Ordinal Utility Differences

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September 8, 2023

Abstract

It is widely held that under ordinal utility, utility differences are ill-defined. Allegedly, for these to be well-defined (without turning to choice under risk or the like), one should adopt as a new kind of primitive quaternary relations, instead of the traditional binary relations underlying ordinal utility functions. Correlatively, it is also widely held that the key structural properties of quaternary relations are entirely arbitrary from an ordinal point of view. These properties would be, in a nutshell, the hallmark of cardinal utility. While much is obviously true in these two tenets, this note explains why, as stated, they should be abandoned. Any ordinal utility function induces a rich quaternary relation. There is such a thing as ordinal utility differences. Furthermore, this induced quaternary relation respects, apart from completeness, the most standard structural properties of quaternary relations. These properties are, from an ordinal point of view, anything but arbitrary; from a quaternary perspective only completeness should be considered the hallmark—if any—of cardinal utility. These facts are explained to be especially relevant to the critical appreciation of the ordinalist methodology.

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I am most indebted to Marcus Pivato for particularly helpful discussions. For valuable comments, criticisms, or suggestions, I am also thankful to an anonymous editor and two anonymous reviewers for *Social Choice and Welfare*, Franz Dietrich, Yorgos Gerasimou, Ivan Moscati, Peter Wakker, and John Weymark. All errors and omissions are mine. Funding from the DFG project 426170771 and the University of Oxford is gratefully acknowledged.

1 Introduction

It is widely held that, under ordinal utility, utility differences are ill-defined. To mention but a few recent examples, Abdellaoui and co-authors write: "...then utility is ordinal ..., which implies that utility differences are not meaningful" (Abdellaoui *et al.*, 2007, p. 359). Reflecting on the history of decision theory, Moscati similarly comments: "it is easy to see that utility differences are not preserved under ordinal utility" (Moscati, 2018, p. 44). As illustrated, this claim is found most easily in economics, where the diffuse methodological doctrine known as ordinalism, of which it is part and parcel, has had and still exerts considerable influence (e.g., Moscati, 2018, Chap. 5). But more generally, the claim also seems to be left unchallenged in abstract measurement theory, to which it ultimately belongs (see esp. Krantz *et al.*, 1971; Narens, 1985). Allegedly, for utility differences to be well-defined (without them being induced in the usual way by decision-making under risk, choice over time, or some other sufficiently rich additively separable structure), one must primitively assume them to be so. This is effectively what one does when to reach the classical difference representation by a cardinal utility function (or some generalization thereof), one starts from quaternary relations instead of the more traditional binary relations underlying ordinal utility functions; see especially Köbberling, 2006 and the references therein.

In this context, it is also widely held—this and the previous claim are in effect two sides of the same coin—that the most distinctive structural properties of quaternary relations are entirely arbitrary from an ordinal point of view. To mention only one example, consider Concatenation (formally stated below), a suitable adaptation to utility measurement of the fact that the measurement of length should be well-behaved with respect to the physical concatenation of the rigid rods, the length of which one would be trying to measure. Upon discovering the crucial role of that property in the measurement of utility differences, Samuelson writes: "There is absolutely no a priori reason why the individual's preference scale ...should obey this arbitrary restriction" (Samuelson, 1938, p. 70). Pivato concurs: "Concatenation ...impl[ies] that our preferences are not merely ordinal" (Pivato, 2015, p. 197).

My main contribution will be to explain why, in their unqualified versions above, these two tenets should be abandoned. Specifically I will show that a binary relation representable by an ordinal utility function induces a rich, if incomplete, quaternary relation. This is for elementary reasons, as can be seen by considering at this stage only the simplest pattern pattern possible: If $v(a) \geq v(b) \geq v(c)$, then it must be that $v(a) - v(c) \geq v(a) - v(b)$ for v and any strictly increasing transformation thereof. (This is notwithstanding the undeniably important fact that $v(a) - v(b) \geq v(b) - v(c)$ might hold together with $w(a) - w(b) < w(b) - w(c)$ for some strictly increasing transformation w of v.) In a nutshell, then, there is such a thing as ordinal utility differences. Moreover, apart from completeness, the quaternary relation defined based on the above and similar considerations respects—starting with Concatenation, for instance—all the structural properties necessary to the classical utility difference representation, or its most standard generalizations. In particular, tapping on representation theorems for incomplete relations, one can show that this induced quaternary relation always admits a multi-utility difference representation (formally defined below). This implies that the key structural properties of quaternary relations are, from an ordinal point of view, anything but arbitrary. This also reveals that in making the canonical progression from an ordinal to a cardinal utility function, completeness is the only additional quaternary assumption one truly needs; starting from the baseline of ordinality, all the other structural assumptions are, in fact, already in place. These elementary clarifications contribute not only to general measurement theory—since the focus on utility is, at the end of the day, merely for the sake of concreteness—but also, as I will emphasize most, to the critical appreciation of ordinalism in economics.

