Could the truths of mathematics have been different?

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Abstract

Could the truths of mathematics have been different than they in fact are? If so, which truths could have been different? Do the contingent mathematical facts supervene on physical facts, or are they free floating? I investigate these questions within a framework of higher-order modal logic, drawing sometimes surprising connections between the necessity of arithmetic and analysis and other theses of modal metaphysics: the thesis that possibility in the broadest sense is governed by a logic of $\mathsf{S5}$, that what is possible holds in some maximally specific possibility, and that every property can be rigidified. The investigation will distinguish sharply between *platonic contingency*—contingency about whether particular abstract "platonic" mathematical objects are arranged in a certain way (e.g. in a natural number or real number structure)—from a deeper variety of *structural contingency* concerning what holds of objects whenever they *are* arranged in that way.

Consider the following examples of mathematical claims:

1. If there are four apples and three pears in the bowl, and no apple is a pear, then there are seven apples or pears in the bowl.

2. The ratio between the diameter and circumference of a circle in Euclidean space is π.

*Thanks to ...

3. Every positive whole even number can be expressed as the sum of two primes.

4. Every collection of reals with strong measure 0 is countable.

5. There is no tree of uncountable height such that every anti-chain and every branch is at most countable.

6. Any two uncountable collections of reals have the same cardinality.

The question with which we are concerned is whether the truths of mathematics could have been different than they in fact are. Moreover, if there are contingent mathematical truths, which ones?

Is it possible, for instance, that the ratio between the diameter and circumference of a Euclidean circle be something other than the number π?[1] If so, what would it be like to live in a Euclidean world where that ratio is different — wouldn't something go horribly wrong? If you think the answer here is obvious, what about the more abstract mathematical claims, such as 4, 5 and 6 above, which also can have interpretations in Euclidean physical space, but are independent of our most widely accepted mathematical and physical theories?

Before I proceed, some clarifications are in order. When I ask whether the truths of mathematics could have been different, I mean to ask whether they could have been different with respect to the *broadest* notion of possibility; i.e. could they have been different with respect to any genuine modality whatsoever. What does the qualification to "genuine" modalities amount to here? This is not the place to give a fully satisfactory answer (I have attempted to do so elsewhere[2]); just note that if we are not careful, our question can be read in a way that makes it trivially true. Some uses of the word 'could' in English are purely epistemic: in this purely epistemic sense it is obvious that Goldbach's conjecture could be true, and could be false, in virtue of the fact that we do not know which it is. A proper account of modality should not count these purely epistemic uses of 'could' as expressing genuine modalities—that something is up with them is already illustrated by the fact that Leibniz's law has the appearance of failing in their

[1] One could certainly imagine that the geometry of physical space were different in such a way that the corresponding ration of a physical circle be different from π — what we are envisaging is a world where space is Euclidean and this happens.

[2] Bacon (2018a), Bacon and Zeng (2022).

scope[3], suggesting that these claims are covertly metalinguistic, or sensitive to modes of presentations, and are not speaking directly about the mathematical facts.[4] A seemingly separate question, that I will leave to one side, is whether mathematics is *metaphysically* contingent. Perhaps this is the very same as the question of its contingency in the broadest sense. However, if it turned out that broad possibility turned out to be quite permissive, some may instead be tempted to take the term 'metaphysical possibility' to refer to a more familiar restricted notion of of possibility according to which various post-Kripkean theses hold, including the necessity of mathematics. This is a question of philosophical terminology that I consider a side issue.

While we will take the existence of some pretty strange possibilities seriously, we will assume that one special class of statements are broadly necessary: those expressed by theorems of classical logic. When higher-order logic is concerned, people sometimes use the term "logic" in a way that includes all sorts of non-obvious things, such as higher-order choice principles or versions of the continuum hypothesis.[5] But here I will reserve the term "classical logic" for logic in a very narrow sense: the statements that can be derived from the classical introduction and elimination rules for the logical connectives and the quantifiers (subjecting the higher-order quantifiers to the analogues of the usual rules for first-order quantification), and some principles for reasoning with complex predicates formed by λ-abstraction. Thus we take things that are logically inconsistent in the narrow sense to be impossible. (We might, in the same spirit, extend this status of broad necessity to certain principles of *modal* logic as well, and we will do so at

[3]Consider the following exchange: "Doesn't it seem as though Clark Kent has the same build as Superman?", "No, I think Superman could be taller than Clark Kent". The second statement seems true given the second speakers state of knowledge, but a lot of priming is needed to get into a context where "No, I think Superman could be taller than Superman" doesn't sound terrible.

[4]Higher-order theories of genuine modalities include Bacon and Zeng (2022), Bacon (2023a) chapter 7. Relevant discussion, especially regarding what counts as a "genuine modality" can be found in Dorr et al. (2021) §8.2-8.3. The details of these theories are not so important here – the key distinction for our purposes is (putting it somewhat imprecisely) that one can infer from a sentence involving a genuine use of a modality another sentence that existentially generalizes into operator position, whereas that move is not as straightforward on the standard pictures about opacity (say, if there are intra-sentential context shifts). Similar accounts can be found in Fritz (MS), Dorr et al. (2021), Roberts [REF], Goodsell and Yli-Vakkuri (MS).

[5]Typically the latter happens when one identifies logic with the theory of some particular class of models.

some points as well.) This constraint is consequential to our investigation. For instance, since our first example appears to be one that formalizes to a theorem of first-order logic, we have a fairly direct route to settling the question of its contingency negatively.

Establishing that a claim is true (or that it is false) by purely logical means, as we have done here, let's us infer that is necessarily true (necessarily false) and so gives us one route to establishing non-contingency. Note, however, that it is possible to establish the non-contingency of a statement without first settling whether that statement is true or false. This is crucial: Gödel's incompleteness theorem tells us that there are some mathematical statements our axiomatic system of logic does not decide to be true or false. However, while the incompleteness theorems preclude us from settling all mathematical statements via the axiomatic method, they do not preclude us from settling the *contingency* of all mathematical statements by the axiomatic method. We will see that broadly logical principles can decide the contingency of many undecidable statements—we can establish that a claim is either necessary or impossible without establishing which it is.

A final point. The reader may be wondering why one needs an extended philosophical discussion of the titular question—after all, the necessity of mathematics is one of a few philosophical theses that enjoys widespread agreement among philosophers.

Well, it is not entirely universal.[6] But even setting aside the occasional voice of dissent, it is valuable in its own right to revisit orthodoxy from time to time. While the necessity of mathematics is an oft repeated claim, it is rare to see positive arguments in its favor, exposing it to the accusation of being dogma rather than established orthodoxy.[7] What is more, even if we end up reaffirming orthodoxy we may learn something in the process. In the same spirit, we will extend an unprejudiced attitude toward many other key components of conventional modal doctrine—including the thesis that

[6]Hartry Field has argued that platonic mathematics is contingent with respect to a broad logical modality (see essays 1 and 3 of Field (1989)), and modality is frequently employed in the study of indefinitely extensible concepts like 'set' and 'ordinal' yielding a limited form of mathematical contingency. The latter sort of view might motivate contingency about how many inaccessible cardinals there are, but not much else; see Linnebo (2013), Studd (2013).

[7]One exception is the result in Leitgeb (2020), although this result is of rather limited interest for reasons we will return to in section 1. Another is Goodsell (2022), which we discuss below.

broad necessity has a logic of $\mathsf{S5}$, the principle that every property can be rigidified and the Leibnizian idea that if something is possible, it is true at a maximally specific possibility. One upshot of our discussion will be that there are non-obvious logical connections between these three ideas. One moral that can be drawn from the discussion is that the orthodox package of $\mathsf{S5}$, possible world metaphysics, and the non-contingency of mathematics, is mutually reinforcing. On the other hand, reasons to doubt some parts of the package may require, or at least reopen consideration of other parts. Once one strays outside the standard package, the logical landscape is intricate and of interest in its own right.

1 Platonic Contingency

The discipline of mathematics is often said to have originated from the general study of patterns in nature.[8] The fact that there are seven goats, that this stick is longer than that one, that this stake fits in that hole, and so on, involve notions such as number, magnitude and shape. And there are various general facts about number, magnitude and shape that do not depend on the particular objects or properties possessing that number, magnitude or shape—for instance, the fact that *if seven of the goats are small and three of the goats are not small, then there are ten goats* is general, and would hold if we replaced *small* and *goat* with *female* and *sheep*, or any other pair of properties. Thus the discipline of mathematics arose from the the need to capture these generalizations.

There are various sorts of facts that philosophers have identified with the "truths of mathematics". Different accounts may even have modal differences—what a nominalist puts forward as the content of a given mathematical claim may not be necessarily equivalent with the content offered by the platonist. Provided the sorts of facts posited by both sides exist, it follows that there are potentially different questions to be asked about the contingency of each sort of fact. The most interesting form of mathematical contingency, in my view, will also involve contingency about the sorts of generalizations about the world that mathematics is supposed to represent. Indeed, when a class of facts comes apart from these generalizations modally, I take this to indicate that those facts are failing to properly correspond to mathematical reality.

[8] See, for instance, Boyer (2011) p1.

I will organize our discussion of mathematical contingency around two important strands of thought on the subject matter of mathematics: *platonism* and *structuralism*. According to the mathematical platonist, the truths of mathematics are about *particular* individuals: numbers, vector spaces, sets and the like—abstract individuals not directly accessible to the senses, or located in space or time. The mathematical structuralist, by contrast, maintains that the statements of mathematics are not about particular individuals. Instead they make make broadly structural claims, which will imply facts about particular individuals when those individuals instantiate the relevant structures. On the version of structuralism we will explore, the mathematical truths *just are* the generalizations alluded to above.[9]

Let us for the sake of argument grant that the platonic abstract objects exist, and that they instantiate mathematical properties and relations, such as *being prime*, *being the successor of*, and so on. Call propositions that ascribe these mathematical properties and relations to platonic mathematical objects the *platonic propositions*. In addition to the platonic propositions, there will also be general propositions stating that any individuals, abstract or otherwise, standing in certain structural relationships to one another will have the analogues of these properties relative to the relations in question: call these the *structural propositions*. When taking mathematical contingency seriously we should not assume these two sorts of propositions are necessarily equivalent. We consequently distinguish two sorts of mathematical contingency: a relatively superficial form of contingency about the behaviour of some particular objects that happen to be of an abstract nature, and a more radical sort concerning what is true about individuals of any sort that are appropriately related to one another.

Let's begin with the first. For there to be platonic mathematical contingency the platonic abstract objects must have at least some mathematical properties contingently. Take a platonic truth such as *2 is less than 7*. There are several states of affairs, consistent by logical lights, that one could posit to witness contingency about this fact.[10] Perhaps the particular abstract object 2 could have been bigger than 7 by switching positions with it in the number series, in the same way that John could have been taller than Mary,

[9]See for instance Hellman (1989), or Parsons (1990) section 3 and 6.

[10]We can witness the consistency by considering a possible worlds model with two worlds, a constant domain of natural numbers, with the extension of $<$ at the first world the usual ordering on those numbers, and its extension at the other world the usual ordering except with 2 and 7 switched (i.e. $1, 7, 3, 4, 5, 6, 2, 8, 9, 10, \dots$).

even if he in fact isn't. Or perhaps there are possibilities where the number 2 is a Roman emperor, and doesn't stand in any of the usual mathematical relations; and so on.

At this point it is worth noting that both Leitgeb (2020) and Goodsell (2022) (sections 3 and 4) offer arguments for the non-contingency of platonic mathematics. But these arguments both take as their starting point the non-contingency of basic mathematical statements, like $2 < 7$, and so do not bear straightforwardly on the sort of platonic contingency we are considering here. Leitgeb also takes for granted the Barcan formula which is rejected by mathematical contingentists such as Parsons (1983), Linnebo (2013), Studd (2013), Berry (2022). (The result in section 5 of Goodsell (2022) is more interesting and we will return to it later.[11])

It is not necessary, nor is it my goal, to establish that such possibilities exist. Note, however, that if it was a necessary truth that 2 is less than 7, then it would seem to be a brute unexplained necessary connection between two distinct individuals. In other branches of science similar posits are generally regarded with suspicion. Why do we think that velocity is the same as the derivative of position with respect to time? Well, if they were different quantities one would be left without an explanation the fact that they modally vary in tandem (whereas if they are the same property there is no mystery). If we are convinced that we shouldn't posit mysterious coincidences like this in physics, we shouldn't regard necessary connections between mathematical objects any differently. On any view short of logicism there will be cases where there is no logical explanation of the connection between 2 and 7.[12] Note that adopting this Humean picture of broad, or "logical"

[11]Another sort of argument for the necessity of mathematics can be found in Yli-Vakkuri and Hawthorne (2020), who appeal to the use of counterfactuals in mathematical reasoning. However this style of argument at most establishes the "counterfactual necessity" of mathematics—it is necessary in the sense that if mathematics had been false then anything would be true.

[12]One way to explain a necessary connection is to show that one sort of individual—velocities, fusions, shadows, etc— are logical constructions out of another—trajectories, points, shadowcasters etc. The logicist might take the position that the platonic 7 is a logical construction out of the platonic 2 (and both ultimately of 0), via logical successor operation. However, this goes beyond our usual notion of logical operation; see the discussion in section 13.1 of Bacon (2023a). There is a different nominalist kind of logicism which replaces first-order quantification over platonic objects with higher-order quantification into the position of numerical quantifiers in which this construction is uncontroversially logical—we'll return to that in section 4.

necessity, does not preclude us from employing restricted modalities that preserve some given class of connections—perhaps the post-Kripkean notion of metaphysical necessity is one such notion preserving essentialist connections and mathematical connections between the platonic objects.[13]

As I have already said, despite being a natural precisification of "mathematical contingency", platonic contingency strikes me as a rather shallow and uninteresting thing to mean by it. For once one takes in earnest the idea that abstract objects can have their mathematical properties contingently, it seems much less clear that the platonic truths really correspond to, or even *modally track*, the sorts of mathematical patterns in nature and elsewhere that we started out with—facts that arguably have a better claim to being the "real" truths that mathematics aims to capture. To dramatize this, imagine that there is a possibility in which the number 7 is a Roman emperor. In this possibility, 7 is not a prime number. Nonetheless, it presumably would not be possible to take seven different pebbles and arrange them correctly into a non-trivial rectangle.[14] What does it mean to take seven different pebbles, when the platonic 7 is not even a number? Well, we can always spell this out using the the vocabulary of first-order quantification and identity: there are pebbles $x_1, ..., x_7$, such that such that x_1 is different from x_2 and x_1 is different from x_3 and .... The claim that seven pebbles are arranged in a non-trivial rectangle could be expressed with a more complicated statement involving quantifiers, identity and predicates expressing spatial relations. This suggests that there is a pattern in nature associated with a mathematical statement, like '7 is prime', which ought to have these claims about pebbles, as well as other applications of arithmetic to the wider world, as logical consequences. To posit strange possibilities for the platonic number 7 is to allow that the properties of the platonic 7 can become unmoored from these patterns. A more interesting, and radical form of mathematical contingency, then, would involve contingency in the patterns themselves.

[13]Certain puzzles of material constitution may similarly push you to think that any individuals can occupy any qualitative role—the most pure form of this position would be formulated as a general claim, without any special exceptions for abstract individuals. See the discussion in Dorr et al. (2021) chapter 14.

[14]Here a trivial rectangle is a single row of seven pebbles, or a single column of seven pebbles.

2 Structural Contingency

According to the structuralist (or at least the version we are focusing on) the contents of mathematical statements are higher-order generalizations whose instances include the sorts of natural patterns we have been interested in. Consider the days, lying in our future, under the temporal ordering. These instantiate the same structure as the platonic numbers under their mathematical ordering: they form a "natural number structure". The day a week from now plays the same structural role as the number 7 in the platonic natural number sequence, and is "prime" in a variant sense spelled out in terms of operations on days analogous to the usual operations on platonic numbers. Many ordinary relations could in principle form a natural number structure: imagine, for instance, that nobody loves Herb, and Herb loves only Mary, and Mary loves only John and, and so on.

Note that in our strange possibility, where the number 7 has become a Roman emperor, the remaining numbers, assuming they have retained their relative positions in the ordering of the numbers, still form a natural number sequence, namely $0, 1, 2, 3, 4, 5, 6, 8, 9, 10, \ldots$. In this sequence the 8 now satisfies the role the 7 used to play, and is "prime" in a natural variant sense that is relevant to the new ordering. So even if it can be contingent whether the particular individual, 7, is prime, it is hard to see how the corresponding structural claim, about the seventh element of any natural number structure, could be contingent; structural contingency can not be obtained on the cheap.

At a first parse the structural content associated with *7 is prime* can be approximated with an infinite list of particular instances of the pattern:

> If Herb loves but is not loved, no two people love the same person and ... , then the first seven lovers cannot be divided non-trivially into equal parts.
>
> If Mary kicks someone but is not kicked, no two people kick the same person and ... , then the first seven kickers cannot be divided non-trivially into equal parts.
>
> $\vdots$

where the first ... stands for the claim that the loved individuals bears the "ancestral" of *loves* to the loved. I.e., for any given loved individual, the unloved lover either loves them, or loves somebody who loves them, or loves somebody who loves somebody who loves them, or .... We can capture the general pattern with a higher-order generalization:

> $\forall R$ and $\forall x$ if x Rs but is not Red, no two things R the same thing and x bears the ancestral of R to anything Red, then the first seven Rers cannot be divided non-trivially into equal parts.

The notion of the ancestral of a relation R used above can also be eliminated in favour of higher-order quantification using Frege's definition (Frege (1879) §79, Frege (1893) Part I §45): x bears any transitive relation extending R to y.

By universal instantiation, we can instantiate R with any binary predicate and x with any name to obtain any sentence in our original list—for by instantiating R and x with *is followed by* and *today* we obtain our first example. The general recipe for determining the structural content of a mathematical claim is then this. Suppose $A(0, S)$ is an arithmetical statement involving only the constant 0 and the successor relation S, such that all of its first-order quantifiers are restricted to the field of S.[15] We can notate the antecedents of the above conditionals as Nat(Herb, loves), Nat(Mary, kicks), and so on—registering that this notation represents a sentence containing a predicate R (loves, kicks, etc.), not a first-order predicate being applied to a name. Thus

Structural Content The structural content of a platonic arithmetical claim $A(0, S)$ is defined as

$$\forall R \forall x(\mathrm{Nat}(x, R) \to A(x, R)).$$

We will notate the structural translation of an arithmetical statement A, as A^*. There is a similar translation available for arithmetical statements belonging to richer languages, perhaps with constants for addition, multiplication, and the ordering on the numbers. When a given selection of arithmetical constants is salient, we will adopt the convention of writing $\mathbf{X}$ for a sequence of variables that match the types of the constants in that language, and $\mathrm{Nat}(\mathbf{X})$ for the appropriate notion of natural number structure for that signature, i.e. with further conditions for the new constants. The variables

[15] In the higher-order context any arithmetical statement is equivalent to one stated only in terms of 0 and successor so this restriction is not really costing us any generality.

and context will always make clear what notion of natural number structure is relevant.[16] Then the structural content of an arithmetical statement $A(\ldots)$ in those constants is just $\forall\mathbf{X}(\mathrm{Nat}\,\mathbf{X} \to A(\mathbf{X}))$ where $\forall\mathbf{X}$ stands for a sequence of quantifiers each binding a single variable in $\mathbf{X}$. (This generalization is important because in a first-order context, addition and multiplication cannot be defined from successor and 0 and so are needed as extra primitives; this notation also allows us hide the signature at play when it is a distraction.)

