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Is logic empirical?

Guido Bacciagaluppi*

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Abstract

The philosophical debate about quantum logic between the late 1960s and the early 1980s was generated mainly by Putnam's claims that quantum mechanics empirically motivates introducing a new form of logic, that such an empirically founded quantum logic is the 'true' logic, and that adopting quantum logic would resolve all the paradoxes of quantum mechanics. Most of that debate focussed on the latter claim, reaching the conclusion that it was mistaken. This chapter will attempt to clarify the possible misunderstandings surrounding the more radical claims about the revision of logic, assessing them in particular both in the context of more general quantum-like theories (in the framework of von Neumann algebras), and against the background of the current state of play in the philosophy and interpretation of quantum mechanics. Characteristically, the conclusions that might be drawn depend crucially on which of the currently proposed solutions to the measurement problem is adopted.

1 Introduction

In 1968 Hilary Putnam published a well-known paper on the question 'Is logic empirical?' (Putnam 1968), which gave rise to much controversy in the 1970s and 1980s. The main claims of Putnam's paper (repeated in Putnam 1974) can be paraphrased as follows:

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(a) Quantum mechanics prompts us to revise our classical logical notions in favour of ‘quantum logical’ ones. This is explained by analogy to geometry, in the sense that also general relativity prompts us to revise our Euclidean (or rather Minkowskian) geometrical notions in favour of Riemannian (or rather pseudo-Riemannian) geometrical notions.

(b) This revision of logic is not merely local, i. e. not merely an instance of a logical system especially suited to a particular subject matter, but it is truly global. Quantum logic is the ‘true’ logic (just as the ‘true’ geometry of space-time is non-Euclidean). Indeed, we have so far failed to recognise that our usual logical connectives *are* the connectives of quantum logic.

(c) Recognising that logic is thus quantum solves the standard paradoxes of quantum mechanics, such as the measurement problem or Schrödinger’s cat.

Of these truly ambitious and indeed exciting claims, the third claim (c) in particular was discussed extensively, and an almost universal consensus was reached (now shared by Putnam, 1994) that a move to quantum logic, even were it otherwise justified, would not resolve the puzzles of quantum mechanics. There have been notable reactions also to Putnam’s first two claims. Yet, with few exceptions (one needs only to recall the masterly paper by Michael Dummett, 1976), the topic seems to be riddled with misunderstandings. Indeed, very few philosophers appear to still consider seriously the possibility that quantum mechanics might have something to say about the ‘true’ logic (I know of only one recent attempt to resurrect this idea, namely by Michael Dickson, 2001, on whose views more below). This chapter aims at clearing such misunderstandings, and at providing a much-needed overall assessment of Putnam’s claims, by updating the debate in the light of the current state of the art in the foundations of quantum mechanics.

As regards Putnam’s claim (a), I take it that it is indeed justified, at least provided one takes ‘quantum logic’ as a *local* logic, suitable to describing a class of propositions in the context of quantum mechanical experiments (or the corresponding class of propositions about properties of quantum mechanical systems). This claim is analogous to the claim that intuitionistic logic is indeed suitable to describing a class of propositions dealing with mathematical constructions. This is distinct from the claim that intuitionistic logic is in fact the logic that underlies all rigorous human thought (and

is thus the ‘true’ logic). Claim (a) understood in this sense, I should think, is relatively uncontroversial, and shall be taken as such for purposes of further discussion. The explanation that quantum logic, suitably defined, has all the main formal properties required of a ‘good’ logic will also fall into this part of the discussion.

Claim (b) is the most controversial one, and its assessment will therefore need the most care. There are two points at issue (both well emphasised already by Dummett). The first point is that motivating a revision of logic does not only require motivating the introduction of some non-classical connectives. An advocate of a revision of logic must show why these connectives do not merely sit alongside the classical connectives, but actually *replace* them. The second point is a Quinean one: such a revision of logic means that, as part of the various revisions to our network of beliefs prompted by the empirical consideration of quantum phenomena, it is possible to choose to make some revisions in our conception of logic. But it is clear that empirical considerations alone cannot *force* us to revise our logic: a distinctly philosophical component will be needed in order to justify whether a revision of logic, as opposed to a revision somewhere else in our network of beliefs, might be desirable. (In the case of geometry, this is the same situation we have known ever since Poincaré. And indeed, we shall note in section 7 that in the interpretation of quantum mechanics one can find a rather close analogy to issues about conventionalism in physical geometry.)

An aspect of claim (b) that is of special importance is the subsidiary claim that the quantum logical connectives are not new connectives that can be defined in terms of the classical ones (and of some additional physical concepts), but that the classical connectives *are* in fact the quantum logical ones in disguise. We shall therefore have to discuss in depth whether there is a sense in which the classical connectives might be reducible to the quantum logical ones, either in some strict formal sense, or in some physical limit. In this context, as we shall see, questions of interpretation in quantum mechanics play an important role. Indeed, most discussions of quantum logic as the ‘true’ logic have taken place, at least implicitly, in the context of the so-called standard interpretation of quantum mechanics. This, however, is the interpretation that is riddled with the usual paradoxes. As we shall see, which alternative approach to the foundations of quantum mechanics one accepts might influence the assessment of whether a global revision of logic is acceptable. Conversely, one might add, one’s views on whether a global revision of logic is acceptable might influence the assessment of which

approach to the foundations of quantum mechanics is most appealing.

We shall proceed as follows. In section 2, we sketch a few basic elements of quantum mechanics that will be needed later. In section 3, we introduce quantum logic (in its lattice-theoretic form) as a local logic of certain experimental propositions; we further discuss the formal properties of such a logic, and mention a few alternative forms of quantum logic. Section 4 introduces the so-called standard interpretation of quantum mechanics, and section 5 assesses Putnam's claims in the context of this interpretation. The claims are found hard to defend, but the standard interpretation itself is not a believable interpretation because it gives rise to the usual paradoxes, at least if one applies it to standard quantum mechanics. Putnam's claims are thus reassessed first, in section 6, in the context of more general quantum-like theories (based on von Neumann algebras), where the classical connectives seem indeed to be reducible to the quantum ones. Then, in section 7, we shall reassess Putnam's claims in the context of the main current approaches to the foundations of quantum mechanics that explicitly address the paradoxes of the standard theory. Our conclusions will be that, while in the case of the approaches known as de Broglie-Bohm theory and as spontaneous collapse theories quantum logic at most can be introduced alongside classical logic, and thus in no way can be construed as replacing it, in the case of the Everett (or many-worlds) approach a case can indeed be made that the classical connectives emerge from the quantum ones.

Before proceeding, I should emphasise that although the title of this chapter may suggest a general treatment of the question of whether logic is empirical, it will deal only with the question of whether considerations related to quantum mechanics may provide an argument for the general claim. (Putnam's original paper (1968) does the same.¹) Of course, if quantum logic provides us with an intelligible global alternative to classical logic, the case for logic being empirical will be strengthened. However, I believe that a comprehensive assessment of the question of whether empirical considerations might prompt us to revise our logic will depend less on the details of the physics and more on the largely conceptual question of whether the notion of logical consequence is *a priori* or is an abstraction from what appear to be valid inferences in our practical use of language.² Indeed, unless

¹As a matter of fact, Putnam's paper was later reprinted with the modified title 'The logic of quantum mechanics'.

²For a recent discussion of the apriorism issue in logic, see e.g. Bueno and Colyvan (2004). Note also that one could very well conceive adopting an apriorist position with

one tends towards the latter position, i. e. unless one thinks that classical logic is already an abstraction from the classical empirical world around us (thus already conceding that ‘logic is empirical’), one will be disinclined to take discoveries in microphysics to be relevant at all to the revisability of logic. We shall not attempt to address this more general question.

2 Quantum mechanics in a nutshell

In the interest of a self-contained presentation, I summarise a few essentials about quantum mechanics that will be needed below. (This section will be rather abstract but elementary.)

In classical mechanics, the state of a system can be represented by a point in (or a subset of, or a probability distribution over) a set called phase space, encoding the positions and momenta of all the particles forming the system. In quantum mechanics, instead, the state of a system is represented by an element in a complex *Hilbert space* (which is a vector space, equipped with a scalar product, that is complete in the norm induced by the scalar product). In particular, this means that for any two states (e. g., for a spin-1/2 system, the states of spin-up and spin-down in direction x), an arbitrary linear combination (or ‘superposition’) is also a possible state:

$$|\varphi\rangle = \alpha|+_x\rangle + \beta|-_x\rangle . \quad (1)$$

Note that the same vector can always be expressed as an appropriate linear combination of vectors in any other basis:

$$|\varphi\rangle = \gamma|+_y\rangle + \delta|-_y\rangle . \quad (2)$$

In quantum mechanics, overall scalar factors do not count, i. e. the vectors $|\varphi\rangle$ and $\varepsilon|\varphi\rangle$ for arbitrary complex ε represent the same state, and by convention all states are usually normalised, i. e. have length 1. Therefore, if the basis vectors are normalised and orthogonal, as in the example above, one has $|\alpha|^2 + |\beta|^2 = |\gamma|^2 + |\delta|^2 = 1$.

A second crucial distinction between classical and quantum mechanics is that, when describing composite systems in quantum mechanics, instead

regard to quantum logic rather than classical logic.

of taking the Cartesian product of the given phase spaces as in classical mechanics, one has to take the *tensor product* of the given Hilbert spaces. For instance, for two spin-1/2 subsystems with Hilbert spaces generated (spanned) by

$$\{|+\frac{1}{x}\rangle, |-\frac{1}{x}\rangle\}, \quad \{|+\frac{2}{x}\rangle, |-\frac{2}{x}\rangle\}, \quad (3)$$

one takes the Hilbert space generated by a basis given by the products of the basis vectors:

$$\{|+\frac{1}{x}\rangle \otimes |+\frac{2}{x}\rangle, |+\frac{1}{x}\rangle \otimes |-\frac{2}{x}\rangle, |-\frac{1}{x}\rangle \otimes |+\frac{2}{x}\rangle, |-\frac{1}{x}\rangle \otimes |-\frac{2}{x}\rangle\} \quad (4)$$

(this construction is independent of the bases chosen for the subsystems).