I hasten to add that I take many—starting with the authors initially quoted—to be aware of some, if not all, of the facts thus highlighted. Surprisingly however (qualifications to follow in Sec. 3), to the best of my knowledge, virtually none has articulated in print the qualifications—the conceptual significance of which they may have missed—under which their claims actually hold. All in all, there appears to be on this specific issue a gap in the literature, one which I hope it will prove useful to fill.

The rest of the paper is organized as follows. Sec. 2 presents and interprets the main result. Sec. 3 further discusses this result and the literature. Sec. 4 briefly concludes.

2 Analysis

2.1 Preliminaries

Let X be a set of options, $\succcurlyeq$, a *binary* relation over X, and R, a *quaternary* relation over X. The symmetrical and asymmetrical parts of $\succcurlyeq$ will be denoted by $\sim$ and $\succ$, respectively, while those of R will be denoted by E and P, respectively. As in the rest of the literature, the default interpretation of $\succcurlyeq$ and R will be that $a \succcurlyeq b$ holds when the decision-maker *prefers* a to b while $abRcd$ holds when she prefers a to b *more intensely* than she prefers c to d.[1]

The key properties used in our analysis are the following ones. First, recall that a *weak*—respectively: a *pre*-—order is a transitive and complete—respectively: reflexive—relation. For brevity, a binary relation $\succcurlyeq$ over X will

[1]For further historical, conceptual, or methodological considerations on quaternary relations, see Moscati, 2018; Baccelli and Mongin, 2016.

be called *ordinally representable* if there exists a function $v : X \to \mathbb{R}$ such that for any $a, b \in X$, $a \succcurlyeq b$ if and only if $v(a) \geq v(b)$. As is well known, this is characterized by the weak order property together with a suitable order-denseness condition (e.g., Krantz *et al.*, 1971, p. 40, Thm. 2). Next, consider the following more specifically quaternary properties.

Neutrality. For all $a, b \in X$: $aaEbb$.

Reversal. For all $a, b, c, d \in X$: if $abRcd$, then $dcRba$.

Concatenation. For all $a, b, c, d, e, f \in X$: if $abRcd$ and $beRdf$, then $aeRcf$.

Co-Concatenation. For all $a, b, c, d, e, f \in X$: if $abRdf$ and $beRcd$, then $aeRcf$.[2]

Separability. For all $a, b, c, d \in X$: if $abRcb$, then $adRcd$.

The next two properties refer to the notion of a *standard sequence*, i.e., a (possibly infinite) sequence $a_0, a_1, \dots \in X$ such that $a_i a_{i-1} E a_{i+1} a_i$ for all i. Such a sequence is said to be *increasing* if $a_1 \succ a_0$ and *decreasing* if $a_1 \prec a_0$. It is said to be *bounded* if there are $w, z \in X$ such that for any a_i in the sequence, $w \succcurlyeq a_i \succcurlyeq z$.

Divisibility. For any finite standard sequence $a_0, \dots a_n \in X$, if $a_n a_0 R a_0 a_0$, then $a_1 a_0 R a_0 a_0$.

Archimedeanity. There is no infinite increasing (respectively: decreasing) bounded standard sequence.

The final two properties are standard richness assumptions. Their different strengths notwithstanding, both conditions force the uncountability of X.

Strong Solvability. For all $a, b, d \in X$, there is a $c \in X$ such that $abEcd$.

Weak Solvability. For all $a, b, c, d, e \in X$ such that $cePab$ and $abPde$ (respectively: $ecPab$ and $abPed$), there is an $f \in X$ such that $feEab$ (respectively: $efEab$).

Here are three representation theorems using the aforementioned properties. Each of them will play a role in the next, main section, hence my spelling them out in some detail in the present one. As only their existence results turn out to matter for the present enquiry, I will focus on these and omit the accompanying uniqueness results. Say that a quaternary relation admits a *utility difference representation* if there exists a function $u : X \to \mathbb{R}$ such that for all $a, b, c, d \in X$, $abRcd$ if and only if $u(a) - u(b) \geq u(c) - u(d)$. The following result is due to Köbberling (2006, Thm. 1).[3]

[2] Co-Concatenation comes from Pivato, 2015, where it is called "Concatenation*".

[3] Köbberling's result is in fact stronger and more elegant than stated here in that she uses the indifference generalizations of some of the weak preference axioms listed next. The difference is not significant for our present purposes, however, and ignoring it usefully simplifies our presentation.