It should be clear why structural contents, unlike platonic contents, can imply functional mathematical claims like those we outlined earlier. Higher-order generalizations directly imply claims about, say, spatial arrangements of pebbles by instantiating the first-order generalizations with pebbles, and the second-order generalizations with the relevant spatial relations. However, although structural propositions *almost* play the right mathematical functional role, there is one wrinkle. Suppose there had only been eight things: me and seven pebbles. In such a possibility the structural translation of "7 is the product of two numbers greater than 1" is vacuously true, because there are no natural number structures, but I still could not arrange those pebbles in a non-trivial rectangle. We will fix this lacuna shortly, and give a more general justification that the structural propositions play the right functional role when there are natural number structures. A more radical form of mathematical contingency, then, would be contingency in the structural propositions, of the form A^*—contingency in the mathematical patterns themselves.

3 Higher-order Generalizations

I have drawn a sharp distinction between higher-order generalizations formed in the language of pure higher-order logic, and platonic statements formulated in terms of special primitives governing platonic mathematical objects. Are these really so different? In the literature on structuralism, a negative answer is often taken for granted. Parsons', for instance, writes:

> 'if the eliminative structuralist uses [higher-order logic], he will

[16]For instance, for a signature that includes addition, $\mathrm{Nat}(0, \mathrm{suc}, \mathrm{add})$ would be defined by conjoining to the statement $\mathrm{Nat}(0, \mathrm{suc})$ the statements $\forall x.\, \mathrm{add}\, x0x$ and$\forall xyz(\mathrm{add}\, xyz \wedge \mathrm{suc}\, yy' \wedge \mathrm{suc}\, zz' \to \mathrm{add}\, xy'z')$, giving the recursive definition of addition from successor and 0, and the statement that addition is function $\forall xyzz'(\mathrm{add}\, xyz \wedge \mathrm{add}\, xyz' \to z = z')$.

> not be able to avoid ontological commitments more uncomfortable on balance than that to mathematical objects, either to Fregean concepts or to multiplicities that are not 'unities'.'[17]

Similar sentiments are advanced in Field (1989) p7. It is sometimes tempting to provide English paraphrases of the structural contents defined above by quantifying over structures, or ω-sequences: the structural content of an arithmetical sentence A becomes 'A is true in every natural number structure'. Indeed, I have lapsed into this way of speaking already. However, if we were to take this seriously we would have gained nothing. For structures, sequences, and the like are just more abstract objects. What's to stop them having their mathematical features contingently, in a way that disconnects *them* from the natural patterns? The higher-order generalizations, by contrast, are immediately and logically connected to the particular instances of a given pattern by universal instantiation.

Of course, if we follow Quine in using higher-order quantification as a different notation for first-order quantification over sets or properties then we do not have a genuine alternative. However this is not the interpretation of higher-order logic employed by many of its contemporary proponents, who use the higher-order quantifiers as devices for forming generalizations in various grammatical positions—roughly as a way of capturing, in a single generalization, an infinite lists of claims differing only in the predicate (name, operator, etc) appearing in that claim, as we explained above.[18] In first-order logic an existential, such as '$\exists x\ x$ is tall' bears a logical kinship with the infinite disjunction 'John is tall or Mary is tall or . . .': whatever the world needs to be like to secure the disjunction, also suffices to secure the existential. A higher-order existential, such as '$\exists R$, Herb Rs Mary', bears the same exact same relationship to the disjunction 'Herb loves Mary or Herb hates Mary or ...'. But it is clear that the world doesn't have to contain relations, structures, or anything like that for this disjunction to be true: it would still be true if, for instance, Herb loves Mary and there are no abstract objects.[19] Having explained what we mean, however, we will revert to our non-literal talk of "structures" as though they were individuals, entrusting

[17] Parsons (1990) p329. §6

[18] See, for instance, Prior (1971), Williamson (2003), Trueman (2020), Bacon (forthcoming). This way of understanding higher-order logic traces back to Frege himself.

[19] A generalization is, of course, a little different from a disjunction. An existential generalization, for instance, is entailed by the disjunction of its instances. But the converse might fail given certain metaphysical views: for instance if there could have been new in-

the reader to infer the higher-order statement intended. In particular we will say "*Mary*, *loves* forms a natural number structure" as short for higher-order statements like Nat(*loves*, *Mary*).

To illustrate the power of higher-order generalizations, and contrast them with their platonic counterparts, let us briefly examine how higher-order claims connect directly with a certain class of arithmetical patterns in nature (thereby partly addressing a long-standing question about the applicability of arithmetic.[20]) Plausibly the need for general arithmetical reasoning originated with elementary patterns naturally expressed with numerical quantifiers, such as:

> If the bowl contains four apples and three pears and the bag contains three apples and four pears, then there are just as many fruit in the bowl as in the bag.

Here the words 'four' and 'three' are determiners, not names. One does not need to posit platonic objects to be their referents in order for them to be meaningful—as we noted earlier, 'four apples are in the bowl' is equivalent to a statement involving first-order quantifiers and identity. The need for higher-order generalizations becomes apparent when we start to notice general patterns. For instance, if we replace 'four' with 'twenty', and 'three' with 'thirty nine' in the above, we also get a truth. Indeed, every instance of the schema

> If the bowl contains N apples and K pears and the bag contains K apples and N pears, then there are just as many fruit in the bowl as in the bag.

where N and K can be replaced by any numerical determiner phrase. To capture this general pattern with a single generalization, the platonist posited special purpose individuals and operations—numbers, addition, and so on—

dividuals that are not F, while all the actual individuals are F, then in that possibility the disjunction of the existential's actual instances is true but the existential false. However, it is only the entailment from the disjunction to the existential we need to justify the ontological innocence of the higher-order generalization: if the disjunction doesn't entail the existence of abstract objects, nothing the disjunction entails can either, by the transitivity of entailment.

[20] cf Hodes (1984). See the introduction of Field (1989) and Goodsell and Yli-Vakkuri (MS) for some related discussion.

and subjected them to the law that for any numbers n and m, $n+m = m+n$.[21] But how does the platonic fact explain the original observations? Notice that we cannot infer from this first-order statement the instances of the above schema by logical means. They do not even strictly imply them if we are liberal about what is possible concerning the platonic objects—the claims about apples and pears would have remained true even if the platonic numbers 4 and 3 had gone AWOL from the platonic natural number structure. By contrast, if it is possible to generalize directly into the grammatical position that numerical quantifiers occupy, we can infer the instances of this schema directly by universal instantiation.[22]

4 The Framework of Higher-order Modal Logic

Since we will be working within the framework of higher-order modal logic, let's say a few things about what that is. As indicated earlier our approach is axiomatic: our basic system will be a neutral system of higher-order modal logic, which we label $\mathsf{H}^{\Box}$ and will informally call the *Background Logic*. We will also outline possible directions we might strengthen this logic to capture substantive principles of modal metaphysics that will later be brought to bear on mathematical contingency.

The Background Logic contains little that can be objected to, and is characterized by four sorts of axioms and rules: (i) the axioms and rules of the classical propositional calculus, (ii) the axioms and rules for classical quantification (these are the usual axioms and rules the first-order quantifiers, and their analogues for all higher-order quantifiers), (iii) a pair of principles governing the λ device, used for turning open formulas into explicit predicates, (iv) the axioms and rules of a normal modal logic—the principle that what is necessary is closed under *modus ponens*, and a rule to the effect that theorems of the Background Logic are also necessary according that logic. The precise details of the language, and the formulation of these logical principles can be found in Appendix A. While the Background Logic does not contain anything particularly contentious, we can also consider strengthening the Background Logic by adding substantive principles of modal metaphysics

[21] This represents a significant change in Frege's approach from the *Grundlagen* to the *Grundgesetze*.

[22] Compare Hodes (1984) and Goodsell and Yli-Vakkuri (MS).

to it.[23] In later sections we will consider three possible ways of strengthening the Background Logic:

1. Strengthening the very minimal modal logic to S4 or S5.
2. Adding "Rigid Comprehension", RC, a comprehension principle stating that every property or relation is coextensive with a modally rigid property or relation.
3. Adding "The Leibniz Biconditionals", LB, a principle saying that every possible proposition is settled (i.e. entailed) by a "world" proposition, and analogues of this principle for properties and relations.

The modal logics S4 and S5 should be familiar to the reader. Rigid Comprehension can be motivated indirectly through the logic of plurals (see Boolos (1984)). Two key principles governing plurals are: (i) for any property F, there are some things, *the* Fs, among which are all and only F individuals, (ii) the property of being one of these things is a modally rigid property. (RC itself, however, cuts out the middle man, and can be stated without reference to plurals.) The Leibniz Biconditionals can be motivated from a key tenet of possible world semantics: that propositions under the entailment order are isomorphic to the subsets of a collection of "worlds" under the subset relation. Propositions corresponding to singletons are what we have above called world propositions, and so any possible proposition will, via the isomorphism, be entailed by a world proposition.

These principles are all components of the conventional theory of modality. We will also explore a final modal principle that articulates a view that is in extreme opposition to the Leibniz biconditionals, implying the existence of lots of possible propositions that are not entailed by any world propositions (the reason behind the name will become clear later):

[23] One substantive issue that the Background Logic takes a stand on the necessitist/contingentist debate (see Williamson (2013)). Necessitism, the thesis that necessarily everything necessarily is something, is a theorem of the Background Logic and corresponding necessitist theses for propositions, properties and relations can also be derived. However, I do not think these consequences have to be understood in a way that is particularly contentious: they are purely devices of generalization pinned down by their logical role, and need not be tied to words like 'exists' or the restricted quantificational idioms of English. There are ways of introducing such generalizing devices even in a contingentist setting (see, for instance, Fine (1979)). Contingentists may wish to supplement our system by adding their preferred contingentist quantifiers, and nothing we say precludes them from doing so. See [ANON].

4. "Mathematical Possibilism", MP. A principle stating that to every complete Boolean algebra that's not "too big", there is a proposition isomorphic to it under the entailment relation.

Precise formulations of these logical principles can be found in Appendix A, and will be discussed in later sections. We adopt the following naming convention for higher-order modal logics:

Convention 4.1. *We denote by* $\mathsf{H}^{\Box}$ *the Background Logic, and the possible extensions as* $\mathsf{H}^{\Box}.\mathsf{4}$*,* $\mathsf{H}^{\Box}.\mathsf{5}$*,* $\mathsf{H}^{\Box}.\mathsf{RC}$*,* $\mathsf{H}^{\Box}.\mathsf{LB}$*,* $\mathsf{H}^{\Box}.\mathsf{MP}$*. If we want to add two or more principles at once we separate them with further dots: for instance* $\mathsf{H}^{\Box}.\mathsf{4}.\mathsf{RC}.\mathsf{MP}$*.*

Within the pure language of higher-order modal logic we can state hypotheses about the contingency of particular structural mathematical contents. In some cases we may find that the Background Modal logic will prove that these contents are not contingent, as for instance, in the case of the structural content of "7 is prime". In other cases we may find that structural mathematical contingency is compatible with this basic higher-order modal logic, and then we can ask how matters change if we assume one of our further principles of modal metaphysics. These questions can be settled using the usual logical methodology of finding axiomatic derivations, and finding models.

As an example of the Background Logic at work, we can state and derive an important connection between the structural arithmetical claims (statements of the form A^*) and the quantifier statements we singled out in the previous section. Following Frege, in the *Grundlagen*, we can provide logical definitions of the zero quantifier, what it is for a quantifier to succeed another, and what it means for a quantifier to be a (finite) numerical quantifier. The zero quantifier, "there are at least 0 Fs" holds vacuously of any property F, the successor of a quantifiers Q means "there is something which Fs and Q other things that F. A finite numerical quantifier is then something that possesses any property applying to the zero quantifier and closed under quantifier successors. Consequently, we have another way of translating platonic arithmetical statements into pure higher-order logic. A given arithmetical statement, $A(0, \mathrm{suc})$ can be mapped into pure higher-order logic by shifting the types: mapping 0 to the zero quantifier, mapping the successor operation to the quantifier operation on quantifiers, replacing first-order variables with variables of quantifier type, and restricting quantification over such variables to a predicate expressing the property of being a numerical quantifier.[24] Call this $A^\dagger$. One can then prove the following theorem in the Background Logic:[25]

Theorem 4.1. *For any second-order arithmetical statement A, the structural content of A, A^*, and the functional content of A, $A^\dagger$ are provably equivalent in the Background Logic, $\mathsf{H}^\Box$, given the existence of a natural number structure.*

This holds because the structure of any given natural number structure is deeply tied to the finite numerical quantifiers. Let 'n' be a first-order variable ranging over individuals a given natural number structure and 'N' a variable of determiner type ranging over finite numerical determiners.[26] For any number n in the structure there are exactly N numbers less than n, for some unique finite numerical quantifier N. Conversely, for each finite numerical quantifier N, there is a unique number n in the structure such that there are exactly N numbers less than n. Clearly the numerical quantifier associated with the successor of n is the quantifier successor of the numerical quantifier associated with n, establishing a pair of mutually involve isomorphisms between an arbitrary natural number structure and the finite numerical quantifiers. This means that if there are any natural number structures, they will all agree with the natural number structure of numerical quantifiers about any arithmetical claim.[27]

Note that the finite numerical quantifiers can form a natural number structure even when there are only finitely many individuals: provided it is possible that there could be any finite number of things, the numerical quantifiers will be distinguishable by a modal property, and thus distinct.[28]

[24]We might call this the *Grundlagen* translation since this is essentially the translation of arithmetical claims found there.

[25]Note that this theorem doesn't extend to any higher-order arithmetical statement. For instance, let M and N be distinct numerical quantifiers. The propositional identity $(N \neq_{(e\to t)\to t} M) =_t \top$ is consistent with the existence of a natural number structure whose Nth and Mth elements, m and n, are such that $(m =_e n) \neq_t \top$.

[26]We use the expression "there are exactly N ..." to mean "there are N ..., but there are not suc N ...".

[27]Note that the existence of a natural number structure is needed to ensure that A^* is not vacuously true, and that quantifier successor is injective.

[28]Even if there *couldn't* be more than a certain finite number of individuals, and the finite numerical quantifiers were finite in number, we would still be able to articulate using them the sense in which 7 is prime. This does not generalize to more complicated

We complained earlier that the platonic propositions do not seem to modally track the functional contents of arithmetical statements. We also saw that structural propositions fail when there aren't any natural number structures. Theorem 4.1 guarantees that this is in a sense the only case, insofar as the functional contents are captured by statements made in terms of numerical quantifiers.

Note that some instances of theorem 4.1 are not particularly illuminating. The functional content of a statement like $3 + 4 = 7$ will itself be a theorem of classical (higher-order) logic, and its immediate implications include only tautological facts like

> If there are three apples in the bowl and four non-apples in the bowl, there are seven things in the bowl.

The structural content of this arithmetical claim is similarly a theorem of the Background Logic. The modal equivalence is thus ensured by both sides being classical theorems, and thus both necessary. However there will be many cases where neither side are theorems of classical logic — such as $\mathrm{Con}(\mathsf{H}^{\Box})^*$ and $\mathrm{Con}(\mathsf{H}^{\Box})^{\dagger}$, the structural and functional translations of the consistency statement for the Background Logic. If we take the possibility of structural contingency seriously, then theorem 4.1 ensures there are still substantive modal correlations between the structural claims and the functional mathematical claims.

5 Arithmetical Contingency

Having finally arrived at an interesting form of mathematical contingency we return to our principal question of whether mathematics could have been different? Is structural mathematical contingency coherent, or does it contain a hidden inconsistency? We will pursue the case of arithmetic in this section; the next will cover the real numbers.

Any inconsistent statement in our minimal background logic will be impossible according to that logic. (This is due to the rule of necessitation: if the Background Logic proves $\neg A$, then it also proves $\Box\neg A$.) Given that inconsistency in this narrow sense suffices for impossibility, there are clearly many

arithmetical statements that we usually take to be true, but one might wonder whether the usual principles of arithmetic actually *should* hold when there is a finite upper bound on how many things there could possibly be.

necessary structural claims. As we noted previously, the dagger translation of the claim '3+4=7' is a classical theorem, and is therefore not contingent. On the other hand some structural arithmetical statements—the structural content of a suitably chosen Gödel sentence, for instance—are logically independent of the Background Logic. Is there any incoherence in assuming that statements like these are contingent? The schema asserting that there is no such structural contingency in a given arithmetical language may be stated as follows:

The Necessity of Arithmetic

$$\Box\forall\mathbf{X}(\mathrm{Nat}(\mathbf{X}) \to A(\mathbf{X})) \lor \Box\forall\mathbf{X}(\mathrm{Nat}(\mathbf{X}) \to \neg A(\mathbf{X}))$$

Where A can be any arithmetical statement in that language, and $A(\mathbf{X})$ is the result of replacing all the constants in that language with the variables in $\mathbf{X}$. Structural arithmetical contingency, then, is an example of an arithmetical sentence, A, which makes this schema false. Pay special note to the fact that the schema is language dependent, and we can consider different versions of it depending on what we count as "arithmetic"; we can vary whether we are talking about the contingency of first or second-order order arithmetic.[29]

An important constraint in the vicinity is a famous result of Dedekind (1888) that any two natural number structures are isomorphic.[30] Dedekind's theorem may be stated and derived in the non-modal fragment of the minimal background logic; i.e. using only the classical quantifier laws, propositional logic, and laws governing λ. Thus its statement and proof belong to pure logic, and do not make reference to any distinctively mathematical notions.[31]

Dedekind's Categoricity Theorem

$$\forall\mathbf{XY}(\mathrm{Nat}(\mathbf{X}) \land \mathrm{Nat}(\mathbf{Y}) \to \mathbf{X} \cong \mathbf{Y})$$

[29] To be explicit, by first-order arithmetic we mean formulas in the signature $0, \mathrm{suc}, <, \mathrm{mult}, \mathrm{add}$ containing only first-order quantifiers, and second-order arithmetic allows second-order quantifiers—the constants $<, \mathrm{mult}, \mathrm{add}$ can be dropped from the signature in this case without loss of expressive power.

[30] Strictly speaking, there are different versions of Dedekind's theorem depending on the signature, and notion of natural number structure for that signature. Dedekind's original result involved the signature $0, \mathrm{suc}$.

[31] This higher-order statement of Dedekind's theorem, and its proof, is a natural way of rendering Dedekind's original argument. At any rate, Dedekind was certainly not working in a background theory of sets.

where $\mathbf{X} \cong \mathbf{Y}$ is short for a higher-order sentence stating that $\mathbf{X}$ and $\mathbf{Y}$ are isomorphic. Since Dedekind's theorem is a theorem of the Background Logic, it is also necessarily true according to the Background Logic. We will thus take it to be necessarily true, mathematical contingency notwithstanding.

A straightforward consequence of Dedekind's theorem is that second-order arithmetical claims must have the same truth value in different natural number structures.[32] That is, for any second-order arithmetical sentence A: $\forall \mathbf{XY}(\mathrm{Nat}(\mathbf{X}) \wedge \mathrm{Nat}(\mathbf{Y}) \to (A(\mathbf{X}) \leftrightarrow A(\mathbf{Y}))$. Since this statement is thus also derivable from classical principles, it is also necessary in the Background Logic. So we can also put a $\Box$ in front of the consequence above.

$$\Box\forall \mathbf{XY}(\mathrm{Nat}(\mathbf{X}) \wedge \mathrm{Nat}(\mathbf{Y}) \to (A(\mathbf{X}) \leftrightarrow A(\mathbf{X})))$$

It is initially tempting to think that Dedekind's theorem automatically rules out arithmetical contingency. For if no two actual natural number structures can disagree about the first-order arithmetical truths how could a possible natural number structure disagree with an actual one either? This line of thought might perhaps be persuasive for anyone inclined towards the Lewisian view of modal reality. Lewis (1986) maintains that whatever is possible is in fact instantiated somewhere in the Lewisian plurality of concrete worlds. On this picture any two possible natural number structures are both in fact simultaneously instantiated somewhere in the Lewisian pluriverse. One can sensibly talk about relations between individuals belonging to different worlds—just as we can make sense of relations between individuals on different planets, say—and so Dedekind's theorem can be applied.