The fundamental consequence of taking tensor products to describe composite systems is that some states of the composite are not product states, e. g. the so-called singlet state of two spin-1/2 systems:

$$\frac{1}{\sqrt{2}} \left(|+\frac{1}{x}\rangle \otimes |-\frac{2}{x}\rangle - |-\frac{1}{x}\rangle \otimes |+\frac{2}{x}\rangle \right). \quad (5)$$

Such non-factorisable states are called *entangled* (the property of being entangled is also independent of the bases chosen for representation in the component systems³). If the state of a composite system is entangled, then the subsystems are evidently not described separately by vectors in their respective Hilbert spaces. This is a characteristic trait of quantum mechanics (Schrödinger, 1935, p. 555, called it ‘not ... *one* but rather *the* characteristic trait of quantum mechanics’), and it is related to the Einstein-Podolsky-Rosen paradox, the Bell inequalities, quantum non-locality *et cetera*.

How can this be? The key, and the third crucial ingredient in our brief summary of quantum mechanics besides Hilbert spaces and tensor products, is the *phenomenology of measurement*. In classical mechanics one can idealise measurements as testing whether a system lies in a certain subset of its phase space. This can be done in principle without disturbing the system, and the result of the test is in principle fully determined by the state of the system. In quantum mechanics we are empirically confronted with the following situation. (i) Measurements can be idealised as testing whether the system lies in a certain (norm-closed) subspace of its Hilbert space — a subset which, in particular, is closed under linear combinations. (ii) A measurement in general disturbs a system: unless the state of the

³Incidentally, it is not independent on the choice of the *subsystems* into which the system is decomposed.

system is either contained in or orthogonal to the tested subspace, the state is projected (‘collapsed’) onto either the tested subspace or its orthogonal complement. (c) This collapse process is indeterministic, and the relevant probabilities are given by the squared norms of the projections of the state on the given subspace and its orthogonal complement, respectively.

For example, take the initial state (1) and test for spin-up in direction x (test for the subspace P_{+x}): the final state will be $|+x\rangle$ with probability $|\alpha|^2$, or $|-x\rangle$ with probability $|\beta|^2$. Now take the singlet state (5) as the initial state and test for $P_{+x}^1 \otimes P_{+x}^2$: the test will come out negative with probability 1, and the state will be undisturbed, since it lies in a subspace orthogonal to the tested one. (The same will be the case if one tests for $P_{-x}^1 \otimes P_{-x}^2$.) Test instead for $P_{+x}^1 \otimes P_{-x}^2$ (or for $P_{-x}^1 \otimes P_{+x}^2$): the result (in both cases) will now be $|+x^1\rangle \otimes |-x^2\rangle$ or $|-x^1\rangle \otimes |+x^2\rangle$, each with probability 1/2. Weaker correlations will be observed if spin is measured along two different directions on the two subsystems. Entanglement thus introduces what appear to be irreducible correlations between results of measurements (even carried out at a distance), and this for a generic pair of tests.

The last two elements of quantum mechanics that we shall also refer to are the Schrödinger equation and the notion of (self-adjoint) operator as an observable quantity.

The Schrödinger equation describes the time evolution of quantum state vectors. It is a linear and unitary equation, i. e. it maps linear combinations into linear combinations, and it preserves the norm (length) of vectors. In its most familiar form, it is a differential equation for the quantum states represented as complex (square-integrable) functions on configuration space (the space of positions of all particles), the so-called Schrödinger waves or wave functions.

Operators, specifically self-adjoint operators (which by the spectral theorem can be decomposed uniquely — in the simplest case — into a sum of projectors onto a family of mutually orthogonal subspaces) play two roles in quantum mechanics. On the one hand they mathematically generate Schrödinger-type evolutions, on the other hand they can be conveniently used to classify simultaneous experimental tests of families of mutually orthogonal subspaces. A system will test positively to only one of these tests, and to this test will be associated the measured value of the corresponding observable. Instead of being understood as specifying probabilities for

results of tests, quantum states can be thus equivalently understood as specifying expectation values for observables.

3 Quantum logic in a nutshell

3.1 Quantum logic as a logic of experimental propositions

The easiest way to introduce the concepts of quantum logic is in terms of a logic of ‘experimental propositions’. That is, one can define explicitly some non-classical connectives for a certain special class of propositions, relating to idealised quantum mechanical tests. These connectives will be arguably well suited for the limited subject matter at hand. If as a result one obtains a logical system satisfying certain formal requirements, we shall say that one has introduced a local non-classical logic. This is meant to be uncontroversial. Indeed, it should be relatively uncontroversial that (provided the formal requirements are indeed met) such a procedure is legitimate, although there may still be scope for disagreement as to how useful the introduction of such a logic is. In order to go on to assess Putnam’s further claims it is essential, at least for the sake of argument, that one accept that in this sense different logics may be more adapted to different subject matters.⁴

The prime example for such a procedure is Kolmogoroff’s (1931) interpretation of intuitionistic logic as a calculus of mathematical tasks (*Aufgabenrechnung*). In this framework, each mathematical proposition p stands for solving the corresponding mathematical task. The classical negation of p (not solving the task) is not itself a mathematical task, so the chosen set of propositions is not closed under classical negation. Instead, showing that a task is impossible to solve is again a mathematical task. This justifies introducing a *strong* negation, for which the law of excluded middle $p \vee \neg p$ breaks down. On this basis, one can set up a logical system, which is just the system of intuitionistic logic. More radical claims are not engaged with at this stage. (Indeed, one can argue that this is the correct and only way of interpreting intuitionistic logic, thus safeguarding the primacy of classical

⁴What we sketch here is quantum logic as descriptive of the empirical behaviour of certain experiments (albeit idealised ones). One can of course also introduce quantum logic abstractly and axiomatically based on the notion of a ‘yes-no’ test. This is the approach of the so-called ‘Geneva school’ of quantum logic (see e.g. Jauch and Piron 1969).

logic.)⁵

In the quantum context, let us define experimental propositions as (suitable equivalence classes of) statements of the form: ‘The system passes a certain test with probability 1’. From the discussion in the previous section, we recognise that these propositions are in bijective correspondence to closed subspaces of the Hilbert space of the system. The classical negation of such a proposition is not an experimental proposition in this sense. Instead, the proposition stating that the system passes with probability 1 the test corresponding to the orthogonal complement of the given subspace is an experimental proposition. This, again, can be taken to define a strong negation. In the quantum case, however, we go further than in the case of Kolmogoroff’s task logic. Indeed, even the classical disjunction of two experimental propositions p and q , corresponding to the set-theoretic union of the two subspaces, is not itself a subspace in general, thus it is not an experimental proposition. Instead, the proposition corresponding to the (closed) span of the two subspaces P and Q (the smallest closed subspace containing both the subspaces P and Q) is an experimental proposition, and we can introduce a corresponding ‘quantum logical’ disjunction. This proposition corresponds to the most stringent test that will be passed with probability 1 if the tests corresponding to P and Q will. The classical conjunction of p and q , corresponding to the intersection of the two subspaces P and Q , is itself an experimental proposition, so in this sense there is no need to introduce a separate quantum logical conjunction. The closed subspaces of a Hilbert space are ordered by inclusion and form a lattice (i.e. suprema and infima are pairwise always well defined), which is further orthocomplemented under the orthogonal complement defined via the scalar product. The quantum logical connectives correspond to the supremum, infimum and orthocomplement in this lattice.

As a consequence of the introduction of the quantum logical connectives, it is not the law of excluded middle that fails, but (one half of) the *distributive law*: the proposition $p \wedge (q \vee r)$ is generally weaker than the proposition $(p \wedge q) \vee (p \wedge r)$. This can be trivially seen by taking the subspaces Q and R to be two rays spanning a plane, and P to be a ray lying in the same plane but non-collinear with either Q or R . In that case, $p \wedge (q \vee r)$ corresponds to the same subspace as p , but both $p \wedge q$ and $p \wedge r$ correspond to the zero sub-

⁵There are further analogies between intuitionistic logic and quantum logic that could be brought to bear on the issue of the revision of logic. Both logics, for instance, allow for classical modal translations (see, respectively, Gödel 1933, and Dalla Chiara 1981).

space, and so does their quantum logical disjunction. Propositions that thus engender violations of distributivity are called *incompatible*; more precisely, two propositions p and q are called compatible iff the sublattice generated by (the subspaces corresponding to) p , $\neg p$, q and $\neg q$ is distributive.⁶

3.2 Formal properties of the logic

So far, what we have described is a *semi-interpreted language* (Van Fraassen, 1970). We have taken a propositional language, and we have fixed a class of structures that are intended as models of the language, namely the class of lattices of subspaces of Hilbert spaces (henceforth: Hilbert lattices). A model in this sense will be a mapping of the propositions onto the subspaces of some Hilbert space, such that (syntactic) conjunctions shall be mapped to intersections, disjunctions to (closed) spans and negations to orthogonal complements of the corresponding subspaces.

In order to say that we are introducing a logic in the formal sense (even a local one), we must have at least also a notion of logical consequence and of logical validity, and presumably other formal properties as well, such as soundness and completeness results for some appropriate logical calculus.

With this in mind, let us return to the classical case. Also in the classical case, we could define a semi-interpreted language by defining a model of the language in terms of subsets of some set, and mapping the logical connectives to the corresponding set unions, set intersections, and complements within the set (these are the lattice operations for the subset ordering relation). Every such lattice of subsets is a distributive lattice (also called a Boolean lattice or Boolean algebra), and conversely every distributive lattice is representable as the lattice of subsets of some set.

One can turn this semi-interpreted language into a logic by defining truth valuations as (orthocomplemented-lattice) homomorphisms from an arbitrary Boolean algebra onto the two-element algebra $\{0, 1\}$, and defining the notion of logical validity by taking the class of all Boolean algebras as reference class. That is, a sentence in the language will be a logical truth, iff it is true under every truth valuation of every model. The logic characterised

⁶There is more than one definition of compatibility in the literature, but this is immaterial for the purposes of this paper. Furthermore, they all coincide in the important case of orthomodular lattices.

by this notion of logical validity can be axiomatised, is sound and complete, and is of course the usual classical logic.