Theorem 1 (**Köbberling**). *Assume Weak Solvability holds. Then, R admits a utility difference representation if and only if it is a weak order respecting Neutrality, Concatenation, Separability, and Archimedeanity.*

Next, say that a quaternary relation admits a *preference intensity representation* if there exists a function $u : X^2 \to \mathbb{R}$ such that for all $a, b, c, d \in X$ *(i)* $abRcd$ if and only if $u(ab) \geq u(cd)$; *(ii)* if $u(ab) \geq u(cd)$, then $u(dc) \geq u(ba)$; and *(iii)* if $\min\{u(ab), u(bc)\} \geq u(dd)$, then $u(ac) \geq \max\{u(ab), u(bc)\}$. Such representations can be seen to nest utility difference representations while being nested by the baseline ordinal representations over pairs (a quaternary relation over X being, equivalently, a binary relation over X^2). The following result is due to Gerasimou (2021, Thm. 1).[4]

Theorem 2 (**Gerasimou**). *Assume X finite. Then, R admits a preference intensity representation if and only if it is a weak order respecting Reversal and Separability.*

Finally, consider a different kind of generalization of the benchmark utility difference representation introduced above. Let A denote any linearly ordered Abelian group. While it is naturally not the only one, $\mathbb{R}$ equipped with the usual order and operations will turn out to be, for us, the only relevant example. Call a *weak utility function* any function $u : X \to A$ such that for all $a, b, c, d \in X$, if $abRcd$, then $u(a) - u(b) \geq u(c) - u(d)$.[5] Say that a quaternary relation admits a *multi-utility difference representation* if there exists a set U of weak utility functions such that for all $a, b, c, d \in X$, $abRcd$ holds if and only if $u(a) - u(b) \geq u(c) - u(d)$ holds for all $u \in U$. The following very general result is due to Pivato (2013a, Thm. 2.4).[6]

Theorem 3 (**Pivato**). *R admits a multi-utility difference representation if and only if it can be embedded in a quaternary relation that is a pre-order respecting Reversal, Concatenation, Co-Concatenation, Divisibility, and Strong Solvability.*

2.2 Main Result

Building on the elementary observation made in the introduction, our analysis is based on the following definition. The general underlying intuition is that just like one can investigate what binary relation is induced by a well-behaved quaternary relation (e.g., Köbberling, 2006, Lemma 6), one can ask, and answer, the converse question.[7]

[4] See also Gerasimou, 2022.

[5] Here I adopt Pivato's terminology despite its departing somewhat from established usage elsewhere in the literature (see esp. the part inspired by Richter, 1966; Peleg, 1970).

[6] Further see Pivato, 2015; 2013b. Incidentally, I am not aware of any other work that would, like these papers, be directly focused on the representation of incomplete quaternary relations. However see, in a different decision-theoretic tradition, Jansen *et al.*, 2018.

[7] Let $\succcurlyeq'$ denote the binary relation induced by quaternary relation R. A natural definition for $\succcurlyeq'$ is $a \succcurlyeq' b$ if and only if $abRba$. It is readily verified that under Def. 1, $\succcurlyeq' = \succcurlyeq$.

Definition 1. *Given $\succcurlyeq$ an ordinally representable binary relation, let R denote the quaternary relation defined by $abRcd$ if and only if either of 1.-3. hold:*

1. *$a \succcurlyeq b$ and $d \succcurlyeq c$;*
2. *$a \succcurlyeq c \succcurlyeq d \succcurlyeq b$;*
3. *$d \succcurlyeq b \succcurlyeq a \succcurlyeq c$.*

In a nutshell, clause 1. states that ab is a positive preference difference while cd is a negative one. Clause 2. (respectively: 3.) states that both ab and cd are positive (respectively: negative) preference differences but the former nests (respectively: is nested by) the latter. In all cases, then, preference difference ab is rightfully considered larger than preference difference cd.

This simple insight leads to our main result, stated next. It exploits only the weak order properties of the ordinally representable $\succcurlyeq$ postulated in Def. 1.

Proposition 1. *Given $\succcurlyeq$ ordinally representable, under Def. 1, R is a pre-order respecting Neutrality, Reversal, Concatenation, Co-Concatenation, Separability, Divisibility, and Archimedeanity.*

Proof. See the Appendix. □

From Prop. 1, the corollaries stated next immediately follow. Only the last of these corollaries exploits the full representability assumption made in Def. 1.[8]

Corollary 1.

1. *Given $\succcurlyeq$ ordinally representable, under Def. 1, R respects, apart from Completeness, all the necessary conditions of Thm. 1.*
2. *Given $\succcurlyeq$ ordinally representable, under Def. 1, R respects, apart from Completeness, all the conditions of Thm. 2.*
3. *Given $\succcurlyeq$ ordinally representable, under Def. 1, R respects all the conditions of Thm. 3.*

Proof. See the Appendix. □

[8] Prop. 1 contributes more than Cor. 1.2 to the appreciation of Thm. 2. In particular, it also shows that a quaternary relation induced by an ordinally representable binary relation must respect Co-Concatenation. Yet a quaternary relation may satisfy all the conditions of Thm. 2—and even, in addition, Concatenation—but violate Co-Concatenation. (Although such was not its initial purpose, this much can be established by Example 1 in Gerasimou, 2021.) The lesson may be that preference intensity representations are, in fact, too general to help capture the progression from ordinal to—in a nutshell—cardinal utility.

