Non-equilibrium in Stochastic Mechanics

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Abstract. The notion of non-equilibrium, in the sense of a particle distribution other than $\rho = |\psi|^2$, is imported into Nelson's stochastic mechanics, and described in terms of effective wavefunctions obeying non-linear equations. These techniques are applied to the discussion of non-locality in non-linear Schrödinger equations.

1. Introduction

The ideal of quantum mechanics as an emergent theory is well represented by Nelson's stochastic mechanics, which aims at recovering quantum mechanics from an underlying stochastic process in configuration space. More precisely, as we sketch in Section 2, Nelson starts from a time-reversible description of a diffusion process in configuration space, and then introduces some (time-symmetric) dynamical conditions on the process, leading to the Madelung equations for two real functions R and S, which are implied by the Schrödinger equation for $\psi = Re^{iS/\hbar}$.

The resulting theory has a number of similarities with de Broglie and Bohm's pilot-wave theory (although in the latter ψ is interpreted as a fundamental quantity). Indeed, particle trajectories in Nelson's stochastic mechanics can be intuitively thought of as de Broglie–Bohm trajectories with a superimposed white noise; and several concepts and techniques from pilot-wave theory can be easily and usefully imported into stochastic mechanics (although this is not usually discussed explicitly). One of these concepts, that of non-equilibrium, is the topic of the present paper.

Non-equilibrium in pilot-wave theory is defined as a situation in which the particle distribution is not equal to $\rho = R^2 = |\psi|^2$. In Section 3 we discuss non-equilibrium in pilot-wave theory and how it makes sense even in stochastic mechanics.

In Section 4 we develop a formalism for describing non-equilibrium in stochastic mechanics in terms of effective wavefunctions and effective non-linear Schrödinger equations related to but different from the ones describing equilibrium.

In Section 5, this formalism is applied to the discussion of non-locality in the context of non-linear Schrödinger equations. Specifically, we shall find a class of entangled solutions of the equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + V\psi + \frac{\hbar^2}{m}\frac{\Delta|\psi|}{|\psi|}\psi \tag{1}$$

(with no interaction terms in V), possessing a *local* hidden variables model.

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Some open questions conclude the paper (Section 6). One should note in particular that the theory of non-equilibrium in stochastic mechanics is potentially richer that the corresponding one in pilot-wave theory. The techniques developed in this paper, however, do not yet allow to discuss the genuinely novel cases (which, among other things, might be useful for the study of causally symmetric models of quantum mechanics).

2. Nelson's stochastic mechanics

In this section, we give a brief summary of Nelson's theory (largely after Nelson 1966 and Davidson 1979), with particular reference to aspects we shall need later. We consider only one particle for simplicity (the general case is analogous).[2]

Nelson (1966) considers the following stochastic differential equations,

$$d\mathbf{x}(t) = \mathbf{b}(\mathbf{x}(t), t)dt + d\mathbf{w}(t) , \tag{2}$$

$$d\mathbf{x}(t) = \mathbf{b}_*(\mathbf{x}(t), t)dt + d\mathbf{w}_*(t) , \tag{3}$$

where $\mathbf{b}(\mathbf{x}(t), t)$ is the mean forward velocity and $\mathbf{b}_*(\mathbf{x}(t), t)$ the mean backward velocity of the particle, and $\mathbf{w}(t)$ and $\mathbf{w}(t)_*$ are suitable Wiener processes with mean square fixed by the diffusion coefficient $\nu > 0$.

These equations provide a time-symmetric kinematics for describing diffusion processes. In pilot-wave terminology they can be thought of as forward and backward stochastic 'guidance equations' for the particle.

One has

$$D\mathbf{x} = \mathbf{b} \quad \text{and} \quad D_*\mathbf{x} = \mathbf{b}_* , \tag{4}$$

with the forward and backward stochastic derivatives D and D_* given by

$$D\mathbf{x}(t)\Big|_{\mathbf{x}(t)=\mathbf{x}} = \lim_{\varepsilon \to 0^+} E_t \left[\frac{\mathbf{x}(t+\varepsilon) - \mathbf{x}(t)}{\varepsilon} \Big| \mathbf{x}(t) = \mathbf{x} \right] \tag{5}$$

and

$$D_*\mathbf{x}(t)\Big|_{\mathbf{x}(t)=\mathbf{x}} = \lim_{\varepsilon \to 0^+} E_t \left[\frac{\mathbf{x}(t-\varepsilon) - \mathbf{x}(t)}{-\varepsilon} \Big| \mathbf{x}(t) = \mathbf{x} \right] . \tag{6}$$

($E_t[. |\mathbf{x}(t) = \mathbf{x}]$ is the expectation value at time t conditional on the value of the process being $\mathbf{x}$.[3])

Now take an arbitrary solution ρ of the forward Fokker–Planck (FP) equation

$$\frac{\partial \rho}{\partial t} = -\text{div}(\mathbf{b}\rho) + \nu\Delta\rho . \tag{7}$$

If we define an osmotic velocity

$$\mathbf{u}_\rho := \nu \frac{\boldsymbol{\nabla}\rho}{\rho} , \tag{8}$$

and current velocity

$$\mathbf{v}_\rho := \mathbf{b} - \mathbf{u}_\rho , \tag{9}$$

ρ satisfies a continuity equation:

$$\frac{\partial \rho}{\partial t} = -\text{div}\left[\left(\mathbf{v}_\rho + \nu\frac{\boldsymbol{\nabla}\rho}{\rho} \right) \rho \right] + \nu\Delta\rho = -\text{div}(\mathbf{v}_\rho\rho) . \tag{10}$$

[2] For a much more comprehensive overview, see Nelson (1985). An introduction with an eye to conceptual questions is given by Bacciagaluppi (2005).