In order to extend this treatment to quantum logic, we need to extend the notion of a truth valuation to non-distributive lattices. Homomorphisms of the entire lattice onto $\{0, 1\}$ will not do, because in general there are no such total homomorphisms (Jauch and Piron 1963). More precisely, Jauch and Piron show that any so-called orthomodular lattice (in particular any Hilbert lattice) admits total homomorphisms onto $\{0, 1\}$ iff it is distributive.⁷ Note that this means that any form of quantum logic *must give up bivalence*. Thus, to insist that every proposition is indeed always true or false (as a matter of logic!) would be question-begging, and, at least for the sake of argument, the failure of bivalence must not be taken as a reason for rejecting the whole framework out of hand.

Instead, one can define workable truth valuations as *partial homomorphisms* onto $\{0, 1\}$, i. e. homomorphisms q defined on some proper (ortho-complemented) sublattice \mathcal{Q} of a given lattice \mathcal{L} , provided one requires also that such a partial homomorphism be *filtered*, i. e. for all $a \in \mathcal{Q}$ and $b \in \mathcal{L}$,

$$a < b, q(a) = 1 \quad \Rightarrow \quad b \in \mathcal{Q} \text{ and } q(b) = 1, \quad (6)$$

and *maximal*, i. e. have no proper extensions. The intuition behind these properties is that as many propositions as possible should be true or false under a truth valuation (maximality) and, in particular, a proposition that is weaker than a true proposition should also be true (filtering). Both properties are of course trivial for total homomorphisms on Boolean lattices. Note also that any partial homomorphism has both a canonical filtered extension and — by an application of Zorn's lemma — a maximal extension. A maximal partial homomorphism is always filtered.

A useful characterisation of truth valuations is the following. For any partial homomorphism q , let \mathcal{S} denote the subset of all $s \in \mathcal{Q}$ such that $q(s) = 1$ (the set of all true propositions). The set \mathcal{S} is a non-empty proper subset of \mathcal{Q} , closed under conjunctions. Together with property (6), this means that it is a so-called *filter*; and maximality of q means that \mathcal{S} is a maximal filter, so-called *ultrafilter*. Truth valuations q are thus in bijective correspondence with ultrafilters \mathcal{S} on the lattice. Note that \mathcal{S}^\perp is the set of all false propositions, and $\mathcal{Q} = \mathcal{S} \cup \mathcal{S}^\perp$.

⁷A quick proof for the special case of Hilbert lattices is given in Bell (1987, pp. 5–6).

Given the above definition of truth valuation, one can now proceed with quantum logic as with classical logic and define a notion of logical validity and logical consequence by fixing a suitable reference class of non-distributive lattices. Quantum mechanics (if assumed to be strictly true) tells us that the world is one specific (only partially known) Hilbert lattice, but the corresponding notion of logic will need to be general enough to cover all possible Hilbert lattices.⁸ Admittedly, the choice of reference class is not as obvious as in the case of Boolean algebras, and there is some trade-off involved in the choice. One could choose the class of all Hilbert lattices, but it is unclear to date whether the resulting logic is axiomatisable. On the other hand, one can choose more general classes of lattices as reference class, for instance the class of all orthocomplemented lattices or the more restrictive class of all orthomodular lattices. These yield axiomatisable logics that are both sound and complete (see e. g. Dalla Chiara and Giuntini 2002, section 6). Note that the logic of all Hilbert lattices, the logic of all orthomodular lattices and the logic of all orthocomplemented lattices are indeed all distinct, i. e. they have different sets of logical truths.

Choosing the logic of all Hilbert lattices would more properly characterise the ‘logic of quantum mechanics’. On the other hand, even if one takes a reference class more general than that of all Hilbert lattices, one can still argue that quantum phenomena have prompted the adoption (at least locally) of a non-classical logic. (Also, as mentioned in section 6, quantum theories of systems with infinitely many degrees of freedom seem to require a larger reference class.) The choice of orthomodular lattices seems particularly attractive, since in an orthomodular lattice there is a unique conditional reducing to the standard conditional for compatible propositions; the resulting connective has some unusual features, but these can be explained in analogy to counterfactual connectives, as is reasonably intuitive in a logic that gives up bivalence (Hardegree 1975).

In any case, the resulting logic is strictly weaker than classical logic, since the reference class that defines logical validity is extended beyond the class of Boolean algebras. Irrespectively of the details of the choice, we shall take it that such a notion of quantum logic provides us with a basis for discussing Putnam’s claims, the interest of which after all lies primarily in the idea that empirical considerations might force us to give up classical logic, and not

⁸Similarly, general relativity (if assumed to be strictly true) tells us that the world is one specific (only partially known) Lorentzian manifold, but the corresponding notion of geometry will cover all possible Lorentzian or pseudo-Riemannian manifolds.

(or only in the second place) in the details of which logic should replace it.

3.3 Alternative frameworks

As an aside, let us remark that we have presented above merely one possible framework for introducing a quantum logic, and that others have been proposed. We should mention two in particular.

First, one could choose a different idealisation for quantum mechanical experiments, in order to include more realistic measurements (described technically by positive-operator-valued measures rather than projection-valued measures). This leads one to consider, instead of the lattice of projections (equivalent to the lattice of subspaces), the poset (partially ordered set) of positive operators. This in turn prompts the introduction of fuzzy quantum logics and other quantum logics that generalise the lattice-theoretic approach (see e.g. Dalla Chiara and Giuntini 2002, sections 11–16). More general poset-theoretical structures arise also as the logics associated with theories of quantum probability, as in the test space approach of Foulis and Randall (1981).

Second, one can focus on a different general aspect of quantum mechanical experiments, namely their incompatibility; and instead of introducing apparently new logical connectives, one can restrict the use of the usual connectives to pairs of compatible propositions. This is the partial Boolean algebra approach to quantum logic (Kochen and Specker 1965a, b, 1967), which also gives rise to logical systems with nice formal properties. The partial Boolean algebra approach and the poset-theoretical approach overlap, unsurprisingly, in that so-called transitive partial Boolean algebras are canonically equivalent to so-called coherent orthomodular posets (Finch 1969, Gudder 1972), so that the corresponding logics are the same.

Note that the partial Boolean algebra approach may present advantages to the advocate of a global revision of logic, because the implied revision of logic appears to be more modest (although in a sense equivalent), and because it is easier to argue that the meaning of the logical connectives has remained the same. One does not construct new connectives that must somehow turn out to be the usual ones in disguise. One merely needs to argue that our usual connectives can be applied only to propositions that are compatible, and that it is an empirical matter, settled in the negative

by quantum mechanics, whether all propositions are indeed so. We shall not attempt to develop here this line of argument, merely note that Putnam himself switched to using at least the formalism of partial Boolean algebras in some later publications (notably Friedman and Putnam 1978). We shall keep to talking of quantum logic in the lattice-theoretic approach, because most of the discussion about Putnam's suggested revision of logic has been in the context of this approach and of the corresponding failure of distributivity.

4 Standard interpretation and measurement problem of quantum mechanics

It is certainly an empirical fact that, if one defines experimental propositions as in the previous section, the resulting lattice fails to be distributive, and a fact that is characteristic of quantum mechanics. If all physics were classical, then the lattice of experimental propositions defined in this way would be distributive. It may also be reasonable to want to define a local non-distributive logic for dealing with such experimental propositions. However, it is not clear at this stage why this logic should be even a candidate for a revised global logic. If one takes a 'naive' *instrumentalist* position, then quantum mechanics just provides us with the means of calculating the probabilities for the results of our experiments. The resulting procedure is certainly different from that in any classical framework, but there seems to be little need to revise anything but our algorithmic procedures for predicting experimental results. If one adopts a subtler Bohrian position, then the language of classical physics becomes a prerequisite for the description of quantum experiments, so that the very formulation of quantum mechanics would seem to require classical logic. Clearly, more than empirical considerations are needed in order to mount a case for the revision of logic at the global level. In particular, a strong *opposition* to the instrumentalist or Bohrian position is necessary in order to reject the overall package that includes classical logic and an instrumentalist or Bohrian reading of quantum mechanics.

In this section we shall sketch the 'naive' *realist* interpretation of quantum mechanics. This interpretation, variously referred to as 'standard' or 'orthodox' or 'von Neumann-Dirac', is problematic, because it gives rise to the usual paradoxes, but it is usually taken as the starting point for further

discussion and elaboration of other subtler approaches to quantum mechanics. It is thus, so to speak, the default realist position in the foundations of quantum mechanics. And in fact, it is the interpretation of choice (at least implicitly) also for discussions of Putnam's claims on the revision of logic. (Other realist approaches, and their implications for Putnam's claims, will be discussed in section 7.)

The standard interpretation consists in the following assignment of (intrinsic) *properties* to quantum systems. A quantum system has a certain property iff it passes with probability 1 a corresponding experimental test (in the sense of the previous section). Properties assigned in this way are thus in bijective correspondence to the closed subspaces of Hilbert space. What can it mean to assign such properties to a physical system?

The case of one-dimensional subspaces is relatively straightforward: a one-dimensional subspace (ray) is the set of all scalar multiples of a given vector, and these all describe the same quantum state. So, saying that a quantum system has a certain one-dimensional property corresponds to saying that its state is a certain vector in the Hilbert space.

The case of multi-dimensional properties is more difficult, but it is also quite crucial. In this case, one should think of entangled systems, where there is a vector describing the composite system but no vector describing each subsystem separately. The composite system will thus be assigned a one-dimensional property, but not the subsystems. Nevertheless, if two systems are entangled there are always *multi*-dimensional tests (in general non-trivial) on the individual subsystems, for which the subsystems will test positively with probability 1. Therefore, according to the standard interpretation, the subsystems are assigned the corresponding multi-dimensional properties. Unless one accepts some form of holism, in which only the composite system is assigned intrinsic properties, one is forced to generalise the notion of properties for the individual subsystems to include also multi-dimensional ones.

The motivation behind the standard interpretation can be phrased in the language of dispositions. Quantum mechanical systems exhibit a range of dispositional properties in the context of experimental tests, some of which are *sure-fire* dispositions. The standard interpretation suggests that these sure-fire dispositions (whether one-dimensional or multi-dimensional) support an inference to real, objective properties of the quantum system, which

are revealed by idealised tests.