But if we do not adopt this fundamentally amodal worldview, it is hard to emulate this sort of reasoning—it involves making comparisons across logical space that do not seem to be legitimate once we take modality seriously. Dedekind's theorem tells us that, necessarily, no second-order arithmetical claim can differ between two natural number structures. We cannot compare, say, an actual natural number structure with a merely possible one that doesn't exist yet, for then the isomorphism needed to make the comparison may not exist yet either. Helping ourselves, temporarily, to a possible worlds way of talking we might say that Dedekind's theorem is an *intra*-world constraint: the relations are being compared, and are natural number structures relative to a single world. What we would need to rule out arithmetical

[32] Curiously, this does not extend to arbitrary higher-order arithmetical statements. It is fairly easy to construct models there are two natural numbers structures, R and S, such that $\forall xy(x \neq y \to x \neq y =_t \top)^S \leftrightarrow \forall xy(x \neq y \to x \neq y =_t \top)^R$ fails.

contingency, by contrast, would be an *inter*-world version of Dedekind's theorem, letting us compare natural number structures taken from different worlds. One strategy for getting around this is to strengthen Dedekind's theorem, and the above consequence, by interlacing the universal quantifiers with an extra modal operator:

$$\Box\forall\mathbf{X}(\mathrm{Nat}(\mathbf{X}) \to \Box\forall\mathbf{Y}(\mathrm{Nat}(\mathbf{Y}) \to \mathbf{X} \cong \mathbf{Y}))$$

$$\Box\forall\mathbf{X}(\mathrm{Nat}(\mathbf{X}) \to \Box\forall\mathbf{Y}(\mathrm{Nat}(\mathbf{Y}) \to (A(\mathbf{X}) \leftrightarrow A(\mathbf{Y}))))$$

However, these stronger claims stand little chance of being true, let alone derivable. Suppose that people are arranged, under the loving relation, in a natural number sequence (i.e. Nat(John, loves)). Had people been arranged under the kicking relation in a natural number sequence, we have absolutely no guarantee that the lovers and kickers can be correlated in a one-to-one fashion in a way that preserves successors, because we have no guarantee that the lovers would be still arranged in the same way. We can bring the difficulties of establishing The Necessity of Arithmetic without making any substantive modal assumptions into sharper relief by demonstrating once and for all that no such derivation is possible: structural arithmetical contingency is consistent in our Background Logic. Indeed, it is consistent with $\mathsf{H}^{\Box}.\mathsf{4}$ and the thesis of Intensionalism, according to which propositions, properties and relations are individuated by necessary equivalence (this system is equivalent, modulo definitions, to the system "Classicism".[33])

Theorem 5.1. *There is a first-order arithmetical sentence, $A(0, \mathrm{suc}, <, \mathrm{add}, \mathrm{mult})$, namely the Gödel sentence for Classicism ($\mathsf{H}^{\Box}.\mathsf{4}$ and Intensionalism), and a model of Classicism in which the former is structurally contingent. I.e. the model makes*

$$\Diamond\exists\mathbf{X}(\mathrm{Nat}(\mathbf{X}) \land A(\mathbf{X})) \land \Diamond\exists\mathbf{X}(\mathrm{Nat}(\mathbf{X}) \land \neg A(\mathbf{X}))$$

true.

Recall that, given Theorem 4.1, the contingency of these structural claims patterns with contingency about the corresponding claims about finite numerical quantifiers. So if there is structural arithmetical contingency the notion of finiteness itself, as encoded by quantifier phrases of the form "there

[33] See Bacon and Dorr (forthcoming), Bacon (2023a) chapters 6-8

are N things", must be modally flexible. In particular, it will turn out that it must be possible that there are more finite numerical quantifiers than there in fact are, using the definition of *being a finite numerical quantifier* from the previous section. This means that, for some higher-order property F, there could have been a finite numerical quantifier that is F despite there being no finite numerical quantifier that is possibly F. It might be tempting to gloss this as saying that it is possible that there are "non-standard" numerical quantifiers, by analogy with non-standard models of arithmetic that contain non-standard numbers above the standard numbers. However this gloss is misleading—there is no non-trivial distinction between standard and non-standard finite quantifiers, in actuality or at the possibilities where arithmetic is different. The numerical quantifiers at the possibility in question are standard finite quantifiers in the exact same sense as the actual finite numerical quantifiers are: they satisfy a principle of induction, apply to a property only if it is Dedekind finite, and so on. Yet we can also say things that capture the idea that the finite numerical quantifiers could have been different than the actual finite numerical quantifiers, such as statements of the form:

> It's possible that there is a finite numerical quantifier that ..., but no finite numerical quantifier possibly ....

Here ... can be filled in by some property that only merely possible finite quantifiers can satisfy — perhaps, the property of coding a proof of the inconsistency of ZFC.[34]

The above line of thought establishes that the property *being a finite numerical quantifier* is not modally rigid—there could have been more of them than there in fact are. Indeed, if this fails to be rigid in this way, then there cannot be *any* other way to rigidly single out the finite numerical quantifiers either. For if there were a property, X, rigidly picking out the actual numerical quantifiers, then it is not only true, but necessary that X applies to the 0 quantifier and is closed under quantifier successor. So, necessarily, if Q is a finite numerical quantifier—i.e. it possesses any property

[34]There is a way to say that the new finite numerical quantifiers are greater than any actual finite numerical quantifier. That is, we have that for any finite numerical quantifier, N, it is necessary that every quantifier that ... is greater than N. But any given claim of this form is consistent with the new quantifiers simply being a normal finite numerical quantifier greater than N. We have no direct way to say that all the finite quantifiers that ... are greater than all the actual finite numerical quantifiers at once.

that applies to the 0 quantifier and is closed under quantifier successor—then it possesses X, establishing that the finite numerical quantifiers cannot outstrip the actual finite numerical quantifiers after all.

These informal remarks can be turned into a proof that there is no structural arithmetical contingency from a substantive principle of modal metaphysics:[35]

Rigid Comprehension Every property (relation, etc.) is coextensive with a rigid property (relation, etc.)

Here we officially understand F to be rigid when there is no modal difference between the possible existence of Fs that are G and the actual existence of Fs that are possibly G. (Incidentally, Rigid Comprehension plays a rather critical role in this paper. While $\mathsf{S5}$ and possible world assumptions are often given center stage in extant discussions of modal metaphysics, my own sense is that these principles are further toward the periphery of the web of modal doctrines, and can easily be revised without doing too much violence elsewhere. The revisions that would be necessary to accommodate failures of Rigid Comprehension, however, strike me as much more thorough going.)

We thus have the following theorem, slightly generalizing a result from (Goodsell (2022), Corollary 14).[36]

Theorem 5.2 (Goodsell)**.** *In* $\mathsf{H}^{\Box}.\mathsf{RC}$*, one can prove*

$$\Box\forall\mathbf{X}(\mathrm{Nat}(\mathbf{X}) \rightarrow A(\mathbf{X})) \vee \Box\forall\mathbf{X}(\mathrm{Nat}(\mathbf{X}) \rightarrow \neg A(\mathbf{X}))$$

Whenever $A(0, \mathrm{suc}, \mathrm{mult}, \mathrm{add}, <)$ *is a sentence of first-order arithmetic in the signature* $0, \mathrm{suc}, \mathrm{mult}, \mathrm{add}, <$.

[35] There are some subtleties involving the notion of rigidity employed here, but they necessary for the wider point. They are relevant only if we take seriously the idea that distinct individuals could have been identical. See Bacon and Dorr (forthcoming), of rigidity and a statement of Rigid Comprehension in the context of $\mathsf{H}^{\Box}$. This notion of rigidity is found in Parsons (1983).

[36] Goodsell's result is about first-order arithmetic in the signature $<, 0$. Hardly any interesting arithmetical claims can actually be expressed in this language, due to the fact that one cannot define addition and multiplication from 0 and $<$ in first-order logic. One way to patch this up is to use the richer notion of a natural number structure that includes among its data operations representing addition and multiplication (Goodsell has communicated to me other ways to patch up the argument here). Theorem 5.2 below slightly generalizes Goodsell's argument in using a weaker background higher-order logic (Goodsell uses Classicism); but in detail it is the same argument.

The idea, informally, is to use Dedekind's theorem to show that every natural number structure is isomorphic to a "modally inflexible" natural number structure in which arithmetical statements are necessarily true if true at all, and then use the necessity of Dedekind's theorem to establish that, necessarily, any natural number structure agrees with with the inflexible natural number structure about what is true. I say, here, that a structure is "modally inflexible" iff it is defining properties and relations are rigid, and the individuals in the fields of these relations are necessarily distinct (the latter clause is needed because $\mathsf{H}^{\Box}.\mathsf{RC}$ is weak enough to be neutral about the necessity of distinctness).

The extent to which this result rules out arithmetical contingency will depend on what we count as "arithmetic". Goodsell's result tells us that the first-order arithmetical truths are not contingent, but it does not extend to sentences of second-order arithmetic—a curious break from Dedekind's "intra-world" theorem, which applies to both. The model construction in appendix B.1 can be modified to show the compatibility of the contingency of a second-order arithmetical statement (the existence of $0^{\#}$) with a strengthening of $\mathsf{H}^{\Box}.5.\mathsf{RC}$.[37]

What reasons do we have to believe that Rigid Comprehension is true? One route to Rigid Comprehension appeals to the behaviour of plural terms in English—expressions like 'the horses in the stable', 'those things' and the like. On the one hand, it seem as though plural expressions have rigid membership conditions. Suppose we encounter some people, and John is one of them—then it seems that he couldn't have failed to be one of those people, and similarly, there couldn't have been more of those particular people than there in fact are. That is to say, if tt is a plural term, then the property *is one of the* tt is rigid.[38] On the other hand, it seems that for any predicate, F, we can form a plural expression, 'the Fs', which is coextensive with F in the sense that something is one of the Fs if and only if it is F. Thus Rigid Comprehension is ensured: if F is any property whatsoever, *being one of the* Fs is the rigid property coextensive with it.

[37]In this model $\Box\mathsf{RC}$ fails. But if conjecture 6.1 is true, then we would expect this sort of contingency to also be consistent with $\Box\mathsf{RC}$.

[38]The rigidity of plural membership is one of the axioms in Linnebo's modal plural logic (Linnebo (2013)). Dummett (Dummett (1991), p.93) suggests we reduce plural quantification to second-order quantification which, outside of a Fregean/extensionalist context, would seem to require some sort of restriction to rigidity. See also Dorr et al. (2021) §1.5 for some related discussion.

Might we resist this argument? Recently Salvatore Florio and Øystein Linnebo have argued that some concepts—like the notion of set, ordinal and even the notion of a thing—are "extensionally indefinite", in the sense that they do not have a definite extension captured by something like a plural term or a set.[39] They therefore deny the plural comprehension principle, that for any F there are some things, the xx, such that something is one of the xx if and only if it is F. So, for instance, the property *being a set*, is extensionally indefinite, and so we must deny that there are some things which are all and only the sets. This blocks the argument for Rigid Comprehension from plural logic.

Furthermore, their picture may also give us positive reasons to doubt Rigid Comprehension. As we have already noticed, it looks as though plurals and rigid properties are in one-to-one correspondence—*being one of* ... and *the* ... *s* are mutual inverses when the latter is restricted to rigid properties—so we can faithfully paraphrase plurals in terms of quantification over rigid properties *provided* both Plural and Rigid Comprehension hold. However, it's tempting to think that this correspondence persists even in contexts where both can fail—i.e. Plural and Rigid Comprehension must fail in the same way. We can also take their picture as a starting point for developing a theory of rigid properties that falls short of Rigid Comprehension but is still powerful enough for many other purposes. It is extremely natural to identify Florio and Linnebo's notion of extensional definiteness with rigidity (or perhaps, a variant of rigidity expressed with a definiteness operator).[40] We may also enrich this with a logic of definiteness allowing us to recover some of the principles of "Critical Plural Logic" that Linnebo and Florio take to be valid, such as that the disjunction of two extensionally definite properties is also extensionally definite.[41]

[39] Florio and Linnebo (2021).

[40] See the discussion of rigidity in chapter 10 of Florio and Linnebo (2021). This is not the only possibility for analysing extensional definiteness: Linnebo (2013) gives a modal Cantorian analysis in terms of the possibility of those things existing together, and Linnebo (2018) explores a Dummettian analysis instead exploits intuitionist logic.

[41] See §12.5 of Florio and Linnebo (2021). Note that if we do not assume the necessity (or at least definiteness) of distinctness the conjunction of two rigid properties may not be rigid. If a and b are distinct but possibly identical, $\lambda x(x = a \wedge x = b)$ is empty but might not have been, and so is not rigid. Yet $\lambda x.x = a$ and $\lambda x.x = b$ are rigid. Some of the principles of critical plural logic require special further assumptions about definiteness beyond the definiteness of distinctness. For instance, their principle of union corresponds, in the present context, to the principle that any (possibly indefinite) second-order property

Florio and Linnebo target mathematical notions like *being a set*, or *being an ordinal*. They do not, by contrast, question the notion of a *finite number*. Yet the instances of Rigid Comprehension needed to prove Theorem 5.2 involve only natural number structures. Might the notion of natural number be extensionally indefinite? Cantor and Aristotle famously had diverging views on this question. Cantor thought not, maintaining that any sequence of ordinals can be completed.[42] In the *Physics* Aristotle maintains that there are arbitrarily large finite quantities, but not any infinite quantities encompassing them. Aristotle primarily applied his views to natural number structures found in nature—such as days ordered chronologically, or sequences of physical magnitudes ordered by their size (Aristotle (350 B.C.E) 203b15)—so a certain sort of Aristotelian may have independent reason to deny all the instances of Rigid Comprehension needed to get this argument going.[43] We may substantiate this by coming up with an Aristotelian model of arithmetical contingency—indeed, we already have as Theorem 5.1 provides us a model of exactly this sort.

We know, given Goodsell's result, that Rigid Comprehension must fail in this model. The model is also Aristotelian in the following way: the first-order domain consists of numbers $0, 1, 2, \ldots$, and for any finite collection of those numbers, there is a rigid property of being one of them, but there is no rigid property coextensive with all the numbers.[44] Indeed, the model can

circumscribing some definite properties has a definite union. While any *definite* property of definite properties can be shown to have a definite union in a minimal logic of definiteness, their stronger principle needs to added by hand; similar points apply to their principle of separation.

[42]See Cantor (1883). Cantor required that completable sequences of ordinals had to be indexable by an already existing ordinal, or else we encounter the Burali-Forti Paradox. Cantor's original theory of ordinals was a bit unclear about this point — he originally presented it as a pair of of inconsistent "Principles of Generation", letting you take successors and arbitrary limits of ordinals— and then added to that a further "Principle of Limitation" that might more charitably be taken to be a qualification of the limit principle, rather than a separate claim.

[43]See Linnebo and Shapiro (2017) for an explicitly modal articulation of Aristotle's position (although see Rosen (2021), Bacon (2023b) for a non-modal alternative).

[44]There is a sense of 'finite' in which the existence of finite rigid properties is guaranteed just by logical considerations. To be finite is to be a property G which possesses every property of properties which (i) applies to all empty properties, and (ii) applies to $\lambda x.Fx \vee x = y$ whenever y is not F and F is a property it applies to. The argument is essentially an induction, using the fact that if F is rigid then so is $\lambda x.Fx \vee y$. Further principles may be needed to extend this argument to other notions of finiteness, such as Dedekind

also be viewed as a model of Linnebo and Florio's Critical Plural Logic minus their principle of infinity, by interpreting the plural quantifiers in terms of quantification over rigid properties.

Might there be another route to the conclusion that there isn't structural arithmetical contingency? One that doesn't go through the Rigid Comprehension principle. Goodsell's result is fairly neutral about the modal logic of $\Box$—on his interpretation $\Box$ represents a determinacy operator. Perhaps stronger assumptions about the logic of broad necessity could close the gap. Several modal principles seem to fall straight out of the notion of broad necessity. First, in virtue of being the strongest necessity, $\Box$ should be at least as strong as aletheic necessity, *it is true that* ... $(\lambda p.p)$: so $\Box$ should be factive, $\forall p(\Box p \to p)$, and necessarily so: $\Box\forall p(\Box p \to p)$. Similarly, in virtue of being the strongest necessity, it should be as strong as the composite necessity of being *broadly necessarily broadly necessary* $(\lambda p.\Box\Box p)$. So $\Box$ should necessarily satisfy the $\mathsf{S4}$ axiom: $\Box\forall p(\Box p \to \Box\Box p)$. This suffices to establish that all theorems of $\mathsf{S4}$ are true of broad necessity. We call this system $\mathsf{H}^{\Box}.\mathsf{4}$.

These further modal principles do not rule out structural arithmetical contingency, for we saw by theorem 5.1 that there is a model of Classicism which contains all the theorems of $\mathsf{S4}$ and arithmetical contingency.

The $\mathsf{S5}$ principle, unlike the two principles discussed above, does not fall directly out of the concept of necessity in the highest degree. There is a tempting argument that it does, but this argument relies on a subtly fallacious use of possible world model theory. The thought rests on the Leibnizian idea that the broadest necessity must correspond to quantification over *all* possible worlds. Now, in a certain *model theory* that treats $\Box$ like a universal quantifier, the corresponding condition secures the validity of $\mathsf{S5}$. However the validity of the object language principle that broad necessity is truth in all possible worlds does not straightforwardly correspond to the claimed model theoretic condition (that in a given model, $\Box A$ is true at a world iff it A is true at every world in the model). It is perfectly consistent to keep this principle as stated while admitting failures of $\mathsf{S5}$ if there is contingency about which things are worlds (indeed, the models of theorem 5.1 above validates the "Leibniz biconditionals", which we'll discuss further in section 6, while also invalidating $\mathsf{S5}$).[45]

With that all said, we might simply take $\mathsf{S5}$ as a substantive metaphysical

finiteness (being injectible into property whose extension you property contain).

[45]See also the discussion in Bacon (2018a) §5.4.

posit and see where it leads.[46] We can obtain the modal higher-order logic $\mathsf{H}^{\Box}.\mathsf{5}$ from $\mathsf{H}^{\Box}.\mathsf{4}$ by adding the following axiom

Brouwer's principle $\Box\forall p(p \to \Box\Diamond p)$

Now, without Rigid Comprehension, it is very hard to compare the extensions of relations across possibilities, *even* in the context of a modal logic of $\mathsf{S5}$. For all we've said, it could be that relations witness one class of extensions in one world, but those same relations "miss out" some extensions at other possibilities, allowing for contingency in the structural arithmetical propositions.[47] In fact it's possible to come up with a model of second-order modal logic in which $\mathsf{S5}$ is true, and there is structural arithmetical contingency. I do not describe this in the appendix but the construction is relatively simple, and given in outline in the footnote.[48]

Theorem 5.3. *There is a model of second-order logic with a* $\mathsf{S5}$ *modal operator in which there is contingency about some first-order arithmetical statement.*

It is rather striking, then, that this situation does not hold in full higher-order logic. Recall that, given certain existence assumptions, structural contingency is equivalent to contingency in the structure of the numerical quantifiers. However, it is possible to show, in a logic of $\mathsf{S5}$, that there cannot be contingency about the structure of the finite numerical quantifiers. An easy induction establishes that every finite numerical quantifier is necessarily

[46] Alternatively, we might make other substantive posits that imply that broad necessity satisfies $\mathsf{S5}$. For instance, Williamson (2016) suggests the principle that every modality has a converse, in analogy with the tense operations corresponding 'will' and 'was'. This principle can be formalized in a higher-order framework (see Bacon and Zeng (2022).