A few further clarifications on Prop. 1 may be helpful. First, while the result focuses on the quaternary preference, or the ordering of first-order differences, induced by a primitive binary preference, the underlying insight holds more generally in that it applies to higher-order differences as well.[9] To mention only one example, an ordinally representable binary $\succcurlyeq$ also induces a well-behaved *octonary* relation T over differences of differences, or second-order differences. Indeed it is readily checked that for $abcdTefgh$ to hold—i.e., for it to be the case that the difference between preference differences ab and cd is larger than that between that between preference differences ef and gh—, it suffices that $a \succcurlyeq e \succcurlyeq g \succcurlyeq c \succcurlyeq d \succcurlyeq h \succcurlyeq f \succcurlyeq b$ holds. (Compare Def. 1.2.) Similar remarks apply to preference differences of higher-order still.

Second, under Def. 1, R must indeed violate Completeness. Specifically, assuming $\succcurlyeq$ non-trivial in the sense that there are $a, b, c \in X$ such that $a \succ b \succ c$, by definition, neither $abRbc$ nor $bcRab$ can hold; i.e., consecutive strict preference differences must be R-incomparable. More generally, considering the 24 possible linear orders $\succ$ over a set of 4 elements $\{a, b, c, d\}$, it may be checked that under Def. 1, the preference differences ab and cd are R-incomparable in exactly $1/3$ of the logically possible cases, specifically when they are either consecutive (as for example in $a \succ b \succ c \succ d$) or overlapping (as for example in $a \succ c \succ b \succ d$).[10] It is exactly the ordering of such differences that imposing Completeness would force one to determine. In addition, under Def. 1, R will generally violate either solvability condition.[11] Nevertheless, as Cor. 1.3 shows, R can be embedded in a richer quaternary relation respecting Strong Solvability.

Third, while Cor. 1.3 is arguably the most revealing implication of Prop. 1, it is one that should be appreciated with care. It states a sufficient condition—being induced by an ordinally representable $\succcurlyeq$—for multi-utility difference representability; but that condition is certainly not necessary.[12] Besides, the exact representation at stake could, with the benefit of hindsight, be deemed trivial. This is because it may be verified to be, more specifically, the real-valued multi-utility difference representation wherein $abRcd$ holds if and only if $u(a) - u(b) \geq u(c) - u(d)$ holds for all $u \in U$, with U the set of *all* the

[9]On which further see Basu, 1982, Remark 2; Richter and Wong, 2019.

[10]This is checked in the Appendix as Observation 1.

[11]More specifically, under Def. 1, Strong Solvability requires general $\succcurlyeq$-indifference. (This may be seen most directly by taking the special case of Strong Solvability where $d = a$.) On the other hand, the consequent of the implication in Weak Solvability requires certain binary indifferences which a strict order, for instance, could not satisfy.

[12] Indeed an example comes readily to mind from the literature on social welfare functionals (e.g. Bossert and Weymark, 2004). The quaternary relation induced by Cardinal Unit-Comparable utility functions (as in the classical axiomatizations of Utilitarianism) also respects the conditions in Thm. 3. However, famously, this informational basis also makes utility levels interpersonally non-comparable, which implies that this quaternary relation is not inducible by an ordinally representable binary order. (Further see fn. 13.)

ordinal utility functions representing the inducing $\succcurlyeq$. Yet this fact should not mislead one into misappreciating the key point here, which is not how restrictive the representation is, but rather that it exists at all in the first place. Indeed, its sheer existence vividly illustrates that one should rectify the twin false impressions that all utility differences are ordinally ill-defined and that the most distinctive quaternary properties are ordinally arbitrary.

