomorphism, and thereby we obtain bijections $S_0 \cong X_{\mathbb{T}_{\neq}}$ and $S_1 \cong G_{\mathbb{T}_{\neq}}$ which commute with source, target, composition, and embedding of identities maps. Remains to show that the topologies correspond. Clearly, any basic open $\langle \vec{a} \rangle \subseteq S_0$ corresponds to the open set $\langle [\vec{x} \mid \top], \vec{a} \rangle \subseteq X_{\mathbb{T}_{\neq}}$. We show that, conversely, any basic open $\langle [\vec{x} \mid \phi], \vec{a} \rangle \subseteq X_{\mathbb{T}_{\neq}}$ corresponds to an open set of S_0 by induction on $[\vec{x} \mid \phi]$. First, $\langle [\ \mid \top], a \rangle$ is $X_{\mathbb{T}_{\neq}}$ if $a = \star$ and empty otherwise, where $\{\star\}$ is the distinguished terminal object of $\mathbf{Sets}$, and $\langle [\vec{x} \mid \top], \vec{a} \rangle \cong \langle \vec{a} \rangle$. Next, $\langle [x, y \mid x = y], a, b \rangle$ corresponds to $\langle a \rangle \subseteq S_0$ if $a = b$, and the empty set otherwise. Similarly, $\langle [x, y \mid x \neq y], a, b \rangle$ corresponds to $\langle a, b \rangle \subseteq S_0$ if $a \neq b$ and the empty set otherwise. Now, suppose $\langle [\vec{x} \mid \phi], \vec{a} \rangle$ corresponds to an open set $U \subseteq S_0$. Then $\langle [\vec{x}, y \mid \phi], \vec{a}, b \rangle \cong U \cap \langle b \rangle$. Next, if $\langle [x, \vec{y} \mid \phi], a, \vec{b} \rangle$ corresponds to an open set $U_a \subseteq S_0$ for each $a \in \mathbf{Sets}_\kappa$, then $\langle [\vec{y} \mid \exists x. \phi], \vec{b} \rangle \cong \bigcup_{a \in \mathbf{Sets}_\kappa} U_a$. Finally, if $\langle [\vec{x} \mid \phi], \vec{a} \rangle$ and $\langle [\vec{x} \mid \psi], \vec{a} \rangle$ correspond to open sets $U, V \subseteq S_0$, then $\langle [\vec{x} \mid \phi \wedge \psi], \vec{a} \rangle \cong U \cap V$, and $\langle [\vec{x} \mid \phi \vee \psi], \vec{a} \rangle \cong U \cup V$. Therefore, $S_0 \cong X_{\mathbb{T}_{\neq}}$ as spaces. For the spaces of arrows, it remains only

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to observe that open subsets of the form $\langle a \mapsto b \rangle \subseteq S_1$ correspond to open subsets of the form

$$\begin{pmatrix} - \\ a \mapsto b \\ - \end{pmatrix} \subseteq G_{\mathbb{T}_{\neq}}$$

and we can conclude that $\mathbb{S}$ is a topological groupoid isomorphic to $\mathbb{G}_{\mathbb{T}_{\neq}}$ in **Gpd**. $\qquad\qquad\dashv$

The topos $\mathrm{Sh}(\mathbb{S})$ of equivariant sheaves on $\mathbb{S}$, therefore, classifies decidable objects, as $\mathrm{Sh}(\mathbb{S}) \simeq \mathrm{Sh}(\mathbb{G}_{\mathbb{T}_{\neq}}) \simeq \mathrm{Sh}(\mathcal{C}_{\mathbb{T}_{\neq}})$, where the last equivalence is by Theorem 1.6.11.

Corollary 2.3.6 *The groupoid $\mathbb{S}$ of small sets is coherent.*

Corollary 2.3.7 *There is an equivalence of toposes,*

$$\mathbf{Sets}^{\mathrm{Fin}_i} \simeq \mathrm{Sh}(\mathcal{C}_{\mathbb{T}_{\neq}}) \simeq \mathrm{Sh}(\mathbb{S})$$

where Fin_i is the category of finite sets and injections.

PROOF $\mathbf{Sets}^{\mathrm{Fin}_i} \simeq \mathrm{Sh}(\mathcal{C}_{\mathbb{T}_{\neq}})$ by [14, VIII, Exc.7–9]. $\qquad\qquad\dashv$

Definition 2.3.8 We fix the generic decidable object, $\mathcal{U}$, in $\mathrm{Sh}(\mathbb{S})$ to be the definable sheaf $\langle [\![x \mid \top]\!]_{X_{\mathbb{T}_{\neq}}} \to X_{\mathbb{T}_{\neq}} \cong S_0, \theta_{[x \mid \top]} \rangle$, which we also abbreviate as $\mathcal{U} = \langle U \to S_0, \theta_U \rangle$ (see the following remark).

Remark 2.3.9 Without reference to $\mathbb{T}_{\neq}$, we can characterize $\mathcal{U}$ as the following equivariant sheaf: $U \to S_0$ is the set over S_0 such that the fiber over a set $A \in S_0$ is the set A (i.e. $U = \coprod_{A \in S_0} A$), and the action by the set S_1 of isomorphisms is just applying those isomorphisms to the fibers. Thus, forgetting the topology, $\mathcal{U}$ is simply the inclusion $\mathbb{S} \hookrightarrow \mathbf{Sets}$. The topology on U is the coarsest such that the projection $U \to S_0$ is continuous and such that for each $a \in \mathbf{Sets}_\kappa$ the image of the section $s_a : \langle a \rangle \to U$ defined by $s_a(A) = a$ is an open set. It is straightforward to verify that this is just an alternative description of $\langle [\![x \mid \top]\!]_{X_{\mathbb{T}_{\neq}}} \to X_{\mathbb{T}_{\neq}} \cong S_0, \theta_{[x \mid \top]} \rangle$.