[3] The definition of $D\mathbf{x}(t)$ thus involves the forward transition probabilities from time t to times $t + \varepsilon$, while the definition of $D_*\mathbf{x}(t)$ involves the backward transition probabilities from time t to times $t - \varepsilon$.

If it is further the case that

$$\mathbf{b}_* = \mathbf{v}_\rho - \mathbf{u}_\rho \ , \tag{11}$$

the same ρ satisfies also the analogous backward equations:

$$\frac{\partial \rho}{\partial t} = -\text{div}(\mathbf{b}_*\rho) - \nu\Delta\rho = -\text{div}\left[\left(\mathbf{v}_\rho + \nu\frac{\boldsymbol{\nabla}\rho}{\rho}\right)\rho\right] - \nu\Delta\rho = -\text{div}(\mathbf{v}_\rho\rho) \ . \tag{12}$$

Note that if only (2) is given, the time reversal (3) is not unique. From *any* solution ρ of the forward FP equation, one could define a backward mean velocity $\mathbf{b}_* = \mathbf{v}_\rho - \mathbf{u}_\rho$ and construct a time reversal of (2). (We shall use this in Section 4.) But if $\mathbf{b}_*$ is given, the single-time distribution ρ of the process will need to solve both FP equations and will thus be unique, because the corresponding current and osmotic velocities are fixed by

$$\frac{1}{2}(\mathbf{b} + \mathbf{b}_*) = \mathbf{v}_\rho \quad \text{and} \quad \frac{1}{2}(\mathbf{b} - \mathbf{b}_*) = \mathbf{u}_\rho = \nu\frac{\boldsymbol{\nabla}\rho}{\rho} \ , \tag{13}$$

and ρ is normalised.

As yet, $\mathbf{b}$ and $\mathbf{b}_*$ (which define the dynamics of the process) are left unspecified. Nelson's aim is to find natural constraints that will yield the Madelung equations for R and S, with R^2 defining the distribution of the process and with

$$\mathbf{v} = \frac{1}{m}\boldsymbol{\nabla}S \ . \tag{14}$$

The latter can be justified for instance from the variational approach of Guerra and Morato (1983), further motivated in Nelson (1985). Alternatively, note that $\mathbf{v}$ being a gradient implies

$$D\mathbf{b} = D_*\mathbf{b}_* \ . \tag{15}$$

This was pointed out already by de la Peña and Cetto (1982, eq. (16)).[4] Thus we can also justify setting $\mathbf{v}$ equal to a gradient as a simple way of enforcing the time-symmetry condition (15) — which we now see is an identity in Nelson's stochastic mechanics.

Given (14) and writing $R^2 := \rho$ we obtain the first Madelung equation (the continuity equation),

$$\frac{\partial R^2}{\partial t} = -\text{div}\left(\frac{1}{m}(\boldsymbol{\nabla}S)R^2\right) = -\frac{1}{m}(\Delta S)R^2 - \frac{1}{m}\boldsymbol{\nabla}S\boldsymbol{\nabla}R^2 \ . \tag{16}$$

By defining further the *mean stochastic acceleration* as

$$\mathbf{a} := \frac{1}{2}\left(D\mathbf{b}_* + D_*\mathbf{b}\right) \ , \tag{17}$$

and imposing 'Newton's law',

$$m\mathbf{a} = -\boldsymbol{\nabla}V \ , \tag{18}$$

Nelson obtains also the Hamilton–Jacobi–Madelung (HJM) equation

$$\frac{\partial S}{\partial t} = -\frac{1}{2m}(\boldsymbol{\nabla}S)^2 - V + \frac{\hbar^2}{2m}\frac{\Delta R}{R} \ . \tag{19}$$

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[4] We give an explicit proof in the Appendix.