[47] Of course, this talk of "missing out" isn't really legitimate without Rigid Comprehension.

[48] Start with two extensional Henkin models of second-order logic with the same infinite domain of individuals which disagree about some structural arithmetical truth (some sentence of the form A^*). This is possible due to the combination of Gödel's incompleteness theorem and Henkin's completeness theorem (see Gödel (1931), Henkin (1950)); the latter rests on the fact that in these models the second-order quantifiers needn't range over arbitrary subsets of the domain of individuals. The modal model of second-order logic is then obtained by having two mutually accessible worlds. Properties are modeled as functions from worlds to extensions, but we only allow functions that take the first world to an extension in the first model, and the second world to an extension in the second model.

a finite numerical quantifier.[49] In $\mathsf{S5}$, it also follows that if N is not a finite numerical quantifier, it is necessarily not one. For suppose N was possibly a finite numerical quantifier. Then it is possibly necessarily one, by the previous argument and normality, and by Brouwer's principle it follows that N is a finite numerical quantifier after all. So there is no contingency about which things are finite numerical quantifiers. By a similar argument, it is possible to show that the numerical ordering of the numerical quantifiers is non-contingent allowing us to establish the non-contingency of arithmetic according to the numerical quantifiers. This in turn precludes contingency in *any* natural number structure, given the isomorphism described above (or alternatively obtained by Dedekind's theorem). Thus we obtain the following complement to theorem 5.2:

Theorem 5.4. *In* $\mathsf{H}^{\Box}.5$ *(*$\mathsf{H}^{\Box}.4 + \mathsf{B}$*), one can derive*

$$\Box\forall\mathbf{X}(\mathrm{Nat}(\mathbf{X}) \rightarrow A(\mathbf{X})) \vee \Box\forall\mathbf{X}(\mathrm{Nat}(\mathbf{X}) \rightarrow \neg A(\mathbf{X}))$$

Whenever $A(0, \mathrm{suc}, <, \mathrm{add}, \mathrm{mult})$ *is a sentence of first-order arithmetic in the signature* $0, \mathrm{suc}, <, \mathrm{add}, \mathrm{mult}$.

Thus far we have focused on statements of first-order arithmetic. When we move to the setting of second-order arithmetic the situation is slightly different. There are statements of second-order arithmetic whose contingency is consistent with $\mathsf{C5}$, and given a certain conjecture the contingency of this statement is consistent with $\mathsf{C.RC}$. However, since second-order arithmetic is in a sense equivalent to first-order analysis with a notion of natural number we leave these facts for the next section.

[49] It is sufficient to show that the property of being necessarily a finite numerical quantifier applies to the 0 quantifier and is closed under quantifier successor. It then follows that if N is a finite numerical quantifier—i.e. it has any property applying to 0 and closed under successor—then it in particular has *being necessarily a finite numerical quantifier*. Necessitation tells us that it is necessary that the 0 quantifier has any property applying to the 0 quantifier and closed under quantifier successor, since this can be established by logic alone. Suppose N is necessarily a finite numerical quantifier. Then, by definition of a finite numerical quantifier, it is necessary that any property applying to 0 and closed under successor applies to N; and thus, necessarily, any such property applies to the successor of N. Thus the successor of N is also necessarily a finite numerical quantifier.

6 Analytical Contingency

We now turn to analytic contingency—contingency about the real numbers. As before, we are less interested in contingency about the nature of the platonic real numbers, but rather about contingency about what holds of things that are arranged in the same structure that platonists take their real numbers to be in fact arranged. Namely, the structure of a *complete ordered field*. This means that the platonic reals are equipped with notions of addition, multiplication, 1, 0, and a relation $<$ that behave nicely with respect to each other: they satisfies the axioms of an "ordered field".[50] Moreover, they satisfy a completeness property, that can be specified by a higher-order generalization:

> For any F, applying to real numbers, if there exist a real no greater than every F, there exists a largest such real.

The platonic reals are not the only structure that instantiates the properties of a complete ordered field. Plausibly, the structure of times under the chronological ordering, with 0AD and 1AD playing the role of the additive and multiplicative units also satisfies these conditions.

Following our previous conventions, we will say that a real number structure consists of data $\mathbf{X}$, consisting of entities of appropriate types representing 0, 1, $<$, multiplication, and addition. It will also be useful to include in our notion of a real number structure a property singling out the natural numbers as a special kind of real number. We will write $\mathrm{Real}(\mathbf{X})$ for the claim that all but the last component of $\mathbf{X}$ form a complete ordered field, and that the last component, the naturals, is the smallest subproperty containing 0 and closed under adding 1. Now the necessity of analysis may be formulated as a schema

The Necessity of Analysis

$$\Box\forall\mathbf{X}(\mathrm{Real}\,\mathbf{X} \to A(\mathbf{X})) \vee \Box\forall\mathbf{X}(\mathrm{Real}\,\mathbf{X} \to \neg A(\mathbf{X}))$$

Where A is any sentence of analysis. As before this schema is language dependent; we can consider the instances of the schema where A is a first-order sentence in the signature described above, or we can extend it to second-order

[50] These axioms include principles like $(a + b).c = a.c + b.c$, $a < b \to a + c < b + c$ and so on.

sentences in that signature. (Curiously, every instance of the no contingency schema with respect to first-order language in the signature that omits the natural number predicate—the signature with 0, 1, $<$, multiplication, and addition—can be derived in the non-modal fragment of $\mathsf{H}^{\Box}$. Tarski (1949) has shown that all the truths in that language are derivable from the condition that they form a real number structure.)

We have an analogue of of Dedekind's theorem for complete ordered fields, due to E.V. Huntington.[51] He showed that the condition of being a complete ordered field characterize the real number structure up to isomorphism:

Huntington's Categoricity Theorem

$$\forall \mathbf{X} \forall \mathbf{Y} (\text{Real}(\mathbf{X}) \wedge \text{Real}(\mathbf{Y}) \to \mathbf{X} \cong \mathbf{Y})$$

For reasons we have covered in the arithmetical context, Huntington's theorem does not directly rule out analytic contingency. Without further posits, we have no way to compare merely possible real number structures with actual ones. Indeed, given theorem 5.1, we cannot rule out arithmetical contingency using $\mathsf{H}^{\Box}$alone, and since we are counting claims about the natural numbers as special cases of claims about the reals we cannot rule out first-order analytic contingency.

One might expect analytic contingency to disappear in the presence of Rigid Comprehension or Brouwer's principle, as it did in the arithmetical case. Let us begin by examining the situation with Rigid Comprehension. For any actual real number structure, we can find a modally inflexible real number structure isomorphic to it, using Huntington's theorem and Rigid Comprehension. We can compare this structure with any real number structure at any other possibility. And due to its rigidity this comparison is tantamount to a comparison with the actual real number structure we started with. It is tempting to think that we can then conclude that actual and possible real number structures cannot disagree.

However, here lies a key disanalogy between the arithmetical and analytic cases. In the former case, we can show that any modally inflexible natural number structure is necessarily a natural number structure, for the only way it could fail to have the inductive property is if its extension could have expanded, which cannot happen in inflexible structures. By contrast, we cannot show that an inflexible real number structure is necessarily a

[51] References Huntington (1903).

real number structure. The sticking point is the completeness property. A modally inflexible real number structure may be complete, but fail to be complete if there could have been properties whose extensions pick out collections of reals that no property in fact picks out—in particular these new extensions may pick out bounded segments of the structure that have no least upper-bound. This could happen in a couple of ways. Perhaps there could have been new properties—properties that don't in fact exist, an idea that has been explored thoroughly in the higher-order contingentism literature.[52] But this is overkill—it could just be that actually existing properties could have had new extensions, different from the extension of any actual property, which could have caused failures of completeness, even in inflexible real number structures.

What sorts of analytic claims can consistently be claimed to be contingent in a background logic with Rigid Comprehension? Clearly certain analytic statements, such as the arithmetical statements, are not candidates for contingency. However, there are many analytic statements that have proved especially elusive to mathematicians leading some to suspect that they have no determinate truth value.[53] The most famous example of this is Cantor's continuum problem, which asks whether there is an uncountable collection of reals that cannot be put in one-to-one correspondence with the real numbers. These notions can all be spelled out in second-order logic, so the structural content of the continuum hypothesis says that every real number structure makes CH true, and is a prime candidate for structural analytic contingency. Indeed, there is a more general division of mathematical statements. Some mathematical statements, like those of arithmetic, when suitably formalized in the language of set-theory cannot have their truth values changed by Cohen's method of forcing.[54] Other mathematical statements, when so formalized, can, like the continuum hypothesis. I conjecture that for any analytic statement that can be changed by forcing, it is consistent in $\mathsf{H}^{\Box}.\mathsf{RC}$ that its structural content is contingent. In the case of CH this means:

Conjecture 6.1. *There is a model of* $\mathsf{C}.\Box\mathsf{RC}$ *(Classicism and* $\Box$*Rigid Comprehension) in which there are failures of The Necessity of Analysis in the*

[52]Fine (1977), Stalnaker (2012), Fritz and Goodman (2016), Fritz (2023).

[53]If they are indeterminate—i.e. contingent with respect to the determinacy modality–they will, of course, also be contingent with respect to the broadest modality, whatever that is.

[54]Cohen (1966).

language of second-order analysis. Specifically, there can be structural contingency about the continuum hypothesis:

$$\Diamond\exists\mathbf{X}(\mathrm{Real}(\mathbf{X}) \wedge \mathrm{CH}(\mathbf{X})) \wedge \Diamond\exists\mathbf{X}(\mathrm{Real}(\mathbf{X}) \wedge \neg\,\mathrm{CH}(\mathbf{X}))$$

What about the situation if, instead of Rigid Comprehension, we assume $\mathsf{S5}$? In the arithmetical case we showed that a *particular* natural number structure comprised of the numerical quantifiers was rigid, providing us with a fixed meter stick to compare natural number structures across logical space. In the case of the real numbers we do not seem to have anything analogous to that. One can certainly construct real number structures at higher-types from the numerical quantifiers but we have no way to show that these structures are rigid. Indeed, here we have a consistency result:

Theorem 6.1. *There is a model of* $\mathsf{C5.RC}$ *(Classicism with* $\mathsf{S5}$ *and Rigid Comprehension) in which there are failures of The Necessity of Analysis in the language of second-order language of analysis. Specifically, one can construct models in which there is structural contingency about the continuum hypothesis:*

$$\Diamond\exists\mathbf{X}(\mathrm{Real}(\mathbf{X}) \wedge \mathrm{CH}(\mathbf{X})) \wedge \Diamond\exists\mathbf{X}(\mathrm{Real}(\mathbf{X}) \wedge \neg\,\mathrm{CH}(\mathbf{X}))$$

The model here is a model of Rigid Comprehension, but not $\Box$Rigid Comprehension.[55] Finally, one can ask if the combination of $\mathsf{S5}$ and $\Box$Rigid Comprehension together could rule out structural analytic contingency. We will return to this question at the end of this section.

(Note that our running example, the continuum hypothesis, is a *second-order* statement about the real numbers. It's natural to wonder if there are any contingent first-order statements about the reals. Here matters are a bit more delicate, and will depend on which statements we count as analytic. As mentioned already, if we restrict ourselves to statements formulated using the predicates of an ordered field ($<$, add, mult, 0, 1) then we are quite expressively limited, and there is no contingency. But if we include a predicate singling out the naturals of a real number structure then a variant of the

[55] Note that he model here can easily be augmented to validate a contingency schema, positing structural contingency about all second-order statements of analysis that can be changed by forcing.

model of theorem 6.1 establishes the consistency in $\mathsf{C5}$ of the contingency of a statement of first-order analysis.[56])

Thus analytic contingency is more resilient—it is harder to rule out contingency about the continuum hypothesis than any first-order statement about natural numbers. However, there may be further substantive principles of modal metaphysics that we could add to Rigid Comprehension (or to Brouwer's principle) to rule out even analytic contingency. Here I'll focus on the following principle, inspired by a Leibnizian metaphysics which ties possibility to the existence of *complete* possibilities—possible states of affairs that settle the truth values of all propositions. Variant principles exist for properties and relations.

The Leibniz Biconditionals

A proposition is broadly possible if and only if it is strictly implied by a world proposition.

A property is broadly possible (i.e. possibly instantiated) if and only if it is strictly implied by a world property.

⋮

Here we say that a proposition is a world proposition if it is broadly possible and, for any other proposition, it strictly implies that proposition or its negation. A world property is defined similarly, understanding strict implication between properties F, G to mean $\Box\forall x(Fx \to Gx)$, and the possibility of a property F to mean $\Diamond\exists x.Fx$. Analogous notions for relations are introduced in a similar manner.

The Leibniz Biconditionals, along with the necessitation of Rigid Comprehension, $\Box\mathsf{RC}$, rule out analytic contingency. We essentially fix the failed

[56]Recall that by Tarski (1949) all truths about the real numbers stateable in the smaller signature are derivable, using logic alone, from the axioms of a complete ordered field, so there obviously cannot be any structural contingency in that case. However, once you have a predicate for the natural numbers you can encode second order quantification over natural numbers using first-order quantification over real numbers. While this isn't enough to state the continuum hypothesis (that would need third-order quantification over naturals), there *are* statements whose truth values can be changed through forcing in second-order arithmetic (such as the existence of a non-constructible set of natural numbers, if we assume $V = L$). I am indebted here to Noah Schweber's response to a question I asked on math.stackexchange: (https://math.stackexchange.com/users/28111/noah schweber).

argument above by using the Leibniz Biconditionals to show that any inflexible real number structure is necessarily complete (and thus necessarily a real number structure), allowing us to proceed as in the arithmetical case. For if it is possible that there is some failure of completeness (a bounded property with no least upperbound) in an inflexible real number structure, a Leibnizian metaphysician will say that there is a *maximally specific* property, singling out a specific possible failure of completeness. That is, a third-order world property W that applies to only one property characterizing a possible failure of completeness in our inflexible real number structure. But now we can ask what elements of our inflexible real number structure *would* have fallen under the unique W property, if there had been any W properties, and it may be shown that these constitute an actual failure of completeness, contradicting the assumption that we started with a real number structure. A full proof may be found in appendix C.2.

Theorem 6.2. *In* $\mathsf{H}^{\Box}.\Box\mathsf{RC}.\mathsf{LB}$ *one can derive all instances of The Necessity of Analysis in the language of second-order analysis:*

$$\Box\forall\mathbf{X}(\mathrm{Real}\,\mathbf{X} \rightarrow A(\mathbf{X})) \vee \Box\forall\mathbf{X}(\mathrm{Real}\,\mathbf{X} \rightarrow \neg A(\mathbf{X}))$$

What reasons do we have to accept the Leibnizian picture? One reason is that it is an entrenched principle of modal metaphysics, instated after the advent of possible world semantics. However, Humberstone (1981) has laid the foundations for an alternative to possible world semantics which does not assume the Leibnizian metaphysics, and it is not obvious that possible world semantics has any distinctive advantage over it.[57] Moreover, since we are already in the business of questioning orthodox positions in modal metaphysics, such as the necessity of mathematics, it would be nice to see a more thorough defense of the principle.

One such defence might appeal to the principle that there ought to be a conjunction of all true propositions—which we might identify with a greatest lowest bound of the truths under the entailment order—thus committing us to at least one world proposition. And if it is necessary that there is a conjunction of all the truths, it seems there ought to be a world proposition witnessing any broadly possible proposition. However this argument contains some subtle gaps that need to be fixed. For all its obviousness, we will need some extra posit to ensure that, necessarily, a greatest lower bound of truths

[57] See Holliday (forthcoming) for an overview of recent work on this.

is itself true. $\Box$Rigid Comprehension is one posit that would ensure this.[58] Even granting this, the move from "necessarily there is a true world proposition" to "for any possible proposition, there is a world proposition entailing it" is not straightforward in a context where we allow Brouwer's principle to fail and there could have been new propositions that don't in fact exist. However, by putting these ideas together one can show that $\mathsf{H}^{\Box}.\mathsf{5}.\Box\mathsf{RC}$ contains the Leibniz biconditionals.[59] This provides us with an alternative way to motivate the Leibniz biconditionals—rather than reaching straight away for heavy duty theoretical posits, like possible worlds, one can directly appeal to a principle of modal logic, $\mathsf{S5}$, and the necessity of Rigid Comprehension. (Of course, if one already had any reason to doubt the $\mathsf{S5}$ principle for broad necessity, or $\Box$Rigid Comprehension, this argument for a Leibnizian modal metaphysics holds no sway.)

Since the combination of $\mathsf{S5}$ and $\Box$Rigid Comprehension imply the Leibniz Biconditionals, this combination also implies The Necessity of Second-order Analysis, by theorem 6.2:

Corollary 6.1. $\mathsf{H}^{\Box}.\mathsf{5}.\Box\mathsf{RC}$ *entails all instances of The Necessity of Analysis in the language of second-order analysis.*

$$\Box\forall\mathbf{X}(\mathrm{Real}\,\mathbf{X} \to A(\mathbf{X})) \vee \Box\forall\mathbf{X}(\mathrm{Real}\,\mathbf{X} \to \neg A(\mathbf{X}))$$

This answers the question we raised earlier in the affirmative: the combination of a $\mathsf{S5}$ modal logic and $\Box$Rigid Comprehension does rule out analytic contingency.

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[58]One can show that any greatest lower bound of the truths is necessarily equivalent to the claim that every truth* is true, where truth* is the rigidification of truth. It is easy to see the latter is true.

[59]This is a strengthening of a result in Bacon and Dorr (forthcoming). It is the left-to-right direction of LB that is the tricky case. First, RC implies that there's a true world proposition, namely the proposition that every truth* is true, where truth* is the rigidification of truth; $\Box$RC thus implies that this consequence is necessary. Suppose that p is possible. Then it is possible that p and there is a true world proposition w. By the Barcan formula, there is a w such that it's possible that p and w is a true world proposition. But in $\mathsf{S5}$ w must in fact be a world proposition. For if w doesn't entail q, it necessarily doesn't entail q by $\mathsf{S5}$. Since w is possibly a world proposition this means it must possibly entail $\neg q$, and thus, by $\mathsf{S5}$, it actually entails $\neg q$; so w is a world as required. So we have a world proposition such that possible w and p, which means there is a world proposition that entails p, because a world proposition is compossible with a proposition only if it entails it.

The combination of the Leibniz Biconditionals and $\Box$Rigid Comprehension is thus a substantive hypothesis of modal metaphysics which ensures that mathematical contingency is very limited. How limited? For all way have said this package, $\mathsf{LB}+\Box\mathsf{RC}$, is compatible with *some* sort of mathematical contingency from some mathematical domain richer than analysis.[60] Might the package allow for set-theoretic contingency, for example, or contingency in some yet richer mathematical language?