But now, enter the paradoxes, specifically the measurement problem and Schrödinger’s cat (which we shall take here as two examples of the same problem). If we take quantum mechanics to be correct and universally valid, then one can easily construct examples in which the dynamics of the theory, the Schrödinger equation, will lead to entanglement between microscopic and macroscopic systems, e. g. between a quantum system being measured and the corresponding measuring apparatus, or between a microscopic system and a cat, in such a way that macroscopically distinguishable states of the apparatus (different readings) or the cat (alive or dead) are correlated with different states of the microscopic system. In such cases, on the standard interpretation, only the multi-dimensional subspaces spanned by the macroscopically distinguishable states correspond to properties of the macroscopic system, and these do not correspond to the macroscopic states we witness (the different readings, the cat alive or dead).⁹

One could say, paradoxically, that the cat is neither alive nor dead, but this formulation trades on an ambiguity: this statement would be paradoxical if ‘dead’ were understood as ‘not alive’ in the classical sense, but if it is understood as ‘not alive’ in the sense of the strong negation of quantum logic (assuming that the live and dead states span the Hilbert space of the cat), then the statement makes perfect sense since in this case ‘dead’ is strictly stronger than the classical ‘not alive’.

⁹Furthermore, this problem cannot be lifted by modelling the states of the apparatus as statistical distributions over microscopic states. If the dynamics is the unitary Schrödinger dynamics, one cannot reproduce the correct measurement statistics for all initial states, unless the state of the apparatus depends on the state of the system to be measured. This result was known already to von Neumann — indeed it prefaces his discussion of measurement in quantum mechanics (von Neumann 1932, section VI.3). For a modern, more general discussion, see e. g. Brown (1986) and references therein.

5 Quantum logic and the standard interpretation of quantum mechanics

5.1 Quantum logic as a logic of properties

The obvious interest of the standard interpretation, from the point of view of quantum logic, is that it allows one to apply the quantum logical structures introduced for experimental propositions also to propositions about intrinsic properties of a quantum system. Thus one speaks of the property lattice of the system, or of the lattice of ‘testable’ propositions about the system.

This move from experimental propositions to properties of a system is explicitly made for instance by Jauch and Piron (1969), who further propose that a quantum state should be understood as the set (in fact the ultrafilter) of true properties about the system. They thus propose, in effect, that a quantum state should be understood as a truth valuation on the lattice of properties of a quantum system. And, indeed, quantum states in the sense of rays in the Hilbert space are in bijective correspondence to the ultrafilters of true propositions they generate (by assigning them probability 1 upon measurement). Thus, truth valuations on the Hilbert lattice of quantum propositions encode all the information about quantum mechanical expectation values.

Abstractly, the introduction of quantum logic for testable propositions is possible simply because there is a bijective correspondence between the experimental propositions and the testable propositions (both being in bijective correspondence with the closed subspaces of the Hilbert space). A closed subspace of the Hilbert space will now represent a proposition about an intrinsic property of the relevant quantum system, and the closed span, intersection and orthogonal complement of such subspaces will correspond to the quantum logical disjunction, conjunction and negation of the respective propositions.

Concretely, the standard interpretation introduces properties corresponding to one-dimensional subspaces P , Q etc., and properties corresponding to multi-dimensional subspaces such as the span of P and Q . The novelty of these properties lies in the fact that under all possible truth valuations, whenever P obtains or whenever Q obtains, also the property corresponding to their span obtains. The interpretation of this property as a quantum

logical disjunction $p \vee q$ allows one to interpret such relations between propositions as relations of *logical consequence*.

By considering quantum logic at the level of intrinsic properties of physical objects, we make a further step in the direction of Putnam's proposals. Indeed, at least as regards the more modest claim (a), the fact that this logic in general is a non-distributive lattice is clearly an empirical fact. The fact that it is best understood as a semi-interpreted language, and the fact that this language has a number of properties that justify calling it a logic in the formal sense, have been discussed above. In this sense thus, it should be relatively uncontroversial that quantum phenomena give us empirical grounds for introducing a logic adapted to the world of physics that is non-distributive and hence non-classical. I take it that what we have called above Putnam's claim (a) is thus both intelligible and justifiable.

The claim that is controversial is claim (b), that this gives us further reasons to revise logic *tout court*, i. e. that this logic of testable quantum mechanical propositions, or logic of quantum mechanics, is in fact the 'true' logic and that we have failed to recognise so far that our usual, apparently classical connectives are in fact the connectives of quantum logic. We turn at last to this controversial point, for the time being in the context of the standard interpretation of quantum mechanics.

5.2 The revision of logic

If we have successfully introduced the quantum logical connectives in the context of propositions about material properties of physical objects, is this not *ipso facto* saying that 'the logic' of the world is quantum? Surely quantum mechanics is a theory that applies to all material objects, so that the resulting quantum logic is not a local but a global logic?

As mentioned in the introduction, the fact that one may justify the introduction of non-classical connectives does not yet mean that logic has been revised. The crucial point is whether these connectives have been introduced *alongside* the classical connectives, or whether they *replace* them (in an appropriate sense). As we shall see now, the standard interpretation is neutral with regard to this question. Indeed, one has a choice between two opposing views.

On the one hand, it is perfectly possible to interpret the properties assigned to systems in the standard interpretation as elementary properties in the sense of classical logic. Indeed, a quantum logical disjunction $p \vee q$ classically must be an elementary proposition: it is not a classical disjunction of terms that include p and q , although one might be tempted to think that it is the disjunction of all one-dimensional subspaces contained in the span of P and Q . As a matter of fact, this is not true: if it were, in the case of entangled systems there would be a quantum state that describes an individual subsystem, contrary to what quantum mechanics says. On top of the elementary propositions, however, one can perfectly well consider complex ones, constructing them by applying the classical connectives to this new quantum set of elementary propositions, e. g. the disjunction of all one-dimensional subspaces contained in the span of P and Q can be considered alongside with the quantum disjunction $p \vee q$ itself. The quantum aspect is physical and lies in the determination of the elementary properties, while the logic remains classical.

This position has admittedly some disadvantages. Complex propositions in general are not directly testable, i. e. verifiable with probability 1. This is simply because neither the set union of P and Q nor the set complement of P are subspaces of the Hilbert space. This would thus be an empirical limitation characteristic of quantum mechanics. More importantly, perhaps, the above relation between p or q and the quantum logical $p \vee q$ cannot be analysed as logical consequence. That is, there are triples of propositions p, q, r such that whenever p or q hold, also r holds; but since r is elementary, this relation of consequence cannot be analysed as logical consequence.

This, however, may seem a small price to pay in order to refrain from revising our logic. And in fact, we shall see in section 7 that this is arguably the position most naturally associated with the approaches to quantum mechanics known as spontaneous collapse theories.

The opposite position consists in maintaining that the properties assigned in the standard interpretation are *all* the possible properties of a physical system: there is *no* property corresponding to the classical disjunction of p and q , or to the classical negation of p . An equivalent way of saying this is that if p and q are the propositions that some physical quantities take certain values, say ' $A = 4$ ' and ' $B = 9$ ', then there is no meaningful physical quantity that can encode the classical disjunction ' $A = 4$ or $B = 9$ '. Note that there is no quantum mechanical observable that encodes it. Indeed, in

disanalogy to classical physics, the operator $(A - 4)(B - 9)$ in general does not represent a quantum mechanical quantity, because in general the operators A and B do not commute (so that $(A - 4)(B - 9)$ is not self-adjoint). But if there is no meaningful physical quantity whatsoever that represents a classical disjunction, insisting that the properties of the standard interpretation are elementary would mean that the vast majority of complex propositions constructed from elementary propositions about quantum systems are meaningless.

If one drops altogether the possibility of using the classical connectives to form complex propositions, one can instead interpret some of the testable propositions as complex propositions in the sense of quantum logic. In so doing one removes the mismatch between logical propositions and physical propositions (indeed, all propositions are testable propositions), and one ensures that the consequence relation described above between testable propositions and their quantum disjunctions is indeed a relation of logical consequence.

This is presumably the best case that can be made for a revision of logic in the context of the standard interpretation of quantum mechanics. It is not made explicitly by Putnam, although some of it must be implicit in his discussion; it is present more or less in Dickson (2001), who explicitly denies the ‘empirical significance’ of classical disjunctions and negations. Still, it appears that if one follows this line of argument, the quantum logical connectives have *supplanted* the meaningless classical ones. What about the claim that the quantum logical connectives are the *same* as the classical connectives? Indeed, since every physical system is a quantum system, we seem to have arrived at the conclusion that an ‘everyday’ disjunctive proposition about any physical object whatsoever is meaningless. But Putnam’s claim that the classical and quantum connectives are the same is surely meant in the sense that we should be able to gain a better understanding of our usual everyday classical connectives by realising that they are indeed quantum logical.

What is missing from the above is an explanation of why classical logic appears to have been so effective until now. One needs to explain how, if the true logic is non-distributive, it is still possible for the connectives to behave *truth-functionally* in special cases. This would give rise to the possibility of abstracting classical logic (empirically!) from our everyday use, and of applying it in the appropriate circumstances (as Putnam undoubtedly did

in the act of writing his famous paper).

Putnam's claim (c), that adopting quantum logic will solve the paradoxes of quantum mechanics, can be understood as an attempt to fill this gap: the quantum logical point of view does indeed explain, according to Putnam, why the world appears classical to us. Indeed, for Putnam the main advantage of a revision of logic is precisely that it will solve the paradoxes of quantum mechanics. We shall briefly discuss now how Putnam argues for this point and why his arguments are justly regarded as flawed. One other author at least, namely Dickson, attempts to argue that, although the quantum connectives are the true connectives, they behave classically when applied to the everyday, macroscopic realm. As we shall also see, his attempt appears to fail on ultimately similar grounds.