3 Discussion

To the best of my knowledge, only three authors previously, and independently, pointed out that one could fruitfully investigate the quaternary relation induced by a well-behaved binary relation. The closest preexisting reference is Pivato, 2015, Ex. 6.3—an illustration made in the context of a systematic inquiry into incomplete interpersonal comparisons of welfare (including also Pivato, 2013a; 2013b; and the original full working paper 2013c).[13] Pivato's definition, relative to a general weak order $\succcurlyeq$, also has 1. but compacts 2. and 3. as the single condition 4. $a \succcurlyeq c$ and $d \succcurlyeq b$. (Under the constraint of 1., 4. is equivalent to 2. and 3.) Def. 1 is in this regard mathematically less elegant but, for our conceptual purposes, more transparent. Pivato proves, by an indirect argument (based on the consideration of the "Suppes-Sen grading principle"; see Pivato, 2013c, Prop. 9.1), that under Def. 1 R is a pre-order respecting Reversal, Concatenation, and Co-Concatenation. In this respect our proof is merely more direct and self-contained. But it also adds that (among other properties) Divisibility holds as well, which—in light of Thm. 3 and Cor. 1.3—leads to the following significant difference. Indeed Pivato comments in passing that "[the quaternary preference induced by a weak order] does not admit a multiutility [difference] representation" (Pivato, 2015, p. 212). Here, Pivato may have had in mind some complications arising from non-representable weak orders; or (as the rest of his Ex. 6.3 can be taken to indicate) he may have just meant to suggest that even when such a representation exists, it is not exploited by the "net gain" approach his paper is focused on promoting. Either way, Cor. 1.3 indicates how his comment could be qualified. Finally and not unrelatedly, because Pivato's main focus is elsewhere, the larger conceptual elaboration offered here is absent from his analysis. This is epitomized by his remark already quoted on p. 1, which—with hindsight—is not entirely consonant with his own findings.

The two other authors who—unbeknownst to Pivato or one another—have previously investigated our question are Blau (1975) and Mayston

[13] In this regard, continuing with the illustration touched upon in fn. 12, one may note that Prop. 1 shows that under the "Ordinal Full Comparability" informational basis, not just the utility levels, but also some utility differences are interpersonally comparable. While this observation may well have been part of those that motivated Pivato's inquiry, it cannot be found in the papers referred to above, to the best of my attention.

(1974; 1982) in the context of their reflection on Sen's and Arrow's impossibility theorem, respectively.[14] Fundamentally, their underlying intuition is identical to ours. But—here again, because their main focus is elsewhere—the technical and conceptual elaboration remain ours. In particular, transitivity is the only structural property of the induced quaternary relation which they consider. Some more details about each author now follow.

To circumvent Sen's impossibility of a Paretian Liberal, Blau proposes a "non-meddlesomeness" domain restriction excluding that "each of the two persons fee[l] more strongly in opposing the other's private decision than he does about his own private decision" (1975, p. 396). To cash out the prerequisite notion of intensity, Blau (p. 397) proposes the definition according to which $abRcd$ if and only if $a \succcurlyeq c \succ d \succcurlyeq b$ with either $a \succ c$ or $d \succ b$.[15] He does not further elaborate on his observation that some notion of intensity thus proves to be ordinally well-defined. He is particularly keen on insisting that on his approach, "there is no *measurement* of preference intensity" (p. 397; emphasis in original).

Independently, Mayston, 1974 proposed the definition consisting only of clause 2 in Def. 1.[16] This was in the context of an original critical reflection on the assumptions of Arrow's impossibility theorem, focused the straightjacket—in Mayston's view—imposed by the Independence of Irrelevant Alternatives (IIA) condition, according to which the social ranking of options a and b can depend on individual welfare information only at a and b.[17] His strikingly lucid comments are worth quoting extensively: "Ordinality, unlike cardinality and utilitarianism, does not assert that such secondary preference relations ... can be made for all pairs of alternatives in a set S. However, it does imply that some pairs can be ordered in this way, by the individual himself. Given that this information is there, we then have the additional question of whether we, the analysts, need to make use of it in deriving consistent welfare measures" (1974, p. 28). In Mayston's view, this last question should be answered positively, and—as illustrated by the possibility theorem his 1974 book is working towards—the IIA condition generalized

[14] Saari's work on these impossibility theorems (esp. Saari, 1991; 1998) should also be mentioned. But unlike Blau or Mayston, Saari does not explicitly use quaternary relations.

[15] To show that R thus defined is a strict partial order (Blau, 1975, Remark 2), the assumption that $c \succ d$ is unnecessary. More generally, Blau's definition could deliver Concatenation and Co-Concatenation, but (obviously not Reflexivity and also) not Reversal. For further discussions or applications of Blau's notion of ordinal intensity, see in particular Sen, 1976, p. 221-223, Sen and Foster, 1973, Sec. A.7.5, and Sen, 2001, p. 73-74.

[16] Mayston claims (1974, p. 28, fn. 1) that thus defined, R is a pre-order. Transitivity holds unproblematically. But assuming $\succcurlyeq$ non-trivial in the sense that there are $a, b \in X$ such that $a \succ b$, Reflexivity cannot hold. Indeed, for these specific a and b it cannot hold that $baRba$, for by definition this would imply $b \succcurlyeq a$. Mayston's definition could deliver Concatenation and Co-Concatenation but (not Reflexivity, then, and also) not Reversal.