2.3.10 Formal Sheaves

We use the groupoid $\mathbb{S}$ of (small) sets to recover an object in $\mathbf{dCoh}_\kappa$ from a coherent groupoid by considering the set $\mathrm{Hom}_{\mathbf{CohGpd}}(\mathbb{G}, \mathbb{S})$ of morphisms into $\mathbb{S}$. (Consider the analogy to the propositional case, where the algebra of clopen sets of a Stone space is recovered by homming into the discrete space 2.) First, however, a note on notation and bookkeeping: because we shall be concerned with functors into $\mathbf{Sets}_\kappa$—a subcategory of $\mathbf{Sets}$ which is not closed under isomorphisms—we fix certain choices *on the nose*, instead of working up to isomorphism or assuming a canonical choice as arbitrarily given. Without going into the (tedious) details of the underlying bookkeeping, the upshot is that we allow ourselves to treat (the underlying set over G_0 and action of) an equivariant sheaf over a groupoid, $\mathbb{G}$ as a functor $\mathbb{G} \longrightarrow \mathbf{Sets}$ in an intuitive way. In particular, we refer to the definable set $[\![\vec{x} \mid \phi]\!]^{\mathbf{M}}$ as the fiber of $[\![\vec{x} \mid \phi]\!]_{X_\mathbb{T}} \to X_\mathbb{T}$ over $\mathbf{M}$, although that is not strictly speaking the fiber (strictly speaking the fiber is, according to our definition, the set $\{\mathbf{M}\} \times [\![\vec{x} \mid \phi]\!]^{\mathbf{M}}$). Moreover, we chose the induced inverse image functor $f^* : \mathrm{Sh}(\mathbb{H}) \longrightarrow \mathrm{Sh}(\mathbb{G})$ induced by a morphism, $f : \mathbb{G} \longrightarrow \mathbb{H}$, of topological groupoids so that, for $A \in \mathrm{Sh}(\mathbb{H})$ the fiber over $x \in G_0$ of $f^*(A)$ is the same set as the fiber of A over $f_0(x) \in H_0$. For example, and in particular, any morphism of topological groupoids $f : \mathbb{G} \longrightarrow \mathbb{S}$ induces a geometric morphism $f : \mathrm{Sh}(\mathbb{G}) \longrightarrow \mathrm{Sh}(\mathbb{S})$ the inverse image part of which takes the generic decidable object $\mathcal{U}$ of 2.3.8 to an (equivariant) sheaf over $\mathbb{G}$,

$$
\begin{array}{ccc}
A & \longrightarrow & U = [\![x \mid \top]\!]_{X_{\mathbb{T}_{\neq}}} \\
\downarrow & \lrcorner & \downarrow \\
G_0 & \xrightarrow{\quad f_0 \quad} & S_0
\end{array}
$$

such that the fiber A_x over $x \in G_0$ is the same set as the fiber of U over $f_0(x)$, which is the set $f_0(x) \in S_0 = \mathbf{Sets}_\kappa$. We hope that this is sufficiently intuitive so that we may hide the underlying book-keeping needed to make sense of it. With this in mind, then, we make the following stipulation.

Definition 2.3.11 For a coherent groupoid $\mathbb{G}$, let $\mathrm{Form}(\mathbb{G}) \hookrightarrow \mathrm{Sh}(\mathbb{G})$ be the full subcategory consisting of objects of the form $f^*(\mathcal{U})$ for all $f : \mathbb{G} \longrightarrow \mathbb{S}$ in $\mathbf{CohGpd}$. Such objects will be called *formal sheaves*.

Observe that:

Lemma 2.3.12 *For a coherent groupoid* $\mathbb{G}$*, a morphism* $f : \mathbb{G} \longrightarrow \mathbb{S}$ *of topological groupoids is in* **CohGpd** *if (and only if) the classified object* $f^*(\mathcal{U}) \in \mathrm{Sh}(\mathbb{G})$ *is compact.*

PROOF Assume that $f^*(\mathcal{U})$ is compact. By Corollary 2.3.7 we have that $\mathcal{C}_{\mathbb{T}_{\neq}}$ is a site for $\mathrm{Sh}(\mathbb{S})$. Write $\mathcal{S}$ for the image of the full and faithfull inclusion of $\mathcal{C}_{\mathbb{T}_{\neq}}$ into $\mathrm{Sh}(\mathbb{S})$. We have that $\mathcal{U}$ is in $\mathcal{S}$, as the image of the object $[x \mid \top]$ in $\mathcal{C}_{\mathbb{T}_{\neq}}$. Since $\mathcal{U}$ is decidable, so is $f^*(\mathcal{U})$. Therefore, since $f^*(\mathcal{U})$ is compact, it is coherent. Now, as an inverse image functor, f^* is coherent and since $\mathrm{Sh}(\mathbb{G})$ is a coherent topos, this means that for any A in $\mathcal{S}$ we have that $f^*(A)$ is coherent, and therefore, in particular, compact. Finally, for any compact object E in $\mathrm{Sh}(\mathbb{S})$ there is a cover $e : A_1 + \ldots + A_n \twoheadrightarrow E$, where $A_1, \ldots, A_n$ are objects of $\mathcal{S}$, and f^* takes this to a cover $e : f^*(A_1) + \ldots + f^*(A_n) \twoheadrightarrow f^*(E)$, whence $f^*(E)$ is compact. Hence f^* takes compact objects to compact objects, so $f : \mathbb{G} \longrightarrow \mathbb{S}$ is in **CohGpd**. ⊣

The formal sheaves on a coherent groupoid can be characterized directly:

Lemma 2.3.13 *An equivariant sheaf* $\mathcal{A} = \langle A \to G_0, \alpha \rangle$ *on a coherent groupoid* $\mathbb{G}$ *is formal just in case:*

(i) $\mathcal{A}$ *is compact decidable;*

(ii) *each fibre* A_x *for* $x \in G_0$ *is an element of* $\mathbf{Sets}_\kappa$*;*

(iii) *for each set* $a \in \mathbf{Sets}_\kappa$*, the set* $\langle \mathcal{A}, a \rangle = \{x \in G_0 \mid a \in A_x\} \subseteq G_0$ *is open, and the function* $s_{\mathcal{A},a} : \{x \in G_0 \mid a \in A_x\} \to A$ *defined by* $s(x) = a$ *is a continuous section; and*

(iv) *for any* $a, b \in \mathbf{Sets}_\kappa$*, the set*

$$\langle \mathcal{A}, a \mapsto b \rangle = \{g : x \to y \mid a \in A_x \wedge \alpha(g, a) = b\} \subseteq G_1$$

is open.