If this is so, then we are left with the following situation. There is a coherent, perhaps even a reasonably convincing case to be made that a non-classical logic is well adapted to a world in which quantum mechanics under the standard interpretation is true. But this world is hugely different from our own. This is precisely what the measurement problem and Schrödinger's cat highlight. Indeed, in such a world it would not seem possible for any intelligent beings to develop at all, let alone beings capable of formulating any kind of logic (let alone quantum mechanics). If the argument does not apply to our world (or at least to a possible world similar to ours), then it loses most of its interest.

5.3 Putnam and the paradoxes

The seemingly logical paradox of Schrödinger's cat, that the cat is neither alive nor dead, trades as we have mentioned on the ambiguity between classical and quantum logical terms. Putnam's way of resolving the paradox is to choose a strictly quantum logical reading: 'dead' is interpreted as 'not alive' in the quantum logical sense of orthocomplementation in the lattice, and the cat is then indeed alive or dead, but in the sense of the quantum logical disjunction. Putnam, however, seems to want to go further, namely he claims that, since the cat is alive or dead (quantum logically), there is a *matter of fact* about the biological state of the cat.

To make the point clearer, let us take an example adapted from Putnam himself (1968, pp. 184–185). Consider an n -dimensional Hilbert space and

take an orthonormal basis $|x_1\rangle, \dots, |x_n\rangle$ in the Hilbert space, which one can associate with a family of tests, or equivalently with some observable X . Denoting the propositions corresponding to the one-dimensional projectors onto the basis vectors as x_1, \dots, x_n , the following is a true proposition under all truth valuations:

$$x_1 \vee \dots \vee x_n . \tag{7}$$

Its truth, however, is understood by Putnam as meaning that the observable X has indeed a value corresponding to one of the x_i . As the reasoning is independent of the particular choice of basis, Putnam concludes that the system possesses values for all such observables. He then interprets measurements as simply revealing those preexisting values, thus proposing that the measurement problem of quantum mechanics is solved by a move to quantum logic.

This is rather bewildering, since, as we have seen in section 3, quantum logic comes equipped with a well-defined semantics, which underlies the quantum logical notion of consequence. And we have seen that truth valuations in this semantics are such that the proposition $x_1 \vee \dots \vee x_n$ can be true without any of the x_i being true. *Any* quantum state that is a non-trivial linear combination of the basis vectors will define such a truth valuation; and in the case of entangled systems, we have seen that a quantum logical proposition can be true *without any* of the one-dimensional projections spanning it being true. To be fair, at the time of Putnam's 1968 paper, the semantics of quantum logic was not fully developed as yet, but the reasoning implied in the paper seems to be technically in error, since he appears to be using a different semantics from that required in quantum logic.

A more charitable reading (perhaps more in line with his later papers, e.g. Putnam 1981), takes Putnam as distinguishing between a quantum level, obeying quantum logic, and a 'hidden' level obeying classical logic. It has in particular been suggested that Putnam's proposals can be analysed in terms of a so-called non-contextual hidden variables theory (Friedman and Glymour 1972), which however confronts them with the standard problems facing such approaches, notably the no-go theorem by Kochen and Specker (1967). Perhaps more plausibly, it has also been suggested to analyse Putnam's proposals in terms of a so-called contextual hidden variables theory (Bacciagaluppi 1993), which however confronts them with the proofs of non-locality for this kind of approaches, specifically those by Heywood and Redhead (1983) and by Stairs (1983). In either case, however, Putnam would seem to be backing away from the proposal that quantum logic is the

global logic. (For Putnam's most recent views on the subject, see Putnam 1994.) In section 7, we shall return to the issue of quantum logic in hidden variables approaches, namely in the context of the most successful of these, pilot-wave or de Broglie-Bohm theory.

Dickson's (2001) attempt to explain how classical logic is effective despite quantum logic supposedly being the global logic, proceeds along slightly different lines. Dickson points out that in the macroscopic realm, when talking about measurement results or cats, we apply logic always to a *distributive sublattice* of all (quantum logical) propositions. As it stands, however, this argument is inconclusive. The sublattice generated by the propositions x_1, \dots, x_n in Putnam's example is distributive, but this fact does not guarantee that the logical connectives will behave truth-functionally, and that is what is at stake. Again to be fair, Dickson suggests that the proper framework for discussing Putnam's claims is that of the more general quantum-like theories based on the formalism of von Neumann algebras. And we shall see in the next section that in that framework the connectives can indeed be shown to behave truth-functionally in certain cases.

As far as the claim concerns the usual formalism of quantum mechanics, however, it may be that Dickson falls prey to a common fallacy. Admittedly, it is a fact that we cannot construct in practice an experiment that would test for a macroscopically entangled state (in particular because of the phenomenon called decoherence), and that at the macroscopic level the only tests we have available are all compatible (so that the corresponding experimental propositions form a distributive lattice). And this fact has often been trumpeted as showing that the measurement problem does not arise. But this practical impossibility is irrelevant to the point that macroscopically entangled states (under the standard interpretation) are incompatible with macroscopic objects having the properties they appear to have, nor does it show that such states do not arise in practice. It thus seems that Dickson's argument fails to improve on Putnam's attempt.

6 Quantum logic and classical propositions

Before proceeding further and enquiring into the status of quantum logic in realist approaches to quantum mechanics other than the standard interpretation, let us dwell in more detail on the question of what it could mean for

the quantum logical connectives to be the same as the classical connectives.

There is an interesting way of making the case that the meaning of the connectives is indeed the same in classical and quantum logic, namely to argue that it is always given in terms of the supremum, infimum and orthocomplement of the lattice: the conjunction of two propositions is the weakest proposition that implies both propositions, their disjunction is the strongest proposition that is implied by either, and the negation of a proposition is its orthocomplement in the lattice. Until empirical evidence for quantum mechanics was obtained, we used to believe that all lattices of propositions we could ever consider would be distributive. We used to believe that the universe of sets was the correct framework for abstract semantics, because we believed it was rich enough to describe the physical world. But, so the argument goes, it has turned out that it is only the ‘universe of Hilbert spaces’ that is rich enough for that purpose. (This line of thought presupposes of course that one has already accepted that the logic should be read off the structure of the lattice of empirical propositions.)

The trouble with this suggestion is that, although at this more general level the quantum and classical connectives can thus be said to be the same, still, if the actual lattice of properties is a Hilbert lattice (of dimension greater than 1), the connectives will just not behave truth-functionally, so that the quantum connectives do not seem to reduce to the classical ones in everyday macroscopic situations. This is precisely the problem facing Dickson: can one have the (unique) logical connectives behave truth-functionally when applied to some propositions in the lattice but not to others? We shall see in the present section that, if, as is quite standardly done, one defines logical consequence through a reference class of lattices that is larger than the class of all Hilbert lattices (which, as noted above, is not known to lead to an axiomatisable logic), in particular if one considers quantum logic to be the logic of all orthocomplemented lattices or of all orthomodular lattices, then there is a rigorous sense in which the connectives interpreted in these non-distributive lattices (i. e. the standard quantum logical connectives) can behave truth-functionally in certain cases. Thus, at least in this more abstract setting, there are situations in which one could arguably ‘mistake’ the logic to be classical.

Recall that two propositions p and q are compatible iff the lattice generated by p , $\neg p$, q and $\neg q$ is distributive. For a subset \mathcal{A} of an orthocomplemented lattice \mathcal{L} , denote by \mathcal{A}^c the set of propositions compatible with

all propositions in \mathcal{A} . If one considers lattices \mathcal{L} more general than Hilbert lattices, the set \mathcal{L}^c (the so-called centre of the lattice) may be non-trivial, i. e. there may exist propositions (other than the trivially true proposition) that are compatible with all propositions in the lattice. Such propositions are called *classical* propositions. Now, it is a theorem that under any truth valuation on \mathcal{L} , a classical proposition is *always* true or false.

Indeed, let q be a truth valuation from \mathcal{L} onto $\{0, 1\}$, defined on an orthosublattice $\mathcal{Q} = \mathcal{S} \cup \mathcal{S}^\perp$ of \mathcal{L} , where \mathcal{S} is the ultrafilter of propositions made true by q . For any ultrafilter \mathcal{S} in \mathcal{L} ,

$$(\mathcal{S} \cup \mathcal{S}^\perp)^c \subset (\mathcal{S} \cup \mathcal{S}^\perp) \quad (8)$$

(Raggio 1981, Appendix 5, Proposition 3). A classical proposition, being compatible with any $a \in \mathcal{L}$, is obviously contained in $(\mathcal{S} \cup \mathcal{S}^\perp)^c$ for any set \mathcal{S} . Therefore, for any truth valuation q , $c \in \mathcal{Q}$, i. e. $q(c) = 1$ or $q(c) = 0$. QED.

It now follows, just as in the classical case, that if a lattice contains classical propositions, the lattice-theoretical connectives applied to the classical propositions will behave truth-functionally, in particular for any two classical propositions a and b , and any truth valuation q that makes $a \vee b$ true, q will make a true or b true.

Indeed, let q be any truth valuation with $q(a \vee b) = 1$. Since a and b are classical, by the above they are both either true or false under q . But if $q(a) = 0$ and $q(b) = 0$, then, since q is homomorphic, $q(a \vee b) = 0$, contrary to assumption. Therefore, if a and b are classical,

$$q(a \vee b) = 1 \quad \Rightarrow \quad q(a) = 1 \text{ or } q(b) = 1, \quad (9)$$

for any truth valuation q . QED.

Note that the fact that a certain proposition a is classical depends on the lattice \mathcal{L} chosen as a model of the logic. Specifically, it depends on the relation of a with all the other propositions in the chosen model. It thus depends on the *meaning* of a . We see that the quantum logical connectives can indeed behave truth-functionally in certain models, but depending on the meaning of the propositions involved. Classical logic appears to be valid in special cases, but the additional inferences one can make in these cases are not logical inferences: they are not based on the propositional *form* of the

statements involved, they are based instead on the fact that the statements have a *classical content*.

If the lattice of properties in our world is the lattice of projections of some Hilbert space, our world does not contain classical propositions (*pace* Putnam and Dickson). On the other hand, at least some lattices that are more general than Hilbert lattices appear to be physically motivated. Indeed, generalisations of quantum mechanics that allow in general for classical propositions exist, and are required to treat systems with infinitely many degrees of freedom, such as in quantum field theory or in quantum statistical mechanics (when taking thermodynamic limits).