My suspicion is that this package is just as inhospitable to other forms of mathematical contingency as it is to analytic contingency. My reason for thinking this is that most mathematical theories admit an interpretation—not necessarily the intended interpretation, but this doesn't matter—where its primitives are defined from a "small ZFC relation"—a relation satisfying the conjunction of the axioms of second-order ZFC in which there are no inaccessibles—and its axioms are true under those definitions.[61] If there could be structural contingency about some statement of the mathematical theory T in a way that was compatible with the existence of small ZFC relations, then there would also have to be structural contingency what holds in all small ZFC relations. However, we have an analogue of Dedekind and Huntington's theorem for set-theory: Zermelo's theorem, a special case of which tells us that any two small ZFC relations are isomorphic.[62] The situation with respect to structural set-theoretic contingency is analogous to the case of analytic contingency: in the presence of $\Box$Rigid Comprehension and the Leibniz Biconditionals one can show that inflexible small ZFC relations are necessarily small ZFC relations, and that there cannot be contingency about what is true in a small ZFC relation in the set-theoretic signature.[63]

[60]A sufficient condition for an interpreted mathematical language to be richer than the language of analysis is if it cannot be interpreted in the language of analysis, in the sense that there is a meaning preserving translation from one language to the other. For if there were such a translation, any sentence of the mathematical language would express the same proposition as a sentence of analysis. And given $\mathsf{LB}+\Box\mathsf{RC}$ no such proposition will be contingent. (NB: interpretability in the sense just defined is not to be confused with the proof-theoretic notion of interpretability).

[61]If the theory T concerns very large mathematical objects this interpretation might not be possible, but usually there is a specific kind of inaccessible that would suffice to interpret the theory, and a similar argument can be run.

[62]Unlike Dedekind and Huntington's theorems, Zermelo's theorem does not pin down ZFC relations down up to isomorphism, but it does pin them down up to a given "height" of the set-theoretic hierarchy.

[63]There are some related results in Bacon (2024), although the setting there is a modal

7 Mathematical Possibilism

We have explored a Leibnizian modal metaphysics in which there is no mathematical contingency. Might there be an equally powerful, but opposing axiom of modal metaphysics that implies that there is as much mathematical contingency as possible? According to this picture, any remotely plausible theory of a given mathematical structure (naturals, reals, etc.) that mathematicians could cook up, should be possible in the broadest sense. Here we will introduce an axiom, Mathematical Possibilism, meeting this description.[64]

First, note that the idea that mathematics is contingent could be seen as a special case of a much more general idea: that broad possibility is as liberal as logical consistency (cf. Bacon (2023a) §8.3).[65] There are different ways one might spell this out. If our standard of consistency is just consistency in the Background Logic, or indeed any recursively axiomatizable theory, then any structural arithmetical statement independent of that theory—such as its consistency statement—will be contingent.[66] However, given Goodsell's result about the Necessity of Arithmetic (theorem 5.2), this ultra liberal conception of logical possibility is incompatible with $\Box$Rigid Comprehension, and so I am tempted to look elsewhere.

Other liberal theories of possibility can be formulated that are compatible with $\Box\mathsf{RC}$.[67] Here I will focus on a principle that, in some sense, is the

higher-order set-theory, and so is slightly different.

[64]What follows in this section is a summary of technical work that will be published in a more appropriate venue; various important details and proofs cannot be presented here for reasons of space.

[65]There are some issues that need to be clarified: consistency is language relative property. We would want to restrict attention to logically perfect languages, in the sense of Russell (1940), to rule out counterexamples involving logically impossible propositions expressed using terms that hide the true logical structure. The proposition that some female foxes are not female foxes can be expressed in a non-logically-perfect language by a logically consistent sentence 'some vixens are not female foxes', where the contradictory form is hidden by using a simple term 'vixen' for a logically complex property.

[66]Bacon (2020), Bacon (2023a) §8.3, §13.2, §18.5-6, Bacon and Dorr (forthcoming), Bacon and Fine (manuscript) discuss theories of logical necessity in the higher-order setting with different standards of consistency being sufficient for possibility.

[67]Several theories of logical possibility are discussed in Bacon (2023a) section 8.3. One theory of logical possibility that is compatible with $\Box\mathsf{RC}$ is explored in Bacon (2020), however that theory implies that there isn't any contingency in statements of pure logic, and so needs to be modified for our purposes.

antithesis of the Leibniz Biconditionals. It tells us that there are broadly possible propositions whose truth is incompatible with things being a maximally specific way: there are propositions that are "atomless" under the entailment relation. However there are different *ways* that a proposition could be atomless, corresponding to the mathematical fact that there are lots of non-isomorphic complete atomless Boolean algebras. Our principle says that there are propositions corresponding to every complete atomless Boolean algebra.

Mathematical Possibilism For any small complete Boolean algebra at type σ, B, there exists a proposition P which, under the entailment relation, is isomorphic to B.

This is a schema, with one instance for each type σ.[68] A complete Boolean algebra consists of a property of type σ things, the *elements*, equipped with operations on the elements representing the Boolean operations, satisfying the axioms of a complete Boolean algebra. A complete Boolean algebra is small if it has fewer elements than there are propositions. The qualification involving smallness in Mathematical Possibilism is, of course, necessary to avoid Cantorian paradoxes.

A straightforward consequence of Mathematical Possibilism is that it entails the axiom of infinity. In particular, it implies that the finite numerical quantifiers over propositions form a natural number structure, and consequently that there exist real number structures (for instance, one constructed from Dedekind cuts) at a sufficiently high type, ρ. Thus in the presence of Mathematical Possibilism one can raise, without vacuity, the question of the contingency of the continuum hypothesis.

What is more, in the presence of $\Box$Rigid Comprehension and a modal version of the axiom of choice, Mathematical Possibilism implies there is all the mathematical contingency we could hope for in the presence of $\Box$Rigid Comprehension. As a proof of concept, this package of principles implies the contingency of CH

Theorem 7.1. *$\Box$Rigid Comprehension, Intensional Choice and Mathematical Possibilism entail structural contingency about the continuum hypothesis:*

$$\Diamond\exists\mathbf{X}(\mathrm{Real}(\mathbf{X})\wedge \mathrm{CH}(\mathbf{X})) \\ \wedge \Diamond\exists\mathbf{X}(\mathrm{Real}(\mathbf{X}) \wedge \neg\, \mathrm{CH}(\mathbf{X})).$$

[68]The full strength of the schema can in fact be obtained from the instance where σ is $t \to t$.

A proof of this theorem is to presented elsewhere.

It remains, then, to explain, and briefly motivate, the modal version of choice. The standard version of higher-order choice is intraworld: it says you can select one thing from the extension of each instantiated property. By contrast, an interworld version of choice says that, at any world, you can select one thing from the extension of a property that is instantiated at every world. But we can generalize this latter principle by replacing quantification over worlds with modal operators. This yields a stronger principle that is applicable even in the absence of world propositions:

Intensional Choice If F is necessarily instantiated, then there exists a G entailing F that is necessarily uniquely instantiated.

Finally, in order for this current vision of widespread mathematical contingency to be interesting, we need some guarantee that the principles appealed to in theorem 7.1 are jointly consistent.

Conjecture 7.1. $\Box$*Rigid Comprehension, Intensional Choice and Mathematical Possibilism are consistent (in the Background Logic).*

While I believe this conjecture to be true, I have not been able to verify it.

8 Platonic Contingency Revisited

We have, now, a clearer picture of what sorts of modal metaphysics permit mathematical contingency. To admit structural contingency, either you must think that certain properties and relations can't be rigidified, reject S5 or you must reject the existence of world properties allowing you to single out particular merely possible entities. (This is an inclusive disjunction—depending on what sort of contingency you want, you may have to accept multiple disjuncts.)

Let us, then, briefly apply what we have learned to the matter of platonic mathematical contingency. We have noted already that there is a relatively cheap sort of platonic contingency in which platonic objects have their defining mathematical features contingently: the natural numbers could have failed to form a natural number structure by becoming physical objects, for instance. One could rule out this cheap sort of platonic contingency by introducing a further posit to the effect that mathematical objects have their basic mathematical properties necessarily. Thus, for instance, we might insist that the the platonic natural numbers necessarily form a natural number structure under the usual relations on platonic numbers—successorhood, and so on—and that these basic properties and relations are necessary features of the platonic numbers; we might make a similar posit about the platonic reals.

As we have seen, however, such stipulations cannot rule out platonic contingency on their own. For if there is *structural* contingency—contingency about what is true in natural number structures or real number structures in general—there will be contingency too in the platonic natural number structure and the platonic real number structure.

9 The Supervenience of Mathematics

In this final section we turn to the question of supervenience. Do the mathematical facts—structural, platonic, or otherwise—supervene on non-mathematical facts, such as facts about the physical world? Or can they, in some sense, float free from the physical world? Of course, if mathematics is necessary, then it is clear that the mathematical facts trivially supervene on any collection of facts. But if there is mathematical contingency, the issue is not trivial.

Some preliminary points. There are some variants of the supervenience question that are also clearly of interest. Here I will treat supervenience on the physical, but it should be clear that a wide variety of non-mathematical facts could be be substituted for that in our discussion without greatly changing the wider points.Without loss of generality we might, for instance, add the mental facts to the supervenience base (if they are not already physical facts). Views that reduce the mathematical to the practices of mathematicians, or to ideal thinkers, may regard this as an important addition (Brouwer (1981)). One can also vary the sort of mathematical propositions that are supervenient—structural, platonic, or otherwise. I will continue to focus on the structural and the platonic mathematical propositions, but we will also pay attention to the other sorts as well. We may also wish to distinguish different varieties of supervenience depending on what sort of necessity is operative—the mathematical might supervene on other facts with respect to nomic necessity, say, but fail to so supervene with respect to broad necessity. Finally, for these questions to be well-posed we need to have a distinction between mathematical and non-mathematical facts, or mathematical and physical facts, to work with. We may yet encounter reasons to doubt this distinction is as sharp as it might at first seem, but we will run with it for now and see where it leads.

Certain positions in the philosophy of mathematics are naturally paired with the supervenience thesis. An obvious candidate would be certain brands of reductive nominalism, which simply *identify* mathematical truths with facts about the physical world in some fashion—be it the practices of mathematicians, the structure of physical quantities, physical inscriptions of sentences and numerals, or what have you. For this sort of nominalist there couldn't be a change in the mathematical without a change in the physical. However, the supervenience thesis is not only attractive to nominalists. Certain stripes of platonism seem also closely aligned with supervenience. Consider someone who is willing to quantify over platonic mathematical objects, but treats them as, in some sense, "metaphysically lightweight". It is commonly thought that certain individuals—holes, directions, the game of chess, and so on—exist and have properties in a derivative way, in virtue of the existence and properties of more basic objects. Just as a perforated lump of cheese is metaphysically prior to the holes in the cheese, physical arrangements or structures may similarly be prior to the abstract objects that those physical structures instantiate. According to one way of cashing out priority talk, the platonic mathematical facts are *grounded* in the physical propositions, or, according to another, they are *metaphysically definable* from the physical individuals, properties and relations.[69] In either case, we should expect the derivative objects and properties to supervene on the more basic ones; in this case it is the platonic mathematical facts that supervene on the physical.

Apart from its attractiveness, the supervenience thesis also has applications: for instance, one could use it to ensure the determinateness of mathematics with respect to a suitable modality expressing determinacy. For if the physical facts are determinate, and the mathematical supervenes on the physical with respect to the broadest modality (which, by definition, subsumes any determinacy modalities) then the mathematical must also be

[69] One limiting case of the latter variant is that the platonic objects are logical objects, à la Frege (1893), and are metaphysically definable from nothing (making them "pure" in a sense I have employed elsewhere). This makes the platonic objects vacuously definable from any class of properties and relations.

determinate.[70]

Supervenience also has its detractors. Hartry Field has argued that putative platonic mathematical facts are both contingently false, and conservative over the physical facts in a modal sense. Thus they will fail to supervene on the physical in a fairly radical way—while many platonic propositions regarded as true by platonists are actually false, they are nonetheless compatible with the actual physical facts, with respect to a broad logical modality, or else the actual physical facts would entail everything by conservativity.[71]

If mathematics does supervene on the physical, how exactly do changes in the mathematical track changes in the physical? One proposal, suggested by remarks in Field (1998) (in the context of indeterminacy rather than broad contingency), grounds the mathematical in the properties of particular physical structures. An undecidable arithmetical statement, for instance, can be given a physical interpretation in terms of the sequence of days, starting from today, on certain cosmological assumptions—such as the infinitude of days.[72] To evaluate this strategy, we should distinguish two ways in which there could be contingency about what is true in the structure of physical days. The first way is if the structure of time was contingent—for instance, the sequence of days could have been finite, or it could have been "non-standard" under the chronological ordering. This means that the days do not form a natural number structure.[73] Another way for there to be contingency in this physical structure, compatible with it being a natural number structure, is if there

[70]I am here supposing that determinacy is a kind of modality, rather than a metalinguistic feature of words (compare Bacon (2018b), Schiffer (2010)). Field (1998) explores a similar mode of argumentation, grounding the determinacy of arithmetical claims in physical structures, but is primarily concerned with securing the determinacy of reference for *words*, like 'finite', in terms of the determinacy of physical predicates like 'days'; than of the properties themselves.

[71]Field (1989), p139

[72]We can also extend this idea to analytic statements: the continuum hypothesis may be interpreted as about instants of time under the assumption that they form a complete ordered field, and so on. The continuum hypothesis is a second-order statement about the reals, but there are first-order statements about the reals that are undecidable–such as a sentence of first-order real analysis coding up the statement that there exists a non-constructible set of natural numbers.

[73]Note that if the sequence of days were non-standard this means that there are some 'standard' days, with the following properties: (a) the day representing 0 is standard, (b) the successor of a standard day is standard, but (c) not every future day is standard. We can see that the days no longer form a natural number structure, since standardness witnesses a failure of the induction property.

were structural arithmetical contingency. For a given arithmetical sentence A, it could be contingent whether A is true in the structure of days *while they form a natural number structure*, if there was structural contingency about whether A held in natural number structures.

The former sorts of possibility illustrate that the structural mathematical claims (at least) do not supervene, with respect to the broadest modality, on the physical in anything like the way that the recipe above suggests. We cannot pin the arithmetical, say, to any particular physical structure, for presumably any given physical objects could, in the broad sense of 'could', have failed to be structured in that way. One might have thought that there are still some broadly mathematical claims that are tied to the physical structure of time. For instance, if the days formed a non-standard structure then one could create a physical "Turing machine" that could tell whether an arbitrary Turing machine would halt within a standard number of days, an impossibility if time was standard. However on reflection such possibilities do not really represent a difference in *mathematics*—Turing's theorem about abstract Turing machines, mathematical entities, still holds. It is only in the presence of a mathematical-to-physical bridge principle, the Church-Turing thesis (what is physically computable is exactly what is computable on an abstract Turing machine), that the mathematics would have to change to accommodate these physical possibilities, and it is the Church-Turing thesis that is clearly failing here.

That said, our discussion so far does suggest that the supervenience thesis may be hard to avoid with respect to more restricted modalities. Let's consider the idea that the structural mathematical claims nomically supervene on the physical (i.e. supervenes with the force of physical necessity). We will assume that it is physically necessary that the sequence of days, starting from today, is ordered in a natural number structure. Let $A(\mathrm{suc}, 0)$ be any arithmetical claim, with structural content $\forall Sy(\mathrm{Nat}(S, y) \to A(S, y))$. Now, whenever R is a physical relation, *later than*, and x a physical object, *today*, then $A(R, x)$ is also a physical statement, since it is expressed entirely in terms of physical predicates, physical names and logical expressions. Moreover, $A(R, x)$ strictly implies $\forall Sy(\mathrm{Nat}(S, y) \to A(S, y))$, since *later than*, *today* is physically necessarily a natural number structure, and by Dedekind's theorem it's necessary that any two natural number structures agree about A. The converse implication is trivial, so that every structural arithmetical claim is physically necessarily equivalent to a physical proposition, securing nomic supervenience.

$$\Box_{\text{Phys}}(\forall Sy(\text{Nat}(S,y) \to A(S,y)) \leftrightarrow A(\text{later than}, \text{today}))$$

We have argued that the structural mathematical claims do not supervene, with respect to *broad* necessity, on particular physical structures, like the structure of days, in the way described. But this does not preclude them from supervening on the physical in some other way. As we noted initially, it's not entirely clear how to draw the line between the physical and the mathematical—there are different versions of the supervenience question depending on how we precisify the notion of a 'physical proposition'. Given the interpretation of higher-order logic we offered in section 3 the most natural thing to mean by 'physical proposition', in my view, counts the structural mathematical propositions as physical propositions; to my mind, this secures the most salient precisification of the supervenience thesis. It's common for philosophers to introduce the idea of a physical proposition in terms of the language it can be expressed in: a proposition that can be expressed in the language of physics, using only physical non-logical constants and logical expressions. This was essentially the line of reasoning we used to establish that A(later than, today) expressed a physical proposition. Since structural mathematical propositions—statements of the form $\forall Xy(\text{Nat}(X,y) \to A(X,y))$—are stated in purely logical terms they clearly meet this criteria. But the thought doesn't need to be expressed in metalinguistic terms. If you start with some physical individuals, properties and relations, then anything metaphysically definable from them—i.e. anything you can make from them by applying logical operations—is also physical.[74]Many philosophers have posited a distinctive project of *metaphysical analysis* (as distinct from linguistic analysis), in which metaphysical reduction takes the form of metaphysical definition of one sort of entity in terms of another. This opens the way for an attractive form of reductive nominalism. For in the absence of any platonic mathematical objects, it's quite natural to identify all mathematical propositions with structural mathematical propositions (see Hellman (1989)), which in turn just are physical propositions on this precisification of 'physical'.

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A Preliminaries

A.1 Language

We work in a simply typed higher-order modal language: there are two base types, e and t, and given any types σ and τ there is a functional type $(\sigma \to \tau)$. We omit type brackets when they are associated to the right, and will write '$M : \sigma$' as short for 'M is a term of type σ' or 'M, of type σ,'.

The terms of language are defined as follows. For each type σ, there will be infinitely many variables of that type. We typically represent these with upper and lower case letters towards the end of the latin alphabet, like X, Y, Z and x, y, z. Occasionally we will use more suggestive names like 'suc' and 'add' for variables depending on their function. Whenever M is a term of type $\sigma \to \tau$ and N a term of type σ, (MN) is a term of type τ and whenever M is a term of type τ and x a variable of type σ, $(\lambda x.M)$ is a term of type $\sigma \to \tau$. Finally we have primitive terms for the logical constants: $\forall_\sigma : (\sigma \to t) \to t$, $\to$: $t \to t \to t$, and $\Box : t \to t$. We may introduce $\exists_\sigma, \bot, \wedge, \vee, \leftrightarrow, =_\sigma$ as abbreviations in any of the standard ways. For instance, $\bot$ may be identified with $\forall_{(t\to t)\to t}\forall_t$, $=_\sigma$ with $\lambda xy\forall_\sigma X(Xx \to Xy)$.

I will adopt some further conventions.[75] We adopt infix notation for the binary logical connectives and identity. λs immediately following a quantifier are omitted. Given a term $P : \sigma \to t$ we write $\forall^P_\sigma$ for $\lambda X \forall_\sigma x(Px \to Xx)$, and $\exists^P_\sigma$ for $\lambda X\exists_\sigma x(Px \wedge Xx)$. We use $\vec{x}$ for sequences $x_1...x_n$. $\lambda\vec{x}, \forall\vec{x}$ etc. stand for strings of λs or quantifiers — e.g. the first amounts to $\lambda x_1 \lambda x_2 \ldots$ — and $R\vec{x}$ stands for $Rx_1 \ldots x_n$. $\vec{\sigma} \to \tau$ stands for $\sigma_1 \to \sigma_2 \to \ldots \to \tau$. $M[N/x]$ is the result of replacing every free occurrence of v in M with N provided no free variable in N becomes bound.

The languages we consider may contain further non-logical constants. As usual logics and theories will be identified with sets of terms of type t.

A.2 Formalizing mathematical notions in higher-order logic

In this section we show how to formalize various familiar mathematical notions in higher-order logic. For the sake of readibility definitions will be given in ordinary English, and we will only provide explicit definitions in the

[75] I am following the conventions of Bacon (2023a).