PROOF Let a morphism $f : \mathbb{G} \longrightarrow \mathbb{S}$ in **CohGpd** be given, inducing a geometric morphism $f : \mathrm{Sh}(\mathbb{G}) \longrightarrow \mathrm{Sh}(\mathbb{S})$ such that the inverse image preserves compact objects. Then $f^*(\mathcal{U})$ is a compact decidable object with fibers in $\mathbf{Sets}_\kappa$; the set $\langle f^*(\mathcal{U}), a \rangle = f_0^{-1}(\langle a \rangle) \subseteq G_0$ is open; the continuous section $\langle a \rangle \to U$ defined by $M \mapsto a$ pulls back along f_0 to yield the required section;

and the set $\langle f^*(\mathcal{U}), a \mapsto b \rangle = f_1^{-1}(\langle a \mapsto b \rangle) \subseteq G_1$ is open. So $f^*(\mathcal{U})$ satisfies conditions (i)–(iv).

Conversely, suppose that $\mathcal{A} = \langle A \to G_0, \alpha \rangle$ satisfies conditions (i)–(iv). Define the function $f_0 : G_0 \to S_0$ by $x \mapsto A_x$, which is possible since $A_x \in \mathbf{Sets}_\kappa$ by (ii). Then for a subbasic open set $\langle a \rangle \subseteq S_0$, we have

$$f_0^{-1}(\langle a \rangle) = \{ x \in G_0 \mid a \in A_x \} = \langle \mathcal{A}, a \rangle$$

so f_0 is continuous by (iii). Next, define $f_1 : G_1 \to S_1$ by

$$g : x \to y \quad \mapsto \quad \alpha(g, -) : A_x \to A_y.$$

Then for a subbasic open $\langle a \mapsto b \rangle \subseteq S_1$, we have

$$f_1^{-1}(\langle a \mapsto b \rangle) = \{ g \in G_1 \mid a \in A_{s(g)} \wedge \alpha(g, a) = b \} = \langle \mathcal{A}, a \mapsto b \rangle$$

so f_1 is continuous by (iv). It remains to show that $f^*(\mathcal{U}) = \mathcal{A}$. First, we must verify that what is a pullback of sets:

$$
\begin{array}{ccc}
A & \longrightarrow & U \\
\downarrow & \lrcorner & \downarrow \\
G_0 & \xrightarrow{\ f_0\ } & S_0
\end{array}
$$

is also a pullback of spaces. Let $a \in A$ with $V \subseteq A$ an open neighborhood. We must find an open box around a contained in V. Intersect V with the image of the section $s_{\mathcal{A},a}(\langle \mathcal{A}, a \rangle)$ to obtain an open set V' containing a and homeomorphic to a subset $W \subseteq G_0$. Then we can write V' as the box $W \times_{S_0} \langle [x, y \mid x = y], a \rangle$ for the open set $\langle [x, y \mid x = y], a \rangle \subseteq U$. Conversely, let a basic open $\langle [x, \vec{y} \mid \phi], \vec{b} \rangle \subseteq U$ be given, for ϕ a formula of $\mathbb{T}_{\neq}$. We must show that it pulls back to an open subset of A. Let $a \in A_z$ be given and assume that a (in the fiber over $f_0(z)$) is in $\langle [x, \vec{y} \mid \phi], \vec{b} \rangle$. Now, since $\mathcal{A}$ is decidable, there is a canonical interpretation of $[x, \vec{y} \mid \phi]$ in $\mathrm{Sh}(\mathbb{G})$ obtained by interpreting $\mathcal{A}$ as the single sort, and using the canonical coherent structure of $\mathrm{Sh}(\mathbb{G})$. Thereby, we obtain an object

$$\mathcal{B} := [\![x, \vec{y} \mid \phi]\!]^{\mathcal{A}} \hookrightarrow \mathcal{A} \times \ldots \times \mathcal{A} \xrightarrow{\pi_1} \mathcal{A}$$

in with an underlying open subset $B \subseteq A \times_{G_0} \ldots \times_{G_0} A \xrightarrow{\pi_1} A$. One can verify that $\mathcal{B}$ satisfies conditions (i)–(iv), se the proof of Lemma 2.3.15 below.

Let $W \subseteq B$ be the image of the continuous section $s_{\mathcal{B},a,\vec{b}}(\langle \mathcal{B}, a, \vec{b}\rangle)$. Then the pullback of $\langle [x, \vec{y} \mid \phi], \vec{b}\rangle$ along f_0 is the image of W along the projection $\pi_1 : \mathcal{A} \times \ldots \times \mathcal{A} \longrightarrow \mathcal{A}$, which is an open subset of A. $\dashv$

The logically definable objects in the category of equivariant sheaves on the groupoid of models and isomorphisms of a theory are readily seen to be a (guiding) example of objects satisfying conditions (i)–(iv) of Lemma 2.3.13, so we have:

Lemma 2.3.14 *For any $\mathcal{C}_\mathbb{T}$ in $\mathbf{dCoh}_\kappa$, the canonical interpretation functor $\mathcal{M}^\dagger$ of 1.6.11 factors through* $\mathrm{Form}(\mathbb{G}_\mathbb{T})$,

$$\mathcal{M}^\dagger : \mathcal{C}_\mathbb{T} \longrightarrow \mathrm{Form}(\mathbb{G}_\mathbb{T}) \hookrightarrow \mathrm{Sh}(\mathbb{G}_\mathbb{T})$$

Next, we show that the formal sheaves on a coherent groupoid form a decidable coherent category.

Lemma 2.3.15 *Let $\mathbb{G}$ be an object of $\mathbf{CohGpd}$. Then* $\mathrm{Form}(\mathbb{G}) \hookrightarrow \mathrm{Sh}(\mathbb{G})$ *is a (positive) decidable coherent category.*

PROOF We verify that $\mathrm{Form}(\mathbb{G})$ is closed under the relevant operations using the characterization of Lemma 2.3.13. By Remark 2.3.2, it suffices to show that conditions (ii)–(iv) of Lemma 2.3.13 are closed under finite limits, images, and finite coproducts.

Initial object. Immediate.

Terminal object. The canonical terminal object, write $\langle X' \to X, \alpha\rangle$, is such that the fiber over any $x \in G_0$ is $\{\star\} \in \mathbf{Sets}_\kappa$, whence the set $\{x \in G_0 \mid a \in X'_x\}$ is X if $a = \star$ and empty otherwise. Similarly, the set $\{g : x \to y \mid a \in X'_x \wedge \alpha(g, a) = b\} \subseteq G_1$ is G_1 if $a = \star = b$ and empty otherwise.