Mathematically, these theories are based on more abstract algebras of observables than the algebra of (self-adjoint) operators on a Hilbert space. For the purposes of quantum logic, the most interesting class of such algebras is that of so-called von Neumann (or W^* -) algebras, which can be represented as certain subalgebras of operators on Hilbert space. Von Neumann algebras can be generated by their projections, so that one can again reduce all statements about observables to statements about projections (i. e. to yes-no tests).¹⁰ The lattices of projections of von Neumann algebras are always orthomodular lattices. (Indeed, historically, the study of orthomodular lattices developed out of the study of von Neumann algebras.) Therefore, unless one insists on characterising quantum logic by the class of all Hilbert lattices, lattices of projections of von Neumann algebras are already included in the models of the most usual varieties of quantum logic, and they are thus a *bona fide* source of examples for the behaviour of the usual quantum logical connectives. Incidentally, we note that J. von Neumann is also associated both with the standard interpretation of quantum mechanics (rightly or wrongly), through his book *Mathematische Grundlagen der Quantenmechanik* (von Neumann 1932), and with the first proposal that quantum mechanics should be interpreted in terms of a non-distributive logic, clearly stated in his paper with Birkhoff four years later (Birkhoff and von Neumann 1936).

When we say that general lattices of projections of von Neumann algebras include classical propositions, the intuition behind it is that there is a

¹⁰Indeed, Raggio (1981) has proved that if \mathcal{L} is the projection lattice of a W^* -algebra \mathcal{M} , there is a bijective correspondence between truth valuations on \mathcal{L} and pure normal states on \mathcal{M} , in the sense of normalised positive linear functionals. That is, truth valuations indeed encode all the information about expectation values of observables in the algebra.

breakdown in the linear structure of the state space of a physical system. Indeed, defining a classical observable as an observable C such that propositions of the form ‘ C has value α ’ are classical, superpositions of states in which a classical observable has different values simply do not exist (one says that such states are separated by a superselection rule).

The framework of von Neumann algebras is general enough to include both quantum and classical physics, and intermediate theories besides. For instance, one can build algebras that are tensor products of a standard quantum system and a purely classical system, and for which there are no states entangling the quantum system and the classical system (Raggio 1988, see also Baez 1987). Although it is generally believed that such theories would be rather *ad hoc*, they do allow one to describe a world in which the measurement problem of quantum mechanics does not arise, a world in which all measuring apparatuses (as well as cats) are made up of classical observables.

It is instructive to see explicitly how the truth-functionality of the quantum logical connectives would apply to a measurement scenario if the ‘pointer’ observable of a measuring apparatus were assumed to be a classical observable C . Suppose the apparatus measures a non-classical observable B taking, say, the two values ± 1 . Now assume that at the end of an (ideal) measurement the following proposition is true (in obvious notation):

$$(B = 1 \wedge C = 1) \vee (B = -1 \wedge C = -1) , \quad (10)$$

where \wedge and \vee denote the infimum and supremum in the lattice. We can now show from the fact that C is classical that the disjunction in (10) is truth-functional, i. e. under any truth valuation q that makes (10) true, $(B = 1 \wedge C = 1)$ or $(B = -1 \wedge C = -1)$ are also true.

Let q be such a truth valuation, i. e.

$$q\left((B = 1 \wedge C = 1) \vee (B = -1 \wedge C = -1)\right) = 1 . \quad (11)$$

We need to show that

$$q(B = 1 \wedge C = 1) = 1 \quad \text{or} \quad q(B = -1 \wedge C = -1) = 1 . \quad (12)$$

Because q is filtered, we have

$$q(C = 1 \vee C = -1) = 1 . \quad (13)$$

Since C is classical,

$$q(C = 1) = 1 \quad \text{or} \quad q(C = -1) = 1, \quad (14)$$

by (9). Suppose for instance that $q(C = 1) = 1$. Since q is a homomorphism, we have that

$$q\left[\left((B = 1 \wedge C = 1) \vee (B = -1 \wedge C = -1)\right) \wedge C = 1\right] = 1. \quad (15)$$

But now, the propositions $B = \pm 1, C = \pm 1$ are all mutually compatible, so that we can distribute over \vee in (15), yielding

$$q(B = 1 \wedge C = 1) = 1. \quad (16)$$

Analogously, if $q(C = -1) = 1$ we obtain $q(B = -1 \wedge C = -1) = 1$. QED.

Note in particular that the truth-functionality has spread to propositions that include non-classical terms. (This appears to be related to the results by Bub and Clifton (1996) on maximal truth-value assignments in a Hilbert lattice compatible with a certain ‘preferred’ observable being assigned definite values.)

If such examples do not describe the actual physics, however, what have we gained in showing that the connectives can sometimes behave truth-functionally? We should perhaps distinguish two questions: (i) Can we envisage worlds, perhaps merely inspired by quantum mechanics and sufficiently close to our own, in which we would consider revising our logic? (ii) Is our world such a world?

In a world as the above, one could indeed maintain that the only meaningful propositions are the propositions in the lattice, since the lattice is general enough to include propositions for which classical logic holds. Thus, a generalised quantum mechanics together with the standard interpretation could arguably meet the objections detailed in the previous section against a revision of logic. We can thus make a case that logic is empirical because there is a *possible* world in which we might be prompted by empirical considerations to revise our logic (question (i)). This is different from establishing that in *our* world we may have good reasons for a revision of logic (question (ii)). Note that while Putnam’s ultimate aim was to show that logic is indeed empirical, his actual claim was that we have reasons to revise our logic in this world.

In order to proceed further with question (ii), and thus address the revision of logic in Putnam's own terms, we shall have to return to standard quantum mechanics. However, we shall have to consider approaches to the foundations of quantum mechanics other than the standard interpretation, in particular approaches that have some credible claim to providing solutions to the puzzles of the standard theory.

7 Quantum logic in other approaches to quantum mechanics

We now leave the standard interpretation of quantum mechanics. While reverting to the standard formalism of quantum mechanics in this section, we shall discuss the status of the claims about quantum logic in the context of other approaches to the foundations of quantum mechanics, approaches that do propose solutions to the puzzles presented to us by quantum mechanics and in particular propose to explain why classical logic is effective (whether or not it be the true logic) in a world in which quantum mechanics is indeed true. The approaches we shall discuss in turn are (i) the pilot-wave theory of de Broglie and Bohm, (ii) spontaneous collapse theories, and (iii) the Everett or many-worlds interpretation. (The presentation of these approaches will necessarily be rather condensed.)

7.1 De Broglie-Bohm theory

The pilot-wave theory of de Broglie and Bohm is a very well-known and well-understood approach to the foundations of quantum mechanics. The theory, as presented by Louis de Broglie at the fifth Solvay conference in October 1927 (de Broglie 1928), is a new dynamics for n -particle systems, described in configuration space (which encodes only the positions of the particles) rather than in phase space. The motion of the particles is determined by a field of velocities defined by the phase S of the complex wave function. At least as regards particle detections, the theory can clearly predict both interference and diffraction phenomena: around the zeros of the wave function, the phase S will behave very irregularly, so one can at least qualitatively expect that the particles will be driven away from regions of configuration space where the wave function is small (as is indeed the case).

In fact, it was the qualitative prediction of electron diffraction and its experimental detection that established the significance of de Broglie's matter waves even before his detailed theory of 1927.

The measurement theory for observables other than functions of position was worked out in general by David Bohm, who rediscovered the theory a quarter of a century later (Bohm 1952). Indeed, it is at first puzzling how such a theory of particles in motion may effectively reproduce the collapse process and the rest of the full phenomenology of quantum mechanics. In modern terminology, what Bohm showed in general is that in situations such as measurements, the wave function of the total system *decoheres*, that is different components of the wave function effectively cease to interfere, because they are in fact separated in configuration space by regions with very small wave function. This has as a consequence that the configuration of the system is effectively trapped inside one of the components. This component alone will be relevant at later times for the dynamics of the system, so that the particles behave as if the wave function had collapsed.

Quantitatively, the statistical predictions of quantum mechanics are reproduced if the positions of the particles are distributed according to the usual quantum distribution. As was known to de Broglie, the velocity field preserves the form of the particle distribution if at any time it is given by the quantum distribution. Intuitively, this is some kind of time-dependent equilibrium distribution, and there is indeed a deep analogy between the statistical aspects of de Broglie-Bohm theory and classical statistical mechanics. Under the assumption of non-equilibrium distributions, the theory instead yields novel predictions as compared to quantum mechanics. Furthermore, pilot-wave theory is explicitly non-local, as any hidden variables theory must be in order to recover the quantum mechanical violations of the Bell inequalities. Finally, the theory can be easily modified to include spin; and various generalisations aiming to cover quantum electrodynamics and other field theories have been proposed. Incidentally, J. S. Bell contributed decisively to the theory's current revival (Bell 1987, *passim*).

As regards quantum logic, it is obvious that since de Broglie-Bohm theory reproduces the phenomenology of quantum mechanics, physical systems have the same dispositions to elicit measurement results in pilot-wave theory as in standard quantum mechanics. The introduction of the connectives at the level of the experimental propositions therefore goes through unaltered. At the level of the intrinsic properties of a system, however, it should be

clear that configuration-space properties obey classical logic no less than phase-space properties in classical physics. Indeed, de Broglie-Bohm theory can be viewed as a theory that is entirely classical at the level of kinematics (particles moving in space and time), and which is quantum only as regards its ‘new dynamics’ (as the title of de Broglie’s paper reads). Thus, the way de Broglie-Bohm theory explains the effectiveness of classical logic at the macroscopic level is that it is already the logic that is operative at the hidden (‘untestable’) level of the particles.

Indeed, the emergence of the classical world around us in de Broglie-Bohm theory happens as follows. At the level of the wave function, a process of decoherence ensures that macroscopically different components will develop that will not generally reinterfere, e. g. it ensures that the ‘live’ and ‘dead’ components of the state of the cat do not reinterfere. What turns these different components into different classical alternatives, however, is the fact that the configuration of the system is located only in one of these different components, and this is already a matter of classical logic. The cat is (classically) either alive or dead, because the particles that compose it are (classically) either in the live component or the dead component of the quantum state. (Decoherence further ensures that they will stay there over time, but this is irrelevant to the point at hand.)