$\Diamond_{\vec{z}} := \lambda R\lambda\vec{z}.\neg\Box\neg R\vec{z}$	$\subseteq_{\vec{\sigma}} := \lambda XY\forall_{\vec{\sigma}}\vec{z}(X\vec{z} \to Y\vec{z})$
$\sim_{\vec{\sigma}} := \lambda XY.(X \subseteq_{\vec{\sigma}} Y \wedge Y \subseteq_{\vec{\sigma}} X)$	$\leq_{\vec{\sigma}} := \lambda XY.\Box X \subseteq_{\vec{\sigma}} Y$
$\mathrm{Rig}_{\vec{\sigma}} := \lambda X\Box\forall_{\vec{\sigma}\to t}Y(\Box\forall^X_{\vec{\sigma}}\vec{z}.Y\vec{z} \leftrightarrow \forall^X_{\vec{\sigma}}\vec{z}.\Box Y\vec{z})$	$\mathrm{World}_{\vec{\sigma}} := \lambda R(\Diamond_{\vec{\sigma}}R \wedge \forall S(R \leq_{\vec{\sigma}} S \vee R \leq_{\vec{\sigma}} \neg_{\vec{\sigma}}S)$
$\mathrm{Ub}^{\preceq} := \lambda Xy.\forall z(Xz \to z \preceq y)$	$\mathrm{Lub}^{\preceq} := \lambda Xy.\,\mathrm{ub}\,Xy \wedge \forall z(\mathrm{ub}\,Xz \to y \preceq z)$
$\mathrm{Dom}_\sigma := \lambda Rx.\exists_\sigma y.(Rxy \vee Ryx)$	$\mathrm{Trans}_\sigma := \lambda R\forall_\sigma xyz(Rxy \wedge Ryz \to Rxz)$
$\mathrm{Ancest}_\sigma := \lambda Sxy\forall R(\mathrm{Trans}\,R \wedge S \subseteq_\sigma R \to Rxy)$	$\mathrm{Fun}_\sigma := \lambda S\forall_\sigma xyy'(Sxy \wedge Sxy' \to y =_\sigma y')$
$F : X \to Y := \forall^X x\exists^Y!y.Fxy$	$F : X \xrightarrow{1-1} Y := \forall^X xx'y.(Fxy \wedge Fx'y \to x = x')$
$\mathrm{PO} := \lambda PR.PR$ is a partial order	$\mathrm{Lattice} := \lambda PR.P, R$ is a lattice
$\mathrm{Compl} := \lambda PR.PR$ is a complemented lattice	$\mathrm{Dist} := \lambda PR.PR$ is a distributive lattice
$\mathrm{BA}_\sigma := \lambda PR.PR$ is a Boolean algebra	$\mathrm{CBA}_\sigma := \lambda PR.PR$ is a complete Boolean algebra

Table 1: Abbreviations

language of higher-order logic when the required definition is not obvious.

We begin with some order-theoretic notions. A *partial order* at type σ consists of a property, $P : \sigma \to t$, and a relation $\preceq : \sigma \to \sigma \to t$ which is transitive, reflexive and antisymmetric with respect to the type σ entities satisfying P. P entities are called *elements* in the partial order. A partial order $P, \preceq$ has meets and joins when any two elements have a greatest lower bound and a least upper bound in the partial order, in which case we call $P, \preceq$ a *lattice*. A lattice is *complete* when for any property F there is a greatest greatest lower bound and least upper bound of the Fs in P. We will sometimes write $a \sqcap b$ and $a \sqcup b$ for the (unique) meet and join of a and b: note that in using this notation we are not treating $\sqcap$ itself as a $\sigma \to \sigma \to \sigma$ term—rather $a \sqcap b$ is a syntactically simple term introduced by existential instantiation. A lattice is *distributive* when $a \sqcap (b \sqcup c)$ and $(a \sqcap b) \sqcup (a \sqcap c)$ are the same. A Boolean algebra $P, \preceq$ is a *complemented* distributive lattice: for every element, a, there is another element b such that $a \sqcup b$ is the greatest element of the lattice and $a\sqcap$ is the least element. A *well-order* at type σ is total partial order such for every property $F : \sigma \to t$, if there are any F elements, there is a $\preceq$-least F element. The ancestral of a relation R holds between x and y when every transitive relation extending R holds between x and y ($\mathrm{Ancestral} := \lambda Sxy\forall R(\mathrm{Trans}\,R \wedge S \subseteq_\sigma R \to Rxy)$).

Given terms $F : \sigma \to \tau \to t$, and $X : \sigma \to t, Y : \tau \to t$, we write $F : X \to Y$ to mean that F is a functional relation between X and Y: every X bears F to a unique Y. $F : X \xrightarrow{1-1} Y$ means that this relation is one-one: no two Xs bear F to the same Y, and $F : X \xrightarrow{\text{onto}} Y$ means that it is onto: for any Y there is some X that bears F to that Y, and $F : X \xrightarrow{\text{bij}} Y$ if it is both one-one and onto. We use '**P**' to stand for a sequence of variables

‘$P : \sigma \to t, \preceq_P : \sigma \to \sigma \to t$’ and ‘$\mathbf{Q}$’ for ‘$Q : \tau \to t, \preceq_Q : \tau \to \tau \to t$’. If $\mathbf{P}$ and $\mathbf{Q}$ are partial orders, then we write $\mathbf{P} \cong \mathbf{Q}$ iff the partial orders are isomorphic: there exists $F : P \xrightarrow{\text{bij}} Q$ such that for any whenever Fxx' and Fyy', $x \preceq_P y$ if and only if $x' \preceq_Q y'$.

A *natural number structure* at type σ consists of an entity $0 : \sigma$, and a functional one-one relation $\mathrm{suc} : \sigma \to \sigma \to t$ such that: nothing bears suc to 0, and moreover, any relation with 0 in its field that relates x toy y when x is in its field and $\mathrm{suc}\, xy$, contains suc: $\forall R(\mathrm{Dom}\, Rz \wedge \forall x(\mathrm{Dom}\, Rx \wedge \mathrm{suc}\, xy \to Rxy) \to \forall xy(\mathrm{suc}\, xy \to Rxy))$. A *first-order natural number structure* consists of the above, and additionally relations $+, \times, <$ such that .

$$+ := \lambda nmk.\forall R(Rn0n \wedge \forall ii'jj'(\mathrm{suc}\, ii' \wedge \mathrm{suc}\, jj' \wedge Rnij \to Rni'j') \to Rnmk)$$

$$\times := \lambda nmk.\forall R(Rn00 \wedge \forall ii'jj'(\mathrm{suc}\, ii' \wedge \mathrm{add}\, njj' \wedge Rnij \to Rni'j') \to Rnmk)$$

$$< := \lambda nm.\forall R(\forall ij(\mathrm{suc}\, ij \to Rij) \wedge \forall ijk(Rij \wedge Rjk \to Rik) \to Rnm)$$

The *domain* of a natural number structure is the field of $<$. We will write $\mathbf{N}$ to abbreviate a sequence of variables $z : \sigma, S : \sigma \to \sigma \to t$ and we write $\mathrm{Nat}^\sigma\, \mathbf{N}$ for the statement that z and S together form a natural number structure at type σ; the same notation will be adopted for first-order natural number structures.

A *real number structure* at a type σ consists of a total partial order property $R : \sigma \to t, \preceq$, elements $0, 1 : \sigma$ and ternary relations $+, \times : \sigma \to \sigma \to \sigma \to t$ that are functional with domain R representing addition and multiplication. We will write $x + y$ as short for the description for the unique z such that $+xyz$, and similarly for $\times$. Addition and multiplication are commutative and associative and distributive in the sense that $x \times (y + z) = (x \times y) + (x \times z)$. 0 and 1 are the units of $+$ and $\times$ respectively (e.g. $\forall_\sigma x(+x0y \to x = y)$, and every element of R has an additive inverse and every element apart from 0 has a multiplicative inverse—i.e. for each x there is a y such that $x + y = 0$ and for each $x \neq 0$ there is a y such that $x \times y = 1$. Moreover if $x \preceq y$ then $x + z \preceq y + z$ and if $0 \leq x$ and $0 \leq y$, $0 \leq x \times y$. Finally it is complete: for any property of elements F that has an upperbound in R has a least upperbound. A *first-order real number structure* consists of the preceding along with a predicate N such that

$$N := \lambda x.\forall F(F0 \wedge \forall y(Fy \wedge \forall z(+x1z \to Fz) \to Fx)$$

We will write $\mathbf{R}$ for a sequence of variables $R, N, +, \times, 0, 1, <$ of the appropriate types. We write $\mathrm{Real}^\sigma\, \mathbf{R}$ to say that they form a real number structure.

Next some modal notions. A proposition, property or relation P of type $\vec{\sigma} \to t$ is possible$_{\vec{\sigma}}$ when it is possible that there exist entities $\vec{x}$ that instantiate P; P is necessary in the dual case. We say that P entails Q, when $\lambda\vec{z}(R\vec{z} \to S\vec{z})$ is necessary$_{\vec{\sigma}}$. A world proposition (property, relation) is something that is possible, and such that, for any other proposition (property, relation), it entails it or its negation. A property (relation) X is rigid iff the X restricted quantifiers necessarily satisfy the Barcan formula and its converse: $\mathrm{Rig}_{\vec{\sigma}} := \lambda X \Box\forall_{\vec{\sigma}\to t} Y(\Box\forall^X_{\vec{\sigma}}\vec{z}.Y\vec{z} \leftrightarrow \forall^X_{\vec{\sigma}}\vec{z}.\Box Y\vec{z})$.

Quantification over "natural number structures" is strictly speaking a sequence of universal quantifiers, one for each element of the signature of a natural number structure. Thus we will need some notation for representing such sequences.

- We use $\mathbf{N}$ for a sequence of variables with the following types $0 : \sigma, \mathrm{suc} : \sigma \to \sigma \to t$.
- We use $\mathbf{R}$ for a sequence of variables with the following variables $R, N : \sigma \to t, +, \times : \sigma \to \sigma \to \sigma \to t, 0, 1 : \sigma, <: \sigma \to \sigma \to t,$.
- We use $\mathbb{N}$ for the *canonical* natural number structure: the sequence of terms $\mathrm{NumQuant}, 0^Q, \mathrm{suc}_Q, <_Q, +_Q, \times_Q$ defined above.
- We use $\mathbb{R}$ for the canonical real number structure: sequence of terms given in theorem A.2 below.

A.3 Logical systems

Here we state some logics of interest. The minimal system $\mathsf{H}^\Box$ is presented in figure 1. We adopt the usual notation from modal logic for modal principles: $\mathsf{T} := \Box\forall_t p(\Box p \to p), \mathsf{4} := \Box\forall_t p(\Box p \to \Box\Box p)$ and $\mathsf{B} := \Box\forall_t p(p \to \Box\Diamond p)$.

To $\mathsf{H}^\Box$ we can add further principles, listed in figure 2, which we denote by appending their names separated by a dot—e.g. $\mathsf{H}^\Box$.5 for adding $\mathsf{T}, \mathsf{4}$ and B, $\mathsf{H}^\Box.\mathsf{5}.\mathsf{RC}$ including RC, etc.

In the statement of MP, $\mathbf{B}$ stands for a pair of variables $B : \sigma, \preceq: \sigma \to \sigma \to t$ and $\mathbf{P}$ for $P : t, \leq$, recalling that $\leq$ is defined as $\lambda pq.p \wedge q = p$. SmallCBA is the property of being a complete Boolean algebra whose cardinality is no bigger than the number of propositions.

Throughout we will appeal to couple of facts that may be derived in these systems about the existence of natural and real number structures. First we

PC $\vdash A$ whenever A is a tautology.

UI $\vdash \forall_\sigma F \to Fa$.

β $A[(\lambda x.M)N] \leftrightarrow A[M[N/x]]$.

η $A[\lambda x.(Fx)] \leftrightarrow A[F]$, where x is not free in F.

K $\Box(A \to B) \to \Box A \to \Box B$

MP If $\vdash P$ and $\vdash P \to Q$, then $\vdash Q$.

Gen If $\vdash P \to Q$, and v is not free in P, $\vdash P \to \forall v Q$.

Nec If $\vdash A$ then $\vdash \Box A$

Figure 1: The Background Logic, $\mathsf{H}^\Box$

RC $\forall_{\vec{\sigma}\to t} R \exists_{\vec{\sigma}\to t} X.(\mathrm{Rig}\, X \wedge R \sim_{\vec{\sigma}} X)$

B $\Box \forall_t p(p \to \Box \Diamond p)$

LB $\forall_{\vec{\sigma}\to t} P(\Diamond_{\vec{\sigma}} P \leftrightarrow \exists_{\vec{\sigma}\to t} W.(\mathrm{World}_{\vec{\sigma}}\, W \wedge W \leq_{\vec{\sigma}} P))$

MP $\forall \mathbf{B}(\mathrm{SmallCBA}_\sigma\, \mathbf{B} \to \exists_t P.(\mathbf{B} \cong \mathbf{P} \wedge P \neq_t \top))$

Figure 2: Key Modal Principles

define what we will call the *canonical natural number structure*, consisting of the cardinality quantifiers:

$$
\begin{aligned}
0_Q &:= \lambda X.\top \\
\mathrm{suc}_Q &:= \lambda Q\lambda X.\exists y.(Xy \wedge Q\lambda z(Xz \wedge z \neq y) \\
\mathrm{NumQuant} &:= \lambda Q\forall Z((Z0 \wedge \forall P(ZP \to Z(\mathrm{suc}\, P)) \to ZQ) \\
<_{\mathrm{Q}} &:= \lambda PQ\forall Z(Z0(\mathrm{suc}\, 0)\wedge \\
&\qquad \forall P'Q'(ZP'Q' \to (ZP'\,\mathrm{suc}\, Q' \wedge Z\,\mathrm{suc}\, P'\,\mathrm{suc}\, Q')) \to ZPQ) \\
+_{\mathrm{Q}} &:= \lambda xyz.\forall R(\forall w(Rw0w \wedge \forall uv(Rwuv \to Rw(\mathrm{suc}\, u)(\mathrm{suc}\, v))) \to Rxyz) \\
\times_{\mathrm{Q}} &:= \lambda xyz.\forall R(\forall w(Rw00 \wedge \forall uv(Rwuv \to Rw(\mathrm{suc}\, u)(\mathrm{add}\, vw))) \to Rxyz)
\end{aligned}
$$

Let the axiom of potential infinity be the following principle:[76]

Potential Infinity $\forall Q(\mathrm{NumQuant}\, Q \to \Diamond Q(\lambda x.\top))$

Theorem A.1. *Given the axiom of Potential Infinity (in $\mathsf{H}^{\Box}$), the canonical natural number structure is indeed a natural number structure.*

Second we will appeal to the fact that, given the axiom of Potential Infinity, and one of several auxiliary assumptions, there exists a real number structure that can be constructed from the canonical natural number structure, and consists of properties of finite numerical quantifiers. We call this the *canonical real number structure*. The definition of this structure, and the proof that it is a real number structure is rather involved. It exploits the non-obvious, but well-known, fact that you can define operations on the powerset of natural numbers that turns it into a into a complete ordered field.

There is a slight wrinkle with transposing the set-theoretic construction to the higher-order framework: sets, unlike properties, are individuated extensionally. We cannot, then, straightforwardly identify reals with properties of naturals since there would be many coextensive properties corresponding to any given real. There are several work arounds. If we have Rigid Comprehension, we can identify reals with rigid properties of naturals, since these *are* individuated extensionally. Without Rigid Comprehension we don't have any guarantee that there are enough rigid properties to play the role of all the reals. However, if we have the well-ordering principle or some similar choice principle we can instead pick a particular property from in a given

[76]cf. Hodes (1990), Goodsell and Yli-Vakkuri (MS).

equivalence class of coextensive properties to be a representative of a given real.

The Well-ordering Principle $\exists R.\, \mathrm{WO}\, R \wedge \mathrm{Dom}\, R \sim \lambda x.\top$

Thus we have:

Theorem A.2. *Given the axiom of Potential Infinity, and either Rigid Comprehension or the Well-Ordering Principle (in $\mathsf{H}^{\Box}$), it is possible to construct a real number structure at the type $\sigma \to t$, where $\sigma = (e \to t) \to t$ the type of quantifiers. Moreover, it is possible to do so in such a way that every property of canonical natural numbers is coextensive with exactly one element of the real number structure.*

There is one final work around that requires no additional assumptions beyond Potential Infinity. We can define a quasi-real number structure as consisting of the same data as a real number structure with the addition of an equivalence relation $\approx$ to represent identity: so that we have $R, N : \sigma \to t$, $+, \times : \sigma \to \sigma \to \sigma \to t$, $0, 1 : \sigma$, $<, \approx : \sigma \to \sigma \to t$. We then require the operations $+, <, \times, R, N$ to respect the notion of identity in the sense that, e.g., if $+abc$ and $a \approx a', b \approx b', c \approx c'$ then $+a'b'c'$. We also modify the conditions for being a complete ordered field by substituting all occurrences of $=$ with $\approx$, so that, for instance, the commutativity law becomes $+abc \wedge +bac' \to c \approx c'$. The notion of an isomorphism between quasi-real number structure is now a (possibly non-functional) relation which preserves $\approx$ and the other field operations. Quasi-real number structures can be constructed, without additional assumptions, from properties of canonical natural numbers using coextensiveness as the notion of identity. Note that every real number structure is automatically a quasi-real number structure with $\approx := =_\sigma$. Appeals to theorem A.2 can be substituted to appeals to the existence of quasi-real number structures in this paper, but in the contexts we need canonical real number structures we will always either have Rigid Comprehension or a well-ordering available.

B Consistency proofs

B.1 Model of Classicism with first-order arithmetical contingency

Here we prove

Theorem B.1. *There is a first-order arithmetical sentence,* $A(0, \text{suc}, <, \text{add}, \text{mult})$ *and a model of Classicism (*H *with* $\mathsf{S4}$ *and Intensionalism) which is structurally contingent. I.e. the model makes*

$$\Diamond\exists\mathbf{X}(\text{Nat}(\mathbf{X}) \wedge A(\mathbf{X})) \wedge \Diamond\exists\mathbf{X}(\text{Nat}(\mathbf{X}) \wedge \neg A(\mathbf{X}))$$

true.

The proof uses methods described in Bacon (2023a). There is described a class of "modal models" which is sound and complete with respect to Classicism. Among these are models are "extensionally full" models, which have, for every subset of their domain, a property that has that subset as its extension, and satisfies similar conditions for relations (see Dorr (2016) definition 4.5). Extensionally full models with an infinite type e domain are arithmetically standard in the following sense.

Definition B.1. *M is arithmetically standard iff* $M \models \forall\mathbf{X}(\text{Nat}\,\mathbf{X} \to A(\mathbf{X}))$ *if and only if* $A(0, \text{suc}, +, \times, <)$ *is an arithmetical truth.*

Here we use the expression $M \models A$ to mean that the sentence A is true in the model M. We have, by Proposition 18.7 and Corollary 18.4 Bacon (2023a) the following fact:

Theorem B.2. *Given any set of modal models, $\mathcal{C}$, there is an arithmetically standard modal model M such that, whenever $N \in \mathcal{C}$, $N \models A$ where A is closed, $M \models \Diamond A$.*

We may construct a model of first-order arithmetical contingency as follows. Let us first find an arithmetical truth, A, whose structural translation, $\forall\mathbf{X}(\text{Nat}\,\mathbf{X} \to A(\mathbf{X}))$, cannot be derived in Classicism. The consistency statement for Classicism would do. By the completeness theorem there is a modal model N of $\exists\mathbf{X}(\text{Nat}\,\mathbf{X} \wedge \neg A(\mathbf{X}))$. Let $\mathcal{C} = \{N\}$: by theorem B.2 above there is an arithmetically standard model M such that $M \models \Diamond\exists\mathbf{X}(\text{Nat}\,\mathbf{X} \wedge \neg A(\mathbf{X}))$. Moreover $M \models \exists\mathbf{X}(\text{Nat}\,\mathbf{X} \wedge A(\mathbf{X}))$. For an arithmetically standard model must make $\exists\mathbf{X}\,\text{Nat}\,\mathbf{X}$—$\bot$ is not an arithmetical truth, so $M \not\models \forall\mathbf{X}(\text{Nat}\,\mathbf{X} \to \bot)$—and $\forall\mathbf{X}(\text{Nat}\,\mathbf{X} \to A(\mathbf{X}))$ since A is an arithmetical truth. M is a model of $\Diamond\exists\mathbf{X}(\text{Nat}(\mathbf{X}) \wedge A(\mathbf{X})) \wedge \Diamond\exists\mathbf{X}(\text{Nat}(\mathbf{X}) \wedge \neg A(\mathbf{X}))$ as required.