[17] For some of the rare references giving Mayston due credit for his pioneering insights, see Fleurbaey and Mongin, 2005, p. 411-412, and Le Breton and Weymark, 2011, p. 293.

accordingly.[18]

While the preceding authors seem to be the only ones to have explicitly investigated the quaternary relation induced by a well-behaved binary relation, I wish to reiterate and stress that I take many others to be aware of some, if not all, of the relevant underlying facts. To mention only one example, in his investigation of preference intensity representations, Gerasimou discusses the property of *lateral consistency* (Gerasimou, 2021, p. 6), whereby the representing function function $u : X^2 \to \mathbb{R}$ is such that if $u(ab) \geq 0$ and $u(bc) \geq 0$, then $u(ac) \geq \max\{u(ab), u(bc)\}$. Clearly, this encapsulates the crucial elementary observation made on p. 1 (later systematized as clause 2 of Def. 1).[19] In this case like in many others, however, because the author's interests are focused elsewhere, this does not lead to anything like the full Def. 1, Prop. 1, or the accompanying conceptual elaboration.

Given the preceding literature review is, to the best of my knowledge, exhaustive, two facts stand out. First, the most conspicuous absence may be that of abstract representational measurement theorists. This is surprising since many pioneers of this field had a keen interest in the various scales possible (esp. Narens, 1985; Luce *et al.*, 1990). However elementary, insights such as the ones offered here help better appreciate the transition from ordinal to more-than-ordinal—under sufficient richness assumptions: cardinal—scales. Second, the connection made by Mayston with the critical discussion of IIA is particularly illuminating. To start with, it highlights the area—social choice theory—where the economic implications of the facts highlighted here may be most vivid. But the lesson is more general than that. Inasmuch as ordinalism broadly construed runs through consumer theory to social choice theory and beyond (e.g., Baccelli and Mongin, 2016, Sec. 2), it holds not only that no notion of preference intensity is required to recover important economic results, but also that no ordinal notion of intensity is available anyway. In light of the facts highlighted here, for such to be the case, something like the IIA condition must indeed be in place, to forbid—by focusing on options pairs by pairs—that any intensity data be taken into account—which obviously requires enlarging one's attention from pairs to triplets and beyond. The ordinal intensity information is thus not so much absent, as it is ignored. This is a methodological tenet independent from the other ones usually bundled in the ordinalist doctrine, starting with the injunction to rely on choice data exclusively. This further underscores that ordinalism is a multifaceted doctrine, one from which several partial departures can be fruitfully envisaged.

[18]Based on similar considerations, Saari famously motivated the Borda Count. Mayston's proposal is specific to economic environments, in which his analysis is set.

[19]This is notwithstanding the interesting subtleties noted in fn. 8.

4 Conclusion

Any ordinal utility function induces a rich quaternary relation respecting, apart from completeness, all the key structural properties necessary to the classical difference representation by a cardinal utility function (or generalizations thereof). This makes it clear that many utility differences are ordinally well-defined and that the most distinctive quaternary properties are not ordinally arbitrary. Among other further implications, this also helps better appreciate the restrictiveness of the ordinalist ideas diffuse in several branches of economics.

5 Appendix

Proof of Proposition 1

Proof. Reflexivity. By definition, $abRab$ iff either 1) $a \succcurlyeq b$ and $b \succcurlyeq a$ or 2) $a \succcurlyeq a \succcurlyeq b \succcurlyeq b$ or 3) $b \succcurlyeq b \succcurlyeq a \succcurlyeq a$. The completeness of $\succcurlyeq$ thus implies the reflexivity of R.

Transitivity. Assume $abRcd$ and $cdRef$. Thus either 1.1) $a \succcurlyeq b$ and $d \succcurlyeq c$ or 2.1) $a \succcurlyeq c \succcurlyeq d \succcurlyeq b$ or 3.1) $d \succcurlyeq b \succcurlyeq a \succcurlyeq c$, and either 1.2) $c \succcurlyeq d$ and $f \succcurlyeq e$ or 2.2) $c \succcurlyeq e \succcurlyeq f \succcurlyeq d$ or 3.2) $f \succcurlyeq d \succcurlyeq c \succcurlyeq e$. There are nine cases to consider. The case 2.1) + 2.2) immediately gives $a \succcurlyeq e \succcurlyeq f \succcurlyeq b$. The case 3.1) + 3.2) immediately gives $f \succcurlyeq b \succcurlyeq a \succcurlyeq e$. Directly or based on what induced indifferences imply, all the remaining cases yield $a \succcurlyeq b$ and $f \succcurlyeq e$. Thus $abRef$ follows in all cases.

Neutrality. Immediate from the reflexivity of $\succcurlyeq$.

Reversal. By definition, $abRcd$ iff either 1) $a \succcurlyeq b$ and $d \succcurlyeq c$ or 2) $a \succcurlyeq c \succcurlyeq d \succcurlyeq b$ or 3) $d \succcurlyeq b \succcurlyeq a \succcurlyeq c$. By definition, $dcRba$ iff either 1) $d \succcurlyeq c$ and $a \succcurlyeq b$ or 2) $d \succcurlyeq b \succcurlyeq a \succcurlyeq c$ or 3) $a \succcurlyeq c \succcurlyeq d \succcurlyeq b$. Thus, $abRcd$ iff $dcRba$.