Thus, if one takes the pilot-wave approach to quantum mechanics, although quantum logic may be introduced as a local logic at the level of experimental propositions, it cannot be taken as the basis for justifying the everyday use of classical logic, and thus cannot aspire to replace classical logic as the ‘true’ logic.¹¹

Incidentally, de Broglie-Bohm theory lends itself to discussing issues of conventionalism by analogy to the case of geometry (i. e. the choice between revising physics or geometry), as mentioned in section 1. In the case of general relativity, one can take the metric of space-time to be Einstein’s $g_{\mu\nu}$ and the geometry to be curved, or one can take the metric to be the flat

¹¹If one wishes, one can choose a dual ontology for the theory, in which both the configurations and the wave function are properties of the system. In this case, one can argue that quantum logic is applicable also in pilot-wave theory to describe those intrinsic properties of a system that are encoded in its quantum state (which above we have called testable properties). However, one rejects the completeness of the standard interpretation, and at the additional level of the hidden variables one retains classical logic. It is the classical logic of the hidden variables that explains the effectiveness of classical logic at the macroscopic level.

Lorentzian $\eta_{\mu\nu}$ and write

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} , \tag{17}$$

where $h_{\mu\nu}$ is a new physical field, and the geometry is still the flat geometry of special relativity. This treatment (which lies at the basis of some attempts to quantise gravity) is observationally equivalent to standard general relativity only given some constraints on the topology of the phenomenological space-time (i. e. as described by $g_{\mu\nu}$), but given those constraints, the flat geometry underlying the split (17) is unobservable, and thus the choice between the two descriptions is conventional.

Similarly in the case of quantum mechanics: if one accepts some form of Putnam's argument (say, in the context of an Everett interpretation — see below), then keeping quantum mechanics as it is might indeed prompt us to revise our logic. But one can always retain classical logic, and have de Broglie-Bohm theory give a story of why this classical level is 'hidden'. The theory will be observationally equivalent to standard quantum mechanics only given some constraints on the position distribution of the particles ('equilibrium'), but given these constraints the postulated classical level is indeed not directly observable, and, always given the constraints, one could again argue that the choice between the two descriptions is conventional.

7.2 Spontaneous collapse theories

Spontaneous collapse theories are variants of quantum mechanics in which the Schrödinger evolution is modified in order to reproduce the phenomenology of collapse. Such theories are generally stochastic, and the best-known ones are on the one hand theories of the Ghirardi-Rimini-Weber (GRW) type, in which the Schrödinger equation is supplemented by certain discontinuous random transformations (Ghirardi, Rimini and Weber 1986); on the other hand the theory of continuous spontaneous localisation (CSL) and related ones, in which the wave function obeys a stochastic differential equation of a certain type (Pearle 1989). For simplicity, we shall focus on the former.

The original version of the GRW theory consists of the following stochastic evolution of the wave function. For one particle, the Schrödinger equation is supplemented at random times (with a fixed average frequency $1/\tau$) by

a transformation known as a ‘hit’, namely a multiplication with a three-dimensional Gaussian $\alpha_\lambda(q - x)$, with a fixed width λ , centred at some random position x , for which the theory specifies a probability density. For n particles there are n independent such three-dimensional hitting processes supplementing the Schrödinger equation, which greatly increases the frequency of any such hit.

The form of this evolution is the same as that used to describe some of the more general collapses observed in less idealised experiments, in this case an ‘unsharp’ measurement of position. The novelty with respect to the usual notion of collapse, however, is that this evolution is spontaneous, i. e. takes place irrespectively of the presence or absence of a measuring apparatus or any other system. Indeed, the theory is specifically intended to provide an approach to quantum mechanics that makes it universally applicable, in particular both to the microscopic and the macroscopic domains, as well as to their interaction.

The way the GRW theory proposes to solve the measurement problem relies on the entangled form of the wave function in typical measurement situations, with components of the wave function corresponding (in the standard interpretation) to macroscopically different states of the apparatus. For such states, even a single hit will trigger collapse on a macroscopic scale. Thus, at least *prima facie*, spontaneous collapse theories such as GRW embrace the standard interpretation of quantum mechanics but change the dynamics given by the Schrödinger equation, so that the states that do not correspond to our everyday experience are very efficiently suppressed.

This might seem to suggest that in spontaneous collapse approaches one can indeed maintain that quantum logic is the true logic, since (as discussed in section 5) the standard interpretation is compatible with this claim, and since spontaneous collapse removes the objections to the standard interpretation. However, spontaneous collapse theories solve the measurement problem by introducing classical alternatives in the possible evolution of the state, through the stochastic element of the dynamics. Thus, spontaneous collapse theories should be understood according to the alternative position that takes the properties assigned by the standard interpretation to be elementary properties, and complex ones as built from these using classical logic. Each of the alternatives in the evolution of the state will correspond to different elementary properties of the quantum system, but the overall state of the system is a *classical disjunction* of these alternative states.

Thus, also in spontaneous collapse theories (as in pilot-wave theory), the quantum connectives do not provide the basis for the effectiveness of the classical connectives. There is no story explaining that the cat is dead or alive classically because it is dead or alive quantum logically. The cat is *first* fleetingly (if at all) dead or alive quantum logically, then the dynamics intervenes and ensures the cat is dead or alive classically. Either a hit on the dead component takes place or one on the alive component does.

One could say that in spontaneous collapse theories quantum logic is naturally adapted to describing the kinematics of the theory, but that the dynamics of the theory requires classical logic. Thus again, although in spontaneous collapse theories it may be natural to introduce quantum logic as a (local) logic at the level of the testable propositions, it cannot be taken as the basis for justifying the everyday use of classical logic. It might claim a place alongside classical logic, but it cannot replace it.¹²

7.3 Everett's many-worlds

The views associated with Everett's many-worlds are indeed many, and some versions, such as the idea that the material universe literally multiplies whenever a measurement occurs, are out of favour for good reason. The modern version we shall adopt here has been developed over the last fifteen years, mainly through the work of Saunders (1993), Wallace (2003, 2005) and Greaves (2004), and nowadays enjoys a broad though admittedly not universal consensus among philosophers of physics.

The Everett approach is an interpretation of quantum mechanics in the strict sense, in that it takes the theory without any additions or modifications. It takes the ontology of quantum mechanics to be given completely by the wave function (of the universe), but instead of adopting a 'God's eye' perspective on the wave function, it asks what would be an internal perspective on such a universe. The key insight of the interpretation is that through the mechanism of decoherence, the wave function develops components that

¹²More recently, a different way of interpreting spontaneous collapse theories has been proposed, in terms of matter density (Ghirardi, Grassi and Benatti 1995). Analogously to the case of de Broglie-Bohm theory, regardless of whether this matter density is taken as the sole ontology of the theory or as a hidden variable additional to an ontological wave function, the effectiveness of classical logic on the macroscopic scale will again derive from the applicability of classical logic to this matter density.

have a stable identity over extremely long periods of time and that are dynamically independent of each other. It thus makes sense to identify these components as quasi-separate ‘worlds’, and to define an internal perspective as centred on each such world.

When a measurement occurs, each observer develops into generally many successors, indexed by their different measurement results. So, which measurement result obtains is a matter of perspective: from the perspective of the live cat, the atom has not decayed and thereby triggered the smashing of the vial of cyanide; from the perspective of the dead cat, it has.

Further recent work pioneered by Deutsch (1999) and perfected in particular by Wallace (2005) has sought to justify the use of the usual quantum probabilities on the basis of rational decision theory as adapted to such a ‘splitting’ agent. If one accepts Lewis’s Principal Principle as the definition of objective chances, the Deutsch-Wallace results imply that the quantum probabilities are indeed objective in each world.

What about logic? Note first of all that, from the perspective of each world, the standard interpretation of quantum mechanics can be applied, taking the relevant component of the universal wave function to be the quantum state for that world. Note also that, although the description of a world given by the relevant component of the wave function is perspectival, it is no less objective than the description of the universe as given by the total wave function.

Thus, again from the perspective of each world (which is the only perspective that makes sense empirically), quantum logic is well adapted to describe the intrinsic properties of physical systems. The question, as we know by now, is whether classical logic is required separately to make sense of the effectiveness of classical logic on the macroscopic scale, or whether there is a sense in which quantum logic can explain how classical logic can be effective in everyday cases, and therefore how we may have arrived to our classical conception of logic by abstraction from the everyday world.

In the case of the Everett interpretation it now seems that this challenge is met. Indeed, while in general a quantum disjunction does not behave truth-functionally (because the different components of the wave function do not decohere, thus all belong to the same world), there are cases in which it does (because the different components do decohere and thus be-

long to different worlds). In such cases, from the perspective of each world, the disjuncts behave like classical alternatives, one of which is actual, the others counterfactual. Although in every world the properties of all physical systems are in bijective correspondence with subspaces of the Hilbert space, *de facto* unobserved macroscopic superpositions are not the kind of properties that appear in a typical world, unlike the case of the standard interpretation. And this is because interference between different components becomes negligible, and an effective superselection rule arises between the different non-interfering components, thus mimicking the case of von Neumann algebras discussed in section 6. The relation between the quantum and the classical connectives is not a formal relation as we had in the case of von Neumann algebras, but the connectives behave classically in a suitable physical limit.

Thus, while the structure of the intrinsic properties of physical systems supports a non-distributive logic at the fundamental level (even in the individual worlds), one can claim that, unlike the case of pilot-wave theory or spontaneous collapse, the perspectival element characteristic of the Everett interpretation introduces a genuine emergence of the classical connectives from the quantum connectives. In this sense, it is only the Everett interpretation, among the major approaches to quantum mechanics, that is compatible with a revision of logic. One is not forced to accept the overall package, but, while perhaps not entirely as Putnam had articulated it, there is an intelligible sense in which (standard) quantum mechanics may suggest that logic be revised.

8 Conclusion

We hope to have clarified in what sense empirical considerations of quantum phenomena may have a bearing on the issue of the ‘true’ logic. Some of Putnam’s (1968) claims in this regard can be justified, but with qualifications.