As originally shown by Madelung (1926a,b), equations (16) and (19) are implied by Schrö-dinger's equation for $\psi = Re^{iS/\hbar}$. However, as pointed out by Wallstrom (1994), the converse is not true unless S has the right multi-valuedness behaviour.[5]

More generally (Davidson 1979), we can define the mean acceleration as

$$\frac{\alpha}{2}(D\mathbf{b}_* + D_*\mathbf{b}) + \frac{\beta}{2}(D\mathbf{b} + D_*\mathbf{b}_*) , \tag{20}$$

with $\alpha, \beta \geq 0$, $\alpha + \beta = 1$. Imposing Newton's law with the acceleration defined by (20),[6] we now obtain

$$\frac{\partial S}{\partial t} = -\frac{1}{2m}(\boldsymbol{\nabla}S)^2 - V + 2m\nu^2(\alpha - \beta)\frac{\Delta R}{R} . \tag{21}$$

As is easy to see, this HJM equation is implied by a generally *non-linear* Schrödinger equation:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + V\psi + \left(\frac{\hbar^2}{2m} - 2m\nu^2(\alpha - \beta)\right)\frac{\Delta|\psi|}{|\psi|}\psi . \tag{22}$$

We can distinguish three canonical cases:

(1) $\nu \neq 0$ and $\alpha > \beta$: one obtains the linear Schrödinger equation by choosing

$$\nu = \frac{\hbar}{2m\sqrt{\alpha - \beta}} ; \tag{23}$$

(2a) $\nu = 0$ (deterministic case), or **(2b)** $\alpha = \beta$: one has

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + V\psi + \frac{\hbar^2}{2m}\frac{\Delta|\psi|}{|\psi|}\psi , \tag{24}$$

corresponding to the classical Hamilton–Jacobi equation and thus called the 'Schrödinger equation of classical mechanics' (Holland 1993, Sect. 2.6 and references therein);[7]

(3) $\nu \neq 0$ and $\alpha < \beta$: one can choose

$$\nu = \frac{\hbar}{2m\sqrt{\beta - \alpha}} \tag{25}$$

to obtain the non-linear equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + V\psi + \frac{\hbar^2}{m}\frac{\Delta|\psi|}{|\psi|}\psi . \tag{26}$$

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[5] Possibly the most promising strategy for overcoming Wallstrom's objection is the suggestion by L. Smolin (personal communication, Waterloo, Ontario, July 2005) to exploit the fact that quantum mechanical ground states will be typically nodeless under appropriate conditions. Instead, the line of argument suggested in Smolin (2006, Sect. IV) does not go through; see Valentini (2010, footnote 3) and Schmelzer (2011, Sect. 2.5). For two recent approaches to overcoming Wallstrom's objection from the point of view of slightly different theories, see de la Peña *et al.* (2011, Sect. 5.3) and the detailed proposal by Schmelzer (2011).

[6] This is equivalent to the procedure by de la Peña (equation (25) in his (1969)) and de la Peña and Cetto (1975). Note that in their treatment the case $\beta = 1$ is shown to correspond to (classical) Brownian motion.

[7] Note that the description of ensemble motions provided in the case $\nu = 0$ corresponds to the special case of classical mechanics in which an initial momentum field is fixed by the choice of the initial Hamilton–Jacobi function. For a careful discussion of these and related issues, see Holland (1993, Chap. 2, esp. Sects. 2.5 and 2.6). The case $\alpha = \beta$ instead is physically very different, as we shall note again in Section 4.

3. Pilot-wave theory and non-equilibrium

The pilot-wave theory by de Broglie (1928) and Bohm (1952) describes deterministic particle trajectories in configuration space, where the particle velocity at any time is given by the 'guidance equation'

$$\mathbf{v} = \frac{1}{m}\boldsymbol{\nabla}S \tag{27}$$

(or $\mathbf{v}_i = \frac{1}{m}\boldsymbol{\nabla}_i S$ for several particles), S being the phase of Schrödinger's wavefunction. If one considers an ensemble of particles guided by identical wavefunctions and distributed according to $\rho = |\psi|^2$, then this relation will be preserved by the Schrödinger equation (a fact already known to de Broglie). Thus the theory describes very easily any diffraction and interference phenomena with material particles.

In order to account for measurements more general than simple detections, in particular of observables other than position, one has to include also the measuring apparatus into the description. The wavefunction of the combined system wlil develop into a superposition of components that are and — provided no macroscopic reinterference takes place — remain (approximately) non-overlapping in the combined configuration space, and the system and apparatus will thus be trapped inside one of these components, the others being 'empty waves'. This component will be solely responsible for guiding the future motion of the combined system. Thus, the theory recovers an *effective collapse* of the wavefunction.

This analysis of measurement was Bohm's decisive contribution, and can be extended to more general cases of decoherence by the environment, thus arguably allowing pilot-wave theory to reproduce the classical regime of quantum mechanics as understood in the theory of decoherence (see the discussion in Bacciagaluppi 2003).

Since the current velocity $\mathbf{v}$ in stochastic mechanics has the same form as the particle velocity in de Broglie–Bohm theory, many of the results and techniques developed in the context of pilot-wave theory can be straightforwardly imported into stochastic mechanics, even though in the latter theory the wavefunction is not considered a fundamental quantity.[8]

We shall now consider the notion of (quantum, or sub-quantum) non-equilibrium as discussed in pilot-wave theory, and suggest it should also be imported into stochastic mechanics.