Concatenation. Assume $abRcd$ and $beRdf$. Thus either 1.1) $a \succcurlyeq b$ and $d \succcurlyeq c$ or 2.1) $a \succcurlyeq c \succcurlyeq d \succcurlyeq b$ or 3.1) $d \succcurlyeq b \succcurlyeq a \succcurlyeq c$, and either 1.2) $b \succcurlyeq e$ and $f \succcurlyeq d$ or 2.2) $b \succcurlyeq d \succcurlyeq f \succcurlyeq e$ or 3.2) $f \succcurlyeq e \succcurlyeq b \succcurlyeq d$. There are nine cases to consider. The case 1.1) + 1.2) immediately yields $a \succcurlyeq e$ and $f \succcurlyeq c$. In the cases 1.1) + 2.2) and 2.1) + 1.2), either $f \succcurlyeq c$ in which case $a \succcurlyeq e$ and $f \succcurlyeq c$, or $c \succcurlyeq f$ in which case $a \succcurlyeq c \succcurlyeq f \succcurlyeq e$. In the cases 1.1) + 3.2) and 3.1) + 1.2), either $e \succcurlyeq a$ in which case $f \succcurlyeq e \succcurlyeq a \succcurlyeq c$, or $a \succcurlyeq e$ in which case $a \succcurlyeq e$ and $f \succcurlyeq c$. The case 2.1) + 2.2) immediately yields $a \succcurlyeq c \succcurlyeq f \succcurlyeq e$. In the cases 2.1) + 3.2) and 3.1) + 2.2), either $c \succcurlyeq f$ in which case $a \succcurlyeq c \succcurlyeq f \succcurlyeq e$, or $f \succcurlyeq c$; in the latter case, either $a \succcurlyeq f$ in which case $a \succcurlyeq e$ and $f \succcurlyeq c$, or $f \succcurlyeq a$; in the latter case, either $e \succcurlyeq a$ in which case $f \succcurlyeq e \succcurlyeq a \succcurlyeq c$, or $a \succcurlyeq e$ in which case $a \succcurlyeq e$ and $f \succcurlyeq c$. The case 3.1) + 3.2) immediately yields $f \succcurlyeq e \succcurlyeq a \succcurlyeq c$. Thus $aeRcf$ follows in all cases.

Co-Concatenation. Assume $abRdf$ and $beRcd$. Thus either 1.1) $a \succcurlyeq b$ and $f \succcurlyeq d$ or 2.1) $a \succcurlyeq d \succcurlyeq f \succcurlyeq b$ or 3.1) $f \succcurlyeq b \succcurlyeq a \succcurlyeq d$, and either 1.2) $b \succcurlyeq e$ and $d \succcurlyeq c$ or 2.2) $b \succcurlyeq c \succcurlyeq d \succcurlyeq e$ or 3.2) $d \succcurlyeq e \succcurlyeq b \succcurlyeq c$. There are nine cases to consider. The cases 1.1) + 1.2), 2.1) + 2.2), 2.1) + 3.2), 3.1) + 2.2), and 3.1) + 3.2) immediately yield $a \succcurlyeq e$ and $f \succcurlyeq c$. In cases 1.1) + 2.2) and 2.1) + 1.2), either $f \succcurlyeq c$ in which case $a \succcurlyeq e$ and $f \succcurlyeq c$, or $c \succcurlyeq f$ in which case $a \succcurlyeq c \succcurlyeq f \succcurlyeq e$. In cases 1.1) + 3.2) and 3.1) + 1.2), either $e \succcurlyeq a$ in which case $f \succcurlyeq e \succcurlyeq a \succcurlyeq c$, or $a \succcurlyeq e$ in which case $a \succcurlyeq e$ and $f \succcurlyeq c$. Thus $aeRcf$ follows in all cases.

Separability. Assume $abRcb$. Then by definition of R either $a \succcurlyeq b \succcurlyeq c$ or $a \succcurlyeq c \succcurlyeq b$ or $b \succcurlyeq a \succcurlyeq c$. Thus $a \succcurlyeq c$ holds in all cases. Therefore by completeness of $\succcurlyeq$ for any d either $a \succcurlyeq d \succcurlyeq c$ or $a \succcurlyeq c \succcurlyeq d$ or $d \succcurlyeq a \succcurlyeq c$, so that by definition $adRcd$ holds.