What can be said about the status of quantum logic in our world, assuming current approaches to the foundations of quantum mechanics, depends on the details of the chosen approach. In particular, one might justify a revision of logic at most if one chooses an Everett interpretation. Indeed, it is a general lesson in the philosophy of physics, confirmed in the present case, that bold philosophical claims made on the basis of quantum mechan-

ics turn out to be highly dependent on the interpretational approach one adopts towards the theory.

The scenario in which consideration of quantum or quantum-like phenomena might make a revision of logic most appealing is possibly that of von Neumann algebras — thus perhaps vindicating Dickson’s (2001) intuition —, where there is a rigorous sense in which the quantum and classical connectives can be said to be the same and to behave truth-functionally or not according to the meaning of the propositions involved. This possibility is presumably not realised in our world, but whether it is or not is itself an empirical issue, thus lending at least some support to the idea that logic is indeed empirical.

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References

- Bacciagaluppi, G. (1993), ‘Critique of Putnam’s quantum logic’, *International Journal of Theoretical Physics* **32**, 1835–1846.
- Baez, J. (1987), ‘Bell’s inequality for C^* algebras’, *Letters in Mathematical Physics* **13**, 135–136.

- Bell, J. S. (1987), *Speakable and Unspeakable in Quantum Mechanics* (Cambridge: Cambridge University Press).
- Beltrametti, E., and van Fraassen, B. C. (eds.) (1981), *Current Issues in Quantum Logic* (New York: Plenum Press).
- Birkhoff, G., and von Neumann, J. (1936), ‘The logic of quantum mechanics’, *Annals of Mathematics* **37**, 823–843. Reprinted in Hooker (1975), pp. 1–26.
- Bohm, D. (1952), ‘A suggested interpretation of the quantum theory in terms of “hidden” variables, I and II’, *Physical Review* **85**, 166–179 and 180–193.
- de Broglie, L. (1928), ‘La nouvelle dynamique des quanta’, in *Électrons et Photons: Rapports et Discussions du Cinquième Conseil de Physique Solvay* (Paris: Gauthier-Villars), pp. 105–132. Translated in G. Bacciagaluppi and A. Valentini, *Quantum Theory at the Crossroads: Reconsidering the 1927 Solvay Conference* (Cambridge: Cambridge University Press, forthcoming), pp. 375–399.
- Brown, H. (1986), ‘The insolubility proof of the quantum measurement problem’, *Foundations of Physics* **16**, 857–870.
- Bub, J., and Clifton, R. (1996), ‘A uniqueness theorem for “no collapse” interpretations of quantum mechanics’, *Studies in History and Philosophy of Modern Physics* **27B**, 181–219.
- Bueno, O., and Colyvan, M. (2004), ‘Logical non-apriorism and the “law” of non-contradiction’, in G. Priest, J. C. Beall and B. Armour-Garb (eds.), *The Law of Non-Contradiction: New Philosophical Essays* (Oxford: Oxford University Press), pp. 156–175.
- Dalla Chiara, M. L. (1981), ‘Some metalogical pathologies of quantum logic’, in Beltrametti and Van Fraassen (1981), pp. 147–159.
- Dalla Chiara, M. L., and Giuntini, R. (2002), ‘Quantum logics’, in D. Gabbay and F. Guenther (eds.), *Handbook of Philosophical Logic*, vol. 6 (Dordrecht: Kluwer), pp. 129–228.
- Deutsch, D. (1999), ‘Quantum theory of probability and decisions’, *Proceedings of the Royal Society of London* **A 455**, 3129–3137.

- Dickson, M. (2001), ‘Quantum logic is $\text{alive} \wedge (\text{it is true} \vee \text{it is false})$ ’, *Philosophy of Science (Proceedings)* **68**, S274–S287.
- Dummett, M. (1976), ‘Is logic empirical?’, in H. D. Lewis (ed.), *Contemporary British Philosophy*, 4th series (London: Allen and Unwin), pp. 45–68. Reprinted in M. Dummett, *Truth and other Enigmas* (London: Duckworth, 1978), pp. 269–289.
- Finch, P. D. (1969), ‘On the structure of quantum logic’, *Journal of Symbolic Logic* **34**, 275–282. Reprinted in Hooker (1975), pp. 415–425.
- Foulis, D. J., and Randall, C. H. (1981), ‘What are quantum logics and what ought they to be?’, in Beltrametti and Van Fraassen (1981), pp. 35–52.
- Friedman, M., and Glymour, C. (1972), ‘If quanta had logic’, *Journal of Philosophical Logic* **1**, 16–28.
- van Fraassen, B. C. (1970), ‘On the extension of Beth’s semantics of physical theories’, *Philosophy of Science* **37**, 325–339.
- Friedman, M., and Putnam, H. (1978), ‘Quantum logic, conditional probability, and interference’, *Dialectica* **32**, 305–315.
- Ghirardi, G.C., Grassi, R., and Benatti, F. (1995), ‘Describing the macroscopic world: closing the circle within the dynamical reduction program’, *Foundations of Physics* **25**, 5–38.
- Ghirardi, G.C., Rimini, A., and Weber, T. (1986), ‘Unified dynamics for microscopic and macroscopic systems’, *Physical Review D* **34**, 470–491.
- Gödel, K. (1933), ‘Eine Interpretation des intuitionistischen Aussagenkalküls’, *Ergebnisse eines mathematischen Kolloquiums* **4**, 39–40. Reprinted and translated in K. Gödel, *Collected Works*, vol. 1, ed. by S. Feferman, S. Kleene, G. Moore, R. Solovay and J. van Heijenoort (Oxford: Oxford University Press, 1986), pp. 300–303.
- Greaves, H. (2004), ‘Understanding Deutsch’s probability in a deterministic multiverse’, *Studies in History and Philosophy of Modern Physics* **35B**, 423–456.
- Gudder, S. (1972), ‘Partial algebraic structures associated with orthomodular posets’, *Pacific Journal of Mathematics* **41**, 712–730.

- Hardegree, G. M. (1975), ‘Stalnaker conditionals and quantum logic’, *Journal of Philosophical Logic* **4**, 399–421.
- Heywood, P., and Redhead, M. L. G. (1983), ‘Nonlocality and the Kochen-Specker paradox’, *Foundations of Physics* **13**, 481–499.
- Hooker, C. A. (1975), *The Logico-Algebraic Approach to Quantum Mechanics*, vol. 1 (Dordrecht: Reidel).
- Jauch, J. M., and Piron, C. (1963), ‘Can hidden variables be excluded in quantum mechanics?’, *Helvetica Physica Acta* **36**, 827–837.
- Jauch, J. M., and Piron, C. (1969), ‘On the structure of quantal proposition systems’, *Helvetica Physica Acta* **43**, 842–848. Reprinted in Hooker (1975), pp. 427–436.
- Kochen, S., and Specker, E. P. (1965a), ‘Logical structures arising in quantum theory’, in L. Addison, L. Henkin and A. Tarski (eds.), *The Theory of Models* (Amsterdam: North-Holland), pp. 177–189. Reprinted in Hooker (1975), pp. 263–276.
- Kochen, S., and Specker, E. P. (1965b), ‘The calculus of partial propositional functions’, in Y. Bar-Hillel (ed.), *Logic, Methodology, and Philosophy of Science* (Amsterdam: North-Holland), pp. 45–57. Reprinted in Hooker (1975), pp. 277–292.
- Kochen, S., and Specker, E. P. (1967), ‘The problem of hidden variables in quantum mechanics’, *Journal of Mathematics and Mechanics* **17**, 59–88. Reprinted in Hooker (1975), pp. 293–328.
- Kolmogoroff, A. (1931), ‘Zur Deutung der intuitionistischen Logik’, *Mathematische Zeitschrift* **35**, 58–65.
- von Neumann, J. (1932), *Mathematische Grundlagen der Quantenmechanik* (Berlin: Springer).
- Pearle, P. (1989), ‘Combining stochastic dynamical state-vector reduction with spontaneous localization’, *Physical Review A* **39**, 2277–2289.
- Putnam, H. (1968), ‘Is logic empirical?’ in R. Cohen and M. Wartofsky (eds.), *Boston Studies in the Philosophy of Science*, vol. 5 (Dordrecht: Reidel), pp. 216–241. Reprinted as ‘The logic of quantum mechanics’ in H.

- Putnam, *Mathematics, Matter, and Method. Philosophical Papers*, vol. 1 (Cambridge: Cambridge University Press, 1975), pp. 174–197.
- Putnam, H. (1974), ‘How to think quantum-logically’, *Synthese*, **29**, 55–61. Reprinted in P. Suppes (ed.), *Logic and Probability in Quantum Mechanics* (Dordrecht: Reidel, 1976) pp. 47–53.
- Putnam, H. (1981): ‘Quantum mechanics and the observer’, *Erkenntnis* **16**, 193–219.
- Putnam, H. (1994): ‘Michael Redhead on quantum logic’, in P. Clark and R. Hale (eds.), *Reading Putnam* (Oxford: Basil Blackwell), pp. 265–280.
- Raggio, G. A. (1981), *States and Composite Systems in W^* -algebraic Quantum Mechanics*, Diss. ETH No. 6824.
- Raggio, G. A. (1988), ‘A remark on Bell’s inequality and decomposable normal states’, *Letters in Mathematical Physics*, **15**, 27–29.
- Saunders, S. (1993), ‘Decoherence, relative states, and evolutionary adaptation’, *Foundations of Physics* **23**, 1553–1585.
- Schrödinger, E. (1935), ‘Discussion of probability relations between separated systems’, *Proceedings of the Cambridge Philosophical Society* **31**, 555–563.
- Stairs, A. (1983), ‘Quantum logic, realism, and value-definiteness’, *Philosophy of Science* **50**, 578–602.
- Wallace, D. (2003), ‘Everett and structure’, *Studies in History and Philosophy of Modern Physics* **34B**, 87–105.
- Wallace, D. (2005), ‘Quantum probability from subjective likelihood: improving on Deutsch’s proof of the probability rule’, to appear in *Studies in History and Philosophy of Modern Physics*, PHILSCI-00002302.