In pilot-wave theory the (independently postulated) wavefunction ψ has two roles: it guides the particle via $\mathbf{v} = \frac{1}{m}\boldsymbol{\nabla}S$, and it defines the particle distribution via $\rho = |\psi|^2$. At first, it may seem puzzling how the wavefunction can play these two roles simultaneously. Indeed, if one thinks of the wavefunction principally as defining the particle distribution (in a maybe hypothetical ensemble), it will be puzzling to say it also guides the motion of the (actual) particle.[9]

If, however, one thinks of the wavefunction principally in its dynamical role as guiding the particle's motion, then it is perfectly natural to expect the particle distribution to depend on the dynamics — namely, if one understands the distribution as an *equilibrium* distribution. By the same token, non-equilibrium distributions, in which the statistical distribution in the ensemble is not equal to $|\psi|^2$, become perfectly intelligible.

The notion of non-equilibrium in pilot-wave theory has been discussed by several authors over the years. Immediately following his initial papers, Bohm (1953) discussed relaxation of the particle distribution from initial non-equilibrium towards $|\psi|^2$ via random external influences.

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[8] This is true in particular of the notion of effective collapse. This is an obvious move, but usually not considered explicitly in treatments of stochastic mechanics. I conjecture it would, among other things, resolve the puzzle of two-time correlations raised by Nelson (2006).

[9] Indeed, this was precisely the reason for Schrödinger's dismissal of Bohm's 1952 theory, as expressed in a letter to Einstein (Schrödinger to Einstein, [after 18 but before 31 January] 1953, *Archive for the History of Quantum Physics*, microfilm no. 37, sect. 005-012 (manuscript) and 005-013 (carbon copy) (in German)) reproduced in the collection of Schrödinger's correspondence edited by von Meyenn (2011, vol. 2., pp. 673–675).

The next year, Bohm and Vigier (1954) hypothesised relaxation through an intrinsic random noise. This theory was discussed in more detail by Bohm and Hiley (1993, Chap. 9), and leads to trajectories identical to those of stochastic mechanics (!). The differences between this stochasic variant of pilot-wave theory and stochastic mechanics proper are, first and obviously, that Bohm and co-workers take the wavefunction to be fundamental, and, second, that they disregard the backward equations and thus the time-symmetric formulation of the theory. As a matter of fact, they are explicitly concerned with the relaxation of non-equilibrium distributions towards (future) equilibrium, and thus with much the same situation as we shall discuss in the next section, of particles evolving under Nelson's stochastic guidance equation (say, the forward one), but subject to a (say, initial) *non-equilibrium constraint.*

Further important work has focused on the analogy between $|\psi|^2$ in pilot-wave theory and equilibrium in classical statistical mechanics. Major examples are Valentini's (1991a,b) sub-quantum H-theorem, the analysis of equilibrium and of how it relates to uncertainty by Dürr, Goldstein and Zanghì (1992), various studies of relaxation behaviour, e.g. the recent work by Towler, Russell and Valentini (2011) (see also references therein), and various studies of the consequences, signatures and possible residues of non-equilibrium by Valentini (see e.g. Valentini 2007, 2010). The last type of investigations is particularly exciting, because it opens up the possibility of new empirical predictions.[10]

The situation is now perfectly analogous in Nelson's theory. The (independently postulated) vector fields $\mathbf{b}$ and $\mathbf{b}_*$ have two roles: they guide the particle via $\mathbf{v} = \frac{1}{2}(\mathbf{b}+\mathbf{b}_*)$ and $\mathbf{u} = \frac{1}{2}(\mathbf{b}-\mathbf{b}_*)$, and they define the particle distribution via $\mathbf{u} = \nu\frac{\boldsymbol{\nabla}\rho}{\rho}$. The guidance equations are law-like, and so is ρ, but this is unsurprising if we interpret it as the equilibrium distribution of the process, while the actual distribution $\tilde{\rho}$ is contingent and might be different from ρ. Perhaps, however, I am labouring an obvious point. A stochastic process (as in Nelson's theory) is a probability measure over a space of trajectories, and *actual frequencies* neither are nor need always match the probabilities (see the more extensive discussion in Bacciagaluppi 2010).

4. Description of non-equilibrium
Suppose we impose a constraint on the particle distribution at some time t_0,

$$\tilde{\rho}(t_0) \neq |\psi(t_0)|^2 \ . \tag{28}$$

We shall now describe the constrained process for $t \geq t_0$. (The same result can be extended to all times t.[11])

In this case, the time development of the distribution still satisfies the forward FP equation for $t > t_0$,

$$\frac{\partial\tilde{\rho}}{\partial t} = -\mathrm{div}(\mathbf{b}\tilde{\rho}) + \nu\Delta\tilde{\rho} \ , \tag{29}$$

but (always for $t > t_0$) it *no longer* satifies the backward equation,

$$\frac{\partial\tilde{\rho}}{\partial t} = -\mathrm{div}(\mathbf{b}_*\tilde{\rho}) - \nu\Delta\tilde{\rho} \ . \tag{30}$$

Since ρ does satisfy the forward equation, however, we can define:

$$\tilde{\mathbf{u}} := \mathbf{u}_{\tilde{\rho}} = \nu\frac{\boldsymbol{\nabla}\tilde{\rho}}{\tilde{\rho}} \ , \tag{31}$$

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[10] Dürr and co-workers instead take it as an advantage of pilot-wave theory over classical statistical mechanics that there is no obvious non-equilibrium to be observed that would require explanation.