Finite products. We do the binary product $\mathcal{A} \times \mathcal{B}$. The fiber over $x \in G_0$ is the product $A_x \times B_x$, and so it is in $\mathbf{Sets}_\kappa$. Let a set $\langle \mathcal{A} \times \mathcal{B}, c\rangle$ be given. We may assume that c is a pair, $c = \langle a, b\rangle$, or $\langle \mathcal{A} \times \mathcal{B}, c\rangle$ is empty. Then,

$$\langle \mathcal{A} \times \mathcal{B}, \langle a, b\rangle\rangle = \langle \mathcal{A}, a\rangle \cap \langle \mathcal{B}, b\rangle$$

and the function $s_{\mathcal{A} \times \mathcal{B}, \langle a, b\rangle} : \langle \mathcal{A} \times \mathcal{B}, \langle a, b\rangle\rangle \to A \times_{G_0} B$ is continuous by the

following commutative diagram:

$$
\begin{array}{ccccc}
\langle\mathcal{A},a\rangle & \xleftarrow{\;\supseteq\;} & \langle\mathcal{A}\times\mathcal{B},\langle a,b\rangle\rangle & \xhookrightarrow{\;\subseteq\;} & \langle\mathcal{B},b\rangle \\
\downarrow{\scriptstyle s_{\mathcal{A},a}} & & \downarrow{\scriptstyle s_{\mathcal{A}\times\mathcal{B},\langle a,b\rangle}} & & \downarrow{\scriptstyle s_{\mathcal{B},b}} \\
A & \xleftarrow{\;\pi_1\;} & A\times_{G_0}B & \xrightarrow{\;\pi_2\;} & B
\end{array}
$$

Similarly, the set $\langle\mathcal{A}\times\mathcal{B},c\mapsto d\rangle$ is either empty or of the form

$$\langle\mathcal{A}\times\mathcal{B},\langle a,b\rangle\mapsto\langle a',b'\rangle\rangle$$

in which case

$$\langle\mathcal{A}\times\mathcal{B},\langle a,b\rangle\mapsto\langle a',b'\rangle\rangle=\langle\mathcal{A},a\mapsto a'\rangle\cap\langle\mathcal{B},b\mapsto b'\rangle.$$

Equalizers and Images. Let $\mathcal{A}$ be a subobject of $\mathcal{B}=\langle\pi_1:B\to G_0,\beta\rangle$, with $A\subseteq B$, and $\mathcal{B}$ satisfying the properties (ii)–(iv) of Lemma 2.3.13. Then given a set $\langle\mathcal{A},a\rangle$,

$$\langle\mathcal{A},a\rangle=\pi_1(A\cap s_{\mathcal{B},a}(\langle\mathcal{B},a\rangle))$$

and we obtain $s_{\mathcal{A},a}$ as the restriction

$$
\begin{array}{ccccc}
 & & A & \xhookrightarrow{\quad} & B \\
 & \nearrow{\scriptstyle s_{\mathcal{A},a}} & & \nearrow{\scriptstyle s_{\mathcal{B},a}} & \downarrow{\scriptstyle \pi_1} \\
\langle\mathcal{A},a\rangle & \xhookrightarrow{\quad} & \langle\mathcal{B},a\rangle & \xhookrightarrow{\quad} & G_0
\end{array}
$$

Similarly, given a set $\langle\mathcal{A},a\mapsto b\rangle\subseteq G_1$,

$$\langle\mathcal{A},a\mapsto b\rangle=\langle\mathcal{B},a\mapsto b\rangle\cap s^{-1}(\langle\mathcal{A},a\rangle)$$

where s is the source map $s:G_1\to G_0$. We conclude that Form $(\mathbb{G})$ is closed under both equalizers and images.

Binary coproducts. Write binary coproducts in $\mathbf{Sets}_\kappa$ as $X+Y=\{\langle 0,x\rangle,\langle 1,y\rangle\mid x\in X\wedge y\in Y\}$. Then if $\langle\mathcal{A}+\mathcal{B},c\rangle$ is non-empty, c is a pair

$c = \langle 0, a \rangle$ or $c = \langle 1, b \rangle$. If the former, then $\langle \mathcal{A} + \mathcal{B}, \langle 0, a \rangle \rangle = \langle \mathcal{A}, a \rangle$, and the section is given by composition:

$$
\begin{array}{ccc}
A & \xrightarrow{\;\;p_1\;\;} & A + B \\
\Big\uparrow{\scriptstyle s_{\mathcal{A},a}} & & \Big\downarrow \\
\langle \mathcal{A} + \mathcal{B}, \langle 0, a \rangle \rangle = \langle \mathcal{A}, a \rangle & \hookrightarrow & G_0
\end{array}
$$

The latter case is similar, and so is verifying that the set $\langle \mathcal{A} + \mathcal{B}, c \mapsto d \rangle$ is open. $\dashv$

Lemma 2.3.16 *Let $\mathbb{G}$ be a coherent groupoid. Then $\mathrm{Form}(\mathbb{G}) \hookrightarrow \mathrm{Sh}(\mathbb{G})$ has a saturated set of κ-small models.*

PROOF This follows from the fact that the coherent inclusion

$$\mathrm{Form}(\mathbb{G}) \hookrightarrow \mathrm{Sh}(\mathbb{G})$$

reflects covers, since every formal sheaf is compact, and any point, given by an element $x \in G_0$,

$$\mathbf{Sets} \longrightarrow \mathbf{Sets}/G_0 \twoheadrightarrow \mathrm{Sh}(G_0) \twoheadrightarrow \mathrm{Sh}(\mathbb{G}) \twoheadrightarrow \mathrm{Sh}(\mathrm{Form}(\mathbb{G}))$$

yields a coherent functor $\mathrm{Form}(\mathbb{G}) \longrightarrow \mathbf{Sets}_\kappa \hookrightarrow \mathbf{Sets}$, since the value of the point at an equivariant sheaf is the fiber over x, and formal sheaves have fibers in $\mathbf{Sets}_\kappa$. $\dashv$

Lemma 2.3.17 *If $f : \mathbb{G} \longrightarrow \mathbb{H}$ is a morphism of $\mathbf{CohGpd}$, then the induced coherent inverse image functor $f^* : \mathrm{Sh}(\mathbb{H}) \longrightarrow \mathrm{Sh}(\mathbb{G})$ restricts to a coherent functor $\mathrm{Form}(f) = F : \mathrm{Form}(\mathbb{H}) \longrightarrow \mathrm{Form}(\mathbb{G})$,*

$$
\begin{array}{ccc}
\mathrm{Form}(\mathbb{H}) & \xrightarrow{\;\;F\;\;} & \mathrm{Form}(\mathbb{G}) \\
\Big\updownarrow & & \Big\updownarrow \\
\mathrm{Sh}(\mathbb{H}) & \xrightarrow{\;\;f^*\;\;} & \mathrm{Sh}(\mathbb{G})
\end{array}
$$