Divisibility. Consider a standard sequence $a_0, \dots a_n$. We show that the definition of R implies that for any i featured in the standard sequence, $a_i \sim a_{i-1}$. Recall that for any i, by definition of a standard sequence, $a_i a_{i-1} E a_{i+1} a_i$, hence $a_i a_{i-1} R a_{i+1} a_i$ and $a_{i+1} a_i R a_i a_{i-1}$. By definition of R, $a_i a_{i-1} R a_{i+1} a_i$ implies $a_i \succcurlyeq a_{i-1}$ and $a_i \succcurlyeq a_{i+1}$, or $a_i \succcurlyeq a_{i+1} \succcurlyeq a_i \succcurlyeq a_{i-1}$, or $a_i \succcurlyeq a_{i-1} \succcurlyeq a_i \succcurlyeq a_{i+1}$. Hence, $a_i a_{i-1} R a_{i+1} a_i$ implies $a_i \succcurlyeq a_{i-1}$ in all cases. Similarly, $a_{i+1} a_i R a_i a_{i-1}$ implies $a_{i-1} \succcurlyeq a_i$ in all cases. Thus, $a_i a_{i-1} E a_{i+1} a_i$ implies $a_i \sim a_{i-1}$, as claimed. Given this, Divisibility immediately follows. This is because by transitivity, it then holds that $a_n \sim a_0$ and $a_1 \sim a_0$, hence that $a_n a_0 R a_0 a_0$ and $a_1 a_0 R a_0 a_0$, hence that $a_n a_0 R a_0 a_0$ implies $a_1 a_0 R a_0 a_0$.

Archimedeanity. Trivial for, as shown in the proof of Divisibility, under the definition of R, there can be no increasing or decreasing standard sequence.

□

Proof of Corollary 1

Proof.

1. Immediate from Prop. 1 and the fact that Weak Solvability, which will generally be violated by R under Def. 1, is a richness condition unnecessary to the utility difference representation.
2. Immediate from Prop. 1.
3. Immediate from Prop. 1 and the fact that any utility function representing $\succcurlyeq$ establishes, through its extension to the real line, that R can be embedded in a quaternary relation respecting, in addition to the other properties, Strong Solvability.

□

Proof of Observation 1

Observation 1. *Given $\succcurlyeq$ an ordinally representable linear order, for any distinct $a, b, c, d \in X$, under Def. 1, the pairs ab and cd are R-incomparable in 1/3 of the logically possible cases.*

Proof. Each of the 24 possible underlying linear orders is examined below, with the relevant argument (if $\succcurlyeq$ makes the pairs R-comparable) or the categorization (if $\succcurlyeq$ makes the pairs R-incomparable) indicated in parenthesis.

1. $a \succ b \succ c \succ d$: ab and cd are R-incomparable (consecutive preference differences of the same sign)
2. $a \succ b \succ d \succ c$: $abRcd$ (Def. 1, clause 1)
3. $a \succ c \succ b \succ d$: ab and cd are R-incomparable (overlapping preference differences of the same sign)
4. $a \succ c \succ d \succ b$: $abRcd$ (Def. 1, clause 2)
5. $a \succ d \succ c \succ b$: $abRcd$ (Def. 1, clause 1)
6. $a \succ d \succ b \succ c$: $abRcd$ (Def. 1, clause 1)
7. $b \succ a \succ d \succ c$: ab and cd are R-incomparable (consecutive preference differences of the same sign)
8. $b \succ a \succ c \succ d$: $cdRab$ (Def. 1, clause 1)
9. $b \succ c \succ a \succ d$: $cdRab$ (Def. 1, clause 1)
10. $b \succ c \succ d \succ a$: $cdRab$ (Def. 1, clause 1)
11. $b \succ d \succ a \succ c$: ab and cd are R-incomparable (overlapping preference differences of the same sign)
12. $b \succ d \succ c \succ a$: $cdRab$ (Def. 1, clause 3)
13. $c \succ a \succ b \succ d$: $cdRab$ (Def. 1, clause 2)
14. $c \succ a \succ d \succ b$: ab and cd are R-incomparable (overlapping preference differences of the same sign)
15. $c \succ b \succ a \succ d$: $cdRab$ (Def. 1, clause 1)
16. $c \succ b \succ d \succ a$: $cdRab$ (Def. 1, clause 1)
17. $c \succ d \succ a \succ b$: ab and cd are R-incomparable (consecutive preference differences of the same sign)
18. $c \succ d \succ b \succ a$: $cdRab$ (Def. 1, clause 1)

19. $d \succ a \succ c \succ b$: $abRcd$ (Def. 1, clause 1)

20. $d \succ a \succ b \succ c$: $abRcd$ (Def. 1, clause 1)

21. $d \succ b \succ a \succ c$: $abRcd$ (Def. 1, clause 3)

22. $d \succ b \succ c \succ a$: ab and cd are R-incomparable (overlapping preference differences of the same sign)

23. $d \succ c \succ b \succ a$: ab and cd are R-incomparable (consecutive preference differences of the same sign)

24. $d \succ c \succ a \succ b$: $abRcd$ (Def. 1, clause 1)

□

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