[11] This can be done by carrying out the analogous proof for $t \leq t_0$, or — more economically — by exploiting the time-symmetry properties of the resulting equations.

$$\tilde{\mathbf{v}} := \mathbf{b} - \tilde{\mathbf{u}} = \mathbf{v} + \mathbf{u} - \tilde{\mathbf{u}} \; , \tag{32}$$

and

$$\tilde{\mathbf{b}}_* := \tilde{\mathbf{v}} - \tilde{\mathbf{u}} = \mathbf{v} + \mathbf{u} - 2\tilde{\mathbf{u}} \; . \tag{33}$$

As discussed in Section 2, $\tilde{\rho}$ will now satisfy a new effective backward FP equation:

$$\frac{\partial \tilde{\rho}}{\partial t} = -\mathrm{div}(\tilde{\mathbf{b}}_*\tilde{\rho}) - \nu\Delta\tilde{\rho} \; . \tag{34}$$

(Note that if $\mathbf{v}$ is a gradient, given that $\mathbf{u}$ and $\tilde{\mathbf{u}}$ are also gradients, so is $\tilde{\mathbf{v}}$.)

We can now define

$$\tilde{R} := \sqrt{\tilde{\rho}} \; , \tag{35}$$

and

$$\tilde{S} := S + \nu m(\log R^2 - \log \tilde{R}^2) = S + 2\nu m(\log R - \log \tilde{R}) \; , \tag{36}$$

so that

$$\frac{1}{m}\boldsymbol{\nabla}\tilde{S} = \frac{1}{m}\boldsymbol{\nabla}S + \nu\frac{\boldsymbol{\nabla}R^2}{R^2} - \nu\frac{\boldsymbol{\nabla}\tilde{R}^2}{\tilde{R}^2} \; , \tag{37}$$

and a new effective wavefunction $\tilde{\psi} := \tilde{R}e^{i\tilde{S}/\hbar}$. (Note also that S and $\tilde{S}$ have the same multivaluedness behaviour.)

By construction, $\tilde{R}^2$ satisfies the continuity equation with current velocity $\frac{1}{m}\boldsymbol{\nabla}\tilde{S}$:

$$\frac{\partial \tilde{R}^2}{\partial t} = -\mathrm{div}\left(\frac{1}{m}(\boldsymbol{\nabla}\tilde{S})\tilde{R}^2\right) = -\frac{1}{m}(\Delta\tilde{S})\tilde{R}^2 - \frac{1}{m}\boldsymbol{\nabla}\tilde{S}\boldsymbol{\nabla}\tilde{R}^2 \; . \tag{38}$$

We shall now obtain also an effective HJM equation. Partial differentiation of (36) gives

$$\frac{\partial \tilde{S}}{\partial t} = \frac{\partial S}{\partial t} + \nu m\left(\frac{1}{R^2}\frac{\partial R^2}{\partial t} - \frac{1}{\tilde{R}^2}\frac{\partial \tilde{R}^2}{\partial t}\right) \; . \tag{39}$$

And substituting (21), (16) and (38) into (39) yields

$$\begin{aligned}
\frac{\partial \tilde{S}}{\partial t} = &- \frac{1}{2m}(\boldsymbol{\nabla}S)^2 - V + 2m\nu^2(\alpha - \beta)\frac{\Delta R}{R} \\
&- \nu\Delta S - \nu\boldsymbol{\nabla}S\frac{\boldsymbol{\nabla}R^2}{R^2} + \nu\Delta\tilde{S} + \nu\boldsymbol{\nabla}\tilde{S}\frac{\boldsymbol{\nabla}\tilde{R}^2}{\tilde{R}^2} \; .
\end{aligned} \tag{40}$$

Using further the fact that

$$\Delta S = \Delta\tilde{S} - 2\nu m\frac{\Delta R}{R} + 2\nu m\frac{(\boldsymbol{\nabla}R)^2}{R^2} + 2\nu m\frac{\Delta\tilde{R}}{\tilde{R}} - 2\nu m\frac{(\boldsymbol{\nabla}\tilde{R})^2}{\tilde{R}^2} \; , \tag{41}$$

and

$$\boldsymbol{\nabla}S = \boldsymbol{\nabla}\tilde{S} - \nu m\frac{\boldsymbol{\nabla}R^2}{R^2} + \nu m\frac{\boldsymbol{\nabla}\tilde{R}^2}{\tilde{R}^2} \; , \tag{42}$$