PROOF If $\mathcal{A}$ is an object of $\mathrm{Form}(\mathbb{H})$ classified by $h : \mathbb{H} \longrightarrow \mathbb{S}$, then $f^*(\mathcal{A}) = F(\mathcal{A})$ is classified by $h \circ f : \mathbb{G} \longrightarrow \mathbb{S}$ in $\mathbf{CohGpd}$. $\dashv$

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This completes the construction of the 'syntactical' functor:

Definition 2.3.18 The functor

$$\text{Form} : \mathbf{CohGpd} \longrightarrow \mathbf{dCoh}_\kappa^{\text{op}}$$

is defined by sending a groupoid $\mathbb{G}$ to the decidable coherent category

$$\text{Form}(\mathbb{G}) \hookrightarrow \text{Sh}(\mathbb{G})$$

of formal sheaves, and a morphism $f : \mathbb{G} \longrightarrow \mathbb{H}$ to the restricted inverse image functor $f^* : \text{Form}(\mathbb{H}) \longrightarrow \text{Form}(\mathbb{G})$.

2.4 The Syntax-Semantics Adjunction

We now show that the syntactical functor is left adjoint to the semantical functor:

$$\mathbf{dCoh}_\kappa{}^{\text{op}} \underset{\text{Form}}{\overset{\text{Mod}}{\rightleftarrows}} \top \quad \mathbf{CohGpd}$$

First, we identify a counit candidate. Given $\mathcal{D}$ in $\mathbf{dCoh}_\kappa$, we have the 'evaluation' functor

$$\mathcal{Y}_\mathcal{D} : \mathcal{D} \longrightarrow \text{Sh}(\mathbb{G}_\mathcal{D})$$

which sends an object D to the 'definable' equivariant sheaf which is such that the fiber of $\mathcal{Y}(D)$ over $F \in X_\mathcal{D}$ is the set $F(D)$, or more informatively, such that the diagram,

$$
\begin{array}{ccc}
\mathcal{D} & \xrightarrow{\ \mathcal{Y}_\mathcal{D}\ } & \text{Sh}(\mathbb{G}_\mathcal{D}) \\
\eta_\mathcal{D} \downarrow & & \downarrow \cong \\
\mathcal{C}_{\mathbb{T}_\mathcal{D}} & \xrightarrow[\ \mathcal{M}^\dagger\ ]{} & \text{Sh}(\mathbb{G}_{\mathbb{T}_\mathcal{D}})
\end{array}
$$

commutes, using the map $\eta_\mathcal{D}$ and isomorphism $\mathbb{G}_\mathcal{D} \cong \mathbb{G}_{\mathbb{T}_\mathcal{D}}$ from Section 2.1. $\mathcal{Y}_\mathcal{D}$ factors through $\text{Form}(\mathbb{G}_\mathcal{D})$, by Lemma 2.3.14, to yield a coherent functor

$$\epsilon_\mathcal{D} : \mathcal{D} \longrightarrow \text{Form}(\mathbb{G}_\mathcal{D}) = \text{Form} \circ \text{Mod}(\mathcal{D})$$

And if $F : \mathcal{A} \longrightarrow \mathcal{D}$ is an arrow of $\mathbf{dCoh}_\kappa$, the square

$$
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\ \epsilon_\mathcal{A}\ } & \text{Form} \circ \text{Mod}(\mathcal{A}) \\
F \downarrow & & \downarrow \text{Form} \circ \text{Mod}(F) \\
\mathcal{D} & \xrightarrow[\ \epsilon_\mathcal{D}\ ]{} & \text{Form} \circ \text{Mod}(\mathcal{D})
\end{array}
$$

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commutes.

Next, we consider the unit. Let $\mathbb{H}$ be a groupoid in **CohGpd**. We construct a morphism

$$\eta_{\mathbb{H}} : \mathbb{H} \longrightarrow \mathbb{G}_{\mathrm{Form}(\mathbb{H})} = \mathrm{Mod}(\mathrm{Form}(\mathbb{G})).$$

First, as previously noticed, each $x \in H_0$ induces a coherent functor $\mathbf{M}_x : \mathrm{Form}(\mathbb{H}) \longrightarrow \mathbf{Sets}_\kappa$. This defines a function $\eta_0 : H_0 \to X_{\mathrm{Form}(\mathbb{H})}$. Similarly, any $a : x \to y$ in H_1 induces an invertible natural transformation $\mathbf{f}_a : \mathbf{M}_x \to \mathbf{M}_y$. This defines a function $\eta_1 : H_1 \to G_{\mathrm{Form}(\mathbb{H})}$, such that $\langle \eta_1, \eta_0 \rangle$ is a morphism of discrete groupoids. We argue that η_0 and η_1 are continuous. Let a subbasic open $U = (\langle g_1 : \mathcal{A} \to \mathcal{B}_1, \ldots, g_n : \mathcal{A} \to \mathcal{B}_n \rangle, \langle a_1, \ldots, a_n \rangle) \subseteq X_{\mathrm{Form}(\mathbb{H})}$ be given, with $g_i : \mathcal{A} = \langle A \to H_0, \alpha \rangle \longrightarrow \mathcal{B}_i = \langle B_i \to H_0, \beta_i \rangle$ an arrow of $\mathrm{Form}(\mathbb{H})$ and $a_i \in \mathbf{Sets}_\kappa$, for $1 \leq i \leq n$. Form the canonical product $\mathcal{B}_1 \times \ldots \times \mathcal{B}_n$ in $\mathrm{Sh}(\mathbb{H})$, so as to get an arrow $g = \langle g_1, \ldots, g_n \rangle : \mathcal{A} \longrightarrow \mathcal{B}_1 \times \ldots \times \mathcal{B}_n$ in $\mathrm{Form}(\mathbb{H})$. Denote by $\mathcal{C}$ the canonical image of g in $\mathrm{Sh}(\mathbb{H})$ (and thus in $\mathrm{Form}(\mathbb{H})$), such that the underlying set C (over H_0) of $\mathcal{C}$ is a subset of $B_1 \times_{H_0} \ldots \times_{H_0} B_n$. Then