B.2 Model of Classicism, **S5**, **RC** and first-order analytic contingency

We would like to construct a model of the following two claims:

$$\Diamond\exists\mathbf{R}(\mathrm{Real}\,\mathbf{R}\wedge\mathrm{CH}\,\mathbf{R})$$

$$\Diamond\exists\mathbf{R}(\mathrm{Real}\,\mathbf{R}\wedge\neg\,\mathrm{CH}\,\mathbf{R})$$

where CH is:

$$\lambda\mathbf{R}(\forall X\subseteq R(\exists F:R\xrightarrow{1-1}X\vee\exists F:X\xrightarrow{1-1}N))$$

Here $\mathbf{R}$ is short for the sequence of variables $R, N, 0, 1, \mathrm{add}, \mathrm{mult}, <$, with R a unary predicate representing the reals of the structure and N representing the naturals.

Below we construct a set-theoretic model, in a background of ZFC+CH, and offer a sketch of proof that it satisfies the desired properties. Create a full functional model as follows.

- $\mathbb{P}$:= the disjoint sum of the partial order $(\{p\mid p$ is a finite partial function from $\omega_2\times\omega$ to $2\},\supseteq)$ and $(\{@\},\{(@,@)\})$.
- $\mathbb{B}:=RO(\mathbb{P})$, the regular open subsets of $\mathbb{P}$.
- $D^t=\mathbb{B}\times 2$.
- $D^e=\omega$
- $D^{\sigma\to\tau}={D^\tau}^{D^\sigma}$
- $\forall_\sigma$ given by meet in the Boolean algebra, similarly for the logical connectives.

$\mathbb{B}$ is a complete Boolean algebra. Intuitively it consists of a solitary atom, $\{@\}$—which will serve as our actual world—and then a large atomless false proposition $P:=\mathbb{P}\setminus\{@\}$. We will show that according to this model "there exists a real structure in which CH true" is true at the actual world, but false throughout the atomless portion of logical space. We will use $\bigsqcap$ and $\bigsqcup$ to denote the meets and joins of elements in this algebra, and p^c for the complement of p. Observe that $\mathbb{B}$ has the countable chain condition: every set of consistent pairwise incompatible elements in $\mathbb{B}$ is countable.

The meanings of terms are computed relative to variable assignments g, which map each variable of type σ to an element of D^σ:

- $[\![x]\!]^g = g(x)$
- $[\![MN]\!]^g = [\![M]\!]^g([\![N]\!]^g)$
- $[\![\lambda x.M]\!]^g = a \mapsto [\![M]\!]^{g[a/x]}$
- $[\![\forall_\sigma]\!] = f \mapsto \bigsqcap_{a \in D^\sigma} f(a)$
- $[\![\rightarrow]\!] = p \mapsto q \mapsto (p^c \sqcup q)$

A formula A is satisfied by g iff $@ \in [\![A]\!]^g$.

We assume, for simplicity, we are working in a language with a constant of type σ for every element of D^σ. If we also play loose with use and mention (or let the elements of D^σ be their own names), this eliminates various bits of fussing involving variable assignments—we can write $[\![A(a_1, ..., a_n)]\!]$ where $a_i \in D^{\sigma_i}$ instead of $[\![A(x_1, ..., x_n)]\!]^g$ where g is a variable assignment mapping x_i to a_i.

Next we appeal to theorem A.2 which guarantees that, given the well-ordering principle, we can construct a canonical real number structure whose elements are properties of natural numbers, and which includes at least one such property with any given extension on the natural numbers. In the present setting we get the following.

Lemma B.1. *Suppose $[\![r$ is a well-ordering$]\!] = \top$, and let $\sigma = e \to t$. Then there exists terms, $\mathbb{R} := R, N : \sigma \to t, 0, 1 : \sigma, +, \times : \sigma \to \sigma \to \sigma \to t, <: \sigma \to \sigma \to t$ each in a single parameter r, corresponding to reals, naturals, operations of addition and multiplication, 0, 1 such that $[\![\mathrm{Real}\,\mathbb{R}]\!] = \top$ and $[\![\forall_{e\to t} X \exists_{e\to t} Y(RY \wedge X \sim Y)]\!] = \top$.*

In order for this construction to work we need to check that such an r exists:

Lemma B.2. *The Well-Ordering Principle, WO^σ, is necessarily true in M. Indeed, there is a particular element of the model, $r \in D^{\sigma\to\sigma\to t}$, such that $[\![r$ is a well-order$]\!] = \top$.*

Proof. It is sufficient to find a relation, $r \in D^{\sigma\to\sigma\to t}$, such that the semantic value of "r totally orders type σ and is well-founded" in M is $\top$.

Let $<$ be some well-order on D^σ, we may define r as

$$r(a)(b) = \begin{cases} \top & \text{if } a < b \\ \bot & \text{otherwise} \end{cases}$$

It is easily seen that $[\![r \text{ is a total order}]\!] = \top$. It remains to show that $[\![r \text{ is well-founded}]\!] = \top$. It suffices to show $[\![\exists y.fy]\!] \leq [\![\exists y.y \text{ is } f \text{ and } r\text{-minimal}]\!]$ for every $f \in D^{\sigma \to t}$.

Let $f \in D^{\sigma \to t}$. If $[\![\exists y.fy]\!] = \bot$ we are done. If $[\![\exists y.fy]\!] \neq \bot$, it suffices to show that for every $b \leq [\![\exists y.fy]\!]$ there is some $a \in D^\sigma$ such that $[\![a \text{ is } f \text{ and } r\text{-minimal}]\!] \sqcap b \neq \bot$.

Since $[\![\exists y.fy]\!] \neq \bot$, there exists a $d \in D^\sigma$ with $[\![fd]\!] \sqcap b \neq \bot$. Let a be a $<$-minimal element with this feature. Suppose $\bot < b' \leq f(a) \sqcap b$, and $d \in D^\sigma$ with $b' \leq [\![fd]\!]$. Then $[\![fd]\!] \sqcap b \neq \bot$ so $r(a)(d) = \top$ or $a = d$ by the minimality of a, so $[\![rad \vee a = d]\!] = \top$. Since $d \in D^\sigma$ was arbitrary, $f(a) \sqcap b \leq [\![\forall_\sigma y(fy \to ray \vee a = y)]\!]$. This means $f(a) \sqcap b \leq [\![a\ f \text{ and } r\text{-minimal}]\!] \sqcap b \neq \bot$ as required. $\square$

First we show that $\Diamond \exists \mathbf{R}(\text{Real}\,\mathbf{R} \wedge \text{CH}\,\mathbf{R})$ is true in the model. Indeed $\exists \mathbf{R}(\text{Real}\,\mathbf{R} \wedge \text{CH}\,\mathbf{R})$ is true in the model (i.e. holds at @) for we know from lemma B.1 that there are elements of the model, $\mathbb{R}$, such that $[\![\text{Real}\,\mathbb{R}]\!] = \top$. But it can also be shown that that the truth of CH is "absolute" in the model.

Lemma B.3. $\exists \mathbf{R}(\text{Real}\,\mathbf{R} \wedge \text{CH}\,\mathbf{R})$ *is true in* M *if and only if the continuum hypothesis is true.*

M is extensionally full in the sense of Dorr (2016) appendix A4: for any subset $X \subseteq D^\sigma$ there is an element $f \in D^{\sigma \to t}$ such that for all $a \in D^\sigma$, $@ \in [\![fa]\!]$ if and only if $a \in X$. Thus in extensional contexts quantification over properties in the model is equivalent to quantification over sets in the metalanguage. This can be used to show that counterexamples to the higher-order version of CH in the model would be counterexamples to the set-theoretic continuum hypothesis and conversely.

Next we show that $\Diamond \exists \mathbf{R}(\text{Real}\,\mathbf{R} \wedge \neg\,\text{CH}\,\mathbf{R})$ is true in the model. In fact $P \leq [\![\exists \mathbf{R}(\text{Real}\,\mathbf{R} \wedge \neg\,\text{CH}\,\mathbf{R})]\!]$ where P is the atomless portion of logical space, $\mathbb{P} \setminus \{@\}$.

Lemma B.4. *The the semantic value of "there is a real number structure* $R, \ldots$ *at type* $e \to t$ *and an uncountable property of those reals which the reals cannot inject into" is the worldless portion of logical space* P.

Proof. Our strategy is to use lemma B.1 to find a real number structure $\mathbb{R} = R, N, \ldots$ made of properties of natural numbers, and then show that P entails that it does not satisfy CH.

For each $\alpha < \omega_2$ we define we define some special properties of natural numbers, $a_\alpha \in D^{e\to t}$, as follows

$$a_\alpha(x) = \{p \in \mathbb{P} \mid p(\alpha, x) = 1\}$$

Intuitively the a_α are highly contingent properties of natural numbers that are nonetheless necessarily coextensive with some R property, and necessarily no pair of them are coextensive.

Now to define the counterexample to the continuum hypothesis, G. In the worldless regions of logical space, G is uncountable and R cannot be injected into G. $G : D^{e\to t} \to D^t$

$$G(a) = \begin{cases} \top & \text{if } a = a_\beta \text{ for some } \beta < \omega_1 \\ \bot & \text{otherwise} \end{cases}$$

Now, let $c' \in D^{(e\to t)\to(e\to t)\to t}$ be the relation necessarily relates each property (element of $D^{\sigma\to t}$) to the minimal such element coextensive with it, obtained from lemma B.1 (i.e. c' with $[\![c'$ is a choice relation for $\sim]\!] = \top$). We can define $c(a)(b) = [\![a \sim b]\!]$ when $b = a_\alpha$ for some α and $= c'ab$ otherwise— it is easily seen that $[\![c$ is a choice relation for $\sim]\!] \geq P$. By lemma B.1 we have a real structure R, in the parameter c such that $[\![\forall X \exists Y(X \sim Y \wedge RY)]\!] = \top$ and $[\![Ra_\alpha]\!] = \top$ for every $\alpha < \omega_2$.

We first show that for any $g \in D^{(e\to t)\to(e\to t)\to t}$, $[\![g : R \xrightarrow{1-1} G]\!] \subseteq \{@\}$—i.e. g is not injective from R to G at the worldless portion of space. Suppose otherwise, for contradiction. So $b := [\![g : R \xrightarrow{1-1} G]\!]) \sqcap P > \bot$ (we add the conjunct so that we can effectively ignore what g is like at the only world in the algebra). Using the axiom of choice, we may define a function $f : \omega_2 \to \omega_1$ that maps each $\alpha < \omega_2$ to a β which *might* enumerate a real number that is G.

$$f(\alpha) = \beta \text{ where } [\![ga_\alpha a_\beta]\!] \sqcap b > \bot$$

We first show that $f : \omega_2 \to \omega_1$, as claimed. Since $b \leq [\![ga_\alpha a_\beta \to Ga_\beta]\!]$ (i.e. b contains the claim that g has codomain G), and since $Ga_\beta = \bot$ when $\beta \geq \omega_1$, $b \leq [\![\neg ga_\alpha a_\beta]\!]$ when $\beta \geq \omega_1$, i.e. $[\![ga_\alpha a_\beta]\!] \sqcap b = \bot$ and so no $\beta \geq \omega_1$ is in the range of f. Thus $f : \omega_2 \to \omega_1$.

Now pick some $\gamma < \omega_1$ such that $f^{-1}(\gamma)$ is uncountable. There must be such a γ since $\omega_2 > \omega_1$. Now consider the following set:

$$\{[\![ga_\alpha a_\gamma]\!] \sqcap b \mid f(\alpha) = \gamma\}$$

The elements of this set are all non-zero (by the definition of f), pairwise incompatible (by the fact that $b \leq [\![g \text{ is injective}]\!]$), and uncountable by our choice of γ. We then have an uncountable anti-chain in $\mathbb{B}$ which is not possible.

To show that $[\![G \text{ is uncountable}]\!]$ we use a similar strategy, this time finding an injective $f : \omega_1 \to \omega$ for the contradiction.

$\square$

C Derivations

C.1 Proof that there is no first-order arithmetical contingency in $\mathsf{H}^{\square}.5$

Proposition C.1 (Prior)**.** *The necessity of distinctness, and the Barcan and converse Barcan formulas at any type are derivable in* $\mathsf{H}^{\square}.5$*.*

The first is proved in Prior (1955) pp.206-207. Essentially if $\Diamond x = y$ then, by $\square$(the Necessity of Identity) we can infer $\Diamond\square x = y$ from which we obtain $x = y$. The necessity of distinctness follows from the contrapositive of $\Diamond x = y \to x = y$. The second result is also due to Prior–see Prior (1956).

Under the assumption that there is a natural number structure of individuals, the finite numerical quantifiers form a natural number structure with respect to the following definitions

Proposition C.2. *In* $\mathsf{H}^{\square}.5$ *we can derive the following*

1. *Being a numerical quantifier,* NumQuant*, is rigid.*

2. *The relations* $<_Q, +_Q, \times_Q$ *on the numerical quantifiers are rigid.*

Proof. In $\mathsf{S5}$, rigidity of a relation R is equivalent to showing (i) $\forall\vec{x}(R\vec{x} \to \square R\vec{x})$. For (i) implies (ii) $\forall\vec{x}(\neg R\vec{x} \to \square\neg R\vec{x})$, and we can establish rigidity as follows. For any relation Z, we have by the Barcan and converse Barcan formulas $\square\forall\vec{x}(R\vec{x} \to Z\vec{x}) \leftrightarrow \forall\vec{x}\square(R\vec{x} \to Z\vec{x})$. But given (i), and the K axiom, the right-hand-side implies $\forall\vec{x}(R\vec{x} \to \square Z\vec{x})$. And given (ii), $\forall\vec{x}(R\vec{x} \to \square Z\vec{x})$ implies the right-hand-side. Thus $\square\forall\vec{x}(R\vec{x} \to Z\vec{x}) \leftrightarrow \forall\vec{x}(R\vec{x} \to \square Z\vec{x})$. We can then apply universal generalization and necessitation to this argument, to obtain the statement that R is rigid.

So now we prove that every numerical quantifier is necessarily a numerical quantifier by induction. Let Z be the property of necessarily being a numerical quantifier: $\lambda Q.\Box\,\mathrm{NumQuant}\,Q$. We will show that Z applies to 0 and is closed under suc. From the definition of a numerical quantifier that every numerical quantifier has Z.

For the base case note that it is a trivial logical truth that every property that applies to 0 and is closed under suc applies to 0, so this logical truth is necessary. Thus we have $\Box\,\mathrm{Quant}\,0$.

For the inductive step, assume that ZQ, i.e. Q is necessarily a numerical quantifier. It follows that it's necessary any property that applies to 0 and is closed under suc applies to $\mathrm{suc}\,Q$; i.e. it's necessary that $\mathrm{suc}\,Q$ is a numerical quantifier.

The proof of the rigidity of $<_Q, +_Q$ and $\times_Q$ are similar. For the case of $<$, the base case consists in showing $\Box 0 < \mathrm{suc}\,0$ and the inductive step, that if $\Box Q < P$ then also $\Box\,\mathrm{suc}\,Q < \mathrm{suc}\,P$ and $\Box Q < \mathrm{suc}\,P$.

□

Lemma C.1. *In* $\mathsf{H}^{\Box}.5$*, we can prove* $\forall\vec{Q}(\mathrm{NumQuant}\,\vec{Q} \wedge A^{\dagger} \rightarrow \Box A^{\dagger})$ *and* $\forall\vec{Q}(\mathrm{NumQuant}\,\vec{Q} \wedge \neg A^{\dagger} \rightarrow \Box\neg A^{\dagger})$ *for any first-order arithmetical sentence* A*.*

Proof. We prove this by induction on first-order arithmetical sentences. The base cases $Q = P$ and $Q < P$ follow by propositions C.2 and C.1.

The inductive cases for the truth functional connectives are straightforward. The quantificational case follows from the rigidity of NumQuant. □

Lemma C.2. *In* $\mathsf{H}^{\Box}.5$ *if there is a natural number structure of individuals, then, necessarily,* $0, <$ *is a natural number structure on the numerical quantifiers.*

Theorem C.1. *In* $\mathsf{H}^{\Box}.5$*, there is no structural arithmetical contingency:*

$$\forall\mathbf{N}(\mathrm{Nat}(\mathbf{N}) \wedge A(\mathbf{N}) \rightarrow \Box\forall Ry(\mathrm{Nat}(\mathbf{N}) \rightarrow A(\mathbf{N})))$$

where $A(<, 0)$ *is an arithmetical sentence.*

Proof. Suppose that $\mathbf{N}$ is a natural number structure and $A(\mathbf{N})$. Since there is a natural number structure, the axiom of Potential Infinity holds, so we know that the canonical number structure $\mathbb{N}$, consisting of numerical quantifiers, forms a natural number structure. Since $A(\mathbf{N})$, $A^{\dagger}(\mathbb{N})$ by Dedekind's theorem. So by Lemma C.1 $\Box A^\dagger(\mathbb{N})$. Since, given $\mathsf{S5}$, the axiom of potential infinity is necessarily true if true at all, $\mathbb{N}$ is necessarily natural number structure. So we know that necessarily any natural number structure $\mathbf{N}$ at type e will be isomorphic to $\mathbb{N}$ and also make $A(\mathbf{N})$ true. $\Box$

C.2 Proof of no second-order analytic contingency given $\Box\mathsf{RC}$ and LB

Recall that we use $\mathbf{R}$ as short for a sequence of variables $R, N : \sigma \to t, +, \times : \sigma \to \sigma \to \sigma \to t, 0 : \sigma, 1 : \sigma$. We will write '$x$ is an element of the structure $\mathbf{R}$' in the exposition to mean Rx.

Here will prove the following theorem.

Theorem C.2. *In* $\mathsf{H}^\Box.\Box\mathsf{RC}.\mathsf{LB}$ *one can derive all instances of The Necessity of Analysis in the language of second-order analysis. Whenever A is a sentence of second-order analysis:*

$$\Box\forall\mathbf{X}(\mathrm{Real}\,\mathbf{X} \to A(\mathbf{X})) \vee \Box\forall\mathbf{X}(\mathrm{Real}\,\mathbf{X} \to \neg A(\mathbf{X}))$$

The formulas of second-order logic (relative to type σ) are defined as follows

- The formulas $Xy_1 \ldots y_n$ are second-order analytical formulas when $x, y, z : \sigma$ and $X : \sigma \to \ldots \to \sigma \to t$.
- If A and B are second-order then $A \wedge B$ and $\neg A$ are too.
- If A is second-order, then $\forall x(Rx \to A)$ is too.
- If A is second-order, then $\forall X(\forall\vec{x}(X\vec{x} \to \bigwedge_i Rx_i) \wedge \mathrm{Rig}\,X \to A)$ is to.