and hence also

$$\begin{aligned}
(\boldsymbol{\nabla}S)^2 =&(\boldsymbol{\nabla}\tilde{S})^2 - 2\nu m\boldsymbol{\nabla}\tilde{S}\frac{\boldsymbol{\nabla}R^2}{R^2} + 2\nu m\boldsymbol{\nabla}\tilde{S}\frac{\boldsymbol{\nabla}\tilde{R}^2}{\tilde{R}^2} \\
&+ \nu^2 m^2\left(\frac{\boldsymbol{\nabla}R^2}{R^2}\right)^2 - 2\nu^2 m^2\frac{\boldsymbol{\nabla}R^2}{R^2}\frac{\boldsymbol{\nabla}\tilde{R}^2}{\tilde{R}^2} + \nu^2 m^2\left(\frac{\boldsymbol{\nabla}\tilde{R}^2}{\tilde{R}^2}\right)^2 \; ,
\end{aligned} \tag{43}$$

after suitable simplification we finally obtain

$$\frac{\partial \tilde{S}}{\partial t} = -\frac{1}{2m}(\boldsymbol{\nabla}\tilde{S})^2 - V + 2m\nu^2(\alpha - \beta + 1)\frac{\Delta R}{R} - 2m\nu^2\frac{\Delta \tilde{R}}{\tilde{R}} \ . \tag{44}$$

We see that the equation for $\tilde{S}$ is not the same as that for S, but has acquired the extra term

$$2m\nu^2(\alpha - \beta + 1)\left(\frac{\Delta R}{R} - \frac{\Delta \tilde{R}}{\tilde{R}}\right) \ . \tag{45}$$

And the corresponding effective Schrödinger equation for $\tilde{\psi}$ is

$$i\hbar\frac{\partial \tilde{\psi}}{\partial t} = -\frac{\hbar^2}{2m}\Delta\tilde{\psi} + V\tilde{\psi} + 2m\nu^2(\alpha - \beta + 1)\frac{\Delta|\psi|}{|\psi|}\tilde{\psi} + \left(\frac{\hbar^2}{2m} + 2m\nu^2\right)\frac{\Delta|\tilde{\psi}|}{|\tilde{\psi}|}\tilde{\psi} \ . \tag{46}$$

Thus, the *non-equilibrium* process can be equally well described as an *equilibrium* process with a different wavefunction and Schrödinger equation. In general this equation is *not* the same as the one satisfied by ψ (and it is not an autonomous equation for $\tilde{\psi}$, because of the presence of the 'old' quantum potential). In special cases, however, the extra term in the HJM equation vanishes, and ψ and $\tilde{\psi}$ will both satisfy the same (autonomous) equation. This happens iff $2m\nu^2(\alpha - \beta + 1) = 0$, i.e.

(a) if $\nu = 0$ (but *not* if $\alpha = \beta$, although in both cases the Schrödinger equation is the 'Schrödinger equation of classical mechanics'), or

(b) if $\beta = 1$, $\alpha = 0$ and ν is arbitrary.

Case **(b)** corresponds to the non-linear Schrödinger equation

$$i\hbar\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + V\psi + \left(\frac{\hbar^2}{2m} + 2m\nu^2\right)\frac{\Delta|\psi|}{|\psi|}\psi \ , \tag{47}$$

and a non-equilibrium distribution in this case is simply an equilibrium distribution corresponding to a *different* wavefunction. For this special case thus, there is no privileged equilibrium distribution (and a case to be made that the wavefunction is to a large extent epistemic).

5. Non-linearity and non-locality

We now apply the results of the previous section to the discussion of non-locality in the context of theories with non-linear Schrödinger equations.

Rigorous results are available on this issue, specifically the well-known paper by Gisin (1989), to which we shall return below. What these results, however, do *not* show is that one automatically has superluminal signalling whenever one has entangled states obeying non-linear equations. Indeed, we shall now construct a counterexample to this proposition.

First of all, note that for product wavefunctions, and only for product wavefunctions,

$$S(\mathbf{x}_1, \mathbf{x}_2) = S_1(\mathbf{x}_1) + S_2(\mathbf{x}_2) \tag{48}$$

and

$$R(\mathbf{x}_1, \mathbf{x}_2) = R_1(\mathbf{x}_1)R_2(\mathbf{x}_2) \ . \tag{49}$$

In this case, and only in this case,

$$\mathbf{v}_i = \frac{1}{m_i}\boldsymbol{\nabla}_i S(\mathbf{x}_1, \mathbf{x}_2) = \frac{1}{m_i}\boldsymbol{\nabla}_i S_i(\mathbf{x}_i) \tag{50}$$

$(i = 1, 2)$ and

$$\mathbf{u}_i = \nu \boldsymbol{\nabla}_i \log R^2(\mathbf{x}_1, \mathbf{x}_2) = \nu \boldsymbol{\nabla}_i \log R_i^2(\mathbf{x}_i) \ . \tag{51}$$

In particular, if the wavefunction has product form, the Nelsonian dynamics decomposes into independent dynamics for the two particles, and is thus local.

Now take any Schrödinger equation of the form (22), i.e.