$$\begin{aligned}
\eta_0^{-1}(U) &= \{ x \in H_0 \mid \exists y \in \mathbf{M}_x(\mathcal{A}). \, \mathbf{M}_x(g_i)(y) = a_i \text{ for } 1 \leq i \leq n \} \\
&= \{ x \in H_0 \mid \exists y \in A_x. \, g_i(y) = a_i \text{ for } 1 \leq i \leq n \} \\
&= \{ x \in H_0 \mid \langle a_1, \ldots, a_n \rangle \in \mathbf{M}_x(\mathcal{C}) \} \\
&= \{ x \in H_0 \mid \langle a_1, \ldots, a_n \rangle \in C_x \}
\end{aligned}$$

which is an open subset of H_0 by Lemma 2.3.13 since $\mathcal{C}$ is in $\mathrm{Form}(\mathbb{H})$. Thus η_0 is continuous. Next, consider a subbasic open of $G_{\mathrm{Form}(\mathbb{H})}$ of the form $U = (\mathcal{A}, a \mapsto b) \subseteq G_{\mathrm{Form}(\mathbb{H})}$, for $\mathcal{A} = \langle A \to H_0, \alpha \rangle$ in $\mathrm{Form}(\mathbb{H})$. Then

$$\begin{aligned}
\eta_1^{-1}(U) &= \{ g : x \to y \mid a \in \mathbf{M}_x(\mathcal{A}) \wedge (\mathbf{f}_g)_{\mathcal{A}}(a) = b \} \subseteq H_1 \\
&= \{ g : x \to y \mid a \in A_x \wedge \alpha(g, a) = b \} \subseteq H_1
\end{aligned}$$

which is an open subset of H_1, since $\mathcal{A}$ is in $\mathrm{Form}(\mathbb{H})$. Thus η_1 is also continuous, so that $\langle \eta_1, \eta_0 \rangle$ is a morphism of continuous groupoids.

Lemma 2.4.1 *The triangle*

$$(8)$$

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commutes.

PROOF Let $\mathcal{A} = \langle A \to H_0, \alpha \rangle$ in $\mathrm{Form}(\mathbb{H})$ be given, and write $E_{\mathcal{A}} \to X_{\mathrm{Form}(\mathbb{H})}$ for the underlying sheaf of $\mathcal{Y}_{\mathrm{Form}(\mathbb{H})}(\mathcal{A})$. Write $a : \mathbb{H} \to \mathbb{S}$ and $a' : \mathbb{G}_{\mathrm{Form}(\mathbb{H})} \to \mathbb{S}$, respectively, for the **CohGpd** morphisms classifying these objects. Then the triangle

$$
\begin{array}{ccc}
\mathbb{H} & \xrightarrow{\eta_{\mathrm{Form}(\mathbb{H})}} & \mathbb{G}_{\mathbf{Form}(\mathbb{H})} \\
& a \searrow \quad \swarrow a' & \\
& \mathbb{S} &
\end{array}
$$

in **Gpd** can be seen to commute. Briefly, for $x \in H_0$, we have $a(x) = A_x = \mathbf{M}_x(\mathcal{A}) = (E_{\mathcal{A}})_{\mathbf{M}_x} = (E_{\mathcal{A}})_{\eta_0(x)} = a'(\eta_0(x))$ and similarly for elements of H_1. ⊣

It follows from Lemma 2.4.1 that the inverse image functor $\eta^*_{\mathrm{Form}(\mathbb{H})}$ preserves compact objects, and so $\eta_{\mathrm{Form}(\mathbb{H})} : \mathbb{H} \longrightarrow G_{\mathrm{Form}(\mathbb{H})}$ is indeed a morphism of **CohGpd**. It remains to verify that it is the component of a natural transformation. Given a morphism $f : \mathbb{G} \longrightarrow \mathbb{H}$ of **CohGpd**, we must verify that the square

$$
\begin{array}{ccc}
\mathbb{G} & \xrightarrow{\eta_{\mathrm{Form}(\mathbb{G})}} & \mathrm{Mod} \circ \mathrm{Form}(\mathbb{G}) \\
f \downarrow & & \downarrow \mathrm{Mod} \circ \mathrm{Form}(f) \\
\mathbb{H} & \xrightarrow[\eta_{\mathrm{Form}(\mathbb{H})}]{} & \mathrm{Mod} \circ \mathrm{Form}(\mathbb{H})
\end{array}
$$

commutes. Let $x \in G_0$ be given. We chase it around the square. Applying $\eta_{\mathrm{Form}(\mathbb{G})}$, we obtain the functor $\mathbf{M}_x : \mathrm{Form}(\mathbb{G}) \longrightarrow \mathbf{Sets}$ which sends an object $\mathcal{A} = \langle A \to G_0, \alpha \rangle$ to A_x. Composing with $\mathrm{Form}(f) : \mathrm{Form}(\mathbb{H}) \longrightarrow \mathrm{Form}(\mathbb{G})$, we obtain the functor $\mathrm{Form}(\mathbb{H}) \longrightarrow \mathbf{Sets}$ which sends an object $\langle B \to H_0, \beta \rangle$ to the fiber over x of the pullback

$$
\begin{array}{ccc}
f_0^*(B) & \longrightarrow & B \\
\downarrow & & \downarrow \\
G_0 & \xrightarrow[f_0]{} & H_0
\end{array}
$$

which is the same as the fiber $B_{f_0(x)}$. And this is the same functor that results from sending x to $f_0(x)$ and applying $\eta_{\mathrm{Form}(\mathbb{H})}$. For $a : x \to y$ in

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G_1, a similar check establishes that $\eta_1 \circ f_1(a) : \mathbf{M}_{f_0(x)} \to \mathbf{M}_{f_0(y)}$ equals $\eta_1(a) \circ \mathrm{Form}(f) : M_x \circ \mathrm{Form}(f) \to \mathbf{M}_y \circ \mathrm{Form}(f)$. It remains to verify the triangle identities.