Note that there is a copy of second-order logic for any choice of σ, although it is typically assumed that $\sigma = e$. Given a choice of variables $\mathbf{R} = R, N : \sigma \to t, +, \times : \sigma \to \sigma \to \sigma \to t, 0 : \sigma, 1 : \sigma$, we say that a formula is a formula of second-order analysis iff it is second-order and $\mathbf{R}$ appear free, and is a sentence of second-order analysis iff its free variables are exactly $\mathbf{R}$.

Observe that the second-order quantifiers are restricted to rigid properties. This is in line with standard mathematical practice, which treats second-order logic as extensional. However, in the presence of Rigid Comprehension, one could drop the restriction to rigid properties without making a difference to the truth of any formula of second-order analysis. A straightforward induction shows that formulas of second-order analysis cannot distinguish between coextensive properties:

Proposition C.3 (Analytic Extensionality)**.** *In* $\mathsf{H}^\Box$ *one can derive* $\forall\vec{z}(X\vec{z} \leftrightarrow Y\vec{z}) \to A \to A[Y/X]$ *for any second-order analytical formula* A

This fact does not extend to arbitrary formulas of higher-orderese, since in the full language one can formulate intensional notions, such as property identity, which are not part of the language of second-order analysis.

First, a few remarks on the proof strategy. A more straightforward version of the proof to follow is possible if we make the assumption of the necessity of distinctness. First show that the rigidification, $\mathbf{R}$, of any real number structure, obtained by rigidifying R, N, $+$, $\times$ and $<$, is necessarily a real number structure. Then we can show, using the Leibniz Biconditionals, that any sentence about the reals that is true in a rigid real number structure is necessarily true in that structure. It follows by Huntington's theorem that, necessarily, any real number structure is isomorphic to $\mathbf{R}$, and so makes true anything that $\mathbf{R}$ actually makes true.

In the absence of the necessity of distinctness, a rigid real number structure could fail to be a real number structure (if, say, everything in its domain became identical). It will be convenient to use a restricted notion of necessity in this argument defined as

$$\Box_{\mathbb{N}} := \lambda p.\Box(\Diamond\exists\mathbf{N}.\,\mathrm{Nat}\,\mathbf{N} \to p)$$

Using 'necessary', 'possible', 'rigid', 'inflexible' and so on in this new sense, the rigidification of the canonical real number structure will be inflexible due to the fact that it is built out of numerical quantifiers which are $\Box_{\mathbb{N}}$-necessarily distinct. Now we can show that any given real number structure is isomorphic to an inflexible real number structure (the canonical reals), and then proceed as above. We will call a structure $\mathbf{R}$ rigid when $R, N, +, \times, <$ are rigid, and inflexible when additionally, $\forall xy(Rx \to \Box_{\mathbb{N}} x \neq y)$, here defining these modal concepts in terms of $\Box_{\mathbb{N}}$. Note, also, that if $\mathbf{R}$ is rigid with respect to $\Box$ it is also rigid with respect to $\Box_{\mathbb{N}}$, so that Rigid Comprehension implies the variant of that principle involving $\Box_{\mathbb{N}}$.

Once we have shown that the rigidification of the canonical real number structure is inflexible, we show that inflexible real number structures are necessarily complete, and consequently that they are necessarily real number structures (it is of course necessarily an ordered field). This will involve the Leibniz biconditionals.

First, we will need a consequence of Huntington's theorem, that no two real number structures (potentially at different types) can disagree about the truth of second-order analytic statements.

Lemma C.3. *For any second-order analytic statement, A,* $\Box\forall_\sigma \mathbf{R}\forall_\tau \mathbf{S}(\mathrm{Real}^\sigma(\mathbf{R})\wedge \mathrm{Real}^\tau(\mathbf{S}) \to (A^\sigma(\mathbf{R}) \leftrightarrow A^\tau(\mathbf{S}))$

Here A^σ and A^τ are obtained by shifting which type is playing the role of "first-order" variables to σ. We omit the proof. Note that, like Analytic Extensionality, this theorem does not extend to arbitrary formulas, such as those involving intensional notions, second-order identity or third-order quantification. For instance, one real number structure may consist of necessarily distinct elements, while an isomorphic one might not; second-order identity and third-order quantification allow one to construct similar examples.

Next we need to construct an inflexible real number structure—note that we require only inflexibility with respect to $\Box_{\mathbb{N}}$. We will use the canonical real number structure obtained from theorem A.2, where we identify reals with rigid properties of the canonical natural number structure (using Rigid Comprehension). The result of rigidifying this structure we will call $\mathbf{R} = R, N, +_\mathbf{R}, \times_\mathbf{R}, 0_\mathbf{R}, 1_\mathbf{R}, <_\mathbf{R}$.

While this structure is clearly rigid, it needs to be shown that it is inflexible and $\Box_{\mathbb{N}}$-necessarily a real number structure. (Note that the canonical real number structure itself is $\Box_{\mathbb{N}}$-necessarily a real number structure, by applying theorem A.2 and the fact that the axiom of Potential Infinity is $\Box_{\mathbb{N}}$-necessary, but we don't know that the canonical real number structure is rigid.) Why is it inflexible? Because the numerical quantifiers are necessarily distinct with respect to $\Box_{\mathbb{N}}$ the reals—rigid properties of numerical quantifiers—will also be necessarily distinct in the same sense. Of course, without the assumption of Potential Infinity, the numerical quantifiers may not in fact form a natural number structure, and $\mathbf{R}$ may not be a real number structure. Thus we should have:

Lemma C.4. *If the axiom of Potential Infinity holds, then* $\mathbf{R}$ *is an inflexible real number structure.*

Note that if there could have been a real number structure then the axiom of Potential Infinity is true, and if there couldn't have been a real number structure the necessity of analysis holds vacuously. While the above lemma doesn't use Rigid Comprehension, we needed it in our definition of $\mathbf{R}$. Next we show that $\mathbf{R}$ is necessarily a real number structure.

Lemma C.5 (Leibniz Biconditionals)**.** *If the axiom of Potential Infinity holds, then* $\mathbf{R}$ *is* $\Box_{\mathbb{N}}$*-necessarily a real number structure.*

Proof. We first show that if R is a rigid property of necessarily distinct individuals, $\forall xy(Rx \wedge Ry \wedge x \neq y \to \Box_{\mathbb{N}} x \neq y)$, then for any element z or $\mathbf{R}$, $\lambda x.z < x$ and $\lambda x.x < z$ are rigid. Suppose $\Diamond_{\mathbb{N}} \exists x(z' < x \wedge Fx)$. So $\Diamond_{\mathbb{N}} \exists x z'(x < z' \wedge z = z' \wedge Fx)$, which by rigidity implies $\exists x z'.x < z' \wedge \Diamond_{\mathbb{N}}(z = z' \wedge Fx)$. Finally, by the necessity of distinctness, $z' = z$, so $\exists x.x < z \wedge \Diamond_{\mathbb{N}} Fx$. For the other direction, we know that if for some x, $z < x \wedge \Diamond_{\mathbb{N}} Fx$ then it's necessary that $z < x$ by rigidity, so $\Diamond_{\mathbb{N}}(z < x \wedge Fx)$, and so also $\Diamond_{\mathbb{N}} \exists x(z < x \wedge Fx)$ applying existential generalization under $\Diamond_{\mathbb{N}}$. This reasoning is easily necessitated establishing the rigidity of $\lambda x.z < x$. The other case is shown similarly.

Let P be the higher-order property of being a collection or reals that has no least upperbound:

$$P := \lambda X.(X \subseteq R \wedge \neg \exists y.\, \mathrm{lub}\, yX)$$

Suppose, for contradiction, that $\mathbf{R}$ is possibly not complete, that is: $\Diamond_{\mathbb{N}} \exists_{e \to t} X.PX$. By the Leibniz biconditionals there is a world property W, that entails P. Now we may consider the property of being a real x such that W entails applying to x—informally, x would have fallen into the unique W collection of properties if W had been instantiated.

$$Y := \lambda x.\Box_{\mathbb{N}} \forall X(WX \to Xx)$$

By the rigidity of $\leq$, Y consists of only reals (if Yx, W entails $\lambda X(Xx \wedge PX)$, to so x is possibly an $\leq$-real—i.e. stands in $\leq$ to something—and so by rigidity there is something it $\leq$s). By the actual completeness of $\mathbf{R}$, Y has a least upperbound, z. We will show that necessarily, z is the least upper bound of the unique property of reals X that has W, if it exists.

First, we establish that z necessarily an upperbound any X that is W. $\Box_{\mathbb{N}} \forall X(WX \to z \geq X)$ writing $z \geq X$ for $\forall x(Xx \to z \geq x)$. Suppose otherwise, for contradiction: $\Diamond_{\mathbb{N}} \exists X(WX \wedge \exists x(Xx \wedge x > z))$. Applying some logic inside $\Diamond_{\mathbb{N}}$, $\Diamond_{\mathbb{N}} \exists x > z(\exists X(WX \wedge Xx)$. Applying the rigidity of $\lambda x.x > z$

we get $\exists z > x \Diamond_{\mathbb{N}} \exists X(Wx \wedge Xx)$. Since W is a world property, it cannot be consistent with Xx unless it entails it: so $\Box_{\mathbb{N}} \forall X(WX \to Xx)$ which means Yx by definition of Y. The fact that $x > z$ contradicts the assumption that z is an upperbound of Y.

Next we establish that necessarily z is the least upperbound of the X that is W, when such an X exists. $\Box_{\mathbb{N}} \forall X(WX \to \operatorname{lub} zX)$. Suppose for contradiction that $\Diamond_{\mathbb{N}} \exists X(WX \wedge \exists x \geq X.x < z))$. Applying logic under $\Diamond_{\mathbb{N}}$, $\Diamond_{\mathbb{N}} \exists x > z \exists X(WX \wedge x \geq X)$. By the rigidity of $\lambda x.x < z$, we have $\exists x < z \Diamond_{\mathbb{N}} \exists X(WX \wedge x \geq X)$. To complete the contradiction it is sufficient to show that $x \geq Y$, contradicting the assumption that z was the *least* upperbound. So suppose Yy, which means $\Box_{\mathbb{N}} \forall X(WX \to Xy)$. It follows, using the normality of $\Box_{\mathbb{N}}$, that $\Diamond_{\mathbb{N}} x \geq y$. Given the necessity of distinctness, we can infer that in fact $x \geq y$ (for otherwise $x \leq y$ and $x \neq y$, and these must be necessary given the rigidity of $\leq$ and the necessity of distinctness, which is incompatible with $\Diamond_{\mathbb{N}} x \leq y$). Thus $x \geq Y$.

$\Box$

Lemma C.6 (Rigid Comprehension)**.** *Let W be a world property of type $(\sigma \to t) \to t$, and $Z : \sigma \to t$ a rigid property. Then if it possible that W is instantiated by a rigid property $\subseteq Z$, then there is an actual rigid property that could have been identical to the W property:*

$$\Box_{\mathbb{N}} \forall Y(WY \to (\operatorname{Rig} Y \wedge Y \subseteq Z)) \to \exists X(\operatorname{Rig} X \wedge X \subseteq Z \wedge \Box_{\mathbb{N}} \forall Y(WY \to Y = X)$$

Proof. Suppose that $\Box_{\mathbb{N}} \forall Y(WY \to (\operatorname{Rig} Y \wedge Y \subseteq Z))$, and Z is the rigid property given by the assumption. Let X be the rigid property coextensive with $\lambda x.(Zx \wedge \Box_{\mathbb{N}} \forall Y(WY \to Yx)$. Clearly X is necessarily rigid, and $X \subseteq Z$. It suffices to show that W entails being coextensive with X, since W entails rigidity and coextensive rigid properties are identical. There are two inclusions to show.

In order to show that $\Box_{\mathbb{N}} \forall Y(WY \to X \subseteq Y)$ it suffices to show

$$\forall x(Xx \to \Box_{\mathbb{N}} \forall Y(WY \to Yx))$$

since by the rigidity of X, we can conclude $\Box_{\mathbb{N}} \forall Y(WY \to \forall x(Xx \to Yx))$. So let x be an arbitrary X. By the definition of X it follows that that $\Box_{\mathbb{N}} \forall Y(WY \to Yx)$, so the claim follows.

In order to show that $\Box_{\mathbb{N}} \forall Y(WY \to Y \subseteq X)$ it suffices to show

$$\forall x(Zx \to \Box_{\mathbb{N}} \forall Y(WY \to Yx \to Xx)$$

since by the rigidity of Z, we can conclude $\Box_{\mathbb{N}}\forall Y(WY \to \forall x(Zx \to Yx \to Xx))$, which is equivalent to the desired claim, since $\Box_{\mathbb{N}}\forall Y(WY \to \forall x(Yx \to Zx))$. So let x be an arbitrary Z. In the case that x is X, we also have $\Box_{\mathbb{N}}Xx$ by rigidity of X, delivering the desired result, $\Box_{\mathbb{N}}\forall Y(WY \to Yx \to Xx)$. In the case that x is not X, that means $\neg\Box_{\mathbb{N}}\forall Y(WY \to Yx)$ or $\neg Zx$. In fact, the first disjunct must be true, for if $\Box_{\mathbb{N}}\forall Y(WY \to Yx)$ but $\neg Zx$ we have have $\Diamond_{\mathbb{N}}Zx$ since $\Box_{\mathbb{N}}\forall Y(WY \to Y \subseteq Z)$, which contradicts the rigidity of Z. In the former case, the worldliness of W implies $\Box_{\mathbb{N}}\forall Y(WY \to \neg Yx)$ yielding the desired result. $\Box$

Now we can establish:

Lemma C.7 (Rigid Comprehension, Leibniz Biconditionals)**.** *Let* $\mathbf{R}$ *be any inflexible real number structure (e.g. as constructed above). For every second-order analytic statement,* $A(\mathbf{R})$*, with free first-order variables* $\vec{x} = x_1, \ldots, x_n$ *and free second-order variables* $\vec{X} = X_1, ..., X_k$*:*

$$\forall\vec{X}\forall\vec{x}((R\vec{x} \wedge \vec{X} \subseteq R \wedge \mathrm{Rig}\,\vec{X}) \to A \to \Box_{\mathbb{N}}A)$$

where above we write $R\vec{x}$ *for* $Rx_1 \wedge \ldots \wedge Rx_n$*, and* $\vec{X} \subseteq R$ *to mean* $X_1 \subseteq R \wedge \ldots X_n \subseteq R$

Proof. We prove by induction on the structure of second-order analytic sentences, A that both A and its negation satisfy the theorem. Below .

1. $\forall\vec{X}\forall\vec{x}((R\vec{x} \wedge \vec{X} \subseteq R \wedge \mathrm{Rig}\,\vec{X}) \to A \to \Box_{\mathbb{N}}A)$
2. $\forall\vec{X}\forall\vec{x}((R\vec{x} \wedge \vec{X} \subseteq R \wedge \mathrm{Rig}\,\vec{X}) \to \neg A \to \Box_{\mathbb{N}}\neg A)$

Let $\vec{X}$ and $\vec{x}$ be arbitrary entities satisfying $(R\vec{x} \wedge \vec{X} \subseteq R \wedge \mathrm{Rig}\,\vec{X})$.

Atomic sentences have the form $x \leq y$, $x = y$, $x + y = z$, $Xy_1...y_n$, etc. 1 follows from the rigidity of the structure, in the former cases, or the rigidity of X in the last case. 2 follows from rigidity and the necessity of distinctness of $x, y, z, y_1...y_n$.

For conjunctions, suppose $A \wedge B$. We know from the inductive hypothesis that $\Box_{\mathbb{N}}A$ and $\Box_{\mathbb{N}}B$, so $\Box_{\mathbb{N}}(A \wedge B)$. This establish 1. in the case of 2, we have either $\neg A$ or $\neg B$, so by the inductive hypothesis one of these two claims is necessary, and thus so is $\neg(A \wedge B)$. For the negation case 1 is trivial from the IH, and 2 follows trivially from the IH and the equivalence of A and $\neg\neg A$.

For first-order generalizations. These have the form of a restricted quantification over the domain of $\leq$: $\forall x(Rx \to A)$. For 1, By the IH, for an arbitrary x in the domain, $\Box_{\mathbb{N}} A$, i.e. $\forall x(Rx \to \Box_{\mathbb{N}} A)$, so by the rigidity of R, we have $\Box_{\mathbb{N}}\forall x(Rx \to A)$. For 2, assume that universal is false: for some x, Rx and $\neg A$. We know that $\Box_{\mathbb{N}}\neg A$ by the inductive hypothesis, and we have $\Box_{\mathbb{N}} Rx$ by rigidity, so $\Box_{\mathbb{N}}\neg\forall x(Rx \to A)$, as required.

For second-order quantification we need Lemma C.6.

For 2, we will show the contrapositive. Suppose $\Diamond_{\mathbb{N}}\exists X(X \subseteq R \wedge \mathrm{Rig}\, X \wedge A)$. We wish to show $\exists X(X \subseteq R \wedge \mathrm{Rig}\, X \wedge A)$. Given the inductive hypothesis, it suffices to show $\exists X(X \subseteq R \wedge \mathrm{Rig}\, X \wedge \Diamond_{\mathbb{N}} A)$. Applying the Leibniz biconditionals to our assumption we get the existence of a world proposition, $\Box_{\mathbb{N}}\forall Y(WY \to (Y \subseteq R \wedge \mathrm{Rig}\, Y \wedge A[Y/X])$. By Lemma C.6, there is actually a rigid property, $X \subseteq R$ such that $\Box_{\mathbb{N}}\forall Y(WY \to X = Y)$, thus $\Diamond_{\mathbb{N}}(X \subseteq R \wedge \mathrm{Rig}\, x \wedge A)$.

$\Box$

Theorem C.3 (Rigid Comprehension, Leibniz Biconditionals)**.** *For any sentence of second-order analysis, A, with free variables $\mathbf{S}$, we can prove $\Box\forall\mathbf{S}(\mathrm{Real}\,\mathbf{S} \to A) \vee \Box\forall\mathbf{S}(\mathrm{Real}\,\mathbf{S} \to \neg A)$.*

Proof. The proof can be given as follows.

Suppose for contradiction that $\Diamond\exists\mathbf{S}(\mathrm{Real}\,\mathbf{S}\wedge A(\mathbf{S}))\wedge\Diamond\exists\mathbf{S}(\mathrm{Real}\,\mathbf{S}\wedge\neg A(\mathbf{S})$. Since there could have been a real number structure, the axiom of Potential Infinity is true, so $\mathbf{R}$ is an inflexible real number structure by Lemma C.4. Either $A(\mathbf{R})$ or $\neg A(\mathbf{R})$—without loss of generality, assume the former. Then we have that it's $\Box_{\mathbb{N}}$-necessary that $\mathbf{R}$ is a real number structure, by Lemma C.5, $\Box_{\mathbb{N}}$-necessary that $A(\mathbf{R})$ by Lemma C.7, and $\Box_{\mathbb{N}}$-necessary that $\forall\mathbf{S}(\mathrm{Real}\,\mathbf{S} \wedge \mathrm{Real}\,\mathbf{R} \to (A(\mathbf{R}) \leftrightarrow A(\mathbf{S}))$ by Lemma C.3. Thus $\Box_{\mathbb{N}}\forall\mathbf{S}(\mathrm{Real}\,\mathbf{S} \to A(\mathbf{S}))$. But $\Diamond\exists\mathbf{S}(\mathrm{Real}\,\mathbf{S}\wedge\neg A(\mathbf{S}))$ entails $\Diamond_{\mathbb{N}}\exists\mathbf{S}.\neg A(\mathbf{S})$, contradiction. In the case that $\neg A(\mathbf{R})$ the argument is similar.

$\Box$