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + V \psi + \left(\frac{\hbar^2}{2m} - 2m\nu^2(\alpha - \beta) \right) \frac{\Delta |\psi|}{|\psi|} \psi \ . \tag{52}$$

If there are no interaction terms in the potential, this equation preserves product wavefunctions. We can see this by noting that, if the potential decomposes as $V_1(\mathbf{x}_1) + V_2(\mathbf{x}_2)$, the linear Schrödinger equation preserves products. But the quantum potential

$$\left(\frac{\hbar^2}{2m} - 2m\nu^2(\alpha - \beta) \right) \frac{\Delta R}{R} \tag{53}$$

just *is* a potential that decomposes in this way:

$$\frac{\Delta R}{R} = \frac{(\Delta_1 + \Delta_2) R_1 R_2}{R_1 R_2} = \frac{\Delta_1(R_1 R_2)}{R_1 R_2} + \frac{\Delta_2(R_1 R_2)}{R_1 R_2} = \frac{\Delta_1 R_1}{R_1} + \frac{\Delta_2 R_2}{R_2} \ . \tag{54}$$

To simplify the construction of our counterexample, we now specialise to one of the invariant non-linear Schrödinger equations derived above, e.g.

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + V \psi + \frac{\hbar^2}{m} \frac{\Delta |\psi|}{|\psi|} \psi \tag{55}$$

(case of $\alpha = 0$, $\beta = 1$, $\nu = \frac{\hbar}{2m}$).[12]

We take an initial product wavefunction $\psi(\mathbf{x}_1, \mathbf{x}_2, t_0)$. From the above, it follows that the corresponding stochastic mechanics for this system is *local* (and similarly if the two particles separately interact locally with measurement apparatuses).

Now we impose *classical correlations* on the initial distribution:

$$\tilde{\rho}(t_0) \neq \tilde{\rho}_1(t_0) \tilde{\rho}_2(t_0) \ . \tag{56}$$

The resulting effective wavefunction $\tilde{\psi}$ is *entangled*, because the functions

$$\tilde{R}(t_0) = \sqrt{\tilde{\rho}(t_0)} \tag{57}$$

and

$$\tilde{S}(t_0) = S(t_0) + \hbar \log R(t_0) - \hbar \log \tilde{R}(t_0) \tag{58}$$

fail to decompose appropriately. The constrained process is thus described by an entangled $\tilde{\psi}$ satisfying a non-linear Schrödinger equation. But, by construction, *it is a local process with added classical correlations.* This refutes the 'folk' claim that entanglement and non-linearity together imply superluminal signalling.

What about Gisin's result, however? What Gisin (1989) has shown (illustrated even more strikingly in Gisin (1990)) is that *if* a theory with a non-linear Schrödinger equation is such that it reproduces the usual phenomenology of collapse, *then* one can use non-linearity in conjunction

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[12] With slight modifications, the argument will cover all the non-linear Schrödinger equations (22).

with entanglement to obtain signalling.[13] As Gisin is very well aware (private communicaton, Arolla, Switzerland, July 2009), this is a substantive assumption, and his theorem is not meant to apply more generally. Our counterexample thus shows (perhaps unsurprisingly) that the family of non-linear equations we have considered are not good candidates for reproducing the standard collapse phenomenology of quantum mechanics!

6. Open questions

To conclude, let us briefly return to the comparison between stochastic mechanics and pilot-wave theory, and to a few open questions that suggest themselves.

Specifically in the context of non-linear Schrödinger equations, we have seen that an apparent threat of non-locality arises from the entanglement of the effective wavefunction describing the non-equilibrium situation, i.e. from the fact that $\tilde{\mathbf{v}}$ and $\tilde{\mathbf{u}}$ do not decompose in terms of the velocities of the single particles. Locality, however, is *preserved* because their sum $\mathbf{b} = \mathbf{v} + \mathbf{u} = \tilde{\mathbf{v}} + \tilde{\mathbf{u}}$ decomposes appropriately. On the other hand, in a pilot-wave model of the same situation (wavefunction $\tilde{\psi}$ satisfying the given non-linear equation) the motion of the particles depends only on $\tilde{\mathbf{v}}$. Thus, despite the fact that the particle distribution in both models would be the same, it would appear that the de Broglie–Bohm model of the situation would be more non-local than the Nelsonian model.[14]

Further, whether one considers linear or non-linear Schrödinger equations, non-equilibrium stochastic mechanics is a potentially richer theory (including a potentially richer range of novel phenomena) than non-equilibrium pilot-wave theory. This is so because the theory is stochastic. Indeed, in a deterministic theory such as pilot-wave theory one can only consider constraints at a single time: the distribution of trajectories is fixed by the particle distribution at a single instant. By contrast, in a stochastic theory any additional constraint on the distribution of trajectories merely subselects further from the ensemble of all possible trajectories that defines the stochastic process; thus, independent multi-time constraints make perfect sense. In particular, one can impose initial and final constraints on a process and might thereby obtain some qualitatively new phenomena (for instance genuine non-locality even if the unconstrained process is local). Such time-symmetrically constrained processes could provide good test cases for the ideas put forward in particular by Price (1996) about retrocausal hidden variables models for quantum mechanics. The techniques developed in this paper, however, cannot be straightforwardly extended to the case of multi-time constraints.