Lemma 2.4.2 *The triangle identities hold:*

$$
\begin{array}{ccc}
& \mathrm{Form}(\mathbb{H}) & \\
\mathrm{Form}(\eta_{\mathbb{H}}) \uparrow & = & \searrow\, 1_{\mathrm{Form}(\mathbb{H})} \\
\mathrm{Form} \circ \mathrm{Mod} \circ \mathrm{Form}(\mathbb{H}) & \xleftarrow{\ \epsilon_{\mathrm{Form}(\mathbb{H})}\ } & \mathrm{Form}(\mathbb{H})
\end{array}
$$

$$
\begin{array}{ccc}
& \mathrm{Mod}(\mathcal{D}) & \\
\eta_{\mathrm{Mod}(\mathcal{D})} \downarrow & = & \searrow\, 1_{\mathrm{Mod}(\mathcal{D})} \\
\mathrm{Mod} \circ \mathrm{Form} \circ \mathrm{Mod}(\mathcal{D}) & \xrightarrow{\ \mathrm{Mod}(\epsilon_{\mathcal{D}})\ } & \mathrm{Mod}(\mathcal{D})
\end{array}
$$

PROOF We begin with the first triangle, which we write:

$$
\begin{array}{ccc}
& \mathrm{Form}(\mathbb{H}) & \\
\mathrm{Form}(\eta_{\mathbb{H}}) \uparrow & & \nwarrow\, 1_{\mathrm{Form}(\mathbb{H})} \\
\mathrm{Form}(\mathbb{G}_{\mathrm{Form}(\mathbb{H})}) & \xleftarrow{\ \epsilon_{\mathrm{Form}(\mathbb{H})}\ } & \mathrm{Form}(\mathbb{H})
\end{array}
$$

This triangle commutes by the definition of $\epsilon_{\mathrm{Form}(\mathbb{H})}$ and Lemma 2.4.1, as can

be seen by the following diagram:

$$\begin{array}{ccc}
\mathrm{Form}(\mathbb{H}) & \xrightarrow{\ \epsilon_{\mathrm{Form}(\mathbb{H})}\ } & \mathrm{Form}(\mathbb{G}_{\mathbf{Form}(\mathbb{H})}) \\
\cap & & \cap \\
\downarrow & \overset{\displaystyle=}{{}_{\mathcal{Y}_{\mathrm{Form}(\mathbb{H})}}} & \downarrow \\
\mathrm{Sh}(\mathbb{H}) & \xleftarrow[\ \eta^*_{\mathbb{H}}\ ]{} & \mathrm{Sh}\big(\mathbb{G}_{\mathbf{Form}(\mathbb{H})}\big)
\end{array}$$

We pass to the second triangle, which can be written as:

$$\begin{array}{ccc}
\mathbb{G}_{\mathcal{D}} & & \\
\downarrow{\scriptstyle \eta_{\mathbb{G}_{\mathcal{D}}}} & \searrow{\scriptstyle 1_{\mathbb{G}_{\mathcal{D}}}} & \\
\mathbb{G}_{\mathbf{Form}(\mathbb{G}_{\mathcal{D}})} & \xrightarrow[\ \mathrm{Mod}(\epsilon_{\mathcal{D}})\ ]{} & \mathbb{G}_{\mathcal{D}}
\end{array}$$

Let $\mathbf{N} : \mathcal{D} \longrightarrow \mathbf{Sets}$ in $X_{\mathcal{D}}$ be given. As an element in $X_{\mathcal{D}}$, it determines a coherent functor $\mathbf{M_N} : \mathrm{Form}(\mathbb{G}_{\mathcal{D}}) \longrightarrow \mathbf{Sets}$, the value of which at $\mathcal{A} = \langle A \to X_{\mathcal{D}}, \alpha \rangle$ is the fiber $A_{\mathbf{N}}$. Applying $\mathrm{Mod}(\epsilon_{\mathcal{D}})$ is composing with the functor $\epsilon_{\mathcal{D}} : \mathcal{D} \longrightarrow \mathrm{Form}(\mathbb{G}_{\mathcal{D}})$, to yield the functor $\mathbf{M_N} \circ \epsilon_{\mathcal{D}} : \mathcal{D} \longrightarrow \mathbf{Sets}$, the value of which at an object B in $\mathcal{D}$ is the fiber over $\mathbf{N}$ of $\mathcal{Y}_{\mathcal{D}}(B)$, which of course is just $\mathbf{N}(B)$. For an invertible natural transformation $\mathbf{f} : \mathbf{M} \to \mathbf{N}$ in $G_{\mathcal{D}}$, the chase is entirely similar, and we conclude the the triangle commutes.$\dashv$

Theorem 2.4.3 *The contravariant functors* Mod *and* Form *are adjoint,*

$$\mathbf{dCoh}_{\kappa}{}^{\mathrm{op}} \underset{\mathrm{Form}}{\overset{\mathrm{Mod}}{\rightleftarrows}} {\scriptstyle\top} \ \mathbf{CohGpd}$$

where Mod *sends a decidable coherent category* $\mathcal{D}$ *to the semantic groupoid* $\mathrm{Hom}_{\mathbf{dCoh}}(\mathcal{D}, \mathbf{Sets}_{\kappa})$ *equipped with the coherent topology, and* Form *sends a coherent groupoid* $\mathbb{G}$ *to the full subcategory* $\mathrm{Form}(\mathbb{G}) \hookrightarrow \mathrm{Sh}(\mathbb{G})$ *of formal sheaves, i.e. those classified by the morphisms in* $\mathrm{Hom}_{\mathbf{CohGpd}}(\mathbb{G}, \mathbb{S})$.

Notice that if $\mathcal{D}$ is an object of $\mathbf{dCoh}_{\kappa}$, then the counit component $\epsilon_{\mathcal{D}} : \mathcal{D} \longrightarrow \mathrm{Form} \circ \mathrm{Mod}(\mathcal{D})$ is a Morita equivalence of categories, in the sense that

it induces an equivalence $\mathrm{Sh}(\mathcal{D}) \simeq \mathrm{Sh}(\mathrm{Form} \circ \mathrm{Mod}(\mathcal{D}))$. In the case where $\mathcal{D}$ is a pretopos, the counit is, moreover, also an equivalence of categories, since any decidable compact object in $\mathrm{Sh}(\mathcal{D})$ is coherent and therefore isomorphic to a representable in that case. Furthermore, for any $\mathcal{D}$ in $\mathbf{dCoh}_\kappa$, we have that the unit component $\eta_{\mathbb{G}_\mathcal{D}} : \mathbb{G}_\mathcal{D} \longrightarrow \mathbb{G}_{\mathrm{Form}(\mathbb{G}_\mathcal{D})}$ is a Morita equivalence of topological groupoids, in the sense that it induces an equivalence $\mathrm{Sh}(\mathbb{G}_\mathcal{D}) \simeq \mathrm{Sh}(\mathbb{G}_{\mathrm{Form}(\mathbb{G}_\mathcal{D})})$. We refer to the full image of Mod in $\mathbf{Gpd}$ as $\mathbf{SemGpd}$, the category of *semantic groupoids*.

Corollary 2.4.4 *The adjunction of Theorem 2.4.3 restricts to an adjunction*

$$\mathbf{dCoh}_\kappa{}^{\mathrm{op}} \underset{\mathrm{Form}}{\overset{\mathrm{Mod}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} \mathbf{SemGpd}$$

with the property that the unit and counit components are Morita equivalences of categories and topological groupoids respectively.

2.5   Stone Duality for Classical First-Order Logic

Returning to the classical first-order logical case, we can restrict the adjunction further to the full subcategory $\mathbf{BCoh}_\kappa \hookrightarrow \mathbf{dCoh}_\kappa$ of Boolean coherent categories. Unlike in the decidable coherent case, the pretopos completion of a Boolean coherent category is again Boolean, so that $\mathbf{BCoh}_\kappa$ is closed under pretopos completion. Since, as we mentioned in Section 1.1, completing a first-order theory so that its syntactic category is a pretopos involves only a conservative extension of the theory and does not change the category of models, it is natural to represent the classical first-order theories by the subcategory of Boolean pretoposes (see e.g. [15], [16]). We shall refer to the groupoids in the image of the semantic functor Mod restricted to the full subcategory of Boolean pretoposes $\mathbf{BPTop}_\kappa \hookrightarrow \mathbf{dCoh}_\kappa$, as *Stone groupoids*. Thus $\mathbf{StoneGpd} \hookrightarrow \mathbf{SemGpd}$ is the full subcategory of topological groupoids of models of theories in classical, first-order logic (the morphisms are still those continuous homomorphisms that preserve compact sheaves).

Corollary 2.5.1 *The adjunction of Theorem 2.4.3 restricts to an adjunction*

$$\mathbf{BPTop}_\kappa{}^{\mathrm{op}} \underset{\mathrm{Form}}{\overset{\mathrm{Mod}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} \mathbf{StoneGpd}$$

with the property that the unit and counit components are Morita equivalences of topological groupoids and equivalences of pretoposes, respectively.

Moreover, given the obvious notion of 'continuous natural transformation' of topological groupoid homomorphisms, the unit components of the foregoing adjunction can also be shown to be equivalences. Thus we have our main result:

Theorem 2.5.2 *The adjunction of Corollary 2.5.1 is a (bi-)equivalence,*

$$\mathbf{BPTop}_\kappa{}^{\mathrm{op}} \simeq \mathbf{StoneGpd} \tag{9}$$

establishing a duality between the category of (κ-small) Boolean pretoposes and Stone topological groupoids.

Finally, a remark on the posetal case and classical Stone duality for Boolean algebras. By a *coherent space* we mean a compact topological space such that the compact open sets are closed under intersection and form a basis for the topology. A *coherent function* between coherent spaces is a continuous function such that the inverse image of a compact open is again compact. Stone duality can be obtained as a restriction of a contravariant adjunction between the category **dLat** of distributive lattices and homomorphisms and the category **CohSpace** of coherent spaces and coherent functions

$$\mathbf{dLat}^{\mathrm{op}} \underset{\longleftarrow}{\overset{\top}{\longrightarrow}} \mathbf{CohSpace} \tag{10}$$

where, as in Stone duality, the right adjoint is the 'Spec' functor obtained by taking prime filters (or homming into the lattice 2), and the left adjoint is obtained by taking the distributive lattice of compact opens (or homming into the Sierpiński space, i.e. the set 2 with one open point). This adjunction restricts to a contravariant equivalence between distributive lattices and sober coherent spaces, and further to the full subcategory of Boolean algebras, **BA** $\hookrightarrow$ **dLat**, and the full subcategory of Stone spaces and continuous functions, **Stone** $\hookrightarrow$ **CohSpace**, so as to give the contravariant equivalence of classical Stone duality:

$$\mathbf{BA}^{\mathrm{op}} \underset{\longleftarrow}{\overset{\simeq}{\longrightarrow}} \mathbf{Stone} \tag{11}$$

The adjunction (10) can be obtained from the adjunction of Theorem 2.4.3 as follows. A poset is a distributive lattice if and only if it is a coherent

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category (necessarily decidable), and as we remarked after Definition 2.1.1, such a poset always has enough κ-small models, so that

$$\mathbf{dLat} \hookrightarrow \mathbf{dCoh}_\kappa$$

is the subcategory of posetal objects. On the other side, any space can be considered as a trivial topological groupoid, with only identity arrows, and it is straightforward to verify that this yields a full embedding

$$\mathbf{CohSpace} \hookrightarrow \mathbf{CohGpd}.$$

Since a coherent functor from a distributive lattice $\mathcal{L}$ into **Sets** sends the top object in $\mathcal{L}$ to the terminal object 1 in **Sets**, and everything else to a subobject of 1, restricting the semantic functor Mod to **dLat** gives us the right adjoint of (10). In the other direction, applying the syntactic functor Form to the subcategory $\mathbf{CohSpace} \hookrightarrow \mathbf{CohGpd}$ does not immediately give us a functor into **dLat**, simply because the formal sheaves do not form a poset (for instance, by Lemma 2.3.13, the formal sheaves on a coherent groupoid include all finite coproducts of 1). However, if we compose with the functor $\mathrm{Sub}(1) : \mathbf{dCoh}_\kappa \longrightarrow \mathbf{dLat}$ which sends a coherent category $\mathcal{C}$ to its distributive lattice $\mathrm{Sub}_\mathcal{C}(1)$ of subobjects of 1, then it is straightforward to verify that we have a restricted adjunction

$$\mathbf{dLat}^{\mathrm{op}} \underset{\mathrm{Form}_1}{\overset{\mathrm{Mod}}{\underset{\top}{\rightleftarrows}}} \mathbf{CohSpace}$$

where $\mathrm{Form}_1(\mathcal{C}) = \mathrm{Sub}_{\mathrm{Form}(\mathcal{C})}(1)$. Moreover, this is easily seen to be precisely the adjunction (10), of which classical Stone duality for Boolean algebras is a special case. Indeed, again up to the reflection into $\mathrm{Sub}(1)$, the duality (11) is precisely the poset case of the duality (9) between (κ-small) Boolean pretoposes and Stone topological groupoids.

Acknowledgements

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