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[13] Imagine for simplicity an EPR setup with a singlet state. If Alice by measuring spin in direction x or y can create on Bob's side equal up-down mixtures of spin eigenstates in direction x or y, respectively, Bob can then use the non-linearity of the evolution to discriminate between these two different mixtures. Thus, Alice can signal arbitrarily fast to Bob using an ensemble of sufficiently separated EPR pairs.

[14] This also suggests the question of whether entangled wavefunctions with decomposable $\frac{1}{m}\boldsymbol{\nabla}S + \nu\frac{\boldsymbol{\nabla}R^2}{R^2}$ are somehow less intrinsically non-local than generic ones.

Appendix

We now show explicitly that requiring that $\mathbf{v}$ be a gradient is a simple sufficient condition for the time-symmetry condition (15), i.e.

$$D\mathbf{b} = D_*\mathbf{b}_* . \tag{A.1}$$

Note first of all that applying the stochastic derivatives D or D_* to an arbitrary random variable $f(\mathbf{x}(t), t)$ one obtains

$$Df(\mathbf{x}(t), t)\Big|_{\mathbf{x}(t)=\mathbf{x}} = \left[\frac{\partial}{\partial t} + \mathbf{b}(\mathbf{x}, t) \cdot \boldsymbol{\nabla} + \nu\Delta\right] f(\mathbf{x}, t) \tag{A.2}$$

and

$$D_*f(\mathbf{x}(t), t)\Big|_{\mathbf{x}(t)=\mathbf{x}} = \left[\frac{\partial}{\partial t} + \mathbf{b}_*(\mathbf{x}, t) \cdot \boldsymbol{\nabla} - \nu\Delta\right] f(\mathbf{x}, t) . \tag{A.3}$$

By (A.2) and (A.3), condition (A.1) is equivalent to

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{u} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{v} + \nu\Delta\mathbf{v} = 0 . \tag{A.4}$$

Now, by the definition of $\mathbf{u}$, we have

$$\frac{\partial \mathbf{u}}{\partial t} = \frac{\partial}{\partial t}\nu\boldsymbol{\nabla}\ln\rho = \nu\boldsymbol{\nabla}\left(\frac{1}{\rho}\frac{\partial\rho}{\partial t}\right) . \tag{A.5}$$

And using the continuity equation (10),

$$\frac{\partial \mathbf{u}}{\partial t} = -\nu\boldsymbol{\nabla}\left(\frac{1}{\rho}\boldsymbol{\nabla}\cdot(\mathbf{v}\rho)\right) = -\nu\boldsymbol{\nabla}\left(\frac{1}{\rho}(\boldsymbol{\nabla}\cdot\mathbf{v})\rho + \frac{1}{\rho}\mathbf{v}\cdot\boldsymbol{\nabla}\rho\right) =$$
$$= -\nu\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\mathbf{v} + \mathbf{v}\cdot\boldsymbol{\nabla}\ln\rho\right) = -\nu\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{v}) - \boldsymbol{\nabla}(\mathbf{v}\cdot\mathbf{u}) . \tag{A.6}$$

Therefore, (A.4) becomes

$$\nu\left(\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{v}) - \Delta\mathbf{v}\right) + \boldsymbol{\nabla}(\mathbf{v}\cdot\mathbf{u}) - (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{u} - (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{v} = 0 . \tag{A.7}$$

From the identity

$$\boldsymbol{\nabla}(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\boldsymbol{\nabla})\mathbf{b} + (\mathbf{b}\cdot\boldsymbol{\nabla})\mathbf{a} + \mathbf{a}\times(\boldsymbol{\nabla}\times\mathbf{b}) + \mathbf{b}\times(\boldsymbol{\nabla}\times\mathbf{a}) , \tag{A.8}$$

and the fact that

$$\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{v}) = \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{v}) + \Delta\mathbf{v} , \tag{A.9}$$

we can further transform (A.7) to

$$\nu\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{v}) + \mathbf{u}\times(\boldsymbol{\nabla}\times\mathbf{v}) + \mathbf{v}\times(\boldsymbol{\nabla}\times\mathbf{u}) = 0 . \tag{A.10}$$

Finally, since $\mathbf{u}$ is a gradient, we obtain the equivalence of (A.1) and

$$\nu\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{v}) + \mathbf{u}\times(\boldsymbol{\nabla}\times\mathbf{v}) = 0 , \tag{A.11}$$

for which $\mathbf{v}$ being a gradient is obviously a sufficient condition.

If we further impose (18) or some similar dynamical law, $\mathbf{v}$ being a gradient is presumably also a necessary condition for (A.1), since all time derivatives of (A.11) need to vanish, too, and they will generally depend on the arbitrary external potential V.[15]

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[15] The condition $D\mathbf{b}_* = D_*\mathbf{b}$ is similarly equivalent to $\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{v} - \nu\Delta\mathbf{v} = 0$, which assuming (A.1) simplifies to $\nu\Delta\mathbf{v} + (\mathbf{u}\cdot\nabla)\mathbf{v} = 0$.

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